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NOTES ON THE COBORDISM GROUP $U^*(L^n(m))$

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1. Let $U^*(X)$ be the unitary cobordism group of a finite CW complex X. P.S. Landweber [4] and K. Shibata [6] determined the unitary cobordism group of the lens space $L^n(m)=S^{2n+1}/Z_m$. In this paper, we use the structure of the reduced unitary cobordism group of $L^n(m)$ to prove the following

Theorem 1. *If positive integers p and q are relatively prime, there exists an isomorphism*

$$\psi\colon \tilde{U}^{ev}(L^n(p))\oplus\tilde{U}^{ev}(L^n(q)) \rightarrow \tilde{U}^{ev}(L^n(pq)),$$

where $\tilde{U}^{ev}(\cdot)=\sum_i \tilde{U}^{2i}(\cdot)$.

Let $U_*(X)$ be the unitary bordism group of a space X. Denote by BZ_m the classifying space of the group Z_m. Using the duality isomorphism D: $U_*(L^n(m))\cong U^*(L^n(m))$ and the isomorphism $U_k(L^n(m))\cong U_k(BZ_m)$ for $k<2n+1$ [3], we have $U_k(BZ_m)\cong\tilde{U}^{2n+1-k}(L^n(m))$ for $k<2n+1$. Then, Theorem 1 implies the following

Theorem 2. *If p and q are relatively prime, there exists an isomorphism*

$$\psi_*\colon U_{od}(BZ_p)\oplus U_{od}(BZ_q) \rightarrow U_{od}(BZ_{pq}),$$

where $U_{od}(\cdot)=\sum_i U_{2i+1}(\cdot)$.

Using the spectral sequence [3], we obtain

$$U_{2k}(BZ_m)\cong U_{2k}.$$

For a prime p, $U_*(BZ_p)$ was determined in [1] and [3].

Denote by $\tilde{K}(X)$ the reduced Grothendieck group of isomorphism classes of complex vector bundles over X. In [2], Conner and Floyd gave the isomorphism

$$\tilde{K}(X)\cong\tilde{U}^{ev}(X)\otimes_{U^*}Z.$$

Therefore, Theorem 1 implies the following

Theorem 3. (N. Mahammed [5]) *If p and q are relatively prime, there exists an isomorphism*

$$\tilde{K}(L^n(p))\oplus\tilde{K}(L^n(q))\cong\tilde{K}(L^n(pq)).$$

2. In this section we prove Theorem 1. Denote by CP^n the n-dimensional complex projective space and by η the canonical complex line bundle over CP^n. Let $\pi: L^n(P)\to CP^n$ be the natural projection and put

$$x_p=\pi^*c_1(\eta),$$

where $c_1(\eta)$ is the first Chern class of η in the sence of Conner and Floyd [2].

Let $F(,)$ is the formal group law such that

$$F(c_1(\xi),\ c_1(\xi'))=c_1(\xi\otimes\xi')$$

for complex line bundles ξ, ξ' over the same CW complex [7]. For a positive integer m, let $[m]_F(x)\in U^*[[x]]$ be a formal power series defined by the following formulas

$$\begin{aligned}&[1]_F(x)=x\\&[k]_F(x)=F(x,\ [k-1]_F(x)).\end{aligned} \qquad \cdots\cdots\cdots\cdots\cdots(1)$$

In [6], K. Shibata gave the following

Theorem 2.1.

$$U^*(L^n(m))\cong\Lambda_{U^*}(D[pt,\ i])\oplus U^*[[x_m]]/(x_m^{n+1},\ [m]_F(x_m)),$$

where $[pt, i]\in U_0(L^n(m))$ is the bordism class represented by an inclusion map of a point, $\Lambda_{U^*}(\ )$ is the exterior algebra over U^* and $(x_m^{n+1}, [m]_F(x_m))$ denotes the ideal generated by x_m^{n+1} and $[m]_F(x_m)$.

The same result can be obtained also by the method of P.S. Landweber [4] directly.

Considering the following short exact sequence

$$0\to\tilde{U}^*(L^n(m))\to U^*(L^n(m))\to U^*\to 0,$$

it follows from Theorem 2.1 that

$$\tilde{U}^{ev}(L^n(m))\cong\bar{U}^*[[x_m]]/(x_m^{n+1},\ [m]_F(x_m)), \qquad \cdots\cdots\cdots\cdots\cdots(2)$$

where $\bar{U}^*[[x_m]]$ is the kernel of the homomorphism

$$\varepsilon:\ U^*[[x_m]]\to U^*$$

defined by $\varepsilon(\sum_{k=0}^{\infty}a_kx_m^k)=a_0$.

We define a homomorphism

$$\psi\colon \tilde{U}^{ev}(L^n(p))\oplus\tilde{U}^{ev}(L^n(q)) \to \tilde{U}^{ev}(L^n(pq))$$

by $\psi(\overline{P(x_p)}, \overline{Q(x_q)})=\overline{P([q]_F(x_{pq}))+Q([p]_F(x_{pq}))}$, where $\overline{P(x_p)}$, $\overline{Q(x_q)}$ and $\overline{P([q]_F(x_{pq}))+Q([p]_F(x_{pq}))}$ are the classes of the formal power series $P(x_p)\in U^*[[x_p]]$, $Q(x_q)\in U^*[[x_q]]$ and $P([q]_F(x_{pq}))+Q([p]_F(x_{pq}))\in U^*[[x_{pq}]]$ respectively.

Using the associativity of the formal group law, we obtain

$$\begin{aligned}[p]_F([q]_F(x)) &= [q]_F([p]_F(x)) \\ &= [pq]_F(x)\,. \end{aligned} \qquad \cdots\cdots\cdots\cdots\cdots(3)$$

From (2) and (3), it follows that the homomorphism ψ is well defined.

We define the multiplication in $\tilde{U}^{ev}(L^n(p))\oplus\tilde{U}^{ev}(L^n(q))$ by

$$(x, y)\cdot(x', y') = (xx', yy')\,.$$

We prove the following lemma, so that the homomorphism ψ is a ring homomorphism.

Lemma 2.2. *If p and q are relatively prime,* $\overline{[p]_F(x_{pq})}\cdot\overline{[q]_F(x_{pq})}=0$ in $\tilde{U}^{ev}(L^n(pq))$.

Proof. We put

$$I_{p,q} = (x_{pq}^{n+1}, [pq]_F(x_{pq}))\,.$$

We show that $[p]_F(x_{pq})\cdot[q]_F(x_{pq})\in I_{p,q}$. From (3),

$$p[q]_F(x)+\sum_{i=2}^{\infty} a_i\{[q]_F(x)\}^i = [pq]_F(x)\,,$$

$$q[p]_F(x)+\sum_{i=2}^{\infty} b_i\{[p]_F(x)\}^i = [pq]_F(x)\,,$$

where $x=x_{pq}$.

Since p and q are relatively prime, there exist integers a and b such that $ap+bq=1$. Then, we have

$$\begin{aligned}&[p]_F(x)\cdot[q]_F(x)\\ &=a[p]_F(x)\{[pq]_F(x)-\sum_{i=2}^{\infty} a_i\{[q]_F(x)\}^i\}\\ &+b[q]_F(x)\{[pq]_F(x)-\sum_{i=2}^{\infty} b_i\{[p]_F(x)\}^i\}\,. \end{aligned} \qquad \cdots\cdots\cdots\cdots\cdots(4)$$

We put

$$X = [p]_F(x)\,,\quad Y = [q]_F(x)\,,\quad a'_i = aa_i \quad\text{and}\quad b' = bb_i\,.$$

The equation (4) implies

$$XY\{1+(\sum_{i=2}^{\infty} a_i' Y^{i-1}+\sum_{i=2}^{\infty} b_i' X^{i-1})\} = I \in I_{p,q}\,.$$

Therefore,

$$XY = I(1+A+A^2+\cdots) \in I_{p,q}\,,$$

where $A=-(\sum_{i=2}^{\infty} a_i' Y^{i-1}+\sum_{i=2}^{\infty} b_i' X^{i-1})$. q.e.d.

Proposition 2.3. *If p and q are relatively prime, then ψ is epimorphic.*

Proof. Since ψ is the ring homomorphism, we need only to prove the existence of the elements y and z which satisfy $\psi(y, z)=\bar{x}_{pq}$. We put

$$[p]_F(x_{pq}) = \sum_{i=0}^{\infty} c_i x_{pq}^{i+1},\ c_0 = p$$

and

$$[q]_F(x_{pq}) = \sum_{i=0}^{\infty} d_i x_{pq}^{i+1},\ d_0 = q\,.$$

We find series $A=\sum_{i=0}^{\infty} a_i x_{pq}^i$ and $B=\sum_{i=0}^{\infty} b_i x_{pq}^i$ which satisfy

$$x_{pq} = A[p]_F(x_{pq})+B[q]_F(x_{pq})\,,$$

that is, a_i and b_i satisfy the following

$$\begin{aligned}
&pa_0+qb_0 = 1\,,\ (c_0=p \text{ and } d_0=q),\\
&a_1c_0+a_0c_1+b_1d_0+b_0d_1 = 0\,,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\sum_{i=0}^{k} a_{k-i}c_i+\sum_{i=0}^{k} b_{k-i}d_i\\
&= a_kc_0+b_kd_0+\sum_{i=1}^{\infty}(a_{k-i}c_i+b_{k-i}d_i)\\
&= 0\,,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\,.
\end{aligned}$$

Since p and q are relatively prime, there exist a_0 and b_0 which satisfy $1=pa_0+qb_0$. Suppose that a_j and b_j are determined for $j<k$. Put

$$a_k = -a_0\sum_{i=1}^{k}(a_{k-i}c_i+b_{k-i}d_i)$$

and

$$b_k = -b_0\sum_{i=1}^{k}(a_{k-i}c_i+b_{k-i}d_i)\,,$$

then a_k and b_k satisfy the above relation. Therefore.

$$x_{pq} = \sum_{k=0}^{\infty} P_{k,1}x_{pq}^k\,.$$

where

$$P_{k,1}=a_k[p]_F(x_{pq})+b_k[q]_F(x_{pq})\,.$$

Suppose that

$$x_{pq}=\sum_{k=0}^{\infty}P_{k,m}x_{pq}^k\,,$$

where $P_{k,m}$ is a polynomial of $[p]_F(x_{pq})$ and $[q]_F(x_{pq})$ with the coefficients in U^*, and for $k\geqq 1$

$$P_{k,m}=x_{pq}^m Q_{k,m},\quad Q_{k,m}\in U^*[[x_{pq}]]\,.$$

Then, we have

$$\begin{aligned}x_{pq}&=P_{0,m}+\sum_{k=1}^{\infty}P_{k,m}\{\sum_{j=0}^{\infty}P_{j,m}x_{pq}^j\}^k\\&=P_{0,m}+\sum_{k=1}^{\infty}P_{k,m}\{P_{0,m}+\sum_{j=1}^{\infty}P_{j,m}x_{pq}^j\}^k\,.\end{aligned}$$

Put

$$P_{0,m}+\sum_{k=1}^{\infty}P_{k,m}\{P_{0,m}+\sum_{j=1}^{\infty}P_{j,m}x_{pq}^j\}^k=\sum_{k=0}^{\infty}P_{k,m+1}x_{pq}^k\,.$$

Then, we have

$$P_{0,m+1}=P_{0,m}+\sum_{k=1}^{\infty}P_{k,m}(P_{0,m})^k$$

and since $P_{j,m}=x_{pq}^m Q_{j,m}$ for $j\geqq 1$, there exists $Q_{j,m+1}\in U^*[[x_{pq}]]$ such that

$$P_{j,m+1}=x_{pq}^{m+1}Q_{j,m+1}\,,\qquad j\geqq 1.$$

By induction, we have

$$x_{pq}=P_{0,n}+\sum_{k=1}^{\infty}P_{k,n}x_{pq}^k\,,$$

and for $k\geqq 1$

$$P_{k,n}=x_{pq}^n Q_{k,n}\,,\quad Q_{k,n}\in U^*[[x_{pq}]]\,.$$

Therefore,

$$x_{pq}-P_{0,n}\in I_{p,q}=(x_{pq}^{n+1},\ [pq]_F(x_{pq}))\,.$$

Put

$$P_{0,n}=P([p]_F(x_{pq}))+Q([q]_F(x_{pq}))+[p]_F(x_{pq})\cdot[q]_F(x_{pq})\cdot R\,,$$

where $R\in U^*[[x_{pq}]]$.
From Lemma 2.2,

$$x_{pq}-P([p]_F(x_{pq}))-Q([q]_F(x_{pq}))\in I_{p,q}\,.$$

Therefore, we obtain

$$\bar{x}_{pq} = \psi(\overline{Q(x_p)}, \overline{P(x_q)}),$$ q.e.d.

Proposition 2.4. *The order of the group* $\tilde{U}^{2s}(L^n(m))$ *is* m^t, $t=\sum_{i=-s+1}^{n-s}\tau_i$, *where* τ_i *is the number of partitions of i for* $i\geqq 0$ *and* $\tau_i=0$ *for* $i<0$.

Proof. Consider the spectral sequence $E_r^{p,q}$ associated with $\tilde{U}^{2s}(L^n(m))$. There is a filtration

$$\tilde{U}^{2s}(L^n(m)) = J^{0,2s} \supset J^{1,2s-1} \supset \cdots \supset J^{2n+1,2s-2n-1} = 0$$

with $J^{p,q}/J^{p+1,q-1} = \tilde{H}^p(L^n(m); U^q)$. Then, for $1 \leqq s+i \leqq n$,

$$\text{the order of } J^{2s+2i,-2i}/J^{2s+2(i+1),-2(i+1)} = \begin{cases} m^{\tau_i} & \text{if } i \geqq 0, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore, the order of $\tilde{U}^{2s}(L^n(m))$ is m^t, $t=\sum_{i=-s+1}^{n-s}\tau_i$. q.e.d.

From the Proposition 2.4, we have the following

Corollary 2.5. *The order of* $\tilde{U}^{2s}(L^n(p)) \oplus \tilde{U}^{2s}(L^n(q))$ *is equal to that of* $\tilde{U}^{2s}(L^n(pq))$.

Proposition 2.3 and Corollary 2.5 prove Theorem 1.

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