Harada, M.
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ON CATEGORIES OF INDECOMPOSABLE MODULES II

MANABU HARADA

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We have studied the Krull-Remak-Schmidt-Azumaya's theorem from point of view of categories in [4], §3. In this note, we shall study further properties of those categories.

We have defined an additive category $\mathfrak{A}$ induced from a family of completely indecomposable modules $\{M_\alpha\}$, namely whose objects consist of directsums of M_α and studied the quotient category $\mathfrak{A}/\mathfrak{J}$ with respect to the ideal $\mathfrak{J}$ in $\mathfrak{A}$ in [4].

In the first section, we characterize submodules M_0 in an object M in $\mathfrak{A}$, which is in $\mathfrak{A}$ and $M_0 \equiv M$ (mod $\mathfrak{J}$), and show that every such M_0 coincides with M if and only if $\mathfrak{J}$ is the Jacobson radical of $\mathfrak{A}$, (see [7] for the radical).

In the second section, we consider Conditions II and III defined in [4], which are related with exchange property defined in [1]. We change slightly the definition of exchange property in this note and show that every direct summand of objects M in $\mathfrak{A}$ has the exchange property in M if and only if $\mathfrak{J}$ is the Jacobson radical of $\mathfrak{A}$.

In the final section, we restrict ourselves to a case where M_α's are projective. We shall show, in this case, that objects in $\mathfrak{A}$ are closely related to semi-perfect modules defined in [9]. Especially we show that an object $P=\sum_I \oplus P_\alpha$ in $\mathfrak{A}$ is perfect if and only if $\{P_\alpha\}$ is an elementwise T-nilpotent system defined in [4] and P is semi-perfect if and only if $\{P_\alpha\}$ is an elementwise semi-T-nilpotent system.

Let R be a ring with unit element and all modules in this note be unitary right R-modules. An R-module M is called *completely indecomposable* if $\mathrm{End}_R(M) = S_M$ is a (non-commutative) local ring. We assume here that indecomposable modules mean completely indecomposable.

1. Dense submodules

Let M_0 be an R-module and assume $M_0=\sum_I \oplus M_\alpha$, where M_α's are indecomposable modules. We have defined an additive category $\mathfrak{A}$ in [4] from the above decomposition as follows: The objects of $\mathfrak{A}$ consist of some directsums of M_α's and the morphisms of $\mathfrak{A}$ consist of all R-homomorphisms. We denote those morphisms by $[M, N']_R$ and we call $\mathfrak{A}$ is induced from a family $\{M_\alpha\}$. Furthermore, we have defined an ideal $\mathfrak{J}$ in $\mathfrak{A}$ in [4] as follows: Let $N=\sum \oplus M'_B$, $N'=\sum \oplus M'_{\beta'}$ be in $\mathfrak{A}$. $\mathfrak{J}$ consists of all morphisms f in $[N, N']_R$ such that $p_{\beta'}fi_\beta \in [M_\beta, M'_{\beta'}]_R$ is not isomorphic for any β and β', where i_β, $p_{\beta'}$ are injection and projection, respectively and N, N' run through all objects in $\mathfrak{A}$. We know from Theorem 7 in [4], that $\mathfrak{J}$ defines an ideal in $\mathfrak{A}$ and $\mathfrak{A}/\mathfrak{J}$ is a C_3-completely reducible abelian category, (see [2] for the definition of ideals).

When we consider object N and morphism f in $\mathfrak{A}/\mathfrak{J}$, we denote them by $\bar{N}$ and $\bar{f}$. Furthermore, if N is in $\mathfrak{A}$ and $N=N_1\oplus N_2$ as R-modules, then $\bar{N}_i$ means $\mathrm{Im}\,\bar{e}_i$, where e_i is a projection of N to N_i. Let S be a ring. By $J(S)$ we denote the Jacobson radical of S and by $\mathfrak{J}_M$ we denote $\mathfrak{J}\cap \mathrm{End}_R(M)$ for an R-module M.

Let $M\supseteq N$ be objects in $\mathfrak{A}$ and i the inclusion of N to M. If i is isomorphic modulo $\mathfrak{J}$, i.e. $\bar{M}=\bar{N}$, then we call N a *dense submodule* in M.

Proposition 1. *Every dense submodule of M is R-isomorphic to M.*

Proof. Since $\bar{M}=\bar{N}$, M and N have isomorphic direct summands by [4], Corollary 1 to Theorem 7.

Let $\{M_\alpha\}$ be a family of completely indecomposable modules and $\{f_{\alpha i}\}_{i=1}^\infty$ any sequence of non isomorphic R-homomorphisms of $M_{\alpha i}$ to $M_{\alpha_{i+1}}$ in $\{M_\alpha\}$ ($M_{\alpha_{i+1}}$ may be equal to $M_{\alpha i}$). If there exists n, which depends on the above sequence and for any element m in M_{α_1}, such that $f_{\alpha n}f_{\alpha n-1}\cdots f_{\alpha_1}(m)=0$, then we call $\{M_\alpha\}$ a (*elementwise*) *T-nilpotent system* (cf. [4]). If the above condition is satistied for any sequence $\{f_{\alpha i}\}$ such that $M_{\alpha i}\neq M_{\alpha_j}$ if $i\neq i$, then we call $\{M_\alpha\}$ a (*elementwise*) *semi-T-nilpotent system.* In general, a semi-T-nilpotent system is not T-nilpotent.

Let $M=\sum_I \oplus M_\alpha$ and J a subset of I, then we denote a submodule $\sum_J \oplus M_{\alpha'}$ by M_J.

Proposition 2. *Let M and P be in $\mathfrak{A}$ and $\bar{M}\supseteq\bar{P}$. Then there exists a submodule P_0 in M satisfying the following conditions.*

1 *P_0 is an object in $\mathfrak{A}$; $P_0=\sum_I \oplus M_\alpha$.*

2 *P_{0J} is a direct summand of M for any finite subset J of I, (if $\{M_{\alpha'}\}_{J'}$ is T-nilpotent system, J' need not be finite).*

3 *$\bar{P}_0=\bar{P}$.*

Furthermore, if $\bar{P}=\mathrm{Im}\,\bar{e}$ and e is an idempotent in $S_M=\mathrm{End}_R(M)$, then we can choose P_0 in Im *e*.

Proof. Let $P=\sum_I \oplus M_\lambda$. Since $\bar{M}\supseteq\bar{P}$ and $\mathfrak{A}/\mathfrak{J}$ is completely reducible, there exist $i\in[P, M]_R$ and $p\in[M, P]_R$ such that $pi\equiv 1_P \pmod{\mathfrak{J}}$. Let J be a subset of I and i_J, p_J be inclusion and projection, respectively. Since p_Jpii_J is isomorphic, $p_J p i i_J$ is R-isomorphic by [4], Lemma 8 and Theorem 8 if J is finite or $\{M_\lambda\}_J$ is a T-nilpotent system. Hence, $\alpha_J = i i_J$ splits, namely $\operatorname{Im} \alpha_J$ is a direct summand of M. Therefore, i is R-monomorphic. We put $P_0 = \operatorname{Im} i$ as R-module, then P_0 satisfies $1 \sim 3$. If $P = \operatorname{Im} e$, we have a relation $pei \equiv pi \equiv 1_P$ (mod $\mathfrak{J}$). Hence, if we take $\alpha_J = e i i_J$, we know that $\operatorname{Im} ei = P_0 \subseteq eM$.

The following theorem gives a special answer for Condition III in [4].

Theorem 1. *Let M be in $\mathfrak{A}$ and $M = \sum_K \oplus N_\gamma$ as R-module. Then each N_γ contains a submodule P_γ such that P_γ is in $\mathfrak{A}$ and $\sum \oplus P_\gamma$ is a dense submodule of M.*

Proof. Let π_γ be a projection of M to N_γ. Then from Proposition 2 we have P_γ in $\mathfrak{A}$ such that $\bar{P}_\gamma = \operatorname{Im} \bar{\pi}_\gamma = \bar{N}_\gamma$. We shall show $\bar{M} = \sum_K \oplus \bar{P}_\gamma$. Let i_γ, i_γ' and i_γ'' be inculsions of P_γ to M, of P_γ to N_γ and of N_γ to M such that $i_\gamma = i_\gamma'' i_\gamma'$, respectively. Since $\bar{P}_\gamma = \operatorname{Im} \bar{\pi}_\gamma$, there exists an R-homomorphism p_γ of M to P_γ such that $i_\gamma p_\gamma = \pi_\gamma$ (mod $\mathfrak{J}$). Let $\{\bar{f}_\gamma\}$ be an element in $\prod_\gamma [P_\gamma, N]_{\mathfrak{A}/\mathfrak{J}}$, where N is an object in $\mathfrak{A}$ and $f_\gamma \in [P_\gamma, N]_R$. We put $f_\gamma'' = f_\gamma p_\gamma i_\gamma'' \in [N_\gamma, N]_R$ and $f = \prod_K f_\gamma' \in [M, N]_R$. Then we have $f i_\gamma = f i_\gamma'' i_\gamma' = f_\gamma'' i_\gamma' = f_\gamma p_\gamma i_\gamma$. Hence, $f i_\gamma \equiv f_\gamma$ (mod $\mathfrak{J}$). We shall show that $\bar{f}$ does not depend on a choice of representative f_γ. It is sufficient to show that if $f | P_\gamma = f i_\gamma$ is in $\mathfrak{J}$ for all γ, then f is in $\mathfrak{J}$ for any f in $[M, N]_R$. Let $N = \sum_L \oplus M_\delta$; M_δ's are indecomposable. If f is not in $\mathfrak{J}$, there exists an idecomposable direct summand T of M such that $p_\delta' f i_T$ is isomorphic, where i_T: $T \to M$, p_δ': $N \to N_\delta$ are inclusion and projection, respectively. Since $\{\pi_\gamma\}_K$ is summable, $1_M = \sum_K \pi_\gamma$ and $f = \sum_K f \pi_\gamma$. Furthermore, since $\{p_\gamma' f \pi_\gamma i_T\}_K$ is summable and $p_\delta' f i_T = \sum p_\delta' f \pi_\gamma i_T$, there exists a finite subset K' in K such that $\sum_{K-K'} p_\delta' f \pi_\gamma i_T$ is not isomorphic, and $\sum_{K'} p_\delta' f \pi_\gamma i_T$ is isomorphic. Therefore, there exists γ in K' such that $p_\delta' f \pi_\gamma i_T$ is isomorphic. On the other hand $p_\delta' f \pi_\gamma i_T \equiv p_\delta' f i_\gamma \pi_\gamma i_T \equiv 0$ (mod $\mathfrak{J}$), which is a contradiction. Conversely, we take a morphism $\bar{f} \in [M, N]_{\mathfrak{A}/\mathfrak{J}}$ and $f \in [M, N]_R$. Put $f_\gamma = f i_\gamma$, then $\bar{f}_\gamma$ does not depend on a choice of f by Proposition 2 and [4], Lemma 5. Thus, we have shown that $[M, N]_{\mathfrak{A}/\mathfrak{J}} = \prod [P_\gamma, N]_{\mathfrak{A}/\mathfrak{J}}$.

We call such P_γ a *dense submodule of N_γ*.

Theorem 2. *Let M be in $\mathfrak{A}$ induced from a family $\{M_\alpha\}$ of completely indecomposable modules M_α, and $N = \sum_I \oplus M_\rho'$ in $\mathfrak{A}$ be a submodule of M. Then the following statements are equivalent.*

1 *N is a dense submodule of M.*

2 *There exists a finite subset J of I, for any direct summand P of M, such that either $P \cap N_J \neq 0$ or $P \oplus N_J$ is not a direct summand of M.*

3 *N contains Im* $(1-f)$ *for some element f in* $\mathfrak{J}_M$ *and* $\bar{i}_N$ *is monomorphic.*

In those cases $N_{J'}$ *is a direct summand of M for any finite subset J' of I. Furthermore, Im* $(1-f)$ *is always a dense submodule of M.*

Proof. $1\to 2$ Since every direct summand of M contains an indecomposable module by [4], Corollary 1 to Theorem 7, we may assume so is P. Since $\bar{P}$ is a minimal object, $\bar{P}$ is small (cf. [5], Theorem 1.4). Hence, $\bar{P}\subseteq \sum_J \oplus \bar{M}_{\alpha_i}$ for some finite set J. We assume that $P\cap N_J=0$ and $P\oplus N_J$ is a direct summand of M. Let e, f and E be projections of M to P, N_J and $P\oplus N_J$, respectively. We may assume $E=e+f$ and $ef=fe=0$. We denote the inclusion of submodules to M by i. Since $\bar{P}\subseteq\bar{N}_J$, there exists α in $[P, N_J]_R$ such that $i_J\alpha\equiv i_P$ (mod $\mathfrak{J}$). Since $\bar{1}_P=\bar{e}\bar{i}_P$ and $\bar{f}\bar{e}=0$, $\bar{1}_P=\bar{e}\bar{i}_J\bar{\alpha}=\bar{0}$. Hence, $P=0$, which is a contradiction.

$2\to 1$ If $\bar{M}\neq\bar{N}$, there exists an indecomposable module P such that $\bar{P}\oplus\bar{N}_J$ is a direct summand of $\bar{M}$ for any finite subset J of I. Since $\bar{P}\oplus\bar{N}_J$ is a directsum of finite many of minimal objects, there exists a direct summand P_0 of M such that $P_0\cap N_J=0$ and $P_0\oplus N_J$ is a direct summand of M (see the proof of Proposition 2).

$1\to 3$ Let i be an inclusion of N to M. Since $\bar{i}$ is isomorphic, there exists j in $[M, N]_R$ such that $\bar{i}\bar{j}=\bar{1}_M$. Put $f=1-ij$, then $f\in\mathfrak{J}_M$ and $1-f=ij$. Since i is monomorphic, $N\supseteq \text{Im}\,(1-f)$.

$3\to 1$ First we shall show that $N'=\text{Im}\,(1-f)$ is a dense submodule of M. We know from the proof of Proposition 10 in [4] that $1-f$ is monomorphic and hence, N' is in $\mathfrak{A}$. Let i be an inclusion of N' to M and $1-f=i\,(1-f)'$; $(1-f)'\in[M, N']_R$. Since $1\equiv 1-f\equiv i(1-f)'$ and $(1-f)'$ is isomorphic, so is $\bar{i}$. Therefore, N' is a dense submodule. Hence, $M\supseteq N\supseteq \text{Im}\,(1-f)$ implies that $\bar{i}_N$ is epimorphic. In order to get the last part we put $\bar{M}=\bar{N}=\bar{P}$ in Proposition 2, then $N_{J'}$, is a direct summand of M.

Corollary 1. *Let M be an object in* $\mathfrak{A}$ *and P a dense submodule of M. If for a direct summand* $N=\sum_J\oplus M_\alpha'$ *of M in* $\mathfrak{A}$, *J is finite or* $\{M_\alpha'\}$ *is a T-nilpotent system, there exists an automorphism* σ *of M such that* $\sigma(N)$ *is a direct summand of P.*

Proof. P contains a submodule N_1' which is isomorphic to N_1 and is a direct summand of M by Proposition 2; say $M=N_1'\oplus N_2'=N_1\oplus N_2$. Since $N_2'\approx N_2$, we obtain the corollary.

Corollary 2. *Let* $\{M_\alpha\}$ *be a family of completely indecomposable modules and* $\mathfrak{A}$ *the induced additive category from* $\{M_\alpha\}$. *Then the following conditions are equivalent.*

1 $\{M_\alpha\}$ *is an elementwise T-nilpotent system.*

2 $\mathfrak{J}$ *is the Jacobson radical of* $\mathfrak{A}$.

3 *Every dense submodule of any M in* $\mathfrak{A}$ *coincides with M.*

Proof. $1 \leftrightarrow 2$ is obtained in [4], Theorem 8.

$1 \rightarrow 3$ Let N be a dense submodule of M. We know from 1 and Proposition 2 that N is a direct summand of M. Hence, $N=M$ by Theorem 2.

$3 \rightarrow 2$ Let f be in $\mathfrak{J}_M$ and $N=\mathrm{Im}(1-f)$. Since N is a dense submodule by Theorem 2, $N=M$. Therefore, $1-f$ is isomorphic, which implies $\mathfrak{J}_M$ is equal to $J(S_M)$.

REMARK. Let $M=\sum_{i=1}^{\infty} \oplus M_i$ as in Theorem 2. We assume that there exists a sequence $\{f_i\}_{i=1}^{\infty}$ of monomorphisms but not epimorphisms f_i of M_i to M_{i+1}. Then for any finite set J of I there exists a dense submodule N in M such that $N \cap M_J=(0)$. Because, we make use of matrix representation of $[M, M]_R$ and by $\{e_{im}\}$ we denote a system of matrix unites. Put $f=\sum_i \sum_{k=1}^{J} f_{i+k-1} f_{i+k-2} \cdots f_i e_{i+k\,i}$, then f is in $\mathfrak{J}$. Hence, $P=\mathrm{Im}(1-f)$ is a dense submodule and $P \cap M_J=(0)$.

If we use the same argument for any set I, we can give an example in which for some subset J with $|J| \leqslant |I|$ there exists a dense submodule P in M such that $P \cap M_J=(0)$. Furthermore, we can give an example in which there exists a dense submodule P in $M=\sum_{i=1}^{\infty} \oplus M_i$ such that $P \cap M_i \neq (0)$ for all i and $P \neq M$.

In the above corollary, we have a situation $\mathfrak{J}_M=J(S_M)$. In this case we obtain a further result.

Lemma 1. *Let M be in $\mathfrak{A}$ and $\mathfrak{J}_M=J(S_M)$. Then for every direct summand N of M we have $\mathfrak{J}_N=J(S_N)$.*

Proof. Since $\mathfrak{J}_M=J(S_M)$, N is in $\mathfrak{A}$ by [4], Corollary 2 to Theorem 7. Put $N=eM$ for some idempotent e in S_M. Then it is clear that $e\mathfrak{J}_M e=\mathfrak{J}_N$, since $\mathfrak{J}$ does not depend on decompositions of M by [4], Lemma 5. Furthermore, $J(S_N)=eJ(S_M)e$. Hence, $J(S_N)=\mathfrak{J}_M$.

Theorem 3. *Let P be in $\mathfrak{A}$ and $\mathfrak{J}_P=J(S_P)$. Then every idempotent a in $S/J(S)$ is lifted to S.*

Proof. Let a be idempotent modulo $\mathfrak{J}_P$. Then there exist a module P_0 in $\mathfrak{A}$ and $a' \in [P, P_0]_R$, $b' \in [P_0, P]_R$ such that $b'a' \equiv a$ and $a'b' \equiv 1_P \pmod{\mathfrak{J}}$. Since P_0 is isomorphic to a direct summand of P by [4], Theorem 7, $\mathfrak{J}_P=J(S_P)$ by Lemma 1. Hence b' is R-monomorphic and $\varepsilon'=a'b'$ is R-isomorphic on P_0. We may regard P_0 as a direct summand of P via b'; $P=\mathrm{Im}\, b' \oplus Q$. We put $\varepsilon=b'\varepsilon'^{-1}b'^{-1}+1_Q$, then $\varepsilon \equiv 1 \pmod{\mathfrak{J}}$. Put $e=\varepsilon b'a'$, then $e|P_0=1_P$ and $\mathrm{Im}\, e=P_0$. Hence, e is idempotent in S_P and $e=\varepsilon b'a' \equiv b'a' \equiv a \pmod{\mathfrak{J}}$.

Corollary. *Let R be a (non-commutative) local ring such that $J(R)$ is T-nil-*

potent. Let S be the ring of column finite matrices over R with any degree. Then every idempotent in $S/J(S)$ is lifted to S.

Proof. Put $M=\sum_{I\ni\alpha}\oplus R_\alpha$; $R_\alpha\approx R$. Then $S=S_M$ and $J(S_M)=\mathfrak{J}_M$ by [4], Lemma 10.

2 Exchange property

We shall recall Condition II in [4]. Let $M=\sum_I\oplus M_\alpha=\sum_I\oplus N_\beta$ be decompositions of M with indecomposable modules M_α, N_β, Condition II in [4] says that for any subset J of I, there exists a subset J' of I such that $M=\sum_{J'}M_\alpha\oplus\sum_J\oplus N_\beta$. However, this is a special case of exchange property defined in [1]. Furthermore, this property induces Condition III in [4], namely every direct summand of M is in $\mathfrak{A}$. Therefore, we shall define a weaker exchange property than one in [1]. Let M be as above (in $\mathfrak{A}$), and $M=\sum_{I\ni i}\oplus P_i$ be any direct decomposition as R-modules. We call a direct summand N of M has the $|I|$-exchange property in M if $M=N\oplus\sum_{I\ni i}\oplus P_i'$ and $P_i'\subset P_i$ for any decomposition $M=\sum_{I\ni i}\oplus P_i$ with $|I|$-factors. If N has the *$|I|$-exchange property in M* for any cardinal $|I|$, we call N has the *exchange property in M*. It is clear that if N has the exchange property in M, then N is an object of $\mathfrak{A}$. P. Crawley and B. Jónsson have shown in [1], Theorem 7.1 (and [10], Theorem 1) that if M is countably generated for all α in I, then Condition III is satisfied.

In the following we always assume that $M=\sum_I\oplus M_\alpha=N_1\oplus N_2$ with indecomposable modules M_α.

Lemma 2. *If either N_1 is finitely generated or a dense submodule of N_1 is a T-nilpotent system, then N_i is in $\mathfrak{A}$, $(i=1, 2)$.*

Proof. If N_1 is finitely generated, then N_1 is a direct summand of M_J for some finite subset J of I. Hence, $N_1\approx M_{J'}$ for some $J'\subset J$ by Krull-Remak-Schmidt's theorem. Therefore, $M=N_1\oplus\sum\oplus M_{\varphi(\alpha)}$ by [4], Corollary 1 to Theorem 7 (Azumaya's theorem), and hence $N_2\approx\sum\oplus M_{\varphi(\nu)}$ is in $\mathfrak{A}$. Next, we assume that a dense submodule N_0 of N_1 is a T-nilpotent system. Then $N_0=N_1$ by Proposition 2. Hence N_1 is in $\mathfrak{A}$. Since $\mathfrak{A}/\mathfrak{J}$ is completely reducible, $\bar{M}=\bar{N}_1\oplus\sum_{K\ni\beta}\bar{M}_\beta=\sum_{I-K}\bar{M}_\alpha\oplus\sum_K\bar{M}_\beta$ for some K in I. Let p be a projection of M to $\sum_{I-K}M_\alpha$. Since $\bar{p}\,|\,\bar{N}_1$ is isomorphic and $\{M_\alpha\}_{I-K}$ is a T-nilpotent system, $p\,|\,N_1$ is an R-isomorphism of N_1 to $\sum_{I-K}\oplus M_\alpha$. Therefore, $M=N_1\oplus\operatorname{Ker}p=N_1\oplus\sum_K M_\beta=N_1\oplus N_2$. Hence, $N_2\approx\sum\oplus M_\beta$.

The following lemma is a special case of [1], Lemma 3.10 and [10], Proposition 1, however we shall give a proof from point of view of our categories.

Lemma 3. *Let M and N_i be as above. If $N_1=\sum_{i=1}^{n}\oplus M_i'$ and $M_i'\approx M_{\alpha i}$ for all i, then N_1 has the exchange property in M.*

Proof. We assume that $M=N_1\oplus N_2=\sum_{I'}\oplus Q_\alpha$ as R-modules. Then $\bar{M}=\bar{N}_1\oplus\bar{N}_2=\sum_{I'}\oplus\bar{P}_\alpha$, where $P_\alpha=\sum_{J_\alpha}\oplus P_{\alpha j}$ is a dense submodule of Q_α and $P_{\alpha j}$'s are indecomposable. Since $\bar{N}_1=\sum_{i=1}^{n}\oplus\bar{M}_i'$ is a small object in $\mathfrak{A}/\mathfrak{J}$, there exist a finite subset I'' of I and a finite subset J_i' of J_i for $i\in I''$ such that $\bar{N}_1\subseteq\sum_{I''\ni i}\sum_{J_i\ni j}\oplus\bar{P}_{ij}$. We know from Proposition 2, 2) that $\sum_i\sum_j\oplus P_{ij}=P$ is a direct summand of M. Since I'' and J_i' are finite, $\mathfrak{J}_P=J(S_P)$ by [4], Lemma 8. Hence, P contains a direct summand N_1' such that $\bar{N}_1'=\bar{N}_1$, $(P=N_1'\oplus P')$. $M=N_1'\oplus P'\oplus\sum_{I''}\oplus Q_i'\oplus\sum_{I-I''}\oplus Q_\alpha$, where $Q_i=Q_i'\oplus\sum_{J_i-J_i'}\oplus P_{ij}$. Let $p_{N'_1}$ be a projection of M to N_1' in this decomposition. Since $\bar{N}_1=\bar{N}_1'$ and $\bar{N}_1\cap(\bar{P}'\oplus\sum_{I''}\oplus\bar{Q}_i'\oplus\sum_{I-I''}\oplus\bar{Q}_\alpha)=\bar{0}$, (see the proof of Theorem 1), $\bar{p}_{N'_1}\,|\,\bar{N}_1$ is isomorphic. Therefore, $p_{N'_1}|N_1$ is isomorphic as an R-module. Thus, we obtain that $M=N_1\oplus P'\oplus\sum_{I''}\oplus Q_1'\oplus\sum_{I-I''}\oplus Q_\alpha$.

Theorem 4. *Let $M=\sum_I\oplus M_\alpha$ with M_α completely indecomposable, and $N_1=\sum_{I'}\oplus M_\beta'$ be a direct summand of M; $M=N_1\oplus N_2$. If I' is finite or $\{M_\beta'\}$ is a T-nilpotent system, then N_i has the exchange property in M for $i=1, 2$.*

Proof. We know from the assumption and [4], Theorem 8 that $\mathfrak{J}_{N_1}=J(S_{N_1})$. Let $M=N_1\oplus N_2=\sum_J\oplus Q_\alpha$. Then $\bar{M}=\bar{N}_1\oplus\bar{N}_2=\sum\oplus\bar{P}_\alpha$, where P_α is a dense submodule of Q_α. We put $P_\alpha=\sum_{J_\alpha\ni i}\oplus P_{\alpha i}$ $(\in\mathfrak{A})$. Since $\mathfrak{A}/\mathfrak{J}$ is completely reducible, $\bar{M}=\bar{N}_2\oplus\sum_J\sum_{J_\alpha'}\oplus\bar{P}_{\alpha i}$, where J_α' is a subset of J_α. The fact $\bar{N}_1\approx\bar{P}=\sum_J\sum_{J_\alpha'}\oplus\bar{P}_{\alpha i}$ implies $\mathfrak{J}_P=J(S_P)$. Let p_{N_1} be a projection of M to N_1 with $\mathrm{Ker}\,p_{N_1}=N_2$. Then $\bar{p}_{N_1}|\bar{P}$ is isomorphic, and hence $p_{N_1}|P$ is isomorphic as an R-module. Therefore, $M=P\oplus N_2$ and $\sum_{T'_\alpha}\oplus P_{\alpha i}\subseteq Q_\alpha$. We have shown that N_2 has the exchange property. If I' is finite, then N_2 has the exchange property from Lemma 3. Thus, we may assume that $\{M'_\beta\}$ is a T-nilpotent system. Noting that N_2 is an object of $\mathfrak{A}$ by Lemma 2, first we assume $N_1=\sum_K\oplus T_\alpha$, $N_2=\sum_{K'}\oplus T'_\beta$ and $T_\alpha\not\approx T'_\beta$ for any α, β, where T_α and T'_β are indecomposable. We make use of the same notation as above. Then $\bar{M}=\bar{N}_2\oplus\sum_J\oplus\bar{P}'_\alpha$ and $P'_\alpha\oplus P''_\alpha=P_\alpha$. Since $\sum\oplus P'_\alpha\approx N_1$, $\sum\oplus P'_\alpha$ is a direct summand of M by Proposition 2, say $M=\sum(P_\alpha'\oplus P'''_\alpha)$ and $Q_\alpha=P_\alpha'\oplus P_\alpha'''$. Then $\sum_J\oplus P'''_\alpha$ is an object in $\mathfrak{A}$ by Lemma 2. Let p be a projection of M to $\sum_J\oplus P'_\alpha$ with $\mathrm{Ker}\,p=\sum_J\oplus P'''_\alpha$. Then $\bar{p}\,|\,\bar{N}_2$ is isomorphic, since $\bar{N}_1\cap\sum\oplus\bar{P}'''_\alpha=\bar{0}$ and $\bar{p}(\bar{N}_2)=\bar{0}$ by the assumption. Hence, $p\,|\,N_1$ is isomorphic as an R-module, which implies $M=N_1\oplus\mathrm{Ker}\,p=N_1\oplus\sum P'''_\alpha$. Hence, N_1 has the exchange property in M. In general case, we choose all direct components $T'_{\beta'}$ in N_2, which is isomorphic to some T_α in N_1 and put $N_2'=\sum\oplus T'_{\beta'}$; $N_2=N_2'\oplus N_2''$. Then, $N_1'=N_1\oplus N_2'$ satisfies the assumption in the first case. Therefore, $M=N_1'\oplus\sum_J\oplus P'''_\alpha$ and $Q_\alpha=P_\alpha'\oplus P'''_\alpha$. Then $M=N_2'\oplus N_1\oplus\sum_J\oplus P'''_\alpha$. Since N_2' satisfies the assumption in the theorem, $N_1\oplus\sum\oplus P'''_\alpha$ has the exchange property from the beginning case. Therefore, $M=N_1\oplus\sum_J\oplus P'''_\alpha\oplus\sum_J\oplus P^{\mathrm{iv}}_\alpha$, and $Q_\alpha\supseteq P^{\mathrm{iv}}_\alpha$. Thus, we have proved that N_1 has the exchange property in M.

Corollary. *Let $\mathfrak{A}$ be as above. Then the following statements are equivalent.*

1 *Every direct summand of object M in $\mathfrak{A}$ has the $\aleph_0$-exchange property in M.*

2 *Every direct summand of object M in $\mathfrak{A}$ has the exchange property in M.*

3 *$\{M_\alpha\}$ is an elementwise T-nilpotent system.*

Proof. $1\to3$ Let $M=\sum_I\oplus M_\alpha=\sum_{I'}\oplus M'_{\beta'}\oplus\sum_{I''}\oplus M'_{\gamma'}$ be a direct decompositions with $|I'|\leqslant\aleph_0$. Since every direct summand of $\sum_{I'}\oplus M'_{\beta'}$ has the $\aleph_0$-exchange property in M, it has the $\aleph_0$-exchange property in $\sum_{I'}\oplus M'_{\beta'}$. Therefore, Condition II is satisfied for $\sum_{I'}\oplus M'_{\beta'}$, which implies 3 by [4], Lamma 9.

$3\to2$ It is clear from the theorem and Proposition 2.

$2\to1$ It is clear.

Proposition 3 ([1], [3], [6] and [10]). *Let M be in $\mathfrak{A}$ and $M=N_1\oplus N_2$. If N_1 is countably generated, then N_1 is in $\mathfrak{A}$. If every M_α is countably generated, then every direct summand of M is in $\mathfrak{A}$.*

Proof. We make use the argument of the proof of [1], Theorem 7.1. First, we note that for any element x in N_1 there exists a direct summand N_0 of N_1 such that $x\in N_0$ and N_0 is in $\mathfrak{A}$. Because, there exists a finite set J such that M_J contains x. From Theorem 4 we have $M=M_J\oplus N_1'\oplus N_2'$, $N_1=N_1'\oplus N_1''$ and $N_2=N_2'\oplus N_2''$, where $N_1''=(M_J\oplus N_2')\cap N_1$ and $N_2''=(M_J\oplus N_1')\cap N_2$, and $x\in N_1''$. If we use the same argument in [6], then we obtain the proposition.

3 Semi-perfect modules

We shall study further properties of $\mathfrak{A}$ in a case of semi-perfect modules defined by E. Mares in [9]. She has shown that every semi-perfect module is a direct sum of completely indecomposable semi-perfect modules ([9], Corollary 4.4). Let P be an R-module and $J(P)$ the radical of P. If P is semi-perfect, then $J(P)$ is small in P, $[P/J(P), P/J(P)]_{R/J(R)} \approx S_P/J(S_P)$ and $J(P)=PJ(R)$, (see [9], §§ 2-5).

Theorem 5. *Let P be a directly indecomposable projective module. Then P is completely indecomposable if and only if P is semi-perfect,* (cf. [5], the proof of Theorem 2.8).

Proof. If P is semi-perfect, then P is completely indecomposable by [9], Corollary 4.4. Conversely, we assume that so is P. Since $P/J(P)$ is $R/J(R)$-projective, $J([P/J(P), P/J(P)])=0$. From an exact sequence $0 \to [P, J(P)]_R \to S_P \to [P/J(P), P/J(P)]_{R/J(R)} \to 0$ we have $[P, J(P)] \supset J(S_P)$. On the other hand, $J(S_P)$ is a unique maximal ideal in S_P and $[P, J(P)] \neq S_P$. Hence, $\mathfrak{J}_P = J(S_P) = [P, J(P)]_R$. Next, we shall show that $J(P)$ is small in P. Let N be a submodule of P such that $P=J(P)+N$. From the following row exact sequence

$$\begin{array}{ccccc} N & \longrightarrow & N/N\cap J(P) & \longrightarrow & 0 \\ & \nwarrow & \wr\wr & & \\ & f & P/J(P) & & \\ & & \uparrow & & \\ & & P & & \end{array}$$

we have $f: P \to N$, which commutes the above diagram. If $N \neq P, f \in \mathfrak{J}_P$. Hence, $\operatorname{Im} f \subset N \cap J(P)$, which is a contradiction. Finally, we show that $J(P)$ is a unique maximal submodule in P. Put $\bar{P}=P/J(P)$, $\bar{R}=R/J(R)$ and $\bar{S}=S_P/\mathfrak{J}_P$. We define $\mu: \bar{P} \otimes_{\bar{R}} [\bar{P}, \bar{R}]_{\bar{R}} \to \bar{S}$ by setting $\mu(p \otimes f)(p')=pf(p')$. Since $\bar{P} \neq 0$ and $\bar{R}$-projective, $\mu \neq 0$. Furthermore, $\bar{S}$ is a division ring, and hence, μ is isomorphic. $\bar{P}\tau(P)=\bar{P}$ implies that there exists p in $\bar{P}$ such that $\mu(p \otimes [\bar{P}, \bar{R}]) \neq 0$, where τ is the trace map of $\bar{P}$. Hence, $\mu(p \otimes f)\bar{S}=\bar{S}$ for some f in $[\bar{P}, \bar{R}]$. Therefore, $\bar{P}=\bar{S}\bar{P}=\mu(p \otimes f)\bar{S}\bar{P} \subset pf(\bar{P}) \subset p\bar{R} \subset \bar{P}$. Hence, $\bar{P}=p\bar{R} \approx \bar{e}\bar{R}$ for some idempotent $\bar{e}$ in $\bar{R}$. Since $\bar{e}\bar{R}\bar{e}$ is a division ring and $\bar{R}$ is semi-simple, $\bar{P}$ is $\bar{R}$-irreducible by [8], Proposition 1 in p. 65. Hence, $J(P)$ is unique maximal, since $J(P)$ is the radical of P. Thus we have proved that P is semi-perfect by [9], Theorem 5.1.

Now let $\{P_\alpha\}$ be a family of completely indecomposable projective modules, and $\mathfrak{A}$ the induced additive category from $\{P_\alpha\}$. Let $P=\sum \oplus P_\alpha$ and $P'=\sum \oplus P_\beta'$ be in $\mathfrak{A}$ and f in $[P, P']_R$. If $f_{\alpha\beta}=p_\alpha f i_\beta$ is epimorphic, then $f_{\alpha\beta}$ splits and hence $f_{\alpha\beta}$ is isomorphic. Since $J(P_\alpha')$ is unique maximal, $\operatorname{Im} f_{\alpha\beta}$

$\subseteq J(P_\alpha')$ is $f_{\alpha\beta}$ is not isomorphic. Hence, if f is in $\mathfrak{J}$, then $\mathrm{Im} f \subseteq \sum \oplus J(P_\beta') = J(P')$. Conversely, if $\mathrm{Im} f \subseteq J(P')$, then f is in $\mathfrak{J}$. Therefore, $[P, P']_R \cap \mathfrak{J} = [P, J(P')]_R$. Furthermore, $0 \to [P, J(P')]_R \to [P, P']_R \to [P/J(P), P'/J(P')]_{R/J(R)} \to 0$ is exact. Thus, for any object P in $\mathfrak{A}$, many arguments in $\mathfrak{A}/\mathfrak{J}$ concerned with $\bar{P}$ coincide with those as $R/J(R)$-modules. From this reason, we make use of terminologies in $\mathfrak{A}/\mathfrak{J}$, instead of ones as $R/J(R)$-modules, if there are no confusions.

Theorem 6. *Let $\mathfrak{A}$ be an induced category from a family of completely indecomposable projective modules $\{P_\alpha\}$. Then an object $P = \sum_I \oplus P_\gamma$ in $\mathfrak{A}$ is perfect if and only if $\{P_\gamma\}_I$ is an elementwise T-nilpotent system.*

Proof. Let $\mathfrak{J}'$ be a full subcategory in $\mathfrak{J}$ which is induced from $\{P_\gamma\}_I$. If P is perfect, then every object in $\mathfrak{A}'$ is semi-perfect. Hence, $\mathfrak{J}'$ is equal to the Jacobson radical in $\mathfrak{A}'$ by the above remark and [9], Theorem 2.4. Therefore, $\{P_\gamma\}_I$ is a T-nilpotent system by [4], Theorem 8. Conversely, we assume that $\{P_\gamma\}_I$ is a T-nilpotent system. Then $\mathfrak{J}_P = J(S_P)$ for every object P in $\mathfrak{A}'$. We shall show that $J(P)$ is small in P for every object P in $\mathfrak{A}'$. Let $P = Q + J(P)$ for some submodule Q and p_1 a projection of P to P_1, where $P = \sum_{I'} \oplus P_\gamma$. Since $p_1(J(P)) \subset J(P_1)$ and $J(P_1)$ is small by Theorem 5, $p_1(Q) = P_1$. Hence, there exists f in $[P_1, Q]_R$ such that $p_1 f = 1_{P_1}$. Therefore, Q contains an object in $\mathfrak{A}'$ which is a direct summand of P. Let T be the set of such objects in Q and define a pertially order in T by the inclusion. We take a totally ordered subset $Q_1 \subset Q_2 \subset \cdots$ in T. Put $Q_0 = \cup Q_i$, then $Q_0 = \sum \oplus N_\beta; N_\beta \approx P_{\pi(\beta)}$ by Lemma 2. Furthermore, the inclusion i_β: $N_\beta \to P$ is not zero modulo $\mathfrak{J}$, since Q_j is a direct summand of P. Hence, Q_0 is a direct summand of P by the proof of Proposition 2,2. Thus, we have a maximal element P_0 in T. $P = P_0 \oplus U$ and $Q = P_0 \oplus Q \cap U$. Since $P = Q + J(P)$ and $J(P) = J(P_0) \oplus J(U)$, $U = J(U) + U \cap Q$. U is also in $\mathfrak{A}'$ by Lemma 2. If $U \neq 0$, $U \cap Q$ contains an object in $\mathfrak{A}'$ which is a direct summand of U and hence of P. Which contradicts to the maximality of P_0. Therefore, $P = P_0 = Q$. Thus, every object in $\mathfrak{A}'$ is semi-perfect by Theorem 5 and [9], Theorem 5.2.

In the above argument, we have used only facts that P_i are semi-perfect and $\mathfrak{J}_P = J(S_P)$. Hence, from Lemma 1, [4], Corollary 2 to Theorem 7 and [9], Theorem 2.3 we have

Proposition 4. *Let $P = \sum \oplus P_\alpha$ and P_α semi-perfect. Then P is semi-perfect if and only if $\mathfrak{J}_P = J(S_P)$.*

Theorem 7. *Let P be an object in $\mathfrak{A}$ induced from projective, completely indecomposable modules P_α. Then we have the following equivalent conditions.*

1 *P is semi-perfect.*

2 $\mathfrak{J}_P = J(S_P)$.

3 *Every dense submodule of P coincides with P.*

4 $P = \sum_I \oplus P_\alpha$ *in* $\mathfrak{A}$, *then* $\{P_\alpha\}$ *is a semi-T-nilpotent system.*

5 *P satisfies the Condition* II *in* [4].

Proof. $1 \leftrightarrow 2$ is proved in Proposition 4.

$1 \rightarrow 3$ Let N be a dense submodule of P. Then $N \supseteq \mathrm{Im}\,(1\text{-}f)$ for some $f \in \mathfrak{J}_P$. Hence, $P \subseteq N + f(P) \subseteq N + J(P) \subseteq P$ by the remark before Theorem 6. Therefore, $P = N + J(P)$ implies $P = N$, since $J(P)$ is small.

$3 \rightarrow 4$ Let $\{f_i\}$ be a family of non isomorphisms of P_{α_i} to $P_{\alpha_{i+1}}$ ($P_{\alpha_i} \neq P_{\alpha_{i+1}}$). Put $f = \sum (-e_{i+1\,i} f_i)$, where $\{e_{ij}\}$ is a system of matrix units in S_P. Then $\mathrm{Im}\,(1-f)$ is a dense submodule of P. From the assumtion and the argument of Lemma 9 in [4], we know that $\{f_i\}$ is a T-nilpotent sequence.

$2 \rightarrow 5$ is proved in [4], Corollary 2 to Theorem 7.

$5 \rightarrow 4$ is proved in [4], Lemma 9.

$4 \rightarrow 1$. Let

$$P = \sum_{K \ni \alpha} \sum_{I_\alpha \ni \beta} \oplus M_{\alpha\beta} \cdots (*),$$

where $M_{\alpha\beta}$'s are indecomposable and $M_{\alpha\beta} \approx M_{\alpha\beta'}$, $M_{\alpha\beta} \not\approx M_{\alpha'\beta'}$ if $\alpha \neq \alpha'$. First we assume that the cardinal λ_α of $|I_\alpha|$ is finite for all α in K. We put $P_{\alpha(n)\,n} = \sum_{\beta=1}^{n} \oplus M_{\alpha\beta}$, where $n = \lambda_\alpha$, and show that $J(P)$ is small in P. We assume $P = N + J(P)$ for some submodule N of P. Let $p_{\alpha(n)\,n}$ be a projection of P to $P_{\alpha(n)\,n}$. Since λ_α is finite, $J(P_{\alpha(n)\,n})$ is small in $P_{\alpha(n)\,n}$. Hence, $p_{\alpha(n)\,n} | N$ is epimorphic, and there exists $g \in [P_{\alpha(n)\,n}, N]_R$ such that $(p_{\alpha(n)\,n} | N) g = 1_{P_{\alpha(n)\,n}}$. Put $P'_{\alpha(n)\,n} = \mathrm{Im}\, g$. Since $\mathrm{Ker}\, p_{\alpha(n)\,n} = \sum_{\alpha \neq \alpha(n)} \oplus P_{\alpha\beta}$,

$$P = P'_{\alpha(n)\,n} \oplus \sum_{\alpha \neq \alpha(n)} \oplus P_{\alpha\beta} \cdots (**)\,.$$

Now, we assume $N \not\subset M_{\alpha(n)\,i_1}$ and $x_1 \in M_{\alpha(n)\,i_1} - N$. Then $x_1 = x' + \sum y_i$ from (**), where $x' \in P'_{\alpha(n)\,n}$, $y_i \in P_{\alpha\beta}$. From the assumption there exists some $y_i \notin N$, since $P'_{\alpha(n)\,n} \subset N$. Hence, there exists x_2 in $M_{\alpha_i,\,i_2} - N$ such that $y_i = x_2 + \sum z_j$, $z_j \in M_{\alpha_i j}$ ($j \neq i_2$). If we replace (*) by (**), we can find x_3 in $M_{\alpha_j k} - N$ and $P = P'_{\alpha(n)\,n} \oplus P'_{\alpha(m)\,m} \oplus \sum \oplus P_{\alpha\beta}$. Repeating this argument, we have a sequence $\{x_j\}$ so that $x_i \in M_{\alpha_i k_i} - N$, and $f_i(x_i) = x_{i+1}$, where f_i is a projection of P to $M_{\alpha_{i+1} k_{i+1}}$, which is a contradiction. Therefore, $J(P)$ is small. Finally, we shall consider a general case. Let $P = \sum_{\lambda_\alpha \geq \aleph_0} \sum_\beta \oplus M_{\alpha\beta} \oplus \sum_{\lambda_\alpha < \aleph_0} \sum_i \oplus M_{\alpha_i}$ and put $P_1 = \sum_{\lambda_\alpha \geq \aleph_0} \sum_\beta M_{\alpha\beta}$ and $P_2 = \sum_{\lambda_\alpha < \aleph_0} \sum_i \oplus M_{\alpha_i}$. We know from the first case P_2 is semi-perfect. If $\lambda_\alpha \geqslant \aleph_0$ for α, the fact that $\{M_{\alpha\beta}\}$ is semi-T-nilpotent implies from the definition that $\{M_{\alpha\beta}\}$ is a T-nilpotent system. Hence, P_1 is perfect by Theorem 6. Therefore, P is semi-perfect from [9], Corollary 5.3 (see Proposition 6 below).

Corollary 1. *Let P be projective and artinian, then P is perfect. Furthermore, if P' is a directsum of artinian submodules and P' is semi-perfect, then P' is perfect.*

Proof. If P is artinian and projective, then P is in $\mathfrak{A}$ and $\mathfrak{J}_P$ is nilpotent ideal by [5], Theorem 2.8. Hence, for any directsum M of any copies of P, we have $\mathfrak{J}_M=J(S_M)$, since $\mathfrak{J}_M$ is nilpotent. Therefore, M is semi-perfect, and P is perfect. Let $P'=\sum\oplus P_i$; P_i's are artinian and P' be semi-perfect. Then $\{P_i\}$ is a semi-T-nilpotent system from Theorem 7. Furthermore, since J_{P_i} is nilpotent, $\{P_i\}$ is a T-nilpotent system. Hence, P' is perfect from Theorem 6.

Corollary 2. *Let P be a semi-perfect module. Then there exists a maximal one among submodules which are perfect and direct summand of P. Those maximal perfect submodules are isomorphic each other.*

Proof. Let $P=\sum_{\lambda_\alpha<\aleph_0}\sum\oplus M_{\alpha i}\oplus\sum_{\lambda_\alpha\geq\aleph_0}\sum M_{\alpha\beta}$ as in the above proof. If $\mathfrak{J}_{M_{\alpha i}}$ is elementwise T-nilpotent, then $\{M_{\alpha i}, M_{\alpha\beta}\}$ is T-nilpotent, since it is semi-T-nilpotent. Hence, if we chooseevery $M_{\alpha i}$ whose ideal $\mathfrak{J}_{M_{\alpha i}}$ is T-nilpotent, $P_1=\sum'_{\lambda_\alpha<\aleph_0}\sum\oplus M_{\alpha i}\oplus\sum_{\lambda_\alpha\geq\aleph_0}\sum\oplus M_{\alpha\beta}$ is a direct summand of M and perfect, where $\sum'$ runs through all $M_{\alpha i}$ in the above. Put $P=P_1\oplus P_2$. If $P=Q_1\oplus Q_2$, $Q_1\supseteq P_1$ and Q_1 is perfect, then $P=Q_1'\oplus P_1\oplus Q_2$ and $Q_2=Q_2'\oplus P_1$. Since $P_2\approx Q_1'\oplus Q_2$, $Q_1'=(0)$ by the assumption. Hence, P_1 is a desired perfect submodule. Let T_1 be a maximal element as in Corollary 2; $P=T_1\oplus T_2$, then T_2 is in $\mathfrak{A}$. It is clear that T_1, P_1 and T_2, P_2 have the isomorphic direct components, respectively. Hence, $P_1\approx T_1$.

Finally, we shall give some results concerned with ones obtained in [9].

First we shall give another proof of [9], Theorem 5.5.

Proposition 5 ([9]). *Let $\mathfrak{A}$ be as above and P a direct summand of an object M in $\mathfrak{A}$. If $J(P)$ is small in P, then P is in $\mathfrak{A}$.*

Proof. Let $M=P\oplus P_1$ and $P=eM$ for some idempotent e. P contains a dense submodule P_0 with inclusion i such that $if\equiv e \pmod{\mathfrak{J}}$ for some f in $[M, P_0]_R$. Put $e=if+x$, $x\in\mathfrak{J}$. Then $P=P_0+x(P)$ and $x(P)\subset P\cap J(M)=J(P)$. Hence, $P=P_0$.

Proposition 6 ([9], Corollary 5.3). *Let $\{P\}_1^n$ be a finite set of semi-perfect modules. Then $\sum_1^n\oplus P_i$ is semi-perfect.*

Proof. Since $\mathfrak{J}_{P_i}=J(S_{P_i})$ for every i, we can show $\mathfrak{J}_P=J(S_P)$ by using fundamental transformations of matrices (see [4], Lemma 8). Hence, P is semi-perfect from Propostion 4.

Proposition 7 ([9], Theorem 7.2). *If $J(R)$ is right T-nilpotent, then every semi-perfect modules is perfect.*

Proof. Let $P=\sum_I \oplus P_\alpha$ be semi-perfect. Then $\mathfrak{J}_P=[P, J(P)]=[P, PJ(R)]$. Hence, for any $f\in[P_\alpha, P_\beta]\cap\mathfrak{J}$ and $x_\alpha\in P_\alpha, f(x_\alpha)=\sum x_{\beta_i}a_{\beta_i}, a_{\beta_i}\in J(R)$. Therefore, $\{P_\alpha\}$ is T-nilpotent system by the assumption. Hence, P is perfect from Theorem 6.

REMARK. [9], Theorem 5.1 is a special case of Theorem 3.

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