Ryan, P. J.
Osaka J.Math.
8 (1971), 251-259

HYPERSURFACES WITH PARALLEL RICCI TENSOR

PATRICK J. RYAN*)

(Received July 13, 1970)

0. Introduction

The purpose of this paper is to classify those Riemannian manifolds with parallel Ricci tensor which arise as hypersurfaces in real space forms. H. B. Lawson, Jr. [1] performed this classification under the assumption of constant mean curvature. Lawson's result may be divided into two parts-determination of the local geometry on the hypersurface, and a rigidity theorem.

In the following, we prove that no assumption on the mean curvature is necessary unless the dimension is 2 or the hypersurface and the ambient space have the same constant curvature. See Theorem 10.

1. The standard examples

We consider first some special complete hypersurfaces which will serve as models in our discussion. $\tilde{M}$ is the ambient space, M is the hypersurface and f: $M\rightarrow\tilde{M}$ is an isometric immersion. In each of the examples, M is a submanifold of $\tilde{M}$ and f is the inclusion mapping.

For $\tilde{M}=E^{n+1}$, we have as our model hypersurfaces, hyperplanes, spheres, and cylinders over spheres.

For $\tilde{M}=S^{n+1}(\tilde{c})$, we have great spheres, small spheres, and products of spheres. The latter may also be thought of as the intersection of two cylinders over spheres in E^{n+2}.

All of the above are explicitly written out in [2] together with their second fundamental forms. We consider the real hyperbolic space of curvature $\tilde{c}<0$ (which we denote by $H^{n+1}(\tilde{c})$) in more detail here since the analogous facts are omitted from [2].

For vectors X and Y in R^{n+2}, we set $g(X, Y)=\sum_{1=1}^{n+1} X^{i}Y^{i}\text{-}X^{n+2}Y^{n+2}$. For given $\tilde{c}<0$, we define $R=\dfrac{1}{\sqrt{-\tilde{c}}}$. Then

$$H^{n+1}(\tilde{c})=\{x\in R^{n+2}\,|\,g(x, x)=-R^{2} \quad\text{and}\quad x_{n+2}>0\}$$

*) This work has been supported in part by the National Science Foundation under Grant GP-7403.

$H^{n+1}(\tilde{c})$ is connected, simply-connected submanifold of R^{n+2} and it is not too hard to show that the restriction of g to tangent vectors yields a (positive-definite) Riemannian metric for $H^{n+1}(\tilde{c})$. Furthermore, $H^{n+1}(\tilde{c})$ is complete and has constant curvature $\tilde{c}$ in this metric. We thus have a model for real hyperbolic space.

We will be interested in the following hypersurfaces of $H^{n+1}(\tilde{c})$.

(i) $M=\{x\mid x_1=0\}$. In this case, the second fundamental form A is zero, M is totally goedesic and is in fact just $H^n(\tilde{c})$.

(ii) $M=\{x\mid x_1=r>0\}$, $A=\sqrt{c-\tilde{c}}$ I where $\tilde{c}<c<0$ and $c=-\dfrac{1}{r^2}$. M is isometric to $H^n(c)$.

(iii) $M=\{x\mid x_{n+2}=x_{n+1}+R\}$, $A=\sqrt{-\tilde{c}}$ I, M is isometric to E^n.

(iv) $M=\{x\mid x_1^2+x_2^2+\cdots x_{n+1}^2=r^2\}$, $A=\sqrt{c-\tilde{c}}\ I$ and $c=\dfrac{1}{r^2}>0$. M is isometric to S^n (c).

(v) $M=\{x\mid x_1^2+x_2^2+\cdots+x_{k+1}^2=r^2,\ x_{k+2}^2+\cdots-x_{n+2}^2=-(r^2+R^2)\}$. Thinking of R^{n+2} as $R^{k+1}\times R^{n-k+1}$ we see that M is a subset of $H^{n+1}(\tilde{c})$ for any $r>0$ and the inclusion mapping is the product of the imbeddings $S^k(c_1)\to R^{k+1}$ and $H^{n-k}(c_2)\to R^{n-k+1}$. Here $c_1=\dfrac{1}{r^2}$ and $c_2=-\dfrac{1}{r^2+R^2}$.

The second fundamental form may be calculated easily and it is given by $A=\lambda I_k\oplus\mu I_{n-k}$ where $\lambda=\sqrt{c_1-\tilde{c}}$ and $\mu=\sqrt{c_2-\tilde{c}}$. This may be simplified to

$$\lambda=\frac{\sqrt{R^2+r^2}}{rR}\ ;\ \mu=\frac{r\sqrt{r^2+R^2}}{R(r^2+R^2)}$$

Note that $\lambda\mu+\tilde{c}=0$.

The eigenvalues λ and μ may also be expressed in terms of c_1 and c_2 as follows

$$\lambda=\frac{c_1}{\sqrt{c_1+c_2}},\quad \mu=\frac{-c_2}{\sqrt{c_1+c_2}}.$$

We note that in all of the above cases, either of the following is true:

(i) M is umbilic in $\tilde{M}$, that is, A is a constant multiple λ of the identity I, and M is of constant curvature $c=\lambda^2+\tilde{c}$.

(ii) A has exactly two distinct eigenvalues $\lambda>\mu$ at each point and they are constant over M. M is the Riemannian product of spaces of constant curvature

$$c_1=\lambda^2+\tilde{c},\ c_2=\mu^2+\tilde{c}\quad\text{where}\quad\lambda\mu+\tilde{c}=0\,.$$

The converse of the above remarks also holds in the following sense.

Theorem 1. *Suppose $\tilde{M}$ is a real space form and M a hypersurface in $\tilde{M}$. Suppose the principal curvatures are constant and at most two are distinct. Then M*

is congruent to an open subset of one of the standard examples.

Proof. Theorem 2.5 of [2] followed by the arguments of Lemma 2 of [1] give the desired result.

2. The curvature operator

In [2] we considered the action of the derivation $R(X,Y)$ on the algebra of tensor fields of a Riemannian manifold. We recall that if T is a tensor field of type (r,s), and X and Y are vector fields,

$$R(X, Y)\cdot T = \nabla_X\nabla_Y T - \nabla_Y\nabla_X T - \nabla_{[X,Y]}T\,.$$

For brevity of notation, we denote by RT the tensor of type $(r, s+2)$ defined by

$$(RT)(X_1, X_2, \cdots, X_s, X, Y) = (R(X, Y)\cdot T)(X_1, X_2, \cdots, X_s)\,.$$

Concerning hypersurfaces which satisfy $RA=0$ where A is the second fundamental form, we have

Proposition 2. *Let M be a hypersurface in a space of constant curvature $\tilde{c}$. If $RA=0$, then*

$$(\lambda_i\lambda_j+\tilde{c})(\lambda_i-\lambda_j) = 0$$

for all i and j, where $\{\lambda_i\}_{i=1}^n$ are the eigenvalues of A.

Proof. Let $x\in M$ be arbitrary and let $\{e_i\}_{i=1}^n$ be an orthonormal basis for $T_x(M)$ such that $Ae_i=\lambda_i e_i$. For each λ, let $T_\lambda=\{X\,|\,AX=\lambda X\}$.

Since A is symmetric, $T_\lambda\subseteq T_\mu{}^\perp$ whenever $\lambda\neq\mu$. Since $RA=0$, we have that $R(X, Y)$ and A commute for all X and Y. In particular,

$$R(e_i, e_j)(Ae_j) = AR(e_i, e_j)e_j$$
$$\lambda_j R(e_k, e_j)e_j = AR(e_i, e_j)e_j$$

Thus, $R(e_i, e_j)e_j$ is a member of T_{λ_j}, and hence $<R(e_i, e_j)e_j, e_i>=0$ whenever $\lambda_i\neq\lambda_j$. Here $<, >$ denotes the Riemannian metric of M. On the other hand, the Gauss equation

$$R(e_i, e_j) = (\lambda_i\lambda_j+\tilde{c})(e_i\wedge e_j)$$

shows that

$$\langle R(e_i, e_j)e_j, e_i\rangle = \lambda_i\lambda_j+\tilde{c}\,.$$

This completes the proof.

Corollary 3. *A has at most two distinct eigenvalues at each point.*

Corollary 4. *If $RA=0$ is replaced by the stronger condition, $\nabla A=0$, the eigenvalues of A are constant on M.*

Proof. Suppose $\lambda>\mu$ are eigenvalues of A at x. Let y be any point of M. Join x to y by a smooth curve γ and let E_i be the vector field along γ obtained by parallel translation of e_i. We compare AE_i and $\lambda_i E_i$ along γ. They agree at x and if X is the tangent vector to γ, we have

$$\nabla_X(AE_i) = (\nabla_X A)E_i + A(\nabla_X E_i) = 0$$

and

$$\nabla_X(\lambda_i E_i) = \lambda_i \nabla_X E_i = 0\,.$$

By the uniqueness of parallel translation, $AE_i=\lambda_i E_i$ at y. Thus, A has the same eigenvalues at y as it has at x.

Lawson's classification now follows directly from the following proposition which may be found in [1].

Proposition 5. *Suppose the Ricci tensor S is parallel* ($\nabla S=0$) *and trace A is constant on M. Then $\nabla A=0$ on M.*

3. The condition $RS=0$

In order to avoid any assumption about the mean curvature, we first examine hypersurfaces satisfying $RS=0$. We will show that when $\tilde{c}\neq 0$, such hypersurfaces must also satisfy $RR=0$. Since this condition has been examined in [2], we make use of results from this source. Since we are ultimately interested in the condition $\nabla S=0$, we may make use of the constancy of the scalar curvature s to take care of troublesome cases.

Proposition 6. *Let M be a hypersurface in a space of constant curvature $\tilde{c}$. Then $RS=0$ if and only if at each point of M,*

$$(\lambda_i-\lambda_j)(\lambda_i\lambda_j+\tilde{c})(\text{trace } A-\lambda_i-\lambda_j) = 0$$

for $1\leqq i, j\leqq n$.

Proof. Let $\hat{S}$ denote the tensor field of type (1,1) satisfying $\langle\hat{S}X, Y\rangle= S(X, Y)$. Clearly $R\hat{S}=0$ if and only if $RS=0$.

Now $\hat{S}X=(n-1)\tilde{c}X+(\text{trace } A)AX-A^2X$, and thus, $\hat{S}e_j=((n-1)\tilde{c}+m\lambda_j-\lambda_j^2)e_j$. Assuming that $R\hat{S}=0$, we have $R(e_i, e_j)$ commutes with $\hat{S}$. (Here m is, by definition, equal to trace A.)

$$\begin{aligned}\text{Now } &\hat{S}R(e_i, e_j)e_j\\ &= \hat{S}(\lambda_i\lambda_j+\tilde{c})e_i\\ &= (\lambda_i\lambda_j+\tilde{c})((n-1)\tilde{c}+m\lambda_i-\lambda_i^2)e_i\end{aligned}$$

$$\text{But } R(e_i, e_j)\hat{S}e_j = ((n-1)\tilde{c}+m\lambda_j-\lambda_j^2)R(e_i, e_j)e_j$$
$$= ((n-1)\tilde{c}+m\lambda_j-\lambda_j^2)(\lambda_i\lambda_j+\tilde{c})e_i$$

The two quantities are equal if and only if

$$(\lambda_i\lambda_j+\tilde{c})(m(\lambda_i-\lambda_j)-(\lambda_i^2-\lambda_j^2)) = 0$$
$$\text{i.e. } (\lambda_i\lambda_j+\tilde{c})(\lambda_i-\lambda_j)(m-\lambda_i-\lambda_j) = 0\,.$$

Furthermore, if this condition is satisfied, $R(e_i, e_j)$ commutes with $\hat{S}$ and this implies $RS=0$. We denote this condition by **.

Proposition 7. *If $\tilde{c}\neq 0$, $RR=0$ if and only if $RS=0$.*

Proof. We recall from [2] that $RR=0$ if and only if condition * $(\lambda_i-\lambda_j)(\lambda_i\lambda_j+\tilde{c})\lambda_k=0$ is satisfied for distinct i, j, k. Now we assume $RS=0$ and work at a particular point x. Choose $i\neq j$.

Assume for the moment that $\lambda_i=0$, $\lambda_j\neq 0$. Then $\lambda_j=\text{trace } A$. We conclude that all non-zero eigenvalues have the same value, trace A. Thus, there can be only one of them. But rank $A\leqq 1$ implies *.

We must now consider the case rank $A=n$. First, we claim it is impossible for three eigenvalues of A to be distinct. For consider the equations:

$$(\lambda-\mu)(\lambda\mu+\tilde{c})\ \ (\text{trace } A-\lambda-\mu) = 0$$
$$(\mu-\nu)(\mu\nu+\tilde{c})\ \ (\text{trace } A-\mu-\nu) = 0$$
$$(\nu-\lambda)(\nu\lambda+\tilde{c})\ \ (\text{trace } A-\nu-\lambda) = 0$$

In order for these to be satisfied, two factors of the same type must vanish. But this gives a contradiction -e.g., $\lambda\mu+\tilde{c}=\mu\nu+\tilde{c}=0$ implies $\lambda=\nu$. Thus, there are at most 2 distinct eigenvalues, say $\lambda\geqq\mu$ at each point. Assuming for the moment that $(\lambda-\mu)(\lambda\mu+\tilde{c})\neq 0$ at x, we let p and q be the multiplicities of λ and μ respectively at x. Then, as in [2], the same conditions hold in a neighborhood of x. Furthermore, in this neighborhood, trace $A=\lambda+\mu$. This means that $(p-1)\lambda+(q-1)\mu=0$.

But neither λ nor μ is zero and hence p and q are greater than 1. The standard arguments of [2] (*pp.* 372-373) now apply, showing that λ and μ are constants near x and hence, that $\lambda\mu+\tilde{c}=0$. This again implies * and completes the proof.

Proposition 8. *If $\tilde{c}=0$ and s is constant, $RR=0$ and $RS=0$ are equivalent.*

Proof. Our conditions $RR=0$ and $RS=0$ reduce respectively to

$$*\lambda_i\lambda_j\lambda_k(\lambda_i-\lambda_j) = 0$$
$$**\lambda_i\lambda_j(\lambda_i-\lambda_j)\ \ (\text{trace } A-\lambda_i-\lambda_j) = 0.$$

Assuming $**$, let λ and μ be distinct non-zero principal curvatures at x. If ν is a principal curvature distinct from λ and μ, we have

$$\nu(\text{trace } A-\lambda-\nu)=0$$
$$\nu(\text{trace } A-\mu-\nu)=0\,.$$

Since $\lambda\neq\mu$ we must conclude that $\nu=0$. But if this is true, then trace $A=\lambda+\mu$. On the other hand, trace $A=p\lambda+q\mu$, where p and q are the appropriate multiplicities. Thus, $(p-1)\lambda+(q-1)\mu=0$ and hence p and q are greater than 1. Unless, of course, $p=q=1$ in which case $*$ is automatically satisfied.

If $p+q=n>2$, the standard argument of [2] shows that λ and μ are constant near x. Thus, $\lambda\mu+\tilde{c}=0$ which implies that $\lambda\mu=0$, a contradiction. Thus, at most 2 principal curvatures are distinct and $*$ holds.

If $p+q<n$, it is not clear that $*$ is satisfied. However, computing the scalar curvature and using the fact that

$$\lambda=-\frac{q-1}{p-1}\mu$$

we have

$$\begin{aligned}
s&=\tilde{c}+\frac{1}{n(n-1)}((\text{trace } A)^2-\text{trace } A^2)\\
&=0+\frac{1}{n(n-1)}((\lambda+\mu)^2-p\lambda^2-q\mu^2)\\
&=\frac{1}{n(n-1)}(2\lambda\mu-(p-1)\lambda^2-(q-1)\mu^2)\\
&=\frac{-1}{n(n-1)}\mu^2\left(\frac{(q-1)^2}{p-1}+(q-1)+\frac{2(q-1)}{p-1}\right)\\
&=\frac{-(q-1)\mu^2}{n(n-1)(p-1)}(p+q)
\end{aligned}$$

Thus μ is constant and so is λ. But a theorem of E. Cartan ([2], Theorem 2.6) says that at most two principal curvatures can be distinct. This is a contradiction. We must conclude that $p+q=n$ and the proof is complete.

Note that even if s is not assumed to be constant, we must have $s<0$. Thus we have also proved the following proposition, which has been proved by S. Tanno [3] under the assumption of positive scalar curvature.

Proposition 9. *For hypersurfaces in E^{n+1} with non-negative scalar curvature, the conditions $RR=0$ and $RS=0$ are equivalent.*

As a prelude to the next theorem, we note that when $\nabla S=0$, we have also $\nabla\hat{S}=0$, and hence, $\nabla(\text{trace } \hat{S})=\text{trace}(\nabla\hat{S})=0$. Hence, the scalar curvature s will be constant.

4. The main theorem

Theorem 10. *Let M be a hypersurface of dimension* >2 *in a real space form of constant curvature* $\tilde{c}$. *If M is not of constant curvature* $\tilde{c}$ *and if* $\nabla S=0$ *on M, then M is an open subset of one of the standard examples or* $\tilde{c}=0$ *and* $A=2$ *on M.*

Proof. We suppose first that M is simply-connected. Then, a unit normal can be chosen consistently on M and the principal curvatures $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n$ are continuous functions. When $\tilde{c}=0$, $RR=0$ by Proposition 7. The proof of Proposition 4.3 of [2], gives rank $A=n=\dim M$. Now we know that at most two principal curvatures are distinct. Denote the larger one by λ and the other by μ so that $\lambda \geqq \mu$. If $\lambda > \mu$ at some point, then that condition holds locally and λ and μ have the same multiplicities p and $n-p$ nearby. If $1<p<n-1$, the standard argument of [2] shows that λ and μ are locally constant. On the other hand, if $p=1$ or $n-1$, the equation

$$s=\tilde{c}+\frac{1}{n(n-1)}(p(p-1)\lambda^2+(n-p)(n-p-1)\mu^2-2p(n-p)\tilde{c})$$

shows that λ and hence μ are locally constant. On the other hand $\{x \mid \lambda=\lambda_0$ and $\mu=\mu_0\}$ is closed. If $\lambda_0>\mu_0$, we have just shown it is also open.

The alternative to this is that $\lambda=\mu$ at all points and M is umbilic.

Now, we consider the case $\tilde{c}=0$. Again $RR=0$ by proposition 8. As before, λ and μ (where $\mu=0$) have respective multiplicities p and $n-p$. We allow $p=0, 1, 2, \cdots, n$. If $2<p\leqq n$, λ is locally constant since

$$s=\frac{1}{n(n-1)}p(p-1)\lambda^2 .$$

Thus, a fixed value for λ and for p holds on M. If $p\leqq 1$ for all points of M, then M has constant curvature 0. If $p=2$ somewhere, then $p=2$ everywhere.

We now see that the hypothesis of our theorem implies trace A=constant on M. Thus, $\nabla A=0$ and we are finished.

If now M is not simply-connected, let $\hat{M}$ be the simply connected Riemannian covering of M with projection π which is a local isometry. If $f\colon M\to\tilde{M}$ is the immersion defining the hypersurface, $f\circ\pi$ is an isometric immersion of $\hat{M}$ into $\tilde{M}$. By the above, $f(\pi(\hat{M}))$ is just an open subset of one of the standard examples. But $\pi(\hat{M})=M$. This completes the proof.

REMARK. It is possible in this proof to avoid the use of proposition 5 and substitute more delicate topological arguments. However, the proof of proposition 5 is straight-forward and, its use seems the most efficient way of proving the more general result.

UNIVERSITY OF NOTRE DAME

Bibliography

[1] H. B. Lawson, Jr.: *Local rigidity theorems for minimal hypersurfaces*, Ann. of Math. **89** (1969), 187-197.

[2] P. J. Ryan: *Homogeneity and some curvature conditions for Hypersurfaces*, Tôhoku Math. J. **21** (1969), 363-388.

[3] S. Tanno: *Hypersurfaces satisfying a certain condition on the Ricci tensor*, Tôhoku Math. J. **21** (1969), 297-303.

Appendix-Proof of Proposition 5

The Case of Constant Mean Curvature

Proposition 5. *Suppose trace A is constant and $\nabla S=0$ (S is the Ricci tensor). Then $\nabla A=0$.*

Proof. We recall that

$$S(X,\ Y)=(n-1)\tilde{c}\langle X,\ Y\rangle+\langle AX,\ Y\rangle \text{ trace } A-\langle AX,\ AY\rangle .$$

Let $\hat{S}$ be the tensor field of type (1,1) related to S by the formula

$$\langle\hat{S}X,\ Y\rangle=S(X,\ Y) .$$

Then $\nabla\hat{S}=0$ if and only if $\nabla S=0$. Thus, we may consider

$$\hat{S}=(n-1)\tilde{c}I+mA-A^2 .$$

Since $\nabla\hat{S}=0$, we have $\nabla(mA-A^2)=0$. Now

$$\begin{aligned}(\nabla_X A^2)Y &= \nabla_X(A^2Y)-A^2(\nabla_X Y)\\ &= (\nabla_X A)AY+A\nabla_X(AY)-A^2\nabla_X Y\\ &= (\nabla_X A)AY+A(\nabla_X A)Y\end{aligned}$$

That is,

$$\nabla_X A^2=(\nabla_X A)A+A(\nabla_X A) .$$

Thus, $(\nabla_X A)A+A(\nabla_X A)-m\nabla_X A=0$.
Suppose now that $AX=\lambda X$, $AY=\mu Y$. Then

$$(\nabla_X A)\mu Y+A(\nabla_X A)Y-m(\nabla_X A)Y=0 .$$

That is, $(\nabla_X A)Y \in T_{m-\mu}$.
Similarly, $(\nabla_Y A)X \in T_{m-\lambda}$.
But Codazzi's equation says precisely that

$$(\nabla_X A)Y = (\nabla_Y A)X .$$

Now if $\lambda \neq \mu$, both of these vectors are zero. If $\lambda = \mu$, we still have that

$$(\nabla_X A)Y \in T_{m-\mu}$$

so that

$$(\nabla_X A)(\nabla_X A)Y \in T_{m-(m-\mu)} = T_\mu .$$

Thus, if $\mu \neq \frac{m}{2}$, $(\nabla_X A)^2 Y = 0$. Since $\nabla_X A$ is symmetric, we must have $(\nabla_X A)Y = 0$.

Finally, if $\mu = \frac{m}{2}$, we construct the geodesic γ through x with initial tangent vector X and we extend Y by parallel translation along γ. Now,

$$\nabla_X(A^2Y - mAY) = (A^2 - mA)\nabla_X Y .$$

But $\nabla_X Y = 0$ along γ. We conclude that $A^2Y - mAY$ is parallel along γ. The value of this vector at x is $\frac{m^2}{4}Y - m\left(\frac{m}{2}\right)Y = -\frac{m^2}{4}Y$. But the vector $-\frac{m^2}{4}Y$ is also parallel along γ. Hence $A^2Y - mAY = -\frac{m^2}{4}Y$ all along γ. This means that

$$\left(A - \frac{m}{2}I\right)^2 Y = 0 \quad \text{along } \gamma .$$

Again, since $\left(A - \frac{m}{2}I\right)$ is symmetric, we have that $AY = \frac{m}{2}Y$ along γ. Hence, along γ,

$$\begin{aligned}(\nabla_X A)Y &= \nabla_X(AY) - A\nabla_X Y \\ &= \nabla_X\left(\frac{m}{2}Y\right) - 0 \\ &= 0 .\end{aligned}$$

We have shown that $(\nabla_X A)Y = 0$ for any pair of principal vectors X and Y at any point $x \in M$. Since the principal vectors span the tangent space, we have shown that $\nabla A = 0$.