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ON MULTIPLY TRANSITIVE GROUPS X

Dedicated to Professor Keizo Asano on his 60th birthday

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1. Introduction

In this paper we shall prove the following theorems.

Theorem 1. *Let G be a permutation group on $\Omega=\{1, 2, \cdots, n\}$ where $n>4$. Assume that a Sylow 2-subgroup P of the stabilizer of any four points in G satisfies the following two conditions:*

(i) *P is a nonidentity semi-regular group.*

(ii) *P fixes exactly r points.*

Then

(I) *If $r=4$, then $|\Omega|=6$, 8 or 12, and $G=S_6$, A_8 or M_{12} respectively.*

(II) *If $r=5$, then $|\Omega|=7$, 9 or 13. In particular, if $|\Omega|=9$, then $G\leqq A_9$, and if $|\Omega|=13$, then $G=S_1\times M_{12}$.*

(III) *If $r=7$ and $N_G(P)^{I(P)}\leqq A_7$, then $G=M_{23}$.*

In a previous paper [10] we proved that if G is a 4-fold transitive group and a Sylow 2-subgroup P of a stabilizer of four points in G is not the identity, then P fixes exactly four, five or seven points. Therefore the following corollary is an immediate consequence of Theorem 1.

Corollary. *Let G be a 4-fold transitive group on Ω and assume that a Sylow 2-subgroup P of a stabilizer of four points in G is not the identity. For a point t of $\Omega-I(P)$, assume that a Sylow 2-subgroup R of the stabilizer of any four points in $N_G(P_t)^{I(P_t)}$ satisfies the following two conditions:*

(i) *R is a nonidentity semi-regular group.*

(ii) $|I(R)|=|I(P)|$.

Then one of the conclusions in Theorem 1 holds for $N_G(P_t)^{I(P_t)}$. In particular, if t is a point of a minimal P-orbit, then $N_G(P_t)^{I(P_t)}$ satisfies the conditions (i) *and* (ii).

The last assertion of this corollary follows from Lemma 1 of [9].

By using these theorems we have the following

Theorem 2. *Let G be a 4-fold transitive group on $\Omega=\{1, 2, \cdots, n\}$. If a Sylow 2-subgroup of a stabilizer of four points in G is a nonidentity abelian group then G must be one of the following groups: S_6, S_7, A_8, A_9 or M_{23}.*

We shall follow the notations of T. Oyama [9].

2. Proof of Theorem 1

Case I. $|I(P)|=4$.

For any four points i, j, k, l of Ω a Sylow 2-subgroup P of G_{ijkl} fixes exactly these four points. Hence, by a lemma of D. Livingstone and A. Wagner [3. Lemma 6], G is a 4-fold transitive group on Ω. By assumption, P is a nonidentity semi-regular group. Therefore, by a theorem of H. Nagao [6], G is S_6, A_8 or M_{12}.

Case II. $|I(P)|=5$.

First assume $|\Omega|>9$. Let a be an involution of P and $I(P)=\{1, 2, \cdots, 5\}$. Since P is a nonidentity semi-regular group, we may assume that a is of the form

$$a=(1)(2)\cdots(5)(6\ 7)(8\ 9)(10\ 11)\cdots.$$

For any two 2-cycles $(6\ 7)$, $(8\ 9)$ of a, $a\in N_G(G_{6789})$. Hence by Lemma 1 of [10], there is an involution b of G_{6789} commuting with a. Since $|I(b)|=5$, we may assume

$$b=(1)(2\ 3)(4\ 5)(6)(7)(8)(9)\cdots.$$

Since $\langle a, b\rangle<N_G(G_{2367})$, also by Lemma 1 of [10] there is an involution c of G_{2367} commuting with a and b. Since $|I(c)|=5$, c is of the form

$$c=(1)(2)(3)(4\ 5)(6)(7)(8\ 9)\cdots.$$

Then $I(ac)=\{1, 2, 3, 8, 9\}$. Hence $\langle a, c\rangle$ is semi-regular on $\{10, 11, \cdots, n\}$, and so we may assume

$$a=(1)(2)\cdots(5)(6\ 7)(8\ 9)(10\ 11)(12\ 13)\cdots,$$
$$c=(1)(2)(3)(4\ 5)(6)(7)(8\ 9)(10\ 12)(11\ 13)\cdots.$$

Since $\langle a, c\rangle<N_G(G_{10\,11\,12\,13})$, there is an involution d of $G_{10\,11\,12\,13}$ commuting with a and c. Since $|I(d)|=5$ and $I(d)\supset\{10, 11, 12, 13\}$, d fixes exactly one point of $I(a)\cap I(c)=\{1, 2, 3\}$ and so d is $(1)(2\ 3)\cdots$, $(2)(1\ 3)\cdots$ or $(3)(1\ 2)\cdots$. We may assume that $d=(1)(2\ 3)\cdots$ since the proofs in the remaining cases are similar. Therefore d is of the form

$$d=(1)(2\ 3)(4\ 5)(6\ 7)(8\ 9)(10)(11)(12)(13)\cdots.$$

Since $\langle a, d\rangle < N_G(G_{2\,3\,10\,11})$, there is an involution f of $G_{2\,3\,10\,11}$ commuting with a and d. f is one of the following forms:

(i) $f=(1)\,(2)\,(3)\,(4\ 5)\,(6\ 7)\,(8\ 9)\,(10)\,(11)\,(12\ 13)\cdots,$

(ii) $f=(1)\,(2)\,(3)\,(4\ 5)\,(6\ 8)\,(7\ 9)\,(10)\,(11)\,(12\ 13)\cdots.$

If f is of the form (i), then

$$af=(1)\,(2)\,(3)\,(4\ 5)\,(6)\,(7)\,(8)\,(9)\cdots.$$

Thus $|I(af)|>5$, which contradicts the assumption. Hence

$$f=(1)\,(2)\,(3)\,(4\ 5)\,(6\ 8)\,(7\ 9)\,(10)\,(11)\,(12\ 13)\cdots,$$

Then

$$cf=(1)\,(2)\,(3)\,(4)\,(5)\,(6\ 8\ 7\ 9)\cdots.$$

Since $cf\in G_{I(a)}$, four points 6, 7, 8, 9 are contained in the same $G_{I(a)}$-orbit. Since we took 2-cycles (6 7) and (8 9) as arbitrary 2-cycles of a, $G_{I(a)}$ is transitive on $\Omega-I(a)$. Hence for any involution x fixing five points $G_{I(x)}$ is also transitive on $\Omega-I(x)$.

By using this result repeatedly, we prove that G_1 is 4-fold transitive on $\Omega-\{1\}$. $G_{I(a)}$ is transitive on $\{6, 7, \cdots, n\}$, and $G_{I(d)}$ is transitive on $\Omega-\{1, 10, 11, 12, 13\}$. Since $G_1\geqq\langle G_{I(a)}, G_{I(d)}\rangle$, G_1 is transitive on $\Omega-\{1\}$. Similarly since $G_{1\,2\,3}\geqq\langle G_{I(a)}, G_{I(c)}\rangle$, $G_{1\,2\,3}$ is transitive on $\Omega-\{1, 2, 3\}$. Therefore $G_{1\,2}$ is transitive or has two orbits $\{3\}$ and $\{4, 5, \cdots, n\}$ on $\Omega-\{1, 2\}$. Since $\langle a, d\rangle < N_G(G_{6\,7\,10\,11})$, there is an involution g of $G_{6\,7\,10\,11}$ commuting with a and d, Similary to f we have

$$g=(1)(2\ 4)(3\ 5)\,(6)\,(7)(8\ 9)\,(10)\,(11)\,(12\ 13)\cdots.$$

Since $\langle a, g\rangle < N_G(G_{2\,4\,6\,7})$, there is an involution h of $G_{2\,4\,6\,7}$ commuting with a and g. Then h is of the form

$$h=(1)\,(2)\,(4)\,(3\ 5)\,(6)\,(7)\cdots.$$

Hence

$$ch=(1)\,(2)\,(3\ 5\ 4)\cdots.$$

Thus $ch\in G_{1\,2}$ and so $G_{1\,2}$ is transitive on $\Omega-\{1, 2\}$. Therefore G_1 is 3–fold transitive on $\Omega-\{1\}$.

Furthermore $G_{I(c)}$ is transitive on $\{4, 5, 10, 11, \cdots, n\}$ and $G_{I(h)}$ is transitive on $\{3, 5, 8, 9, \cdots, n\}$. Since $G_{1\,2\,6\,7}\geqq\langle G_{I(c)}, G_{I(h)}\rangle$, $G_{1\,2\,6\,7}$ is transitive on $\Omega-\{1, 2, 6, 7\}$ and so G_1 is 4–fold transitive on $\Omega-\{1\}$.

By assumption a Sylow 2-subgroup of $(G_1)_{2\,3\,4\,5}$ is a nonidentity semi-regular group on $\{6, 7, \cdots, n\}$, G_1 must be S_6, A_8 or M_{12} by Theorem of [6]. Since $|\Omega|>9$, $|\Omega|=13$ and $G_1=M_{12}$. Since there is no transitive extension of M_{12}, $G=S_1\times M_{12}$.

Next assume $|\Omega|\leqq 9$. Since $|I(P)|=5$ and $P\neq 1$, $|\Omega|=7$ or 9. Now we consider the case $|\Omega|=9$. Since there is not an involution fixing seven points, G has not a transposition. Assume, by way of contradiction, that G has an odd permutation. Then there is a 2-element in G, which is an odd permutation.

First suppose that G has an element x of order 8. We may assume

$$x=(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)\ (9)\,.$$

Since

$$x^2=(1\ 3\ 5\ 7)\ (2\ 4\ 6\ 8)\ (9)\,,$$

$x^2\in N_G(G_{1\,3\,5\,7})$ and hence x^2 commutes with an involution a of $G_{1\,3\,5\,7}$. a is of the form

$$a=(1)\ (3)\ (5)\ (7)\ (2\ 6)\ (4\ 8)\ (9)\,.$$

Then $a\in N_G(G_{1\,3\,2\,6})$. Hence a commutes with one of the following elements of $G_{1\,3\,2\,6}$:

$$b_1=(1)\ (3)\ (2)\ (6)\ (4\ 8)\ (5)\ (7\ 9)\,,$$
$$b_2=(1)\ (3)\ (2)\ (6)\ (4\ 8)\ (7)\ (5\ 9)\,,$$
$$b_3=(1)\ (3)\ (2)\ (6)\ (4\ 8)\ (9)\ (5\ 7)\,.$$

Then we have

$$xb_1=(1\ 2\ 3\ 8)\ (4\ 5\ 6\ 9\ 7)\,,$$
$$xb_2=(1\ 2\ 3\ 8)\ (4\ 9\ 5\ 6\ 7)\,,$$
$$xb_3=(1\ 2\ 3\ 8)\ (4\ 7)\ (5\ 6)\ (9)\,,$$
$$(xb_1)^5=(xb_2)^5=(1\ 2\ 3\ 8)\ (4)\ (5)\ (6)\ (7)\ (9)\,.$$

Thus if G has an element of order 8, then G has an element consisting of one 4–cycle or one 4–cycle and two 2–cycles.

Suppose that G has an element x consisting of one 4–cycle and two 2–cycles. We may assume that

$$x=(1\ 2\ 3\ 4)\ (5\ 6)\ (7\ 8)\ (9)\,.$$

Since $x\in N_G(G_{1\,2\,3\,4})$, x commutes with an involution a of $G_{1\,2\,3\,4}$. a is one of the following forms:

(i) $a=(1)\ (2)\ (3)\ (4)\ (9)\ (5\ 6)\ (7\ 8)\,,$

(ii) $a=(1)\ (2)\ (3)\ (4)\ (9)\ (5\ 7)\ (6\ 8)\,.$

If a is of the form (i), then

$$xa=(1\ 2\ 3\ 4)\ (9)\ (5)\ (6)\ (7)\ (8)\,.$$

If a is of the form (ii), then $a\in N_G(G_{1\,2\,5\,7})$. Hence a commutes with one of the following elements of $G_{1\,2\,5\,7}$:

$$b_1 = (1)(2)(5)(7)(6\ 8)(3)(4\ 9),$$
$$b_2 = (1)(2)(5)(7)(6\ 8)(4)(3\ 9),$$
$$b_3 = (1)(2)(5)(7)(6\ 8)(9)(3\ 4).$$

Then we have

$$xb_1 = (1\ 2\ 3\ 9\ 4)(5\ 8\ 7\ 6),$$
$$xb_2 = (1\ 2\ 9\ 3\ 4)(5\ 8\ 7\ 6),$$
$$xb_3 = (1\ 2\ 4)(3)(5\ 8\ 7\ 6).$$

Thus

$$(xb_1)^5 = (xb_2)^5 = (xb_3)^{-3} = (1)(2)(3)(4)(5\ 8\ 7\ 6)(9).$$

Hence if G has an element of order 8 or consisting of one 4–cycle and two 2–cycles, then G has an element consisting of one 4–cycle. Therefore we may assume that G has an element x of the form

$$x = (1\ 2\ 3\ 4)(5)(6)(7)(8)(9).$$

Then

$$x^2 = (1\ 3)(2\ 4)(5)(6)(7)(8)(9).$$

Since $x^2 \in N_G(G_{1356})$, x^2 commutes with an involution a of G_{1356}. Then a is of the form

$$a = (1)(3)(5)(6)(2\ 4)(i_1)(i_2\ i_3),$$

where $\{i_1, i_2, i_3\} = \{7, 8, 9\}$. Then we have

$$xa = (1\ 4)(2\ 3)(5)(6)(i_1)(i_2\ i_3).$$

Thus if G has an odd permutation, then G has an element consisting of three 2–cycles.

Therefore finally suppose that G has an element x consisting of three 2–cycles. We may assume that

$$x = (1\ 2)(3\ 4)(5\ 6)(7)(8)(9).$$

Since $x \in N_G(G_{5678})$, x commutes with an involution a of G_{5678}. a is one of the following forms:

(i) $a = (1\ 2)(3\ 4)(5)(6)(7)(8)(9)$.

(ii) $a = (1\ 3)(2\ 4)(5)(6)(7)(8)(9)$.

If a is of the form (i), then

$$xa = (1)(2)(3)(4)(5\ 6)(7)(8)(9).$$

Thus xa is a transposition, which is a contradiction. Thus a must be of the form (ii). On the other hand $x \in N_G(G_{1256})$. Hence x commutes with an involution b of G_{1256}, and b is of the form

$$b = (1)\,(2)\,(5)\,(6)\,(3\ 4)\,(i_1)\,(i_2\, i_3)\,,$$

where $\{i_1, i_2, i_3\} = \{7, 8, 9\}$. Then

$$ab = (1\ 4\ 2\ 3)\,(5)\,(6)\,(i_1)\,(i_2\, i_3)\,.$$

Thus we have

$$x(ab)^2 = (1)\,(2)\,(3)\,(4)\,(5\ 6)\,(7)\,(8)\,(9)\,,$$

which is also a contradiction. Therefore $G \leqq A_9$.

Case III. $|I(P)| = 7$, $N_G(P)^{I(P)} \leqq A_7$.

Let $I(P) = \{1, 2, \cdots, 7\}$. The proof of this case will be given in various steps:

(1) *P is elementary abelian.*

Proof. If P has an element

$$x = (1)\,(2) \cdots (7)\,(8\ 9\ 10\ 11) \cdots,$$

then $x \in N_G(G_{8\,9\,10\,11})$. Hence x normalizes some Sylow 2-subgroup P' of $G_{8\,9\,10\,11}$. By assumption, $x^{I(P)} \in N_G(P')^{I(P')} \leqq A_7$. Thus x has a 2-cycle, contrary to the semi-regularity of P. Therefore P has no element of order 4, whence P is elementary abelian.

(2) $|\Omega| \geqq 15$.

Proof. Let

$$a = (1)\,(2) \cdots (7)\,(8\ 9) \cdots$$

be an involution of P. Then $a \in N_G(G_{1\,2\,8\,9})$. Hence a commutes with an involution b of $G_{1\,2\,8\,9}$. By assumption, $|I(b)| = 7$ and $b^{I(a)} \in A_7$. Hence we may assume

$$b = (1)\,(2)\,(3)\,(4\ 5)\,(6\ 7)\,(8)\,(9)\,(10)\,(11) \cdots.$$

Then we have

$$a = (1)\,(2) \cdots (7)\,(8\ 9)\,(10\ 11) \cdots.$$

Since $\langle a, b\rangle < N_G(G_{4\,5\,8\,9})$, there is an involution c of $G_{4\,5\,8\,9}$ commuting with a and b. By assumption, $|I(c)| = 7$, $c^{I(a)} \in A_7$ and $c^{I(b)} \in A_7$. Hence we may assume

$$c = (1)\,(2\ 3)\,(4)\,(5)\,(6\ 7)\,(8)\,(9)\,(10\ 11)\,(12)\,(13) \cdots.$$

Then we have

$$a = (1)\,(2) \cdots (7)(8\ 9)\,(10\ 11)\,(12\ 13) \cdots,$$

$$ac = (1)\,(2\ 3)\,(4)\,(5)\,(6\ 7)\,(8\ 9)\,(10)\,(11)\,(12\ 13) \cdots.$$

Since ac is an involution and $|I(ac)| \geqq 5$, $|I(ac)| = 7$. Thus ac fixes two more points in $\{14, 15, \cdots, n\}$. Hence $|\Omega| \geqq 15$.

(3) *One of the following holds:*

Case i. $N_G(P)^{I(P)}$ *is transitive.*

(i. i) $N_G(P)^{I(P)}=A_7$.

(i. ii) $N_G(P)^{I(P)}$ *is isomorphic to* $LF_2(7)$, *which will be denoted by* A_7^*.

Case ii. $N_G(P)^{I(P)}$ *has two orbits, say* Δ *and* Γ.

(ii. i) $|\Delta|=1$ *and* $|\Gamma|=6$. $N_G(P)^{I(P)}$ *is* A_6 *on* Γ, *which will be denoted by* A_6.

(ii. ii) $|\Delta|=1$ *and* $|\Gamma|=6$. $N_G(P)^{I(P)}$ *is isomorphic to* A_5 *on* Γ, *which will be denoted by* A_6^*.

(ii. iii) $|\Delta|=2$ *and* $|\Gamma|=5$. $N_G(P)^{I(P)}$ *is* $N_{A_7}(A_5)$, *which will be denoted by* $N(A_5)$.

(ii. iv) $|\Delta|=3$ *and* $|\Gamma|=4$. $N_G(P)^{I(P)}$ *is* $N_{A_7}(A_4)$, *which will be denoted by* $N(A_4)$.

(ii. v) $|\Delta|=3$ *and* $|\Gamma|=4$. $N_G(P)^{I(P)}=N_{A_7^*}(K_4)$ *where* K_4 *is a regular four group on* Γ. $N_{A_7^*}(K_4)$ *will be denoted by* $N(K_4)$.

Proof. Let

$$a=(1)\,(2)\cdots(7)(i\,j)\cdots$$

be an involution of P. For any two points i_1 and i_2 of $I(a)$, $a\in N_G(G_{i_1 i_2 i j})$. Hence there is an involution $x_{i_1 i_2}$ of $G_{i_1 i_2 i j}$ commuting with a. Set $a_{i_1 i_2}=(x_{i_1 i_2})^{I(a)}$, Then

$$a_{i_1 i_2}=(i_1)\,(i_2)\,(i_3)\,(i_4\,i_5)(i_6\,i_7)\,,$$

where $\{i_1, i_2, \cdots, i_7\}=\{1, 2, \cdots 7\}$. Let T be the restriction of the group generated by all involutions of $C_G(a)_{i\,j}$ on $I(a)$. Then $a_{i_1 i_2}\in T$.

(3.1) Suppose that T is transitive. By § 166 of [1], T is A_7 or isomorphic to $LF_2(7)$. If $T=LF_2(7)$, then $T=\langle(1\,2\,3\,6\,4\,5\,7), (2\,3\,4)\,(5\,6\,7), (2\,7\,6\,3)(4\,5)\rangle$.

(3.2) Suppose that T has an orbit of length 1. Let $\{1\}$ be the orbit of length 1 and set $\Gamma=\{2, 3, \cdots, 7\}$. Then for any two points i_1 and i_2 of Γ there is an involution $a_{i_1 i_2}$ of the form

$$a_{i_1 i_2}=(1)\,(i_1)\,(i_2)\,(i_3\,i_4)\,(i_5\,i_6)\,.$$

Thus $\langle a_{i_1 i_2}\rangle$ is a 2-group fixing exactly two points i_1 and i_2 of Γ. Hence from a lemma of D. Livingstone and A. Wagner [3. Lemma 6] T_1 is a doubly transitive group on Γ. Hence from § 166 in [1] T_1 is A_6 or isomorphic to A_5 on Γ. In the second case $T=\langle(2\,3\,4)(5\,7\,6), (3\,4\,5\,7), (3\,7)\,(5\,6)\rangle$.

(3.3) Suppose that T has an orbit of length 2. Let $\{1, 2\}$ be the orbit of length 2 and set $\Gamma=\{3, 4, \cdots, 7\}$. For any point i_1 of Γ there is an involution $a_{1\,i_1}$ of the form

$$a_{1\,i_1}=(1)\,(2)\,(i_1)\,(i_2\,i_3)(i_4\,i_5)\,.$$

Hence from Lemma 6 of [3] T_{12} is transitive on Γ. By §166 in [1] T_{12} is A_5 or a group of order 10 generated by (3 4 5 6 7) and (3 4)(5 7). Assume $|T_{12}|=10$. Then there is an element a_{34} of the form

$$a_{34}=(1\ 2)\ (3)\ (4)\ (j_1)(j_2\ j_3)\,.$$

Set $y=(3\ 4\ 5\ 6\ 7)$. Since $\langle y\rangle$ is the unique Sylow 5-subgroup of T_{12} and $a_{34}\in N_T(T_{12})$, $a_{34}\,ya_{34}=y^r$ where $r=1, 2, 3$ or 4. But this is impossible since $a_{34}\,ya_{34}=(3\ 4\cdots)$. Thus $|T_{12}|\neq 10$. Hence $T_{12}=A_5$ and so $T=N_{A_7}(A_5)$.

(3.4) Suppose that T has an orbit of length 3. Let $\{1, 2, 3\}$ be the orbit of length 3. Set $\Delta=\{1, 2, 3\}$ and $\Gamma=\{4, 5, 6, 7\}$. For any two points i_1 and i_2 of Γ there is an involution $a_{i_1 i_2}$ such that $(a_{i_1 i_2})^\Gamma=(i_1)\,(i_2)\,(i_3\,i_4)$. Hence again by Lemma 6 of [3] T^Γ is doubly transitive. Thus $T^\Gamma=S_4$. Since $T\leqq A_7$, $|T_\Gamma|=1$ or 3. For any point j_1 of Δ there is an involution $a_{j_1 4}$ such that $(a_{j_1 4})^\Delta=(j_1)\,(j_2 j_3)$. Hence similarly T^Δ is transitive on Δ, and so $T^\Delta=S_3$.

First assume $|T_\Gamma|=3$. Then

$$|T_\Delta|=|T|/|T^\Delta|=|T_\Gamma|\cdot|T^\Gamma|/|T^\Delta|=3\cdot|S_4|/|S_3|=12\,.$$

Hence $T_\Delta=A_4$ and $T\leqq N_{A_7}(A_4)$. On the other hand

$$|T|=|T_\Gamma|\cdot|T^\Gamma|=3\cdot|S_4|=|N_{A_7}(A_4)|\,.$$

Thus $T=N_{A_7}(A_4)$.

Next assume $|T_\Gamma|=1$. Then

$$|T_\Delta|=|S_4|/|S_3|=4.$$

Hence T_Δ is a regular four-group of degree 4, which is denoted by K_4. Since $|T_\Gamma|=1$, $T\lneqq N_{A_7}(K_4)$. Since $T\cong T^\Gamma=S_4$, $K_4=\langle(1)\ (2)\ (3)\ (4\ 5)\ (6\ 7), (1)\ (2)\ (3)\ (4\ 6)\ (5\ 6)\rangle$ and $T=\langle(1)\ (2)\ (3)\ (4\ 5)\ (6\ 7), (1\ 2)\ (3)\ (4)\ (6)\ (5\ 7), (1)\ (2\ 3)\ (4)\ (5)\ (6\ 7)\rangle$. Thus $T<A_7{}^*$ and so $T=N_{A_7{}^*}(K_4)$.

(3.5) Suppose that T has an orbit with length greater than 3. Then obviously T is one of the groups above.

Now $T\leqq N_G(G_{I(P)})^{I(P)}$. By Lemma 2 of [10] $N_G(G_{I(P)})^{I(P)}=N_G(P)^{I(P)}$. Hence $T\leqq N_G(P)^{I(P)}\leqq A_7$. Thus $N_G(P)^{I(P)}$ is one of the groups above.

REMARK. Since T is contained in $(C_G(a)_{i\,j})^{I(a)}$ for a 2-cycle $(i\,j)$ of a, we denote T by $\mathfrak{T}_{i\,j}(a)$.

(4) *Let x be an arbitrary involution of G. Then $|I(x)|=7$.*

Proof. Since $|\Omega|$ is odd, $|I(x)|$ is odd. Let x be of the form

$$x=(i\,j)\ (k\ l)\cdots\,.$$

Then x normalizes some Sylow 2-subgroup P' of $G_{i\,j\,k\,l}$. By assumption $x^{I(P')}\in A_7$. Therefore $|I(x)|\geqq 3$. If $|I(x)|\geqq 4$, then $|I(x)|=7$ by assumption.

Suppose by way of contradiction that $|I(x)|=3$. We may assume that x is of the form

$$x=(1)\,(2)\,(3)\,(4\;5)\,(6\;7)\,(8\;9)\cdots.$$

Since $x\in N_G(G_{4567})$, there is an involution a of G_{4567} commuting with x. Since $|I(a)|=7$ and $x^{I(a)}\in A_7$, $I(a)=\{1, 2, \cdots, 7\}$.

First assume that x and a have the same 2-cycle (8, 9) namely

$$a=(1)\,(2)\cdots(7)\,(8\;9)\cdots.$$

Then ax is an involution and $|I(ax)|\supset\{1, 2, 3, 8, 9\}$. Hence $|I(ax)|=7$. Thus x and a have two 2-cycles in common. Therefore we may assume that

$$x=(1)\,(2)\,(3)\,(4\;5)\,(6\;7)\,(8\;9)\,(10\;11)\cdots,$$
$$a=(1)\,(2)\cdots(7)\,(8\;9)\,(10\;11)\cdots.$$

Then $\langle a, x\rangle$ is semi-regular on $\{12, 13, \cdots, n\}$. On the other hand since $\langle a, x\rangle < N_G(G_{4589})$, there is an involution b of G_{4589} commuting with a and x. Since $b^{I(a)}\in A_7$ and $b^{I(ax)}\in A_7$, we may assume that

$$b=(1)\,(2\;3)\,(4)\,(5)\,(6\;7)\,(8)\,(9)\,(10\;11)\cdots.$$

Since $|I(b)|=7$, b fixes exactly two more points of $\{12, 13, \cdots, n\}$. But this is impossible since $b\in C_G(\langle a, x\rangle)$ and $\langle a, x\rangle$ is semi-regular on $\{12, 13, \cdots, n\}$.

Thus a and x have not the same 2-cycle. Therefore we may assume that

$$x=(1)\,(2)\,(3)\,(4\;5)\,(6\;7)\,(8\;9)\,(10\;11)\cdots,$$
$$a=(1)\,(2)\cdots(7)\,(8\;10)\,(9\;11)\cdots.$$

Let $(i_1 j_1)$ be an arbitrary 2-cycle of x other than (4 5). Then x normalizes some Sylow 2-subgroup P' of $G_{45i_1j_1}$. Since $x\in N_G(P')^{I(P')}\leqq A_7$, $I(P')=\{1, 2, 3, 4, 5, i_1, j_1\}$. Hence P' is also a Sylow 2-subgroup of G_{12345}. By the conjugacy of Sylow 2-subgroups of G_{12345} we have that for any other 2-cycle $(i_2 j_2)$ $(\neq(4\;5))$ of x there is an element of G_{12345} which takes $\{i_1, j_1\}$ into $\{i_2, j_2\}$. Therefore the number of G_{12345}-orbits in Ω–$\{1, 2, 3, 4, 5\}$ is one or two. If it is one, then since $P'\leqq G_{12345i_1}$, $|\Omega|-5=|G_{12345}: G_{12345i_1}|$ is odd, which is a contradiction. Hence it must be two and 6 and 7 belong to different orbits of G_{12345}, say T_6 and T_7 respectively. Obviously $|T_6|=|T_7|>1$. Thus G_{1234} is transitive or has three orbits $\{5\}$, T_6, T_7 on $\{5, 6, \cdots, n\}$ since P' is also a Sylow 2-subgroup of G_{1234}.

Now since $\langle a, x\rangle < N_G(G_{891011})$, there is an involution c of G_{891011} commuting with a and x. Since $x^{I(c)}\in A_7$, c fixes $\{1, 2, 3\}$ pointwise. Hence by the same argument as is used above for a x and c have not the same 2-cycle. Since $c^{I(a)}\in A_7$, we have

$$c=(1)\,(2)\,(3)\,(4\;6)\,(5\;7)\,(8)\,(9)\,(10)\,(11)\cdots.$$

Since $\langle x, c\rangle < G_{1\,2\,3}$ and $\{4, 5, 6, 7\}$ is a $\langle x, c\rangle$–orbit, $G_{1\,2\,3}$ is transitive on Ω–$\{1, 2, 3\}$.

Next since $\langle a, c\rangle < N_G(G_{4\,6\,8\,10})$, there is an involution d of $G_{4\,6\,8\,10}$ commuting with a and c. Since $d^{I(a)} \in A_7$, we may assume that

$$d = (1)\,(2\;3)\,(4)\,(6)\,(5\;7)\,(8)\,(10)\,(9\;11)\,\cdots .$$

Then $d \in N_G(G_{1\,2\,3\,4})$. Hence if $G_{1\,2\,3\,4}$ is intransitive on $\{5, 6, \cdots, n\}$, then d must fix the $G_{1\,2\,3\,4}$–orbit $\{5\}$, which is impossible. Thus $G_{1\,2\,3\,4}$ is transitive on $\{5, 6, \cdots, n\}$.

Therefore $G_{1\,2\,3}$ is doubly transitive on $\{4, 5, \cdots, n\}$. Since $G_{1\,2\,3\,4\,5}$ has two orbits of odd length in $\{6, 7, \cdots, n\}$, $G_{1\,2\,3\,4\,6}$ has exactly two orbits of odd length in $\{5, 7, 8, \cdots, n\}$ by the doubly transitivity of $G_{1\,2\,3}$. Since $a \in G_{1\,2\,3\,4\,6}$ and a fixes exactly two points 5 and 7 of $\{5, 7, 8, \cdots, n\}$, 5 and 7 belong to different G_{12346}–orbits, say T_5' and T_7' respectively. Since $d \in N_G(G_{1\,2\,3\,4\,6})$ d fixes two orbits T_5' and T_7' or interchanges them. But this is impossible since d has a 2-cycle (5 7) and fixes a point 8. This contradiction shows that $|I(x)| \neq 3$. Hence $|I(x)| = 7$.

(5) $|\Omega| \geqq 23$ and $|\Omega| - 7 \equiv 0$ (*mod* 8).

Proof. By (2) $|\Omega| \geqq 15$. Let

$$a = (1)\,(2) \cdots (7)\,(8\;9)\,(10\;11)\,(12\;13)\,(14\;15) \cdots$$

be an involution of P. Then there is an involution b of $G_{1\,2\,8\,9}$ commuting with a. Since $|I(b)| = |I(ab)| = 7$, we may assume that b is of the form

$$b = (1)\,(2)\,(3)\,(4\;5)\,(6\;7)\,(8)\,(9)\,(10)\,(11)\,(12\;13)\,(14\;15) \cdots .$$

Since $\langle a, b\rangle < N_G(G_{4\,5\,8\,9})$, there is an involution c of $G_{4\,5\,8\,9}$ commuting with a and b. Since $|I(c)| = |I(ac)| = |I(bc)| = |I(abc)| = 7$, we may assume that c is of the form

$$c = (1)\,(2\;3)\,(4)\,(5)\,(6\;7)\,(8)\,(9)\,(10\;11)\,(12)\,(13)\,(14\;15) \cdots .$$

Suppose $|\Omega| > 15$. Since $\langle a, b, c\rangle$ is an elementary abelian group and every involutions of $\langle a, b, c\rangle$ fix exactly seven points of $\{1, 2, \cdots, 15\}$, $\langle a, b, c\rangle$ is semi-regular on $\{16, 17, \cdots, n\}$. Since $|\langle a, b, c\rangle| = 8$, $|\Omega| = 15 + 8k$ where $k \geqq 1$. Hence

$$|\Omega| \geqq 23 \text{ and } |\Omega| - 7 \equiv 0 \pmod 8.$$

Therefore to complete the proof we must show that $|\Omega| \neq 15$. Suppose by way of contradiction that $|\Omega| = 15$. Since $b^{I(a)}$ and $c^{I(a)}$ are elements of $\mathfrak{T}_{8\,9}(a)$, we may assume that $\mathfrak{T}_{8\,9}(a)$ is one of the following:

(*a*) $\mathfrak{T}_{8\,9}(a) = A_7$ or A_7^*,

(*b*) $\mathfrak{T}_{8\,9}(a) = A_6$ or A_6^*, and its orbits are $\{1\}$ and $\{2, 3, \cdots, 7\}$.

(*c*) $\mathfrak{T}_{8\,9}(a) = N(A_5)$ and its orbits are $\{2, 3\}$ and $\{1, 4, 5, 6, 7\}$, (*d*) $\mathfrak{T}_{89}(a)=N(A_4)$ or $N(K_4)$, and its orbits are $\{1, 2, 3\}$ and $\{4, 5, 6, 7\}$.

First assume that $\mathfrak{T}_{89}(a)\neq A_6{}^*$. Since $b^{I(a)}=(1)(2)(3)(4\ 5)(6\ 7)$, by (3) there is an involution x of $C_G(a)_{89}$ such that x is of the form

$$x=(1)(2)(3)(4\ 6)(5\ 7)(8)(9)\cdots.$$

Then we have

$$bx=(1)(2)(3)(4\ 7)(5\ 6)(8)(9)\cdots.$$

Since $|I(bx)|\geqq 5$, bx is of order $2r$ where r is odd. Hence $y=(bx)^r$ is an involution commuting with b and so $|I(y)|=|I(by)|=7$. Since $y^{I(b)}\in A_7$

$$y=(1)(2)(3)(4\ 7)(5\ 6)(8)(9)(10)(11)(12\ 14)(13\ 15).$$

Then we have

$$ay=(1)(2)(3)(4\ 7)(5\ 6)(8\ 9)(10\ 11)(12\ 15)(13\ 14).$$

Thus ay is an involution fixing exactly three points, which contradicts (4).

Next assume that $\mathfrak{T}_{89}(a)=A_6{}^*$. Since $b^{I(a)}=(1)(2)(3)(4\ 5)(6\ 7)$ and $c^{I(a)}=(1)(2\ 3)(4)(5)(6\ 7)$ belong to $\mathfrak{T}_{89}(a)$, by (3) there is an involution z of $C_G(a)_{89}$ such that z is of the form

$$z=(1)(2)(6)(3\ 5)(4\ 7)(8)(9)\cdots.$$

Since az fixes three points 1, 2, 6 of $\{1, 2, \cdots, 9\}$, az fixes four more points of $\{10, 11, \cdots, 15\}$. Therefore z must be one of the following forms:

(i) $z=(1)(2)(6)(3\ 5)(4\ 7)(8)(9)(10\ 11)\cdots$,

(ii) $z=(1)(2)(6)(3\ 5)(4\ 7)(8)(9)(12\ 13)\cdots$.

If z is of the form (i), then

$$bz=(1)(2)(3\ 5\ 7\ 6\ 4)(8)(9)(10\ 11)\cdots.$$

Hence $(bz)^5$ is of even order and fixes at least nine points, which is a contradiction. If z is of the form (ii), then

$$cz=(1)(2\ 5\ 3)(4\ 7\ 6)(8)(9)(12\ 13)\cdots.$$

Then similary we have a contradiction. Thus $|\Omega|\neq 15$.

(6) *If $|P|\geqq 4$, then $|P|\geqq 8$ and $G_{I(P)}$ is transitive on $\Omega-I(P)$. In particular if $N_G(P)^{I(P)}=A_6{}^*$, $N(A_5)$, $N(A_4)$ or $N(K_4)$, then P and $G_{I(P)}$ have these properties.*

The proof is by steps.

(6.1) *If $N_G(P)^{I(P)}$ is $A_6{}^*$, $N(A_5)$, $N(A_4)$ or $N(K_4)$, then $|P|\geqq 4$.*

Proof. We may assume that if $N_G(P)^{I(P)}=A_6{}^*$, then its orbits are $\{1\}$ and $\{2, 3, \cdots, 7\}$, if $N_G(P)^{I(P)}=N(A_5)$, then its orbits are $\{2, 3\}$ and $\{1, 4, 5, 6, 7\}$

and if $N_G(P)^{I(P)}=N(A_4)$ or $N(K_4)$, then its orbits are $\{1, 2, 3\}$ and $\{4, 5, 6, 7\}$. Let

$$a=(1)(2)\cdots(7)(8\ 9)(10\ 11)(12\ 13)(14\ 15)(16\ 17)(18\ 19)\cdots$$

be an involution of P. Then there is an involution b of G_{4589} commuting with a. By the assumption on the orbits of $N_G(P)^{I(P)}$ we may assume that

$$b=(1)(2\ 3)(4)(5)(6\ 7)(8)(9)(10)(11)(12\ 13)(14\ 15)(16\ 18)(17\ 19)\cdots.$$

Furthermore there is an involution c of $G_{16\,17\,18\,19}$ commuting with a and b. Since $a^{I(c)}\in A_7$ and $b^{I(c)}\in A_7$,

$$c=(1)(4)(5)(2\ 6)(3\ 7)(16)(17)(18)(19)\cdots$$

or

$$c=(1)(4)(5)(2\ 3)(6\ 7)(16)(17)(18)(19)\cdots.$$

Suppose that c is of the first form. If $N_G(P)^{I(P)}=N(A_5)$, $N(A_4)$ or $N(K_4)$, then 2 and 6 belong to different orbits, which is a contradiction. If $N_G(P)^{I(P)}=A_6{}^*$, then $|(N_G(P)^{I(P)})_{145}|=2$, which is also a contradiction. Thus c must be of the second form. Then we have

$$bc=(1)(2)\cdots(7)(16\ 18)(17\ 19)\cdots.$$

Hence $\langle a, bc\rangle$ is a four-group in $G_{I(P)}$. Thus a Sylow 2-subgroup P of $G_{I(P)}$ is of order at least 4.

(6.2) *If* $|P|\geqq 4$, *then* $|P|\geqq 8$ *and* $G_{I(P)}$ *is transitive on* $\Omega-I(P)$.

Proof. Suppose by way of contradiction that $|P|=4$. Since P is a semiregular elementary abelian group, the automorphisum group $A(P)$ of P is isomorphic to S_3. Obviously $A(P)\geqq N_G(P)/C_G(P)$. If $N_G(P)_{I(P)}\geqq C_G(P)$, then $N_G(P)/N_G(P)_{I(P)}$ is a homomorphic image of a subgroup of $A(P)$. But this is impossible since $N_G(P)/N_G(P)_{I(P)}\cong N_G(P)^{I(P)}$ and $A(P)\cong S_3$. Hence $N_G(P)_{I(P)}$ $\ngeqq C_G(P)$. Thus $N_G(P)^{I(P)}\trianglerighteq C_G(P)^{I(P)}\gneqq 1$.

First suppose $N_G(P)^{I(P)}=A_7$, $A_7{}^*$, A_6 or $A_6{}^*$. Then $N_G(P)^{I(P)}$ is a simple group. Hence $N_G(P)^{I(P)}=C_G(P)^{I(P)}$.

Next suppose $N_G(P)^{I(P)}=N(A_5)$, $N(A_4)$ or $N(K_4)$. Then we may assume that $N_G(P)^{I(P)}$ has the orbits mentioned in (6.1). We have also three involutions a, b and c, which are used in the proof of (6. 1). Since $|P|=4$, we may assume that $P=\langle a, bc\rangle$. Then $b^{I(P)}=(1)(2\ 3)(4)(5)(6\ 7)\in C_G(P)^{I(P)}$. Since $b^{I(P)}$ is not contained in a proper normal subgroup of $N_G(P)^{I(P)}$ in these cases, $N_G(P)^{I(P)}=C_G(P)^{I(P)}$.

Now $N_G(P)/N_G(P)_{I(P)}\trianglerighteq(C_G(P)\cdot N_G(P)_{I(P)})/N_G(P)_{I(P)}$. Since $N_G(P)/N_G(P)_{I(P)}\cong N_G(P)^{I(P)}$ and $(C_G(P)\cdot N_G(P)_{I(P)})/N_G(P)_{I(P)}\cong C_G(P)/N_G(P)_{I(P)}\cap C_G(P)=C_G(P)/C_G(P)_{I(P)}\cong C_G(P)^{I(P)}$, $N_G(P)/N_G(P)_{I(P)}=(C_G(P)\cdot N_G(P)_{I(P)})/N_G(P)_{I(P)}$. Hence $N_G(P)=C_G(P)\cdot N_G(P)_{I(P)}$. Thus $N_G(P)/C_G(P)=(C_G(P)\cdot$

$N_G(P)_{I(P)})/C_G(P) \cong N_G(P)_{I(P)}/C_G(P) \cap N_G(P)_{I(P)} = N_G(P)_{I(P)}/C_G(P)_{I(P)}$. On the other hand P is a Sylow 2-subgoup of $N_G(P)_{I(P)}$ and contained in $C_G(P)_{I(P)}$. Hence $|N_G(P)_{I(P)}/C_G(P)_{I(P)}|$ is odd and so $|N_G(P)/C_G(P)|$ is odd. Therefore every 2-elements of $N_G(P)$ belong to $C_G(P)$.

Let

$$a = (1)(2) \cdots (7)(8\ 9) \cdots$$

be an involution of P. For an arbitrary 2-cycle $(i\ j)$ of a other than $(8\ 9)$, there is an involution x of $G_{8\,9\,i\,j}$ commuting with a. Then x normalizes some Sylow 2-subgroup P' of $G_{I(P)}$ containing a. By the argument above $x \in C_G(P')$. Since $|P'|=4$ and x fixes exactly four points $8, 9, i, j$ of $\Omega - I(P')$, P' has an involution

$$a' = (1)(2) \cdots (7)(8\ i)(9\ j) \cdots .$$

Therefore $\langle a, a' \rangle$ is a subgroup of $G_{I(P)}$ and $\langle a, a' \rangle$ is transitive on $\{8, 9, i, j\}$. Since $(i\ j)$ is an arbitrary 2-cycle of a other than $(8\ 9)$, $G_{I(P)}$ is transitive on $\Omega - I(P)$. Since $|\Omega - I(P)| \equiv 0 \pmod 8$ by (5), $|G_{I(P)}| \equiv 0 \pmod 8$. But a Sylow 2-subgroup of $G_{I(P)}$ is of order 4, which is a contradiction. Thus $|P| \geqq 8$.

Next we shall prove that $G_{I(P)}$ is transitive on $\Omega - I(P)$. Let

$$a = (1)(2) \cdots (7)(8\ 9) \cdots$$

be an involution of P. For an arbitrary 2-cycle $(i\ j)$ of a other than $(8\ 9)$, there is an involution x of $G_{8\,9\,i\,j}$ commuting with a. Then x normalizes smoe Sylow 2-subgroup P' of $G_{I(P)}$ containing a. If x commutes with only two elements of P', then by a theorem of H. Zassenhaus [12, Satz 5] P' contains a cyclic group of index 2. Since $|P'| \geqq 8$ and P' is elementary abelian, we have a contradiction. Thus x commutes with some involution of P' other than a. Therefore by the same argument above we have that $G_{I(P)}$ is transitive on $\Omega - I(P)$.

(7) $N_G(P)^{I(P)} \neq N(A_5)$.

Proof. Suppose by way of contradiction that $N_G(P)^{I(P)} = N(A_5)$. We may assume that $N_G(P)^{I(P)}$-orbits are $\{1, 2\}$ and $\{3, 4, \cdots, 7\}$. Let

$$a = (1)(2) \cdots (7)(8\ 9)(10\ 11)(12\ 13)(14\ 15) \cdots$$

be an involution of P. Since $\mathfrak{X}_{8\,9}(a) \leqq N_G(P)^{I(P)} = N(A_5)$, $\mathfrak{X}_{8\,9}(a) = N(A_5)$. Therefore there are involutions

$$b = (1)(2)(3)(4\ 5)(6\ 7)(8)(9) \cdots$$

and

$$c = (1)(2)(3)(4\ 6)(5\ 7)(8)(9) \cdots$$

such that b and c commute with a. Then we have

$$bc = (1)(2)(3)(4\ 7)(5\ 6)(8)(9) \cdots .$$

Since $|I(bc)|\geqq 5$, bc is of order $2r$ where r is odd. Therefore $d=(bc)^r$ is an involution commuting with b. Since $|I(b)|=|I(ab)|=7$, we may assume that

$$b=(1)\,(2)\,(3)\,(4\ 5)\,(6\ 7)\,(8)\,(9)\,(10)\,(11)\,(12\ 13)\,(14\ 15)\cdots.$$

Then since $d^{I(b)}\in A_7$,

$$d=(1)\,(2)\,(3)\,(4\ 7)\,(5\ 6)\,(8)\,(9)\,(10)\,(11)\cdots.$$

Since $\langle b, d\rangle$ is of order 4, $G_{1\,2\,3\,8\,9\,10\,11}$ is transitive on $\Omega-\{1, 2, 3, 8, 9, 10, 11\}$ by (6). Since $N_G(P)^{I(P)}=N(A_5)$, also by (6) $G_{1\,2\cdots 7}$ is transitive on $\Omega-\{1, 2, \cdots, 7\}$. Thus $G_{1\,2\,3}$ is transitive on $\{4, 5, \cdots, n\}$.

On the other hand $\{3, 4, \cdots, 7\}$ is the orbit of $N_G(P)$. Hence $G_{1\,2}$ is transitiveon on $\{3, 4, \cdots, n\}$. Therefore G is transitive on Ω or G-orbits are $\{1, 2\}$ and $\{3, 4, \cdots, n\}$.

Now suppose that G-orbits are $\{1, 2\}$ and $\{3, 4, \cdots, n\}$. There is an involution f of $G_{4\,5\,8\,9}$ commuting with a and b. Since $\{1, 2\}$ is the G-orbit.

$$f=(1\ 2)\,(3)\,(4)\,(5)\,(6\ 7)\,(8)\,(9)\,(10\ 11)\,(12)\,(13)\,(14\ 15)\cdots.$$

Since $G_{1\,4\,5\,8}$ fixes $\{2\}$, a Sylow 2-subgroup of $G_{1\,4\,5\,8}$ is also a Sylow 2-subgroup of $G_{1\,2\,4\,5\,8}$. Since $\langle b, f\rangle<N_G(G_{1\,2\,4\,5\,8})$, there is an involution x of $G_{1\,2\,4\,5\,8}$ commuting with b and f. Let $I(x)=\{1, 2, 4, 5, 8, i_1, i_2\}$. Then

$$b^{I(x)}=(1)\,(2)\,(4\ 5)\,(8)\,(i_1\ i_2)\,,$$

$$f^{I(x)}=(1\ 2)\,(4)\,(5)\,(8)\,(i_1\ i_2)\,.$$

Hence $(i_1\ i_2)=(6\ 7)$ or $(14\ 15)$.

First assume that $(i_1\ i_2)=(6\ 7)$. Then $I(x)=\{1, 2, 4, 5, 6, 7, 8\}$. Since $\{1, 2\}$ is the G-orbit, $N_G(G_{I(x)})^{I(x)}=N(A_5)$. Hence $G_{I(x)}$ is transitive on $\Omega-I(x)$ by (6). On the other hand $G_{1\,2\cdots 7}$ is transitive on $\{8, 9, \cdots, n\}$. Hence $G_{1\,2\,4\,5\,6\,7}$ is transitive on $\{3, 8, 9, \cdots, n\}$. Since a Sylow 2-subgroup of $G_{1\,2\,4\,5\,6\,7}$ is a Sylow 2-subgroup of $G_{1\,2\,4\,5\,6}$ and $|\{3, 7, 8, \cdots, n\}|$ is even, $G_{1\,2\,3\,4\,5\,6}$ has two orbits $\{7\}$, $\{3, 8, \cdots, n\}$ on $\{3, 7, 8, \cdots, n\}$. Since $N_G(P)^{I(P)}=N(A_5)$, there is an element

$$z=(1\ 2)\,(3\ 7)\,(4)\,(5)\,(6)\cdots.$$

Since $z\in N_G(G_{1\,2\,4\,5\,6})$, z fixes the $G_{1\,2\,4\,5\,6}$-orbit $\{7\}$, which is a contradiction.

Next assume that $(i_1\ i_2)=(14\ 15)$. Then $I(x)=\{1, 2, 4, 5, 8, 14, 15\}$. Since $x^{I(b)}\in A_7$ and $x^{I(f)}\in A_7$,

$$x=(1)\,(2)\,(4)\,(5)\,(8)\,(14)\,(15)\,(3\ 9)\,(10\ 11)\,(6\ 7)\,(12\ 13)\cdots.$$

Then we have

$$ax=(1)\,(2)\,(3\ 9\ 8)\,(4)\,(5)\,(6\ 7)\,(10)\,(11)\,(12)\,(13)\cdots.$$

Thus ax is of even order and $|I(ax)|\geqq 8$, which is a contradiction.

Therefore G must be transitive on Ω. Let R be a Sylow 2-subgroup of

$N_G(P)_1$. Since $N_G(P)^{I(P)}=N(A_5)$, R has three orbits of length 1 and one orbit of length 4 on $I(P)$. On the other hand since $|P|\geqq 8$, R-orbits in $\Omega-I(P)$ are of length at least 8. Therefore if Q be a 2-group of G_1 containing R as a normal subgroup, then Q fixes $I(P)$. Since $R_{I(P)}=P$, Q normalizes P. Thus $Q\in N_G(P)_1$ and so $Q=R$, namely R is a Sylow 2-subgroup of G_1. Similarly a Sylow 2-subgroup R' of $N_G(P)_3$ is a Sylow 2-subgroup of G_3. By assumption R' has the orbit $\{1, 2\}$ of length 2. Since G is transitive, G_1 is conjugate to G_3. Hence R is conjugate to R', which is impossible.

Thus there is no group such that $N_G(P)^{I(P)}=N(A_5)$.

(8) $N_G(P)^{I(P)}\neq N(A_4)$ *and* $N(K_4)$.

Proof. Suppose by way of contradiction that $N_G(P)^{I(P)}=N(A_4)$ or $N(K_4)$. We may assume that $N_G(P)^{I(P)}$-orbits are $\{1, 2, 3\}$ and $\{4, 5, 6, 7\}$. Let

$$a=(1)\,(2)\cdots(7)\,(8\;9)\,(10\;11)\cdots$$

be an involution of P. As in the proof of (7) there are commuting involutions b and d in $C_G(a)_{8\,9}$:

$$b=(1)\,(2)\,(3)\,(4\;5)\,(6\;7)\,(8)\,(9)\,(10)\,(11)\cdots,$$
$$d=(1)\,(2)\,(3)\,(4\;7)\,(5\;6)\,(8)\,(9)\,(10)\,(11)\cdots,$$

Let R and R' be Sylow 2-subgroups of $N_G(P)_1$ and $N_G(P)_4$ respectively. Since $N_G(P)^{I(P)}=N(A_4)$ or $N(K_4)$, by the same argument as in the proof of (7) $G_{1\,2\,3}$ is transitive on $\{4, 5, \cdots, n\}$, and R and R' are Sylow 2-subgroups of G_1 and G_4 respectively. Since R fixes exactly one point and R' fixes exactly two points, R and R' are not conjugate in G. Thus G_1 and G_4 are not conjugate in G and hence G is intransitive on Ω.

Therefore G has exactly two orbits $\{1, 2, 3\}$ and $\{4, 5, \cdots, n\}$. Set $\Delta=\{4, 5, \cdots, n\}$. Since $\langle a, b\rangle < N_G(G_{4\,5\,8\,9})$, there is an involution f of $G_{4\,5\,8\,9}$ commuting with a and b. Then we may assume that

$$f=(1)\,(2\;3)\,(4)\,(5)\,(6\;7)\,(8)\,(9)\,(10\;11)\,(12)\,(13)\cdots.$$

Let P' be a Sylow 2-subgroup of $G_{4\,5\,8\,9}$ containing f. Since $\{1, 2, 3\}$ is the G-orbit, $\{1\}$ is a $N_G(P')^{I(P')}$-orbit. Hence $N_G(P')^{I(P')}=A_6$ or $A_6{}^*$.

Since $\{5, 6, 7\}$ is the $N_G(P)_4$-orbit, $\{5, 8, 9, 12, 13\}$ is the $N_G(P')_4$-orbit and $\{8, 9, \cdots, n\}$ is the $G_{I(P)}$-orbit, G_4 is transitive on $\Omega-\{4\}$.

Since $\{4, 5, 6, 7\}$ is the $N_G(P)$-orbit, P is a Sylow 2-subgroup of $G_{4\,5\,6}$ and $|I(P)\cap\Delta|=4$. On the other hand since $\{1\}$ and $\{2, 3, \cdots, 7\}$ are the $N_G(P')$-orbits, P' is a Sylow 2-subgroup of $G_{4\,5\,8}$ and $|I(P')\cap\Delta|=6$. Thus P and P' are not conjugate in $G_{4\,5}$ and hence $G_{4\,5}$ is intransitive on $\Delta-\{4, 5\}$.

Therefore $G_{4\,5}$ has two orbits $\{6, 7\}$ and $\{8, 9, \cdots, n\}$ on $\Delta-\{4, 5\}$. Let P'' be a Sylow 2-subgroup of $G_{4\,5\,6\,8}$. Then P'' fixes one or three points of the G-orbit $\{1, 2, 3\}$. If $I(P'')=\{1, 2, \cdots, 6, 8\}$, then $\{1, 2, 3\}$ is a $N_G(P'')$-orbit.

Hence $N_G(P'')^{I(P'')}=N(A_4)$ or $N(K_4)$. By the same argument as is used for P, $\{6, 8\}$ is a $G_{4\,5}$-orbit, which is a contradiction. Therefore $I(P'')=\{j_1, 4, 5, 6, 8, k_1, k_2\}$, where $j_1\in\{1, 2, 3\}$ and $\{k_1, k_2\}\subset\Omega-\{1, 2, 3, 4, 5, 6, 8\}$. Then $\{j_1\}$ is a $N_G(P'')^{I(P'')}$-orbit. Thus $N_G(P'')^{I(P'')}=A_6$ or $A_6{}^*$. Since P'' has an orbit of length 2 in $\{1, 2, 3\}$ and is semi-regular, $|P''|=2$. Therefore by (6) $N_G(P'')^{I(P'')}=A_6$. Hence $\{6, 8, k_1, k_2\}$ is a $N_G(P'')_{4\,5}$-orbit, which is a contradiction.

Thus we have no group such that $N_G(P)^{I(P)}=N(A_4)$ or $N(K_4)$.

(9) $N_G(P)^{I(P)}\neq A_6{}^*$. *If* $N_G(P)^{I(P)}=A_6$, *then* $|P|=2$.

Proof. If $N_G(P)^{I(P)}=A_6{}^*$, then $|P|\geqq 8$ by (6). Therefore suppose by way of contradiction that $N_G(P)^{I(P)}=A_6$ or $A_6{}^*$ and $|P|\geqq 4$. We may assume that $N_G(P)^{I(P)}$–orbits are $\{1\}$ and $\{2, 3, \cdots, 7\}$. Let

$$a=(1)\,(2)\cdots(7)\,(8\;9)\,(10\;11)\cdots$$

be an involution of P. Since $a\in N_G(G_{2\,3\,8\,9})$, there is an involution b of $G_{2\,3\,8\,9}$ commuting with a. We may assume

$$b=(1)\,(2)\,(3)\,(4\;5)\,(6\;7)\,(8)\,(9)\,(10)\,(11)\cdots.$$

Let P' be a Sylow 2-subgroup of $G_{I(b)}$ containing b.

Assume that G is intransitive on Ω. By (6) $G_{I(P)}$ is transitive on $\{8, 9, \cdots, n\}$, and $\{1\}$, $\{2, 3, \cdots, 7\}$ are $N_G(P)^{I(P)}$–orbits. On the other hand $I(b)=\{1, 2, 3, 8, 9, 10, 11\}$ and $N_G(G_{I(b)})^{I(b)}=A_7$, $A_7{}^*$, A_6 or $A_6{}^*$. Therefore G has two orbits $\{1\}$ and $\{2, 3, \cdots, n\}$. Then $G=G_1$ satisfies the condition (*) of [9], which is a contradiction. Thus G must be transitive on Ω.

Since $|P|\geqq 8$ by (6), a Sylow 2–subgroup of $N_G(P)_1$ is a Sylow 2-subgroup of G_1 and fixes exactly one point. Similarly a Sylow 2-subgroup of $N_G(P)_2$ is a Sylow 2-subgroup of G_2 and fixes exactly three points. Thus G_1 and G_2 are not conjugate in G, which contradicts the transitivity of G. Thus we complete the proof of (9).

(10) *There are four points i, j, k and l of Ω such that a Sylow 2-subgroup of $G_{i\,j\,k\,l}$ is of order at least* 4.

Proof. Suppose by way of contradiction that for any four points i, j, k and l a Sylow 2-subgroup of $G_{i\,j\,k\,l}$ is of order 2. Let

$$a=(1)\,(2)\cdots(7)\,(8\;9)\,(10\;11)\,(12\;13)\,(14\;15)\,(16\;17)\,(18\;19)\cdots$$

be an involution. Since $a\in N_G(G_{8\,9\,10\,11})$, there is an involution b of $G_{8\,9\,10\,11}$ commuting with a. We may assume that

$$b=(1)\,(2)\,(3)\,(4\;5)\,(6\;7)\,(8)\,(9)\,(10)\,(11)\,(12\;13)\,(14\;15)\,(16\;18)\,(17\;19)\cdots.$$

Since $\langle a, b\rangle<N_G(G_{16\,17\,18\,19})$, there is an involution c of $G_{16\,17\,18\,19}$ commuting with a and b. Then $c^{I(a)}$ is one of the following:

(i) $c^{I(a)}=(1)\,(2\ 3)\,(4)\,(5)\,(6\ 7)$,

(ii) $c^{I(a)}=(1)\,(2)\,(3)\,(4\ 5)\,(6\ 7)$,

(iii) $c^{I(a)}=(1)\,(2)\,(3)\,(4\ 6)\,(5\ 7)$.

Assume $c^{I(a)}$ is of the form (i). Since $c^{I(b)}\in A_7$,

$$c=(1)\,(2\ 3)\,(4)\,(5)\,(6\ 7)\,(8)\,(9)\,(10\ 11)\,(16)\,(17)\,(18)\,(19)\cdots.$$

Thus $|I(c)|\geqq 9$, which is a contradiction.

Next assume that $c^{I(a)}$ is of the form (ii). Then $\langle bc, a\rangle$ is a subgroup of $G_{I(P)}$ and of order 4, contrary to the assumption.

Therefore $c^{I(a)}$ must be of the form (iii). Then similarly $c^{I(b)}$ and $a^{I(b)}$ have no 2-cycle in common, $c^{I(ab)}$ and $a^{I(ab)}$ also have no 2-cycle in common. Therefore

$$c=(1)\,(2)\,(3)\,(4\ 6)\,(5\ 7)\,(8\ 10)\,(9\ 11)\,(12\ 14)\,(13\ 15)\,(16)\,(17)\,(18)\,(19)\cdots.$$

On the other hand $\langle a, b\rangle<N_G(G_{4\,5\,8\,9})$. Hence there is an involution d of $G_{4\,5\,8\,9}$ commuting with a and b. Then

$$d=(1)\,(2\ 3)\,(4)\,(5)\,(6\ 7)\,(8)\,(9)\,(10\ 11)\,(12)\,(13)\,(14\ 15)\cdots.$$

Therefore we have

$$cd=(1)\,(2\ 3)\,(4\ 7\ 5\ 6)\,(8\ 11\ 9\ 10)\,(12\ 15\ 13\ 14)\cdots,$$

$$a(cd)^2=(1)\,(2)\,(3)\,(4\ 5)\,(6\ 7)\,(8)\,(9)\cdots(14)\cdots.$$

Thus $a(cd)^2$ is of even order and $|I(a(cd)^2)|\geqq 11$, which is a contradiction. Thus (10) is proved.

(11) $G=M_{23}$.

Proof. By (10) we may assume that $|P|\geqq 4$. Then by (6) and (9) $N_G(P)^{I(P)}=A_7$ or A_7^* and $G_{I(P)}$ is transitive on $\Omega-I(P)$. Hence G is transitive on Ω or has two orbits $\{1, 2, \cdots, 7\}$ and $\{8, 9, \cdots, n\}$. Let

$$a=(1)\,(2)\cdots(7)\,(8\ 9)\,(10\ 11)\cdots$$

be an involution of P. Since $a\in N_G(G_{1\,2\,8\,9})$, there is an involution b of $G_{1\,2\,8\,9}$ commuting with a. We may assume that

$$b=(1)\,(2)\,(3)\,(4\ 5)\,(6\ 7)\,(8)\,(9)\,(10)\,(11)\cdots.$$

By (9) $N_G(G_{I(b)})^{I(b)}=A_7$, A_7^* or A_6. Hence G is transitive on Ω.

Now we may assume that if $N_G(G_{I(b)})^{I(b)}=A_6$ then its orbits are $\{1\}$ and $\{2, 3, 8, 9, 10, 11\}$. Then since $G_{I(P)}$ is transitive on $\{8, 9, \cdots, n\}$, and $\{2, 3, \cdots, 7\}$ is an orbit of $N_G(P)_1$, G_1 is transitive on $\{2, 3, \cdots, n\}$.

Since $b^{I(a)}\in N_G(P)^{I(P)}$, $\{4, 5, 6, 7\}$ is a $N_G(P)_{1\,2\,3}$-orbit. Hence $G_{1\,2\,3}$ is transitive or has two orbits $\{4, 5, 6, 7\}$ and $\{8, 9, \cdots, n\}$ on $\{4, 5, \cdots, n\}$. Set $|P|=2^r$ where $r\geqq 3$. Since $G_{I(P)}$ is transitive on $\Omega-I(P)$ and P is semi-regular, $|\Omega-I(P)|=2^r\cdot s$ where s is odd. On the other hand a Sylow 2-subgroup

Q of $N_G(P)_{123}$ is also a Sylow 2-subgroup of G_{123}. Hence $|Q|=2^r\cdot 4$ and there is at least one Q-orbit T in $\Omega-I(P)$, which is of length 2^r. Let i be a point of T. Then $|Q_i|=4$ and Q_i is a 2-group of G_{123i}. Thus $G_{I(Q_i)}$ is transitive on $\Omega-I(Q_i)$ by (6). Since $i\notin\{4, 5, 6, 7\}$, $I(Q_i)\not\supseteq\{4, 5, 6, 7\}$. Therefore G_{123} is transitive on $\{4, 5, \cdots, n\}$.

Hence this implies that G_{12} is transitive or has two orbits $\{3\}$ and $\{4, 5, \cdots, n\}$ on $\{3, 4, \cdots, n\}$. If G_{12} is transitive on $\{3, 4, \cdots, n\}$, then G is 4-fold transitive on Ω. Since a Sylow 2-subgroup P of G_{1234} is semi-regular, $G=M_{23}$ by a theorem of [8].

Thus to complete the proof of (11) we must show that G_{12} is transitive. Hence suppose by way of contradiction that G_{12} has two orbits $\{3\}$ and $\{4, 5, \cdots, n\}$ on $\{3, 4, \cdots, n\}$. Then $N_G(P)^{I(P)}=A_7{}^*$. Since G is doubly transitive on Ω, any stabilizer of two points in G fixes exactly three points. Therefore $N_G(G_{I(b)})_{12}$ fixes at least three points. Hence $N_G(G_{I(b)})^{I(b)}=A_7{}^*$. On the other hand since $\langle a, b\rangle < N_G(G_{4589})$, there is an involution c of G_{4589} commuting with a and b. We may assume

$$c = (1)\,(2\;3)\,(4)\,(5)\,(6\;7)\,(8)\,(9)\,(10\;11)\cdots.$$

Now b normalizes some Sylow 2-subgroup P' of $G_{I(a)}$ containing a. Since P' is conjugate to P, $|P'|\geqq 8$ and $N_G(P')^{I(P')}=A_7{}^*$. If b commutes with only two elements 1 and a of P', then by a theorem of H. Zassenhaus [12, Satz 5] P' has a cyclic subgroup of order at least 4, which is a contradiction. Therefore there is an involution a' of P' which is different from a and commutes with b. We may assume

$$a' = (1)\,(2)\cdots(7)\,(8\;10)\,(9\;11)\cdots.$$

Since $\langle a', b\rangle < N_G(G_{45810})$, there is an involution c' of G_{45810} commuting with a' and b. Then c and c' fix two points 4,5 and have the same 2-cycle (6 7) in $I(P)$. Since $N_G(G_{I(P)})^{I(P)}=A_7{}^*$, $c^{I(P)}=c'^{I(P)}$. Thus we have

$$c' = (1)\,(2\;3)\,(4)\,(5)\,(6\;7)\,(8)\,(10)\,(9\;11)\cdots.$$

Then

$$(cc')^{I(b)} = (1)\,(2)\,(3)\,(8)\,(9\;11\;10)\,,$$

which is a contradiction since $(cc')^{I(b)}\in A_7{}^*$. Thus we complete the proof.

3. Proof of Theorem 2

By Corollary of [10] $|I(P)|=4$, 5 or 7 and $N_G(P)^{I(P)}=S_4$, S_5 or A_7 respectively. If P is a semi-regular abelian group, then $G=S_6$, S_7, A_8, A_9 or M_{23} by a theorem of [8]. Therefore from now on we assume by way of contradiction that P is not semi-regular.

We shall treat the following three cases separately:

Case I. $|I(P)| = 4$ and $N_G(P)^{I(P)} = S_4$.

Case II. $|I(P)| = 5$ and $N_G(P)^{I(P)} = S_5$.

Case III. $|I(P)| = 7$ and $N_G(P)^{I(P)} = A_7$.

Case I. $|I(P)| = 4$ and $N_G(P)^{I(P)} = S_4$.

Let $|I(P_{t_1 t_2})|$ is the smallest number such that $t_1 \in \Omega - I(P)$ and $t_2 \in \Omega - I(P_{t_1})$. For any four points i, j, k and l of $I(P_{t_1 t_2})$ let P' be a Sylow 2-subgroup of $G_{i\,j\,k\,l}$ containing $P_{t_1 t_2}$. Since P' is abelian, $P' \subseteq N_G(P_{t_1 t_2})$. By minimality of $|I(P_{t_1 t_2})|$ for any point t of $I(P_{t_1 t_2}) - I(P')$ $(P_t')^{I(P_{t_1 t_2})}$ is a semi-regular group $(\geqq 1)$. Thus $N_G(P_{t_1 t_2})^{I(P_{t_1 t_2})}$ satisfies the conditions (i), (ii) and (iii) of the following lemma.

Therefore to complete the proof of this case it is sufficient to prove the following lemma.

Lemma 1. *Let G be a permutation group on $\Omega = \{1, 2, \cdots, n\}$. Assume that a Sylow 2-subgroup P of the stabilizer of any four points in G satisfies the following three conditions:*

(i) $|I(P)| = 4$.

(ii) *P is a non-identity abelian group.*

(iii) *For any point t of $\Omega - I(P)$ P_t is a semi-regular group* $(\geqq 1)$.

Then P is semi-regular.

Proof. For any four points of Ω there is a 2-group fixing exactly these four points by (i). Hence by the lemma of [3] G is 4-fold transitive on Ω. Assume by way of contradiction that P is not semi-regular. Then there is a point t of $\Omega - I(P)$ such that P_t is a non-identity semi-regular group by (iii). By Corollary $N_G(P_t)^{I(P_t)} = S_6$, A_8 or M_{12}. Since P is abelian, $N_G(P_t)^{I(P_t)} \neq M_{12}$. Furthermore since $|I(P_t) - I(P)| = 2$ or 4, t belongs to a P-orbit of length 2 or 4, and a non-identity element of P fixes 4, 6 or 8 points of Ω. Since there is no 4-fold transitive group of degree less than 35 except known one [2. p. 80], the degree of G is not less than 35.

From now on we assume that P is a Sylow 2-subgroup of $G_{1\,2\,3\,4}$.

(1) Suppose that P has exactly one orbit of length 2. We may assume that this orbit is $\{5, 6\}$. Let

$$a = (1)\,(2) \cdots (6)\,(7\ 8) \cdots$$

be an involution of P_5. Since P is abelian, there is an element $(1)(2)(3)(4)(5\ 6) \cdots$ in $C_G(P_5)$. Since $(1)(2)(3)(4)(5\ 6) \in C_G(P_5)^{I(P_5)} \trianglelefteq N_G(P_5)^{I(P_5)} = S_6$, $N_G(P_5)^{I(P_5)} = C_G(P_5)^{I(P_5)}$. Hence $N_G(P_5) = C_G(P_5) \cdot N_G(P_5)_{I(P_5)}$. By the same argument as in the proof of (6.2) in Section 2, every 2-elements of $N_G(P_5)$ belong to $C_G(P_5)$.

Since $a \in N_G(G_{1\,2\,7\,8})$, a normalizes a Sylow 2-subgroup P' of $G_{1\,2\,7\,8}$. By the

4-fold transitivity of G P' has exactly one orbit $\{i_1, i_2\}$ of length 2. Then a fixes $\{i_1, i_2\}$ as a set. Hence a commutes with an involution b of P'_{i_1}. Since $|I(b)|=6$,

$$b=(1)\,(2)\,(7)\,(8)\,(i_1)\,(i_2)\cdots$$

First suppose that a fixes $\{i_1, i_2\}$ pointwise. Then we may assume that $\{i_1, i_2\}=\{3, 4\}$. Thus we have

$$a=(1)\,(2)\cdots(6)\,(7\;8)\cdots,$$
$$b=(1)\,(2)\,(3)\,(4)\,(5\;6)\,(7)\,(8)\cdots.$$

Let P'' be a Sylow 2-subgroup of G_{1234} containing $\langle a, b\rangle$. Since P'' is abelian, $\{5, 6\}$ and $\{7, 8\}$ are P''-orbits of length 2, which is a contradiction.

Next suppose that a has a 2-cycle $(i_1\, i_2)$. We may assume that $(i_1\, i_2)=(9\;10)$. Then

$$a=(1)\,(2)\cdots(6)\,(7\;8)\,(9\;10)\cdots,$$
$$b=(1)\,(2)\,(3\;4)\,(5\;6)\,(7)\,(8)\,(9)\,(10)\cdots.$$

Since $\langle a, b\rangle<N_G(G_{3478})$, $\langle a, b\rangle$ normalizes a Sylow 2-subgroup P''' of G_{3478}. By the same argument above a and b have the same 2-cycle on a P'''-orbit of length 2. We may assume that this P'''-orbit is $\{11, 12\}$. Then $\langle a, b\rangle<C_G(P'''_{11})$ and $I(P''_{11})=\{3, 4, 7, 8, 11, 12\}$. Since P'''_{11} is semi-regular on $\Omega-I(P'''_{11})$ and $I(\langle a, b\rangle)\cap\{\Omega-I(P'''_{11})\}=\{1, 2\}$, $|P'''_{11}|=2$. Hence $|P|=|P'''|=4$. By Theorem 1 of [7] P is elementary abelian. Let c be an involution of P'''_{11}. Then we have

$$a=(1)\,(2)\cdots(6)\,(7\;8)\,(9\;10)\,(11\;12)\cdots,$$
$$b=(1)\,(2)\,(3\;4)\,(5\;6)\,(7)\,(8)\,(9)\,(10)\,(11\;12)\cdots,$$
$$c=(1\;2)\,(3)\,(4)\,(5\;6)\,(7)\,(8)\,(9\;10)\,(11)\,(12)\cdots.$$

Since $\langle b, c\rangle<N_G(G_{1234})$, $\langle b, c\rangle$ normalizes a Sylow 2-subgroup Q of G_{1234} containing a. Then Q is semi-regular on $\{7, 8, \cdots, n\}$, and Q-orbits in $\{7, 8, \cdots, n\}$ are of length 4. Since $I(\langle b, c\rangle)\cap\{7, 8, \cdots, n\}=\{7, 8\}$, $\langle b, c\rangle$ fixes a Q-orbit containing 7 and 8, say $\{7, 8, j_1, j_2\}$. Then there is an involution

$$a'=(1)\,(2)\,(3)\,(4)\,(5\;6)\,(7\;j_1)\,(8\;j_2)\cdots$$

of Q. If b has a 2-cycle $(j_1\, j_2)$, then

$$ba'=(1)\,(2)\,(3\;4)\,(5)\,(6)\,(7\;j_1\;8\;j_2)\cdots.$$

Thus ba' is of order 4 and contained in G_{1256}. Since a Sylow 2-subgroup of G_{1256} is elementary abelian, we have a contradiction. If b fixes $\{7, 8, j_1, j_2\}$ pointwise, then $\{7, 8, j_1, j_2\}=\{7, 8, 9, 10\}$. Then we have

$$ca'=(1\;2)\,(3)\,(4)\,(5)\,(6)\,(7\;j_1\;8\;j_2)\cdots,$$

which is also a contradiction.

Therefore it is impossible that P has only one orbit of length 2.

(2) Suppose that P has at least two orbits of length 2. Then P is an elementary abelian group of order 4 and any involution of P fixes four or six points in Ω. Let r be a number of P-orbits of length 2, and s a number of involutions of P fixing six points. Since for any P-orbit of length 2 there is exactly one involution of P such that it fixes this P-orbit pointwise, $s=r$. Since $r\geqq 2$ and $s\leqq 3$, $r=s=2$ or 3. We may assume that P-orbits of length 2 are $\{5, 6\}$, $\{7, 8\}$, $\cdots$. Then there are two involutions

$$a = (1)\,(2) \cdots (6)\,(7\;8) \cdots ,$$
$$b = (1)\,(2)\,(3)\,(4)\,(5\;6)\,(7)\,(8) \cdots$$

such that $\langle a, b\rangle = P$.

Assume that $r=s=2$. Since $N_G(P_5)^{I(P_5)}=S_6$, there is a 2-element

$$x = (1)\,(2)\,(3\;4\;5\;6) \cdots$$

in $N_G(P_5)$ such that $\langle x, P\rangle$ is a 2-group. Then $x^2 \in N_G(P)$. Since x^2 fixes the P-orbit $\{5, 6\}$, x^2 fixes also the P-orbit $\{7, 8\}$. Thus $\langle x^2, P\rangle$ has exactly three orbits $\{3, 4\}$, $\{5, 6\}$, $\{7, 8\}$ of length 2. Since $x\in N_G(\langle x^2, P\rangle)$ and x takes $\{3, 4\}$ into $\{5, 6\}$, x fixes $\{7, 8\}$ as a set. By taking xa instead of x if necessary, we may assume that

$$x = (1)\,(2)\,(3\;5\;4\;6)\,(7)\,(8) \cdots .$$

Then $\langle x, b\rangle$ is a non-abelian 2-group, which is a contradiction.

Thus $r=s=3$. Then P has one more orbit of length 2, say $\{9\;10\}$. Hence

$$a = (1)\,(2) \cdots (6)\,(7\;8)\,(9\;10) \cdots ,$$
$$b = (1)\,(2)\,(3)\,(4)\,(5\;6)\,(7)\,(8)\,(9\;10) \cdots .$$

Since $P<N_G(G_{5\,6\,7\,8})$, there is an involution c of $G_{5\,6\,7\,8}$ such that $c\in C_G(P)$. By assumption $|I(c)|=6$. Hence $|I(c)\cap I(P)|=2$ or 0.

First assume that $|I(c)\cap I(P)|=2$. Then we may assume that

$$c = (1)\,(2)\,(3\;4)\,(5)\,(6)\,(7)\,(8) \cdots .$$

Since $c^{I(P)}= (1)\,(2)\,(3\;4) \in C_G(P)^{I(P)} \trianglelefteq N_G(P)^{I(P)}=S_4$, $C_G(P)^{I(P)}=N_G(P)^{I(P)}$. By the same argument as in the proof of (6.2) in Section 2, every 2-elements of $N_G(P)$ belong to $C_G(P)$. Hence there is a 2-element

$$y = (1\;3\;2\;4) \cdots$$

in $C_G(P)$ such that $\langle y, c, P\rangle$ is a 2-group. Since $y\in C_G(P)$, y fixes the three P-orbits $\{5, 6\}$, $\{7, 8\}$, $\{9, 10\}$ as a set. Therefore y, ya, yb or yab fixes $\{5, 6, 7, 8\}$ pointwise, and so one of these elements and c generate a non-abelian 2-group of $G_{5\,6\,7\,8}$, which is a contradiction.

Next assume that $|I(c)\cap I(P)|=0$. Then we may assume that

$$c=(1\ 2)\ (3\ 4)\ (5)\ (6)\ (7)\ (8)\ (9)\ (10)\cdots.$$

Since $P<N_G(G_{5678})$, P normalizes a Sylow 2-subgroup P' of G_{5678} containing c. Then $\{9, 10\}$ is a P'-orbit. Furthermore P fixes a P'-orbit containing $\{1, 2\}$. If $\{1, 2\}$ is a P'-orbit then $a\in C_G(P')$. Since $a^{I(P')}=(5)\ (6)\ (7\ 8)$, from the same reason as above we have a contradiction. Therefore the length of the P'-orbit containing $\{1, 2\}$ is 4. Since every P-orbits in $\{11, 12, \cdots, n\}$ are of length 4, the P'-orbit containing $\{1, 2\}$ is $\{1, 2, 3, 4\}$. Then also $a\in C_G(P')$. Hence similarly we have a contradiction.

Thus the minimal P-orbit is of length 4 and any involution of P fixes four or eight points.

(3) Suppose that the minimal P-orbit on Ω-$I(P)$ is of length 4 and P has exactly one orbit of length 4. We may assume that there is an involution

$$a=(1)\ (2)\cdots(8)\ (9\ 10)\ (11\ 12)\cdots$$

in P such that a fixes exactly eight points. Since $a\in N_G(G_{1\,2\,9\,10})$, a normalizes a Sylow 2-subgroup P' of $G_{1\,2\,9\,10}$. By assumption P' has exactly one orbit of length 4. Hence a fixes this P'-orbit, and hence a commutes with an invlution b of P' which fixes exactly eight points. Since $b^{I(a)}\in A_8$ and $a^{I(b)}\in A_8$, we may assume that

$$b=(1)\ (2)\ (3)\ (4)\ (5\ 6)\ (7\ 8)\ (9)\ (10)\ (11)\ (12)\cdots.$$

Since a Sylow 2-subgroup P'' of G_{1234} containing $\langle a, b\rangle$ has not an orbit of length 2, P'' has two orbits $\{5, 6, 7, 8\}$ and $\{9, 10, 11, 12\}$ of length 4, which is a contradiction. Thus P has at least two orbits of length 4.

(4) Suppose that a minimal P-orbit on Ω-$I(P)$ is of length 4 and P has at least two orbits of length 4. Then we may assume that P-orbits of length 4 are $\{5, 6, 7, 8\}$, $\{9, 10, 11, 12\}$, $\cdots$. Since $|P: P_5|=4$ and $|P_5|=2$ or 4, $|P|=8$ or 16. If P has an element of order 4, then this element has a 4-cycle on $\{5, 6, 7, 8\}$ or $\{9, 10, 11, 12\}$. But this is a contracidtion since $N_G(P_5)^{I(P_5)}=N_G(P_9)^{I(P_9)}=A_8$. Thus P is elementary abelian.

First assume that $|P|=16$. Then we may assume that there are three involutions

$$a=(1)\ (2)\cdots(8)\ (9\ 10)\ (11\ 12)\cdots,$$
$$b=(1)\ (2)\cdots(8)\ (9\ 11)\ (10\ 12)\cdots,$$
$$c=(1)\ (2)\ (3)\ (4)\ (5\ 6)\ (7\ 8)\ (9)\ (10)\ (11)\ (12)\cdots$$

in P. Since $c^{I(P_5)}=(1)\ (2)\ (3)\ (4)\ (5\ 6)\ (7\ 8)\in C_G(P_5)^{I(P_5)}\trianglelefteq N_G(P_5)^{I(P_5)}=A_8$, $C_G(P_5)^{I(P_5)}=N_G(P_5)^{I(P_5)}=A_8$. Hence there is an involution

$$d=(1)\ (2)\ (3\ 4)\ (5)\ (6)\ (7\ 8)\cdots$$

in $C_G(P_5)$ such that d is conjugate to c. Then we have

$$cd = (1)(2)(3\ 4)(5\ 6)(7)(8)\cdots.$$

Since $|I(cd)|\geqq 4$, cd is of order 2r where r is odd. Hence $x=(cd)^r$ is an involution commuting with a, b and c. Since $x^{I(c)}\in A_8$,

$$x^{I(c)} = (1)(2)(3\ 4)(i)(j)(k\ l)$$

where $\{i, j, k, l\}=\{9, 10, 11, 12\}$. On the other hand $\langle a, b\rangle$ is regular on $\{9, 10, 11, 12\}$. Therefore $x\notin C_G(\langle a, b\rangle)$, which is a contradiction.

Next assume that $|P|=8$. Then there is involutions

$$a = (1)(2)\cdots(8)(9\ 10)(11\ 12)\cdots,$$
$$b = (1)(2)(3)(4)(5\ 6)(7\ 8)(9)(10)(11)(12)\cdots$$

in P. From the same argument as above there is an involution

$$x = (1)(2)(3\ 4)(5\ 6)(7)(8)\cdots$$

commuting with a and b. Since $x^{I(b)}\in A_8$, we may assume that

$$x = (1)(2)(3\ 4)(5\ 6)(7)(8)(9)(10)(11\ 12)\cdots.$$

If $|I(ab)|=8$, then we have

$$a = (1)(2)\cdots(8)(9\ 10)(11\ 12)(13\ 14)(15\ 16)\cdots,$$
$$b = (1)(2)(3)(4)(5\ 6)(7\ 8)(9)(10)(11)(12)(13\ 14)(15\ 16)\cdots,$$
$$x = (1)(2)(3\ 4)(5\ 6)(7)(8)(9)(10)(11\ 12)(13)(14)(15\ 16)\cdots.$$

Since $|P|=8$, there is an invluiton

$$c = (1)(2)(3)(4)(5\ 7)(6\ 8)(9\ 11)(10\ 12)(13\ 15)(14\ 16)\cdots$$

In P. Then we have

$$cx = (1)(2)(3\ 4)(5\ 7\ 6\ 8)(9\ 12\ 10\ 11)(13\ 16\ 14\ 15)\cdots,$$
$$a(cx)^2 = (1)(2)(3)(4)(5\ 6)(7\ 8)(9)(10)\cdots(15)\cdots.$$

Thus $a(cx)^2$ is of even order and $|I(a(cx)^2)|\geqq 12$, which is a contradiction.

Next if $|I(ab)|=4$, then $\langle a, b\rangle$ is semi-regular on $\{13, 14, \cdots, n\}$. On the other hand x fixes six points of $\{1, 2, \cdots, 12\}$. Hence x fixes exactly two points of $\{13, 14, \cdots, n\}$, contrary to the result that $x\in C_G(\langle a, b\rangle)$. The lemma is proved.

Case II. $|I(P)|=5$ and $N_G(P)^{I(P)}=S_5$.

Let t be a point of Ω-$I(P)$ such that t belongs to the minimal P-orbit. Since $|I(P)|=5$, by Corollary $|I(P_t)|=7$, 9 or 13. If $|I(P_t)|=13$, then $N_G(P_t)^{I(P_t)}=S_1\times M_{12}$, which is a contradiction since P is abelian. Therefore $|I(P_t)|=7$ or 9 and t belongs to a P-orbit of length 2 or 4. From now on we assume that $I(P)=\{1, 2, \cdots, 5\}$.

(1) First we shall ahow that if $|I(P_t)|=9$, then t belongs to a P-orbit of length 4. Assume by way of contradiction that t is a point of a P-orbit of length 2. Set $I(P_t)=\{1, 2, \cdots, 9\}$ and $H=N_G(P_t)^{I(P_t)}$. Since $|P:P_t|=2$, a Sylow 2-subgroup of the stabilizer of any four points in H is of order 2 and $H \nleqq A_9$.

If H_i is transitive on $\{1, 2, \cdots, 9\}-\{i\}$ for any point i of $I(P_t)$, then H is doubly transitive. Since H has an invloution consisting of two 2-cycles, $H=A_9$. This is a contradiction. Therefore we may assume that H_1 is intransitive on $\{2, 3, \cdots, 9\}$.

First assume that H_1 has an orbit of length 1 in $\{2, 3, \cdots, 9\}$. Then we may assume that this orbit is $\{2\}$. Set $\Delta=\{3, 4, \cdots, 9\}$. For any three points i_1, i_2 and i_3 of Δ there is an involution

$$x = (1)(2)(i_1)(i_2)(i_3)(i_4\, i_5)(i_6\, i_7).$$

Thus x fixes exactly these three points i_1, i_2 and i_3. From Lemma 6 of [3] $H_{1\,2}$ is 3-fold transitive on Δ. By § 166 in [1], $H_{1\,2}=A_7$. Hence a Sylow 2-subgroup of $H_{1\,2\,3\,4}$ is of order 4, which is a contradiction.

Next assume that H_1 has an orbit of length 2. Then we may assume that $\{2, 3\}$ is the H_1-orbit. Set $\Delta=\{4, 5, \cdots, 9\}$. For any two points i_1 and i_2 of Δ there is an involution

$$x = (1)(2)(3)(i_1)(i_2)(i_3\, i_4)(i_5\, i_6).$$

Then from the same reason as above, $H_{1\,2\,3}$ is doubly transitive on Δ. On the other hand there is an involution $(1)(2\,3)(j_1)(j_2)(j_3)(j_4)(j_5\, j_6)$. Thus $H_1^{\Delta}=S_6$. Hence there is an involution

$$y = (1)(2)(3)(i_1)(i_2)(i_3\, i_5)(i_4\, i_6).$$

Then $\langle x, y\rangle$ is a 2-group of $H_{1\,2\,3\,i_1}$ and of order 4, which is a contradiction.

For the remaining cases by the same argument as above we have also a contradiction. Thus we complete the proof.

(2) Next we shall show that if t is a point of a P-orbit of length 2, then $|I(P_t)|=7$ and $C_G(P_t)^{I(P_t)}=S_7$. Let t be a point of a P-orbit $\{6, 7\}$. Then by (1) $I(P_6)=\{1, 2, \cdots, 7\}$. For any four points i_1, i_2, i_3 and i_4 of $I(P_6)$ there is a Sylow 2-subgroup P' of $G_{i_1\,i_2\,i_3\,i_4}$ containing P_6. Set $C=C_G(P_6)^{I(P_6)}$. Since P' is abelian, $P'<C_G(P_6)$. Thus C has an involution $(i_1)(i_2)(i_3)(i_4)(i_5)(i_6\, i_7)$. By the same argument as in (1) we have that C is one of the following groups:

(i) If C is transitive on $I(P_6)$, then by Theorem 8.3 and Theorem 13.3 of [11] $C=S_7$.

(ii) If C has two orbits of length 1 and 6, then $C=S_1\times S_6$. We may assume that the C-orbits are $\{1\}$ and $\{2, 3, \cdots, 7\}$.

(iii) If C has two orbits of length 2 and 5, then $C=S_2\times S_5$. We may assume that the C-orbits are $\{1, 2\}$ and $\{3, 4, \cdots, 7\}$.

(iv) If C has two orbits of length 3 and 4, then $C=S_3\times S_4$. We may assume that C-orbits are $\{1, 2, 3\}$ and $\{4, 5, 6, 7\}$.

Since $N_G(P)^{I(P)}=S_5$, there is a 2-element

$$x = (1\ 4)\ (2)\ (3)\ (5) \cdots$$

in $N_G(P)$.

First suppose that $\{6, 7\}^x=\{6, 7\}$. Since P has an element $y=(1)\ (2) \cdots (5)\ (6\ 7) \cdots$, x or xy is of the form $(1\ 4)\ (2)\ (3)\ (5)\ (6)\ (7) \cdots$. Therefore we may assume that

$$x = (1\ 4)\ (2)\ (3)\ (5)\ (6)\ (7) \cdots .$$

Since $\langle x, P_6\rangle < G_{2\,3\,5\,6\,7}$, $x\in C_G(P_6)$. On the other hand $C=C_G(P_6)^{I(P_6)}$ is one of the groups listed above. Hence the points 1 and 4 are contained in the same C-oribt. Thus $C=S_7$.

Next suppose that $\{6, 7\}^x \neq \{6, 7\}$. Set $\{8, 9\}=\{6, 7\}^x$. Since $x^2\in P$, $\{8, 9\}^x=\{6, 7\}^{x^2}=\{6, 7\}$. Hence $x\in N_G(P_{6\,8})$. Set $H=N_G(P_{6\,8})$ and $\Delta=I(P_{6\,8})$. Since $C_G(P_{6\,8})>C_G(P_6)$, $H>\langle x, C_G(P_6)\rangle$. On the other hand C is one of the groups listed above. Therefore x and all elements of C fixing the set $I(P)=\{1, 2, \cdots, 5\}$ generate S_5 on $I(P)$. Thus $N_H(H_{I(P)})^{I(P)}=S_5$. New P^Δ is an elementary abelian group of order 4 and a Sylow 2-subgroup of $(H^\Delta)_{I(P)}$. Hence $N_{H^\Delta}(P^\Delta)^{I(P)}=N_{H^\Delta}(H^\Delta_{I(P)})^{I(P)}=S_5$. Since the automorphism group of P^Δ is a subgroup of S_3 and $N_{H^\Delta}(P^\Delta)^{I(P)}/C_{H^\Delta}(P^\Delta)^{I(P)}$ is a homomorphic image of a subgroup of this automorphism group, $C_{H^\Delta}(P^\Delta)^{I(P)}\geqq A_5$. Since $\{6, 7\}$ is the P^Δ-orbit, there is an element

$$y = (1\ 4)\ (2\ 3)\ (5)\ (6)\ (7) \cdots$$

such that $y^\Delta\in C_{H^\Delta}(P^\Delta)$. Thus $N_G(G_{I(P_6)})^{I(P_6)}\geqq\langle y, C_G(P_6)\rangle^{I(P_6)}=S_7$. Since P_6 is a Sylow 2-subgroup of $G_{I(P)}$, $N_G(P_6)^{I(P_6)}=N_G(G_{I(P_6)})^{I(P_6)}=S_7$. Furthermore $N_G(P_6)^{I(P_6)}\trianglerighteq C$ and C has a transposition. Therefore $C=S_7$.

(3) Suppose that P has exactly one orbit of length 2. Let $\{t_1, t'_1\}$ be the P-orbit of length 2, and let t_2 be a point of the minimal P_{t_1}-orbit on Ω-$I(P_{t_1})$. Since P is abelian, $I(P_{t_1\,t_2})$-$I(P)$ consists of one P-orbit of length 2 and several P-orbits of length at least 4. Thus $|I(P_{t_1\,t_2})|$-$5\equiv 2 \pmod 4$.

Set $H=N_G(P_{t_1\,t_2})$ and $\Delta=I(P_{t_1\,t_2})$. For any four points i_1, i_2, i_3 and i_4 of Δ let P' be a Sylow 2-subgroup of $G_{i_1\,i_2\,i_3\,i_4}$ containing $P_{t_1\,t_2}$. Then $P'\triangleright P_{t_1\,t_2}$ and P'^Δ is a Sylow 2-subgroup of $(H^\Delta)_{i_1\,i_2\,i_3\,i_4}$. Since $|\Delta|-5\equiv 2 \pmod 4$, P'^Δ has exactly one orbit $\{u_1, u_1'\}$ of length 2. By (2) $I(P'_{u_1})\neq\Delta$. Since t_2 is the point of the minimal P_{t_1}-orbit, for any point v of $\Delta-I(P'_{u_1})$ $P_{t_1\,t_2}=P'_{u_1 v}$. Thus $|P^\Delta|=|P'^\Delta|$ and $(P^\Delta)_{u_1 v}=1$. Since $C_G(P'_{u_1})<C_G(P'_{u_1 v})=C_G(P_{t_1\,t_2})<H$ and $C_G(P'_{u_1})^{I(P'_{u_1})}=S_7$ by (2), $C_{H^\Delta}(P'_{u_1})^{I(P'^\Delta_{u_1})}=S_7$.

Thus H^Δ satisfies the conditions (i), (ii) and (iii) of the following lemma.

Hence if we prove the following lemma, then the number of P-orbits of length 2 is greater than 1.

Lemma 2. *Let G be a permutation group on $\Omega=\{1, 2, \cdots, n\}$. Then it is impossible that a Sylow 2-subgroup P of the stabilizer of any four points in G satisfies the following three conditions:*

(i) $|I(P)|=5$ *and* $|P|$ *is constant.*

(ii) *P is an abelian group.*

(iii) *P has exactly one orbit of length 2. Let t be a point of the orbit of length 2, then $C_G(P_t)^{I(P_t)}=S_7$ and P_t is a non-identity semi-regular group.*

Proof. Assume by way of contradiction that G is a counter-example to Lemma 2. Let P be a Sylow 2-subgroup of G_{1234} and $I(P) = \{1, 2, 3, 4, 5\}$. Since P has an orbit of length 2 and some orbits of length at least 4, $|\Omega| \geqq 5+2+4=11$. Let $\{6, 7\}$ be a P-orbit of length 2. By the same argument as in the proof of (1) of Lemma 1, $|\Omega| \geqq 13$ and for an involution

$$a = (1)(2) \cdots (7)(8\ 9)(10\ 11)(12\ 13) \cdots$$

of P_6, there is two commuting involutions

$$b = (1)(2)(3)(4\ 5)(6\ 7)(8)(9)(10)(11)(12\ 13) \cdots,$$
$$c = (1)(2\ 3)(4)(5)(6\ 7)(8)(9)(10\ 11)(12)(13) \cdots$$

in $C_G(a)$. Moreover P is a cyclic group or an elementary abelian group of order 4.

(a) Suppose that P is an elementary abelian group. Then by the same argument as in the proof (1) of Lemma 1, there is an element $(1)(2)(3)(6)(7)(4\ 5)(8\ j_1\ 9\ j_2)\cdots$ in G_{12367} or $(1)(4)(5)(6)(7)(2\ 3)(8\ j_1\ 9\ j_2)\cdots$ in G_{14567}. Since $C_G(P_6)^{I(P_6)}=S_7$, a Sylow 2-subgroup of G_{12367} and a Sylow 2-subgroup of G_{14567} are conjugate to P. But P is an elementary abelian group, which is a contradiction.

(b) Therefore for any four points i, j, k and l a Sylow 2-subgroup of G_{ijkl} is cyclic. Since $C_G(P_6)^{I(P_6)}=S_7$, there is a 2-element

$$x = (1)(2)(3)(4\ 6\ 5\ 7) \cdots$$

in $C_G(P_6)$ such that $\langle x, P\rangle$ is a 2-group and $x^2 \in N_G(P)$. Assume that $\langle x, P\rangle$ has an orbit $\{i_1, i_2, i_3, i_4\}$ of length 4, which is different from $\{4, 5, 6, 7\}$. Since P is cyclic, we may assume that

$$d = (1)(2) \cdots (5)(6\ 7)(i_1\ i_2\ i_3\ i_4) \cdots$$

is the generator of P. If x has a 4-cycle on $\{i_1, i_2, i_3, i_4\}$, then x or x^{-1} is of the the form $(i_1\ i_2\ i_3\ i_4)$ on $\{i_1, i_2, i_3, i_4\}$. Hence

$$x^2 = (1)\,(2)\,(3)\,(4\ 5)\,(6\ 7)\,(i_1\ i_2)\,(i_3\ i_4)\cdots.$$

Thus $x^2 \in C_G(P)$. If x has not a 4-cycle on $\{i_1, i_2, i_3, i_4\}$, then

$$x^2 = (1)\,(2)\,(3)\,(4\ 5)\,(6\ 7)\,(i_1)\,(i_2)\,(i_3)\,(i_4)\cdots.$$

Thus also $x^2 \in C_G(P)$. On the other hand since $C_G(P_6)^{I(P_6)} = S_7$, $N_G(G_{I(P)})^{I(P)} = N_G(P)^{I(P)} = S_5$. Then $(x^2)^{I(P)} = (1)\,(2)\,(3)\,(4\ 5) \in C_G(P)^{I(P)} \trianglelefteq N_G(P)^{I(P)} = S_5$. Hence $N_G(P)^{I(P)} = C_G(P)^{I(P)}$. By the same argument as in the proof of (6.2) in Section 2, every 2-elements of $N_G(P)$ belong to $C_G(P)$. Since $\langle b, c\rangle < N_G(G_{1\,2\,3\,4\,5})$, there is a Sylow 2-subgroup P' of $G_{1\,2\,3\,4\,5}$ such that $a \in P'$ and $\langle b, c\rangle < N_G(P')$. Since P' is conjugate to P, $\langle b, c\rangle < C_G(P')$. Since $I(\langle b, c\rangle) \cap \{8, 9, \cdots, n\} = \{8, 9\}$ and P' is semi-regular on $\{8, 9, \cdots, n\}$, P is of order 2, which is a contradiction.

Therefore $\langle x, P\rangle$ has exactly one orbit of length 4, namely $\{4, 5, 6, 7\}$. Let Q be a 2-group of $G_{1\,2\,3}$ containing $\langle x, P\rangle$ as a normal subgroup. Then Q fixes $\{4, 5, 6, 7\}$. Hence $Q = \langle x, P\rangle$. Thus $\langle x, P\rangle$ is a Sylow 2-sbubgroup of $G_{1\,2\,3}$. For any point i of $\{4, 5, \cdots, n\}$ let P'' be a Sylow 2-subgroup of $G_{1\,2\,3\,i}$. Then similarly a Sylow 2-subgroup Q' of $G_{1\,2\,3}$ containing P'' has exactly one orbit of length 4, which contains i. By the conjugacy of Sylow 2-subgroups of $G_{1\,2\,3}$ there is an element of $G_{1\,2\,3}$ which takes $\{4, 5, 6, 7\}$ into the Q'-orbit containing i. Thus $G_{1\,2\,3}$ is transitive on $\{4, 5, \cdots, n\}$. On the other nand $C_G(P_6)^{I(P_6)} = S_7$. Hence G is 4-fold transitive on Ω. By Theorem 1 of [7] this is a contradiction. Thus lemma is proved.

(4) Suppose that P has at least two orbits of length 2. Let $\{6, 7\}$, $\{8, 9\}\cdots$ be P-orbits of length 2. Then $I(P_6) = \{1, 2, \cdots, 7\}$. Since $|P : P_{6\,8}| = 4$, $P^{I(P_{6\,8})}$ is an elementary abelian group of order 4. For any four points i, j, k and l of $I(P_{6\,8})$ let P' be a Sylow 2-subgroup of $G_{i\,j\,k\,l}$ containing $P_{6\,8}$. Then $|I(P'^{I(P_{6\,8})})| = 5$ and $P'^{I(P_{6\,8})}$ is a Sylow 2-subgroup of $(N_G(P_{6\,8})^{I(P_{6\,8})})_{i\,j\,k\,l}$ of order 4. Set $\Delta = I(P_{6\,8})$, $H = N_G(P_{6\,8})^{I(P_{6\,8})}$ and $P^{\Delta} = Q$. Since $C_G(P_6) < C_G(P_{6\,8}) \leqq N_G(P_{6\,8})$, $C_H(Q_6)^{I(Q_6)} = S_7$.

From now on we deal with H. Then the proof is similar to the proof (2) of Lemma 1. Let r be a number of Q-orbits of length 2 and s a number of involutions of Q. Then $r = s = 2$ or 3.

If $r = s = 2$, then by the same argument as in the proof (2) of Lemma 1 we have a contradiction.

Therefore $r = s = 3$. Hence we may assume that Q has exactly three orbits $\{6, 7\}$, $\{8, 9\}$ and $\{10, 11\}$ of length 2. Then Q has the following two involutions

$$a = (1)\,(2)\cdots(7)\,(8\ 9)\,(10\ 11)\cdots,$$

$$b = (1)\,(2)\cdots(5)\,(6\ 7)\,(8)\,(9)\,(10\ 11)\cdots.$$

Since $|Q| = 4$ and Q is semi-regular on $\{12, 13, \cdots, n\}$, $|\Delta| - 5 \equiv 2 \pmod 4$. Therefore a Sylow 2-subgroup of the stabilizer of any four points in H has exactly one or three orbits of length 2. Since $Q<N_H(H_{6789})$, Q normalizes a Sylow 2-subgroup Q' of H_{6789}. Then Q fixes at least one Q'-orbit of length 2. Thus Q centralizes an involution c of Q' fixing exactly seven points. Since $I(c)\supset\{6, 7, 8, 9\}$, $|I(c)\cap I(Q)|=3$ or 1.

In the case $|I(c)\cap I(Q)|=3$ using the same argument as in the proof (2) of Lemma 1, we have a contradiction.

Hence $|I(c)\cap I(Q)|=1$. Then we may assume that

$$c=(1)(2\ 3)(4\ 5)(6)(7)(8)(9)(10)(11)\cdots.$$

Since $\langle b, c\rangle<N_H(H_{4567})$ and $\langle b, c\rangle<C_H(a)$, $\langle b, c\rangle$ normalizes a Sylow 2-subgroup Q'' of H_{4567} containing a. Then $I(Q'')=\{1, 4, 5, 6, 7\}$. Since $C_H(Q_6)^{I(Q_6)}=S_7$, H_{4567} is conjugate to H_{1234}, and so Q'' is conjugate to Q. Thus Q'' has exactly three orbits of length 2. If $\{8, 9\}$ is a Q''-orbit, then $b\in C_H(Q'')$. Since $|I(b)\cap I(Q'')|=3$, as is shown above, we have a contradiction. Hence the Q''-orbit containing $\{8, 9\}$ is of length 4 say $\{8, 9, i_1, i_2\}$. If $\{8, 9, i_1, i_2\}=\{8, 9, 10, 11\}$, then c belongs to $C_G(Q'')$. Since $|I(c)\cap I(Q'')|=3$, we have also a contradiction. Thus $\{i_1, i_2\}\subset\{12, 13, \cdots, n\}$. Since $\langle a, b\rangle$ is semi-regular on $\{12, 13, \cdots, n\}$ and a has a 2-cycle $(i_1\ i_2)$, b has not a 2-cycle $(i_1\ i_2)$. Thus $\{i_1, i_2\}^b\neq\{i_1, i_2\}$. On the other hand $b\in N_H(Q'')$. Hence $\{8, 9, i_1, i_2\}^b=\{8, 9, i_1^b, i_2^b\}$ is a Q''-orbit, which is a contradiction. Thus the minimal P-orbit is of length 4.

(5) We shall ahow that if t belongs to a P-orbit of length 4, then $|I(P_t)|=9$ and $C_G(P_t)^{I(P_t)}=A_9$ or $S_1\times A_8$. By the argument above the minimal P-orbit on Ω-$I(P)$ is of length 4 and P is abelian. Hence by Corollary $|I(P_t)|=9$ and $N_G(P_t)^{I(t)}\leqq A_9$. Let $I(P_t)=\{1, 2, \cdots, 9\}$. Then there are elements

$$a_1=(1)(2)\cdots(5)(6\ 7)(8\ 9)\cdots,$$
$$a_2=(1)(2)\cdots(5)(6\ 8)(7\ 9)\cdots$$

in P. Since $\langle a_1, a_2\rangle<N_G(G_{6789})\cap C_G(P_6)$, there is a Sylow 2-subgroup P' of G_{6789} such that $\langle a_1, a_2\rangle<N_G(P')$ and $P'>P_6$. Since $P'<C_G(P_6)$, we may assume that there are elements

$$b_1=(1)(2\ 3)(4\ 5)(6)(7)(8)(9)\cdots,$$
$$b_2=(1)(2\ 4)(3\ 5)(6)(7)(8)(9)\cdots$$

in P'. Since $\langle a_1, b_1\rangle<N_G(G_{2367})$, similarly we may assume that there are elements

$$c_1=(1)(2)(3)(4\ 5)(6)(7)(8\ 9)\cdots,$$
$$c_2=(1)(2)(3)(4\ 8)(6)(7)(5\ 9)\cdots$$

in $C_G(P_6)\cap G_{2367}$. Then $C_G(P_6)>\langle a_1, a_2, b_1, b_2, c_1, c_2\rangle$. Hence $C_G(P_6)_1$ is transitive on $\{2, 3, \cdots, 9\}$. Therefore $C_G(P_6)^{I(P_6)}$ is transitive or has two orbits $\{1\}$ and $\{2, 3, \cdots, 9\}$ on $I(P_6)$. If $C_G(P_6)^{I(P_6)}$ is transitive, then $C_G(P_6)^{I(P_6)}$ is doubly transitive.

Since $C_G(P_6)^{I(P_6)}$ has an involution consisting of two 2-cycles, $C_G(P_6)^{I(P_6)}=A_9$. Next suppose that $C_G(P_6)^{I(P_6)}$ is intransitive. Then for any four points of $\{2, 3, \cdots, 9\}$ $(C_G(P_6)^{I(P_6)})_1$ has an involution fixing exactly these four points. Hence from Lemma 6 of [3] $(C_G(P_6)^{I(P_6)})_1$ is 4-fold transitive on $\{2, 3, \cdots, 9\}$. Thus $C_G(P_6)^{I(P_6)}=S_1\times A_8$.

(6) By (4) the minimal P-orbit on $\Omega-I(P)$ is of length 4. Let $|I(P_{t_1\,t_2})|$ be the smallest number such that $t_1\in\Omega-I(P)$ and $t_2\in\Omega-I(P_{t_1})$. Then $|I(P_{t_1\,t_2})|\gneqq 9$. Let R be a Sylow 2-subgroup of $G_{I(P_{t_1\,t_2})}$. Set $H=N_G(R)^{I(R)}$ and $\Delta=I(R)$. Then if a Sylow 2-subgroup of the stabilizer of any four points in H is semi-regular on Δ, then by Theorem 1 $|\Delta|=9$, which is a contradiction. Hence there are four points j, j, k and l of Δ such that a Sylow 2-subgroup Q of $H_{i\,j\,k\,l}$ is not semi-regular on $\Delta-I(Q)$. By the minimality of $|\Delta|$, there is a point t of $\Delta-I(Q)$ such that Q_t is a non-identity semi-regular group. By (3) and (4), $|I(Q_t)|=9$ and t belongs to a Q-orbit of length 4. By (5) $C_H(Q_t)^{I(Q_t)}=A_9$ or $S_1\times A_8$. Therefore by the same argument as in the proof of Lemma 1 we have a contradiction. Thus Case II is proved.

Case III. $|I(P)|=7$ and $N_G(P)^{I(P)}=A_7$.

Let $|I(P_{t_1\,t_2})|$ be the smallest number such that $t_1\in\Omega$-$I(P)$ and $t_2\in\Omega-I(P_{t_1})$. Since P is abelian, $I(P_{t_1\,t_2})$ consists of some P-orbits. By Theorem 1 $|I(P_{t_1})|=23$. Hence $|I(P_{t_1\,t_2})|\gneqq 23$.

Let R be a Sylow 2-subgroup of $G_{I(P_{t_1\,t_2})}$. Set $H=N_G(R)^{I(R)}$ and $\Delta=I(R)$. Let Q be a Sylo w 2-subgroup of the stabilizer of any four points in H. Then Q satisfies the following conditions:

(i) $|I(Q)|=7$

(ii) Q is abelian and $|Q|$ is constant for any four points i, j, k and l.

(iii) For any point t of $\Delta-I(Q)$ Q_t is a semi-regular group$\geqq 1$. If $Q_t\neq 1$, then $N_H(Q_t)^{I(Q_t)}=M_{23}$.

If a Sylow 2-subgroup of the stabilizer of any four points in H is semi-regular, then by Theorem 1 $|\Delta|=23$, which is a contradiction. Hence we may assume that a Sylow 2-subgroup Q of $H_{1\,2\,3\,4}$ is not semi-regular. Therefore there is a point t of the minimal Q-orbit such that $N_H(Q_t)^{I(Q_t)}=M_{23}$ and $|I(Q_t)|=23$.

Let Q' be a Sylow 2-subgroup of $H_{1\,2\,3\,i}$, where $i\in\Delta-\{1, 2, 3\}$. Then by (iii) the minimal Q'-orbit is of length at least 16. Since $N_H(Q_t)^{I(Q_t)}=M_{23}$, a Sylow 2-subgroup of $H_{1\,2\,3}$ containing Q has exactly one orbit of length 4 and the point 4 belongs to this orbit. By the conjugacy of Sylow 2-subgroups of $H_{1\,2\,3}$, a Sylow 2-subgroup of $H_{1\,2\,3}$ containing Q' has exactly one orbit of length 4 which contains i. Thus $H_{1\,2\,3}$ has an element carrying 4 into i, and so $H_{1\,2\,3}$ is transitive on Δ-$\{1, 2, 3\}$. On the other hand $N_H(Q_t)^{I(Q_t)}=M_{23}$. Hence H is 4-fold transitive on Δ. Therefore to prove Case III it is sufficient to prove the following lemma.

Lemma 3. *Let G be a 4-fold transitive group on $\Omega=\{1, 2, \cdots, n\}$, and P a Sylow 2-subgroup of $G_{1\,2\,3\,4}$. Assume that P satisfies the following conditions:*

(i) $P \neq 1$ and $|I(P)|=7$.

(ii) For any point t of $\Delta\text{-}I(P)$ P_t is a semi-regular group $\geqq 1$. Then $G=M_{23}$.

Proof. If P is semi-regular, then by the theorem of [8] $G=M_{23}$. Therefore from now on suppose by way of contradiction that P is not semi-regular. Let $I(P)=\{1, 2, \cdots, 7\}$. The proof will be given in various steps:

(1) *For a point t of $\Omega\text{-}I(P)$ if $P_t \neq 1$, then P_t is an elementary abelian group.*

Proof. The proof is similar to the proof (1) of Case III in Section 2.

(2) *For any point t of $\Omega\text{-}I(P)$ $|I(P_t)| \geqq 23$.*

Proof. This is a direct consequence of Corollary.

(3) *For a point t of $\Omega\text{-}I(P)$ if $P_t \neq 1$, then $|I(P_t)|=23$ and $N_G(P_t)^{I(P_t)}=M_{23}$.*

Proof. This follows from Theorem 1.

(4) *For a point t of $\Omega\text{-}I(P)$ if $P_t \neq 1$, then $|P_t|=2$ or 4 and every 2-elements of $N_G(P_t)$ belong to $C_G(P_t)$.*

Proof. Since $M_{23}=N_G(P_t)^{I(P_t)} \cong N_G(P_t)/N_G(P_t)_{I(P_t)} \trianglerighteq C_G(P_t)\cdot N_G(P_t)_{I(P_t)}/N_G(P_t)_{I(P_t)}$ and M_{23} is a simple group, $N_G(P_t)=C_G(P_t)\cdot N_G(P_t)_{I(P_t)}$ or $C_G(P_t) \leqq N_G(P_t)_{I(P_t)}$. Let $I(P_t)=\{1 ,2, \cdots, 23\}$. Then we may assume that P_t has an involution

$$a=(1)(2)\cdots(23)(24\ 25)\cdots .$$

Since $a \in N_G(G_{1\,2\,24\,25})$, there is an involution b of $G_{1\,2\,24\,25}$ commuting with a. Since $b^{I(a)} \in M_{23}$, $|I(b^{I(a)})|=7$. Hence $|I(b)|=23$ and we may assume that

$$b=(1)(2)\cdots(7)(8\ 9)(10\ 11)\cdots(22\ 23)(24)(25)\cdots(29)\cdots .$$

Thus $|\Omega| \geqq 29$. Since $b \in N_G(a)$, b normalizes a Sylow 2-subgroup Q of $G_{I(a)}$ containing a. Then Q is a semi-regular elementary abelian group on $\{24, 25, \cdots, n\}$. Since $b \in N_G(Q)$ and $|I(b) \cap (\Omega\text{-}I(Q))|=16$, by Lemma of *H.* Nagao [4] $|Q| \leqq 2^{2\cdot 4}=2^8$. On the other hand the automorphism group $A(Q)$ of an elementary abelian group of order 2^r is of order $(2^r-1)(2^r-2)\cdots(2^r-2^{r-1})$.

Suppose that $N_G(Q)_{I(Q)} \geqq C_G(Q)$. Since $N_G(Q)/C_G(Q)$ is a subgroup of $A(Q)$, $N_G(Q)/N_G(Q)_{I(Q)}$ being isomorphic to $N_G(Q)^{I(Q)}=M_{23}$ is a homomorphic image of a subgroup of $A(Q)$. But if $r \leqq 8$, then the order of $A(Q)$ is not divisible by 23, which is a contradiction. Thus $N_G(Q)_{I(Q)} \ngeqq C_G(Q)$. Hence $N_G(Q)=C_G(Q)\cdot N_G(Q)_{I(Q)}$. Therefore by the same argument as in the proof (6.2) of Case III in Section 2 every 2-elements of $N_G(Q)$ belong to $C_G(Q)$.

Since $\langle a, b\rangle < N_G(G_{8\,9\,24\,25})$, there is an involution c of $G_{8\,9\,24\,25}$ commuting with a and b. Since $I(b^{I(a)}) \neq I(c^{I(a)})$ and $b^{I(a)}$ and $c^{I(a)}$ are the commuting involutions of M_{23}. $|I(b^{I(a)}) \cap I(c^{I(a)})|=3$. On the other hand since $c^{I(b)} \in M_{23}$,

$|I(b)\cap I(c)|=7$. Hence $|I(b^{\Omega-I(a)})\cap I(c^{\Omega-I(a)})|=4$.

Now since $\langle b, c\rangle < N_G(C_{I(a)})$, $\langle b,c\rangle$ normalizes a Sylow 2-subgroup Q' of $G_{I(a)}$. Then since Q' is conjugate to Q in $G_{I(a)}$, $\langle b,c\rangle < C_G(Q')$. Since Q' is semi-regular on $\Omega-I(a)$ and $|I(\langle b,c\rangle)\cap(\Omega-I(a))|=4$, $|Q|=|Q'|\leqq 4$.

(5) *Let x be an involution. If* $|I(x)|\geqq 4$, *then* $|I(x)|=23$.

Proof. If $|I(x)|\geqq 4$, then $|I(x)|=7$ or 23. Suppose by way of contradiction that $|I(x)|=7$. Then P has an involution a fixing 7 points and an involution b fixing 23 points. We may assume that $I(b)=\{1, 2, \cdots, 23\}$ and

$$a=(1)(2)\cdots(7)(8\ 9)(10\ 11)\cdots(22\ 23)\cdots.$$

Since $N_G(P)^{I(P)}=A_7$, $G_{1\,2\,3\,4}$ has an element $(1)(2)(3)(4)(5\ 6\ 7)\cdots$. Let Δ be a $G_{1\,2\,3\,4}$-orbit containing $\{5, 6, 7\}$. Since P is a Sylow 2-subgroup of $G_{1\,2\,3\,4}$, Δ is of odd length. Then by the conjugacy of Sylow 2-subgroups of $G_{1\,2\,3\,4}$ Δ is only one $G_{1\,2\,3\,4}$-orbit of odd length in $\{5, 6, \cdots, n\}$.

Now suppose that there is a point i of $\Delta-\{5, 6, 7\}$ such that $P_i\neq 1$. Then $N_G(P_i)^{I(P_i)}=M_{23}$. On the other hand i belongs to Δ, which is of odd length. Hence a Sylow 2-subgroup P' of $G_{1\,2\,3\,4\,i}$ containing P_i is also a Sylow 2-subgroup of $G_{1\,2\,3\,4}$. Since $N_P(P_i)^{I(P_i)}$ and $N_{P'}(P_i)^{I(P_i)}$ are non-identity 2-subgroups of $(N_G(P_i)^{I(P_i)})_{1\,2\,3\,4}$, $I(N_P(P_i)^{I(P_i)})=I(N_{P'}(P_i)^{I(P_i)})$. But $i\notin\{1, 2, \cdots, 7\}$, which is a contradiction. Thus $P_i=1$.

If a and b have a 2-cycle $(i_1\ i_2)$ in common, then we have

$$ab=(1)(2)\cdots(7)(8\ 9)(10\ 11)\cdots(22\ 23)(i_1)(i_2)\cdots.$$

Since $P_{i_1}=P_{i_2}\neq 1$, both i_1 and i_2 are not points of Δ.

Next if a 2-cycle $(i_1\ i_2)$ of a is not a 2-cycle of b, then we may assuem that

$$a=(1)(2)\cdots(7)(8\ 9)(10\ 11)\cdots(22\ 23)(i_1\ i_2)(i_3\ i_4)\cdots,$$
$$b=(1)(2)\cdots(23)(i_1\ i_3)(i_2\ i_4)\cdots.$$

Since $\langle a, b\rangle < N_G(G_{i_1\,i_2\,i_3\,i_4})$, there is an involution c of $G_{i_1\,i_2\,i_3\,i_4}$ commuting with a and b. Since $c^{I(b)}\in M_{23}$, $|I(c)\cap I(b)|=7$. Hence $|I(c)|=23$. Since $a^{I(c)}$ and $b^{I(c)}$ are the commuting elements of M_{23} and $I(b)\supset I(a)$, $I(a^{I(c)})=I(b^{I(c)})=\{1, 2, \cdots, 7\}$. Hence a Sylow 2-subgroup of $G_{1\,2\,3\,4}$ containing a and c fixes $\{5, 6, 7\}$ pointwise. Hence i_1, i_2, i_3 and i_4 do not belong to Δ. Thus $\Delta=\{5, 6, 7\}$.

Now in the proof of Case II of Theorem 2 in [5] we used only the following conditions: In a 4-fold transitive group G an involution a fixes exactly seven points and a $G_{1\,2\,3\,4}$-orbit of odd length is $\{5, 6, 7\}$. Therefore similarly $G=M_{23}$, which is a contradiction. Thus we complete the proof of (5).

(6) *If P is not semi-regular, then we have a contradiction.*

Proof. For a point t of $\Omega-I(P)$ suppose that $P_t\neq 1$. We may assume that

$I(P_t)=\{1, 2, \cdots, 23\}$ and P_t has an involution

$$a = (1)(2)\cdots(23)(24\ 25)\cdots.$$

Since $a \in N_G(G_{1\ 2\ 24\ 25})$, there is an involution b of $G_{1\ 2\ 24\ 25}$ commuting with a. We may assume that

$$b = (1)(2)\cdots(7)(8\ 9)(10\ 11)\cdots(22\ 23)(24)(25)\cdots.$$

Since $b \in N_G(G_{I(a)})$, b normalizes a Sylow 2-subgroup Q of $G_{I(a)}$. Then by (3) and (4) $b \in C_G(Q)$ and $C_G(Q)^{I(Q)}=M_{23}$.

Let x be an arbitrary 2-element of $C_G(Q)$ such that $x^{I(Q)}$ is an involution. Since all involutions in M_{23} are conjugate, there is an involution y of $C_G(Q)$ such that y is conjugate to b and $x^{I(Q)}=y^{I(Q)}$. Then $xy \in Q$. Hence $xy=a' \in Q$, and so $x=a'y$. Since a' is an involution commuting with y, x is also an involution.

Now there is a 2-element

$$z = (1)(2)(3)(4\ 5)(6\ 7)(8\ 10\ 9\ 11)(12\ 14\ 13\ 15)(16\ 18\ 17\ 19)(20\ 22\ 21\ 23)\cdots$$

in $C_G(Q)$. By the argument above z^2 is an involution. Hence $|I(z^2)|=23$ by (5). By the same reason z is an involution since $z^{I(z^2)}$ is an involution, which is a contradiction. Thus we complete the proof.

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