Matsuda, M.
Osaka, J. Math.
8 (1971), 29-32

A NOTE ON THE DEFINING EQUATION OF A TRANSITIVE LIE GROUP

MICHIHIKO MATSUDA

(Received August 17, 1970)

We prove the following theorem: The operations of a transitive Lie group G acting on a manifold M are characterized as solutions of a differential equation on M.

1. Introduction. Let G be a connected Lie group acting differentiably on a C^∞-differentiable manifold M. We assume that the action is transitive. Fix a point o in M. By $D^k(o;M)$ we denote the space of all k-jets of local diffeomorphisms with source o and target anywhere in M. Let H^k be the subset of $D^k(o;M)$ consisting of all k-jets with target o. Then H^k is a Lie group. The space $D^k(o;M)$ is a principal fiber bundle with base M and structural group H^k.

Let K^o be the isotropy subgroup of G at o and G^k be the set of all k-jets of actions of K^o with source o. Then G^k is a Lie subgroup of H^k (Proposition 1). Let P^k be the set of all k-jets of actions of G with source o. Then P^k is an associated fiber bundle with fiber G^k to the principal fiber bundle $G(M, K^o)$. Also P^k is a reduced bundle with structural group G^k of the principal fiber bundle $D^k(o:M)$.

Let $P^k(M)$ be the space of all k-jets of actions of G with source and target anywhere in M. Then $P^k(M)$ is an associated fiber bundle with fiber P^k to the principal fiber bundle $G(M, K^o)$.

Theorem. *There exists an integer, l, such that the following holds: Suppose f is a local diffeomorphism of M defined on a connected domain V. Then f is a restriction of the action of an element g in G to V if and only if $j^l_x(f) \in P^l(M)$ for all x in V.*

REMARK 1. Our theorem was stated in a classical form by Lie in [5] for a Lie algebra of vector fields and proved by E. Cartan in [1] for a local Lie group of transformations.

REMARK 2. For a pseudo-group of infinite dimension, Kuranishi [4] gave a sufficient condition in order that it may be defined by a partial differential equation. Also for an infinite dimensional Lie algebra of vector fields, Singer and

Sternberg [6] gave a sufficient condition in order that it may be defined by a partial differential equation. Our theorem is not contained in their results as a special case.

2. Prolongation of the action of G. The group K^o is a Lie subgroup of G, since it is closed in G. The group G^k is the image of the differentiable homomorphism $j_o^k : g \to j_o^k(g)$ from K^o to H^k.

Proposition 1. *For every k, G^k is a Lie subgroup of H^k and the map j_o^k from K^o to G^k is differentiable.*

Proof. In general the image G' of a differentiable homomorphism j from a Lie group K into a Lie group H has a structure of a Lie subgroup of H such that the map j from K to G' is differentiable. The proof was given by Chevalley [2, p. 119] in the case where K and H are connected. A proof for the general case is given as follows. Let K_c and H_c be the connected components of the identity in K and H respectively. Then the image G'_* of K_c by the homomorphism j is a connected Lie subgroup of H_c. Any inner automorphism $I(g')$ defined by g' in G' maps G'_* into H_c differentiably and its image is G'_* itself. Since G'_* is an integral manifold of the involutive distribution defined by its Lie algebra, $I(g')$ gives a diffeomorphism of G'_* (Chevalley [2, p. 95]). Hence G' has a structure of a Lie subgroup of H such that its connected component of the identity is G'_*.

The group G acts on P^k by $gp^k = j_o^k(gf)$, $p^k = j_o^k(f)$. The action is differentiable and transitive. Let K^k be the isotropy subgroup of G at $o^k = j_o^k(\text{identity})$ and G_{k-1}^k be the set of all k-jets of actions of K^{k-1} on M with source o. Then G_{k-1}^k is a Lie group, and P^k is an associated fiber bundle with fiber G_{k-1}^k to the principal fiber bundle $G(P^{k-1}, K^{k-1})$.

The (k-th)-structure form ω^k on P^k with values in $T_{o^{k-1}}(P^{k-1})$ is difined by

$$\omega^k(p^k ; X^k) = g_*^{-1}(\pi_{k-1}^k)_* X^k, \quad p^k = j_o^k(g), X^k \in T_{p^k}(P^k),$$

where $\pi_{k-1}^k j_o^k(g) = j_o^{k-1}(g)$ (see Guillemin and Sternberg [3]). It is well-defined. The group G leaves ω^k on P^k invariant:

$$\omega^k(gp^k ; g_* X^k) = \omega^k(p^k ; X^k) \qquad \text{for any } g \in G.$$

Since P^k has a structure of an associated fiber bundle to $G(P^{k-1}, K^{k-1})$, we have the inequality $\dim P^{k-1} \leqq \dim P^k \leqq \dim G$. Hence there exists an integer k such that $\dim P^{k-1} = \dim P^k$. We denote the smallest integer k with this property by l.

At every point p^l in P^l the projection π_{l-1}^l gives a diffeomorphism from a neighborhood of p^l to a neighborhood of $\pi_{l-1}^l p^l$. Hence to every vector X^{l-1} in $T_{o^{l-1}}(P^{l-1})$ we can correspond a differentiable vector field $X^l(p^l)$ on P^l by

$\omega^l(p^l; X^l(p^l)) = X^{l-1}$. It is left invariant by G.

The group G acts on P^l transitively. Hence by the uniqueness theorem of a solution of an ordinary differential equation the following proposition holds.

Proposition 2. *Let φ and ψ be two differentiable maps from a connected manifold W into P^l. If they satisfy the relation*

$$\omega^l(\varphi(w);\varphi_* X) = \omega^l(\psi(w);\psi_* X)$$

for all w in W and X in $T_w(W)$, then there exists an element g of G such that the identity $\psi(w) = g\varphi(w)$ holds on W. Every element g of G which maps $\varphi(w_o)$ to $\psi(w_o)$ for a point w_o in W has this property.

Corollary 1. *The group K^l leaves all points in P^l fixed and hence all points in M fixed.*

This follows from the assumption that G is connected; For P^l is connected and we can take P^l as the W in Proposition 2.

Corollary 2. *If the actions of two elements g and g' in G coincide on an open set in M, then their actions coincide on M.*

3. Proof of Theorem. The necessity of the condition is obvious. We prove that it is sufficient. The first step is to prove the theorem for a sufficiently small connected neighborhood U of o. Take U so small that a local cross-section $\varphi : U \ni u \to \varphi_u \in G$ exists. Let us define a map f^l from U to $D^l(o:M)$ by $f^l(u) = j^l_o(f\varphi_u)$. Then it is defferentiable as a map from U to $D^l(o:M)$. By the hypothesis, for every u in U there exists an element g_u in G such that $j^l_u(f) = j^l_u(g_u)$. Hence the image of f^l is contained in P^l.

The Lie subgroup G^l of H^l has countable connected components at most, since the closed subgroup K^o of the connected Lie group G has this property. At every point p^l in P^l we can take a neighborhood U^l of p^l in $D^l(o:M)$ such that the two connected components of p^l in $P^l \cap U^l$ in the topology of P^l and in that of U^l coincide. Hence f^l is differentiable as a map from U to P^l. Let φ^l be a map from U to P^l defined by $\varphi^l(u) = j^l_o(\varphi_u)$. It is differentiable. For any vector X in $T_u(M)$ we obtain the identity

$$(\pi^l_{l-1} f^l)_* X = (g_u)_* (\pi^l_{l-1}\varphi^l)_* X$$

by the definition of jets. Hence we have

$$\omega^l(f^l(u); f^l_* X) = \omega^l(\varphi^l(u); \varphi^l_* X)$$

for all u in U and X in $T_u(M)$. By Proposition 2 there exists an element g in G such that $f^l(u) = g\varphi^l(u)$ on U. By Corollary 1 we have $g_u\varphi_u = g\varphi_u$ modulo K, the isotropy subgroup of G which leaves all points in M fixed. Hence for any u in U, $g_u = g$ modulo K.

The second step is to prove the theorem for a general connected domain V. For every x in V take an element φ_x of G which maps o to x. Then there exists an element h_x in G such that the identity $f(v) = h_x\varphi^{-1}(v)$ holds on U_x, the connected component of x in $\varphi_x U \cap V$. If $U_x \cap U_{x'} \neq \phi$, then by Corollary 2, $h_x\varphi_x^{-1} = h_{x'}\varphi_{x'}^{-1}$ modulo K. Since V is connected, there exists an element g in G such that the identity $f(v) = g(v)$ holds on V.

OSAKA UNIVERSITY

Bibliography

[1] E. Cartan: Théorie des Groupes Finis et Continus et la Géométrie Différentielle traitées par la Méthode du Repère Mobile, Chap. 8, Gauthier-Villars, Paris, 1937.

[2] C. Chevalley: Theory of Lie Groups, I, Princeton Univ. Press, Princeton, 1946.

[3] V. Guillemin and S. Sternberg: *Deformation theory of pseudogroup structures*, Mem. Amer. Math. Soc. **64** (1966).

[4] M. Kuranishi: *On the local theory of continuous infinite pseudo groups*, II, Nagoya Math. J. **19** (1961), 55–91.

[5] S. Lie: Theorie der Transformationsgruppen, I, Chap.11, Teubner, Leipzig, 1888.

[6] I.M. Singer and S. Sternberg: *The infinite groups of Lie and Cartan*, J. Analyse Math. **15** (1965), 1–114.