

SMOOTH STRUCTURES ON $S^p \times S^q$

Dedicated to Professor A. Komatu for his 60th birthday

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1. Introduction

This paper gives the details promised in [8] and [9]. Novikov classified smooth structures modulo one point of the manifolds which are tangentially homotopy equivalent to a product $S^p \times S^q$ of spheres (see [16]). As is shown in §2, every smooth structure on $S^p \times S^q$ has a stably trivial tangent bundle. Therefore, applying the theorem of Novikov, we can classify smooth structures on $S^p \times S^q$ modulo one point. On the other hand, the author determined in his paper [8] the inertia group $I(S^p \times \tilde{S}^q)$ of $S^p \times \tilde{S}^q$. In the present paper, we shall give the complete classification theorem of smooth structures on $S^p \times S^q$ for $p+q \geq 6$ and $q \geq p \geq 1$, by combining above results. Quite a similar argument proves the classification theorem of smooth structures on a sphere bundle over sphere with a cross section. In §6 we shall show some examples and in §8 we shall answer the problem of I. James on H -spaces (see [12]).

The main results of this paper will be Theorem 5.2 in §5, Theorem 7.1 in §7 and Theorem 8.1 in §8.

2. Definitions, notations and a preliminary lemma

All manifolds, with or without boundary, will be oriented and smooth (or *PL*;) in the differentiable case they may have corners, which we shall deal with by *ignoring* them. For oriented and smooth manifolds M_1 and M_2 , $M_1 = M_2$ will mean M_1 is diffeomorphic to M_2 by an orientation preserving diffeomorphism. We shall use the notations and terminologies of [8] and in §4 we shall use the notations of Novikov [16].

Denote by $\mathcal{S}(M)$ the set of smooth structures on M modulo orientation preserving diffeomorphisms.

Define a subset $\mathcal{S}_0(S^p \times S^q)$ of $\mathcal{S}(S^p \times S^q)$ as follows. A smooth structure $(S^p \times S^q)_\alpha$ represents an element of $\mathcal{S}_0(S^p \times S^q)$ if and only if $(S^p \times S^q)_\alpha$ is diffeomorphic to $S^p \times \tilde{S}^q$ for some homotopy sphere \tilde{S}^q ($p \leq q$).

For $M_\alpha, M_\beta \in \mathcal{S}(M)$, define $M_\alpha \sim M_\beta$ if and only if there exists an orientation preserving diffeomorphism $f: M_\alpha \rightarrow M_\beta$ modulo one point. Denote by $\mathcal{S}'(M)$ the quotient set $\mathcal{S}(M)/\sim$ of $\mathcal{S}(M)$.

For $\alpha, \beta \in \Theta_q$, define $\alpha \sim \beta$ if and only if $\alpha = \beta$ or $\alpha = -\beta$. Denote by Θ_q/\sim the quotient set of Θ_q .

For $\alpha, \beta \in \Theta_q$ and for a fixed integer $p \geq 2$, define $\alpha \approx \beta$ if and only if $\alpha - \beta$ or $\alpha + \beta$ embeds in the $(q+p+1)$ -dimensional euclidean space R^{q+p+1} with a trivial normal bundle. Denote by Θ_q^p/\approx the quotient set of Θ_q .

Define a subgroup $G'(q)$ of $G(q) = \pi_{q+N}(S^N)$ (N : large) as follows. A map $f: S^{q+N} \rightarrow S^N$ represents an element of $G'(q)$ if and only if f represents the Pontrjagin-Thom map of some framed imbedding $\tilde{S}^q \times D^N \subset S^{q+N}$. (As is well known, $G'(q) = G(q)$ if q is of the form $2k+1$ or $4k$ and $G(q)/G'(q)$ is 0 or Z_2 if q is of the form $4k+2$.)

Denote by $[N]$ the homology class represented by an oriented manifold N .

Denote by $C(M)$ the concordance classes of smoothing a PL-manifold M .

In the following we shall use the same symbol for an element of a quotient group (or a quotient set) as its representative.

Let a smooth structure M_α on $S^p \times S^q$ be given i.e., assume that there is given a piecewise differentiable homeomorphism

$$f: S^p \times S^q \longrightarrow M_\alpha.$$

Let x_0 (resp. y_0) denote a point of S^p (resp. S^q). Let $R_{x_0}^p$ (resp. $R_{y_0}^q$) denote a small open neighbourhood of x_0 (resp. y_0) in S^p (resp. S^q) which is PL-homeomorphic to the euclidean space R^p (resp. R^q). Since $f(R_{x_0}^p \times S^q)$ ((resp. $f(S^p \times R_{y_0}^q)$) is an open submanifold in M_α , $f(R_{x_0}^p \times S^q)$ (resp. $f(S^p \times R_{y_0}^q)$) has an induced smooth structure

$$\{f(R_{x_0}^p \times S^q)\}_\alpha \quad (\text{resp. } \{f(S^p \times R_{y_0}^q)\}_\alpha).$$

By making use of the Cairns and Hirsch's smoothing theorem [4] and [6], there exists a homotopy sphere \tilde{S}^q (resp. \tilde{S}^p) such that $\{f(R_{x_0}^p \times S^q)\}_\alpha$ (resp. $\{f(S^p \times R_{y_0}^q)\}_\alpha$) is diffeomorphic to

$$R^p \times \tilde{S}^q \quad (\text{resp. } \tilde{S}^p \times R^q).$$

Let d_1 (resp. d_2) be this diffeomorphism. Let x_1 (resp. y_2) denote the origin of R^p (resp. R^q). Then

$$d_1^{-1}(x_1 \times \tilde{S}^q) \quad (\text{resp. } d_2^{-1}(\tilde{S}^p \times y_2))$$

obviously represents a generator $[x_0 \times S^q]$ (resp. $[S^p \times y_0]$) of

$$H_q(S^p \times S^q) \cong H_q(M_\omega) \quad (\text{resp. } H_p(S^p \times S^q) \cong H_p(M_\omega)).$$

We may assume that $d_1^{-1}(x_1 \times \tilde{S}^q)$ and $d_2^{-1}(\tilde{S}^p \times y_1)$ intersect transversally at one point. It follows that there exists a smooth imbedding

$$g : \tilde{S}^p \times D^q \natural D^p \times \tilde{S}^q \longrightarrow M_\omega - \text{Int } D^{p+q}$$

inducing isomorphisms of homology groups

$$g_* : H_*(\tilde{S}^p \times D^q \natural D^p \times \tilde{S}^q) \xrightarrow{\cong} H_*(M_\omega - \text{Int } D^{p+q})$$

where \natural denotes a generalized plumbing of two manifolds obtained as follows. When we regard $\tilde{S}^p \times D^q$ as $D_+^p \times D^q \cup_{\tau_1} D_-^p \times D^q$ and $D^p \times \tilde{S}^q$ as $D^p \times D_+^q \cup_{\tau_2} D^p \times D_-^q$,

$$\tilde{S}^p \times D^q \natural D^p \times \tilde{S}^q$$

denotes the oriented differentiable $(p+q)$ -manifold formed from the disjoint sum $\tilde{S}^p \times D^q \cup D^p \times \tilde{S}^q$ by identifying $D_-^p \times D^q$ with $D^p \times D_+^q$ in such a way that $D_-^p = D^p$ and $D^q = D_+^q$. If $q \geq p \geq 2$, clearly we have

$$\pi_1(\partial(\tilde{S}^p \times D^q \natural D^p \times \tilde{S}^q)) \cong \pi_1(M_\omega - \text{Int } D^{p+q}) \cong \{1\}$$

and we can deduce that

$$\pi_1(M_\omega - \text{Int } D^{p+q} - \text{Int } (\tilde{S}^p \times D^q \natural D^p \times \tilde{S}^q)) \cong \{1\}$$

by the Van Kampen's theorem. Hence $g(\tilde{S}^p \times D^q \natural D^p \times \tilde{S}^q)$ is a deformation retract of $M_\omega - \text{Int } D^{p+q}$. It follows from the Smale's h -cobordism theorem that $\tilde{S}^p \times D^q \natural D^p \times \tilde{S}^q$ is diffeomorphic to $M_\omega - \text{Int } D^{p+q}$ for $p+q \geq 6$, $q \geq p \geq 2$. Applying the similar argument as above, we can show that $\tilde{S}^p \times D^q \natural D^p \times \tilde{S}^q$ is diffeomorphic to $S^p \times S^q - \text{Int } D^{p+q}$. Thus M_ω is diffeomorphic to $\tilde{S}^p \times \tilde{S}^q \# \tilde{S}^{p+q}$ for some homotopy sphere \tilde{S}^{p+q} for $p+q \geq 6$, $q \geq p \geq 2$, here $\#$ denotes the connected sum operation. If $p+q \geq 6$, $q \geq p \geq 2$, \tilde{S}^p embeds in the $(p+q)$ -dimensional euclidean space R^{p+q} with a trivial normal bundle, therefore $\tilde{S}^p \times D^q$ is diffeomorphic to $S^p \times D^q$ (see for example Hsiang, Levine and Szczarba [7]).

In case $p=1$ and $q \geq 5$. The similar argument proves that $S^1 \times D^q \natural D^1 \times \tilde{S}^q$ embeds in $M_\omega - \text{Int } D^{1+q}$ inducing isomorphisms

$$H_*(S^1 \times D^q \natural D^1 \times \tilde{S}^q) \cong H_*(M_\omega - \text{Int } D^{1+q})$$

of homology groups for some homotopy sphere $\tilde{S}^q \in \Theta_q$. Therefore

$$H_*(M_\omega - \text{Int } D^{1+q} - \text{Int } (S^1 \times D^q \natural D^1 \times \tilde{S}^q), \partial(S^1 \times D^q \natural D^1 \times \tilde{S}^q))$$

is trivial by the excision isomorphism theorem.

Using the Poincaré duality theorem, we have that

$$H_*(M_\omega - \text{Int} D^{1+q} - \text{Int} (S^1 \times D^q \mathbin{\mathbb{V}} D^1 \times \tilde{S}^q), \partial(M_\omega - \text{Int} D^{1+q})) \cong \{0\}.$$

Since the pseudo PL-isotopy group $\pi_0(\text{PL } S^{q-1})$ of S^{q-1} is trivial, $S^1 \times D^q \mathbin{\mathbb{V}} D^1 \times \tilde{S}^q$ is PL homeomorphic to the standard plumbing $S^1 \times D^q \mathbin{\mathbb{V}} D \times S^q$. It follows that

$$\partial(S^1 \times D^q \mathbin{\mathbb{V}} D^1 \times \tilde{S}^q)$$

is PL-homeomorphic to S^q . Obviously we have

$$\pi_1(\partial(M_\omega - \text{Int} D^{1+q})) \cong \pi_1(\partial(S^1 \times D^q \mathbin{\mathbb{V}} D^1 \times \tilde{S}^q)) \cong \{1\}$$

and the natural inclusion induces the isomorphism

$$\pi_1(S^1 \times D^q \mathbin{\mathbb{V}} D^1 \times \tilde{S}^q) \cong \pi_1(M_\omega - \text{Int} D).$$

It follows from the Van kampen's theorem that

$$\pi_1(M_\omega - \text{Int} D^{1+q} - \text{Int} (S^1 \times D^q \mathbin{\mathbb{V}} D^1 \times \tilde{S}^q)) \cong \{1\}.$$

Hence $S^1 \times D^q \mathbin{\mathbb{V}} D^1 \times \tilde{S}^q$ is diffeomorphic to $M_\omega - \text{Int} D^{1+q}$ by the Smale's h -cobordism theorem and M_ω is diffeomorphic to $S^1 \times \tilde{S}^q \# \tilde{S}^{1+q}$ for some homotopy spheres $\tilde{S}^q \in \Theta_q$ and $\tilde{S}^{1+q} \in \Theta_{q+1}$.

Combining these results, we have the following lemma:

Lemma 2.1. *If $p+q \geq 6$ and $q \geq p \geq 1$, every smooth structure M_ω on $S^p \times S^q$ is diffeomorphic to $S^p \times \tilde{S}^q \# \tilde{S}^{p+q}$ for some homotopy spheres \tilde{S}^q and \tilde{S}^{p+q} .*

As a corollary we have

Corollary 2.2. *If $p+q \geq 6$ and $q \geq p \geq 1$, every smooth structure M_ω on $S^p \times S^q$ has a stably trivial tangent bundle.*

3. The inertia group of $S^p \times \tilde{S}^q$ for $q \geq p$

The inertia group $I(S^p \times \tilde{S}^q)$ is essential for classifying smooth structures on $S^p \times S^q$. The following lemma is proved in Theorem C of Kawakubo [8].

Lemma 3.1. *Let $K_1 : \pi_p(\text{SO}) \times \Theta_q \rightarrow \Theta_{p+q}$ denote the pairing defined by Milnor-Munkres-Novikov. Then it holds that*

$$I(S^p \times \tilde{S}^q) = K_1(\pi_p(\text{SO}_q), \tilde{S}^q)$$

for $p+q \geq 5$.

4. Diffeomorphism modulo a point

Lemma 4.1. *If $S^p \times \tilde{S}_1^q$ is diffeomorphic to $S^p \times \tilde{S}_2^q$ modulo a point, then $S^p \times \tilde{S}_1^q$ is actually diffeomorphic to $S^p \times \tilde{S}_2^q$ for $p+q \geq 5$ and $q \geq p \geq 1$.*

Proof. When $q \leq p+3$, every $S^p \times \tilde{S}^q$ is diffeomorphic to the product $S^p \times S^q$ of the natural spheres S^p , S^q (see Hsiang, Levine and Szczarba [7]). Therefore we may assume $q > p+3$. If $S^p \times \tilde{S}_1^q$ is diffeomorphic to $S^p \times \tilde{S}_2^q$ modulo a point, then there exists a homotopy sphere \tilde{S}^{p+q} such that $S^p \times \tilde{S}_1^q \# \tilde{S}^{p+q}$ is diffeomorphic to $S^p \times \tilde{S}_2^q$. Let

$$f : S^p \times \tilde{S}_1^q \# \tilde{S}^{p+q} \longrightarrow S^p \times \tilde{S}_2^q$$

be the diffeomorphism, and let

$$f' : S^p \times \tilde{S}_1^q - \text{Int } D^{p+q} \longrightarrow S^p \times \tilde{S}_2^q - \text{Int } D^{p+q}$$

be the restriction of f . Let X be the manifold obtained by attaching two manifolds $W_1 = D^{p+1} \times \tilde{S}_1^q$ and $W_2 = D^{p+1} \times \tilde{S}_2^q$ by the diffeomorphism

$$f' : \partial W_1 - \text{Int } D^{p+q} \longrightarrow \partial W_2 - \text{Int } D^{p+q}.$$

Obviously the boundary ∂X is diffeomorphic to the homotopy sphere \tilde{S}^{p+q} .

Define a diffeomorphism r_1 (resp. r_2) of $D^{p+1} \times \partial D_1^q$ (resp. $D^{p+1} \times \partial D_2^q$) by writing

$$\begin{aligned} r_1(x, y) &= (x, r'_1(y)) \\ (\text{resp. } r_2(x, y) &= (x, r'_2(y))). \end{aligned}$$

Here r'_1 (resp. r'_2) represents the element of $\pi_0(\text{Diff } S^{q-1}) = \Gamma_q$ corresponding to \tilde{S}_1^q (resp. \tilde{S}_2^q) $\in \Theta_q$ by the natural isomorphism $\Gamma_q \cong \Theta_q$ due to Smale. Then it is clear that the manifold $W_1 = D^{p+1} \times \tilde{S}_1^q$ (resp. $W_2 = D^{p+1} \times \tilde{S}_2^q$) can be written as

$$\begin{aligned} W_1 &= D^{p+1} \times D_1^q \cup_{r_1} D^{p+1} \times D_1^q \\ (\text{resp. } W_2 &= D^{p+1} \times D_2^q \cup_{r_2} D^{p+1} \times D_2^q). \end{aligned}$$

Let y_1 (resp. y_2) denote a point of \tilde{S}_1^q (resp. \tilde{S}_2^q). According to Haefliger [5] and the covering isotopy theorem, we can assume that $f'(x, y_i) = (x, y_i)$ for all $x \in S^p$. Hence there exists a natural imbedding

$$g : D^{p+1} \longrightarrow D^{p+1} \times \tilde{S}_2^q,$$

defined by $g(x) = (x, y_2)$. Clearly $D^{p+1} \times y_1 \cup g(D^{p+1}) (= D^{p+1} \times y_1 \cup D^{p+1} \times y_2)$ is the natural sphere. Denote by B_1 the normal disk bundle of $D^{p+1} \times y_1 \cup g(D^{p+1}) = D^{p+1} \times y_1 \cup D^{p+1} \times y_2$ in X . Let $h \in \pi_p(\text{SO}_q)$ be the characteristic map of the bundle B_1 . Regarding B_1 as

$$D^{p+1} \times D_2^q \cup_h D^{p+1} \times D_1^q,$$

we have the following manifold

$$Y = D^{p+1} \times D_2^q \cup_h D^{p+1} \times D_1^q \cup_{r_1} D^{p+1} \times D_1^q$$

which is imbedded in X . Since Y is none other than

$$B_1 \odot D^{p+1} \times \tilde{S}_1^q,$$

∂Y is diffeomorphic to $K_1(h, \tilde{S}_1^q)$ (see Kawakubo [8]). We shall show that Y is diffeomorphic to X . We may assume that Y is contained in $\text{Int } X$. It is obvious that the imbedding $i: Y \rightarrow X$ induces isomorphisms of homology groups

$$i_*: H_*(Y) \xrightarrow{\cong} H_*(X),$$

and that

$$\pi_1(X) \cong \pi_1(Y) \cong \pi_1(\partial Y) \cong \{1\}.$$

(When $p=1$ and $q \geq 5$, this follows from the Van-Kampen's theorem.) According to the Van Kampen's theorem, we shall have

$$\pi_1(X - \text{Int } Y) \cong \{1\}.$$

It follows from the theorem of J.H.C. Whitehead [19] that Y is a deformation retract of X . Now applying the Smale's h -cobordism theorem, we see that X is diffeomorphic to Y ; hence

$$\tilde{S}^{p+q} = \partial X = \partial Y = K_1(h, \tilde{S}_1^q).$$

Since $K_1(\pi_p(\text{SO}_q), \tilde{S}_1^q)$ is exactly the inertia group $I(S^p \times \tilde{S}_1^q)$ (see Lemma 3.1), \tilde{S}^{p+q} belongs to the inertia group $I(S^p \times \tilde{S}_1^q)$. It follows that

$$S^p \times \tilde{S}_1^q = S^p \times \tilde{S}_2^q \# \tilde{S}^{p+q} = S^p \times \tilde{S}_2^q,$$

which completes the proof of Lemma 4.1.

Corollary 4.2. *The set $\mathcal{S}_0(S^p \times S^q)$ is in one-to-one correspondence with the set $\mathcal{S}'(S^p \times S^q)$ by the composition*

$$\mathcal{S}_0(S^p \times S^q) \longrightarrow \mathcal{S}(S^p \times S^q) \longrightarrow \mathcal{S}'(S^p \times S^q)$$

of two natural maps for $p+q \geq 6$ and $p \geq q \geq 1$.

Proof. Since every element of $\mathcal{S}'(S^p \times S^q)$ is represented by $S^p \times \tilde{S}^q$ for some homotopy sphere \tilde{S}^q (see Lemma 2.1.), the composition is surjective. If two elements $S^p \times \tilde{S}_1^q$ and $S^p \times \tilde{S}_2^q$ go into the same element of $\mathcal{S}'(S^p \times S^q)$ by the

composition, i.e.,

$$S^p \times \tilde{S}_q^q \# \tilde{S}^{p+q} = S^p \times \tilde{S}_2^q$$

for some homotopy sphere \tilde{S}^{p+q} , then $S^p \times \tilde{S}_1^q$ is actually diffeomorphic to $S^p \times \tilde{S}_2^q$ by Lemma 4.1. This shows that the composition is injective. The following proposition is a geometric interpretation of the set $\mathcal{S}_0(S^p \times S^q)$ (i.e., of the set $\mathcal{S}'(S^p \times S^q)$).

Proposition 4.3. *The set $\mathcal{S}_0(S^p \times S^q)$ (or $\mathcal{S}'(S^p \times S^q)$) is in one-to-one correspondence with the quotient set Θ_q^p / \approx by*

$$S^p \times \tilde{S}^q \longrightarrow \{\tilde{S}^q\}$$

for $p+q \geq 6$ and $q \geq p \geq 2$.

The proof will be based on the following two lemmas.

Lemma 4.4. *A necessary and sufficient condition for $S^p \times \tilde{S}^q$ and $S^p \times S^q$ to be diffeomorphic is that \tilde{S}^q embeds in R^{p+q+1} with a trivial normal bundle for $p+q \geq 6$ and $q \geq p \geq 2$.*

Lemma 4.5. *Either $S^p \times (\tilde{S}_1^q \# \tilde{S}_2^q)$ or $S^p \times (\tilde{S}_1^q \# (-\tilde{S}_2^q))$ is diffeomorphic to $S^p \times S^q$ if and only if $S^p \times \tilde{S}_1^q$ is diffeomorphic to $S^p \times \tilde{S}_2^q$ for $p+q \geq 6$ and $q \geq p \geq 2$.*

First, assuming these two lemmas, we prove Proposition 4.3. Define a map

$$l : \Theta_q \longrightarrow \mathcal{S}_0(S^p \times S^q)$$

by

$$l(\tilde{S}^q) = S^p \times \tilde{S}^q.$$

Obviously the map l is surjective. It follows from Lemma 4.4 and Lemma 4.5 that l induces a map

$$l' : \Theta_q^p / \approx \longrightarrow \mathcal{S}_0(S^p \times S^q)$$

such that the following diagram commutes:

$$\begin{array}{ccc} \Theta_q & \xrightarrow{l} & \mathcal{S}_0(S^p \times S^q) \\ j \searrow & & \nearrow l' \\ & \Theta_q^p / \approx & \end{array}$$

where j denotes the natural projection. The fact that l' is injective follows also

from Lemma 4.4 and Lemma 4.5. Hence, assuming two lemmas, we obtain Proposition 4.3.

Next, we shall prove these lemmas.

Proof of Lemma 4.4. Since it is easy to prove this Lemma, it is left to the reader.

Proof of Lemma 4.5. Suppose that $S^p \times (\tilde{S}_1^q \# (-\tilde{S}_2^q))$ is diffeomorphic to $S^p \times S^q$, i.e., there exists a diffeomorphism

$$f : S^p \times (\tilde{S}_1^q \# (-\tilde{S}_2^q)) \longrightarrow S^p \times S^q .$$

Let $R : S^p \times S^q \rightarrow S^p \times S^q$ be an orientation preserving diffeomorphism inducing isomorphisms of homology groups

$$R_* : H_*(S^p \times S^q) \xrightarrow{\cong} H_*(S^p \times S^q)$$

with

$$R_*[S^p \times y_0] = -[S^p \times y'_0]$$

and

$$R_*[x_0 \times S^q] = -[x'_0 \times S^q] .$$

(Obviously there always exists such a diffeomorphism.) Making use of this diffeomorphism R (if necessary), we can assume that

$$f_*[S^p \times y_0] = [S^p \times y'_0] .$$

According to Haefliger [5] and the covering isotopy theorem, we can assume that

$$f(x \times y_0) = x \times y'_0 \quad \text{for all } x \in S^p .$$

When we write $S^p \times (\tilde{S}_1^q \# (-\tilde{S}_2^q))$ as

$$S^p \times D^q \cup_{r'_1 - r'_2} S^p \times D'^q$$

where $r'_i : S^{q-1} \rightarrow S^{q-1}$ represents \tilde{S}_i^q ($i=1, 2$) (for the definition of r_i by r'_i see page 169) and $S^p \times S^q$ as

$$S^p \times D^q \cup_{id} S^p \times D'^q ,$$

we may assume that

$$f(S^p \times D^q) = S^p \times D^q$$

and that $g=f|_{S^p \times D^q}$ is a bundle map by the uniqueness of tubular neighbourhoods. Using this bundle map g , we have a diffeomorphism

$$S = g^{-1} \cup g^{-1} : S^p \times D^q \underset{id}{\cup} S^p \times D^q \longrightarrow S^p \times D^q \underset{id}{\cup} S^p \times D^q .$$

Making use of this diffeomorphism S (if necessary), we can assume that

$$f|_{S^p \times D^q} = id .$$

Therefore we can define a diffeomorphism

$$f' : (S^p \times (\tilde{S}_1^q \# (-\tilde{S}_2^q)) - S^p \times \text{Int } D^q) \underset{r_2}{\cup} S^p \times D^q \longrightarrow (S^p \times S^q - S^p \times \text{Int } D^q) \underset{r_2}{\cup} S^p \times D^q$$

by

$$f'|_{S^p \times (\tilde{S}_1^q \# (-\tilde{S}_2^q)) - S^p \times \text{Int } D^q} = f|_{S^p \times (\tilde{S}_1^q \# (-\tilde{S}_2^q)) - S^p \times \text{Int } D^q}$$

and

$$f'|_{S^p \times D^q} = id .$$

Clearly

$$(S^p \times (\tilde{S}_1^q \# (-\tilde{S}_2^q)) - S^p \times \text{Int } D^q) \underset{r_2}{\cup} S^p \times D^q$$

is diffeomorphic to $S^p \times \tilde{S}_1^q$ and

$$(S^p \times S^q - S^p \times \text{Int } D^q) \underset{r_2}{\cup} S^p \times D^q$$

is diffeomorphic to $S^p \times \tilde{S}_2^q$, i.e., f' gives a diffeomorphism of $S^p \times \tilde{S}_1^q$ onto $S^p \times \tilde{S}_2^q$. Suppose that $S^p \times (\tilde{S}_1^q \# \tilde{S}_2^q)$ is diffeomorphic to $S^p \times S^q$. The similar argument proves that $S^p \times \tilde{S}_1^q$ is diffeomorphic to $S^p \times (-\tilde{S}_2^q)$. Since there always exist orientation reversing diffeomorphisms

$$g_1 : S_1^p \longrightarrow S_1^p$$

and

$$g_2 : -\tilde{S}_2^q \longrightarrow \tilde{S}_2^q ,$$

there exists an orientation preserving diffeomorphism

$$g : S^p \times (-\tilde{S}_2^q) \longrightarrow S^p \times \tilde{S}_2^q$$

defined by $g = g_1 \times g_2$. Hence $S^p \times \tilde{S}_1^q$ is diffeomorphic to $S^p \times \tilde{S}_2^q$.

Conversely suppose that a diffeomorphism

$$f : S^p \times \tilde{S}_1^q \longrightarrow S^p \times \tilde{S}_2^q$$

be given. If $f_*[S^p \times y_0] = [S^p \times y'_0]$, then the similar argument as above shows that we can assume

$$f(x \times y_0) = x \times y'_0 \quad \text{for all } x \in S^p.$$

Moreover we can assume that

$$f(D^p \times D^q \cup D'^p \times D^q) = D^p \times D^q \cup D'^p \times D^q$$

and that

$$f|_{D^p \times D^q} = id$$

when we regard $S^p \times \tilde{S}_1^q$ as $(D^p \times D^q \cup D'^p \times D^q) \cup_{r_1} S^p \times D'^q$ and $S^p \times \tilde{S}_2^q$ as $(D^p \times D^q \cup D'^p \times D^q) \cup_{r_2} S^p \times D'^q$. Therefore we can define a diffeomorphism $f' : (S^p \times \tilde{S}_1^q - S^p \times \text{Int } D^q) \cup_{-r_1} S^p \times D^q \longrightarrow (S^p \times \tilde{S}_2^q - S^p \times \text{Int } D^q) \cup_{f(-r_1)} S^p \times D^q$ by

$$f'|_{S^p \times \tilde{S}_1^q - S^p \times \text{Int } D^q} = f|_{S^p \times \tilde{S}_1^q - S^p \times \text{Int } D^q}$$

and

$$f'|_{S^p \times D^q} = id.$$

Clearly

$$(S^p \times \tilde{S}_1^q - S^p \times \text{Int } D^q) \cup_{-r_1} S^p \times D^q$$

is diffeomorphic to $S^p \times S^q$. Since $f|_{D^p \times D^q} = id$, $x_0 \times D^q \cup_{r_2' - r_1'} x_0 \times D'^q$ ($x_0 \in \text{Int } D^p$) is imbedded in

$$(S^p \times \tilde{S}_2^q - S^p \times \text{Int } D^q) \cup_{f(-r_1)} S^p \times D^q$$

with a trivial normal bundle, i.e., $\tilde{S}_1^q \# (-\tilde{S}_2^q)$ embeds in $S^p \times S^q$ with a trivial normal bundle. Hence $\tilde{S}_1^q \# (-\tilde{S}_2^q)$ embeds in R^{p+q+1} with a trivial normal bundle. It follows from Lemma 4.4 that $S^p \times (\tilde{S}_1^q \# (-\tilde{S}_2^q))$ is diffeomorphic to $S^p \times S^q$.

If there exists a diffeomorphism $f : S^p \times \tilde{S}_1^q \rightarrow S^p \times \tilde{S}_2^q$ such that

$$f_*[S^p \times y_0] = -[S^p \times y'_0],$$

then replacing \tilde{S}_2^q by $-\tilde{S}_2^q$, we have an orientation preserving diffeomorphism

$$f' : S^p \times \tilde{S}_1^q \longrightarrow S^p \times (-\tilde{S}_2^q)$$

such that

$$f'_*[S^p \times y_0] = [S^p \times y'_0].$$

The similar argument proves that $S^p \times (\tilde{S}_1^q \# \tilde{S}_2^q)$ is diffeomorphic to $S^p \times S^q$. This completes the proof of Lemma 4.5 and consequently finishes the proof of Proposition 4.3.

The following lemma is a revised form of the classification theorem of Novikov [16] and is useful to compute the set $\mathcal{S}'(S^p \times S^q)$ (for examples Proposition 6.2 and Proposition 6.4).

Lemma 4.6. $\mathcal{S}'(S^p \times S^q)$, hence $\mathcal{S}_0(S^p \times S^q)$, is in one-to-one correspondence with $G'(q)/\sim$ for $p+q \geq 6$ and $q \geq p \geq 2$ where \sim is the restriction to $G'(q)$ of the Novikov's relation.

Proof. In the following we shall use the notations of Novikov [16]. Since $T_N(S^p \times S^q) \cong S^{N+p+q} \vee S^{N+q} \vee S^{N+p} \vee S^N$,

$$\pi_{N+p+q}(T_N) \cong Z + G(p) + G(q) + G(p+q)$$

and the set $A(S^p \times S^q)$ consists of all possible elements of the form $1_{N+p+q} + \alpha$ where $1_{N+p+q} \in Z$, $\alpha \in G(p) + G(q) + G(p+q)$. Novikov has proved the following:

Lemma 4.7. (Lemma 9.1 of [16])

$$B(M_1^n \# \tilde{S}^n) = B(M_1^n) + \kappa_* B(\tilde{S}^n) \quad \text{where} \quad \kappa_* : G(n) \rightarrow \pi_{N+n}(T_N(M^n)).$$

Lemma 4.8. (Lemma 11.4 of [16]) For each homotopy sphere $\tilde{S}^q \in \Theta_q$ the set

$$B(S^p \times \tilde{S}^q) \subset A(S^p \times S^q)$$

contains all elements of the form

$$1_{N+p+q} + \tilde{B}(\tilde{S}^q) + G(p) \quad [\text{mod } G(p+q)],$$

where to the element $1_{N+p+q} + 0$ corresponds the manifold

$$M^{p+q} = S^p \times S^q,$$

and the set $\tilde{B}(\tilde{S}^q)$ represents a coset mod $\text{Im } J$ in the group $G(q)$.

Lemma 4.9. (Theorem 11.5 of [16]) 1) If $q \equiv 2 \pmod{4}$, then each element of the set $A(S^p \times S^q) \pmod{G(p+q)}$ belongs to one of the sets $B(S^p \times \tilde{S}^q)$, $\tilde{S}^q \in \Theta_q$ and there exists the following embedding

$$B(S^p \times \tilde{S}^q) \subset 1_{N+p+q} + \tilde{B}(\tilde{S}^q) + G(p) \quad \pmod{G(p+q)}.$$

For any pair $\tilde{S}^p \in \Theta_p$, $\tilde{S}^q \in \Theta_q$ there exists a smooth sphere $\tilde{S}_1^q \in \Theta_q$ such that

$$B(\tilde{S}^p \times \tilde{S}^q) = B(S^p \times \tilde{S}_1^q) \quad \pmod{G(p+q)}.$$

2) If a manifold M_1^{p+q} is such that

$$B(M_1^{p+q}) \equiv B(\tilde{S}^p \times \tilde{S}^q) \quad \pmod{G(p+q)}$$

for $\tilde{S}^p \in \Theta_p$, $\tilde{S}^q \in \Theta_q$, then the manifold M_1^{p+q} is not combinatorially equivalent to the manifold $M^{p+q} = S^p \times S^q$. 3) If $B(M_1^{p+q}) = B(M_2^{p+q}) \bmod G(p+q)$, then the manifolds M_1^{p+q} and M_2^{p+q} are diffeomorphic modulo a point.

Let A' denote a subset of A consisting of those elements which have the form $1_{N+p+q} + \alpha$ where $1_{N+p+q} \in Z$ and $\alpha \in G(p) + G'(q) + G(p+q)$. Then it follows from these lemmas that every $S^p \times \tilde{S}^q \# \tilde{S}^{p+q}$ has a representative in A' and that every element of $A' \bmod G(p) + G(p+q)$ (i.e., every element of $1_{N+p+q} + G'(q)$) corresponds to some smooth structure $S^p \times \tilde{S}^q$ modulo a point. Therefore we have a well-defined surjective map

$$k : S'(S^p \times S^q) \longrightarrow G'(q)/\sim.$$

If two elements $S^p \times \tilde{S}_1^q$ and $S^p \times \tilde{S}_2^q$ go into the same element, then they are diffeomorphic modulo a point by 3) of Lemma 4.9, i.e., k is injective. Thus $S'(S^p \times S^q)$, (hence $\mathcal{S}_0(S^p \times S^q)$) is in one-to-one correspondence with $G'(q)/\sim$ where \sim is the restriction to $G'(q)$ of the Novikov's relation. This makes the proof of Lemma 4.6. complete.

Proposition 4.10. *The set $\mathcal{S}_0(S^1 \times S^q)$ (hence $S'(S^1 \times S^q)$) is in one-to-one correspondence with the set \mathcal{S}_q/\sim by*

$$S^1 \times \tilde{S}^q \longrightarrow \tilde{S}^q \quad \text{for } q \geq 5.$$

Proof. Let $g : S^1 \times \tilde{S}_1^q \rightarrow S^1 \times \tilde{S}_2^q$ be an orientation preserving diffeomorphism. Since h induces isomorphisms of homology groups, $g_*[S^1 \times y_1]$ is equal to either $[S^1 \times y_2]$ or $-[S^1 \times y_2]$ where y_1 (resp. y_2) denotes a point of \tilde{S}_1^q (resp. \tilde{S}_2^q).

Firstly, suppose that

$$g_*[S^1 \times y_1] = [S^1 \times y_2].$$

It is obvious that $R^1 \times \tilde{S}_1^q$ (resp. $R^1 \times \tilde{S}_2^q$) is the universal covering space of $S^1 \times \tilde{S}_1^q$ (resp. $S^1 \times \tilde{S}_2^q$). Since g is a diffeomorphism, the induced covering space $g^*(R^1 \times \tilde{S}_2^q)$ is also the universal covering space of $S^1 \times \tilde{S}_1^q$. It follows from the uniqueness of the universal covering space that there exists a diffeomorphism

$$\tilde{g} : R^1 \times \tilde{S}_1^q \longrightarrow R^1 \times \tilde{S}_2^q$$

such that the following diagram commutes:

$$\begin{array}{ccc} R^1 \times \tilde{S}_1^q & \xrightarrow{\tilde{g}} & R^1 \times \tilde{S}_2^q \\ \downarrow p_1 & & \downarrow p_2 \\ S^1 \times \tilde{S}_1^q & \xrightarrow{g} & S^1 \times \tilde{S}_2^q \end{array}$$

where p_1 (resp. p_2) denotes the projection of the covering space. Let

$$\tilde{g}_n : R^1 \times \tilde{S}_1^q \longrightarrow R^1 \times \tilde{S}_2^q$$

be the diffeomorphism defined by

$$\tilde{g}_n(t, y) = (\tilde{g}_\varphi(t, y) + n, \tilde{g}_\psi(t, y))$$

where \tilde{g}_φ (resp. \tilde{g}_ψ) denotes the projection of \tilde{g} to the left hand (resp. to the right hand). Since \tilde{S}_1^q is compact, there exists a large integer N such that

$$\tilde{g}_N(\{0\} \times \tilde{S}_1^q) \cap \{0\} \times \tilde{S}_2^q = \emptyset.$$

We may assume that

$$\tilde{g}_N(\{0\} \times \tilde{S}_1^q) \subset [1, \infty) \times \tilde{S}_2^q.$$

Denote by W the manifold bounded by $\{0\} \times \tilde{S}_2^q$ and $\tilde{g}_N(\{0\} \times \tilde{S}_1^q)$. It is easily seen that there exists a deformation retract of $R^1 \times \tilde{S}_2^q$ to W , hence the natural inclusion $i : W \rightarrow R^1 \times \tilde{S}_2^q$ induces a homotopy equivalence. Obviously $\{0\} \times \tilde{S}_1^q \xrightarrow{\tilde{g}_N} R^1 \times \tilde{S}_2^q$ gives a homotopy equivalence and the natural inclusion $\{0\} \times \tilde{S}_2^q \rightarrow R^1 \times \tilde{S}_2^q$ gives a homotopy equivalence. Combining these, we see that the inclusion

$$\begin{aligned} \{0\} \times \tilde{S}_1^q &\xrightarrow{\tilde{g}_N} W \\ \text{(resp. } \{0\} \times \tilde{S}_2^q &\longrightarrow W) \end{aligned}$$

gives a homotopy equivalence, i.e., the triple

$$(W, g_N(\{0\} \times \tilde{S}_1^q), \{0\} \times \tilde{S}_2^q)$$

is an h -cobordism. Making use of the Smale's h -cobordism theorem, we have an orientation preserving diffeomorphism $h : \tilde{S}_1^q \rightarrow \tilde{S}_2^q$.

Secondly, suppose that

$$g_*[S^1 \times y_1] = -[S^1 \times y_2].$$

Then the similar argument proves that there exists an orientation preserving diffeomorphism $h : \tilde{S}_1^q \rightarrow -\tilde{S}_2^q$. Thus a map G of $\mathcal{S}_0(S^1 \times S^q)$ to Θ_q/\sim is well-defined. Obviously G is surjective. Suppose that two smooth structures $S^1 \times \tilde{S}_1^q$ and $S^1 \times \tilde{S}_2^q$ go into the same element by G . If $\tilde{S}_1^q = \tilde{S}_2^q$, then we have clearly a diffeomorphism $f : S^1 \times \tilde{S}_1^q \rightarrow S^1 \times \tilde{S}_2^q$. If $\tilde{S}_1^q = -\tilde{S}_2^q$ by the diffeomorphism $h_2 : \tilde{S}_1^q \rightarrow -\tilde{S}_2^q$, then we have an orientation preserving diffeomorphism

$$h_1 \times h_2 : S^1 \times \tilde{S}_1^q \longrightarrow S^1 \times \tilde{S}_2^q$$

where h_1 denotes an orientation reversing diffeomorphism of S^1 (clearly there exists such a diffeomorphism). Hence, in any case, we have an orientation preserving diffeomorphism

$$f : S^1 \times \tilde{S}_1^q \longrightarrow S^1 \times \tilde{S}_2^q.$$

This shows that G is injective and completes the proof of Proposition 4.10.

Summing up the results in this section, we state these as follows.

Theorem 4.11. *We can identify the following four sets $\mathcal{S}_0(S^p \times S^q)$, $\mathcal{S}'(S^p \times S^q)$, Θ_q^p / \approx and $G'(q) / \sim$ for $p+q \geq 6$, $q \geq p \geq 2$ and we can identify the following three sets $\mathcal{S}_0(S^1 \times S^q)$, $\mathcal{S}'(S^1 \times S^q)$ and Θ_q / \sim for $q \geq 5$.*

5. Smooth structures on $S^p \times S^q$

In this section we shall show the classification theorem. As is shown in §4, we can identify the four sets $\mathcal{S}_0(S^p \times S^q)$, $\mathcal{S}'(S^p \times S^q)$, Θ_q^p / \approx and $G'(q) / \sim$ for $p+q \geq 6$ and $q \geq p \geq 2$. Therefore we can state the classification theorem in four formulas. But we especially take up two formulae. One is a geometric interpretation (Θ_q^p / \approx) and the other is a homotopy theoretic interpretation ($G'(q) / \sim$).

Theorem 5.1. *For $p+q \geq 6$, and $q \geq p \geq 2$, we have*

$$\mathcal{S}(S^p \times S^q) = \{S^p \times \tilde{S}_i^q \# \tilde{S}_{i+j}^{p+q} \mid \tilde{S}_i^q \in \Theta_q^p / \approx, \tilde{S}_{i+j}^{p+q} \in \Theta_{p+q} / K_1(\pi_p(\text{SO}_q), \tilde{S}_i^q)\}.$$

For $p=1$, $q \geq 5$, we have

$$\mathcal{S}(S^1 \times S^q) = \{S^1 \times \tilde{S}_i^q \# \tilde{S}_{i+j}^{1+q} \mid \tilde{S}_i^q \in \Theta_q / \sim, \tilde{S}_{i+j}^{1+q} \in \Theta_{1+q} / K_1(\pi_1(\text{SO}), \tilde{S}_i^q)\}.$$

Theorem 5.2. *For $p+q \geq 6$ and $q \geq p \geq 2$, we have*

$$\mathcal{S}(S^p \times S^q) = \{S^p \times \tilde{S}_i^q \# \tilde{S}_{i+j}^{p+q} \mid \tilde{S}_i^q \in G'(q) / \sim, \tilde{S}_{i+j}^{p+q} \in \Theta_{p+q} / K_1(\pi_p(\text{SO}_q), \tilde{S}_i^q)\}.$$

For $p=1$, $q \geq 5$, we have

$$(*) \mathcal{S}(S^1 \times S^q) = \{S^1 \times \tilde{S}_i^q \# \tilde{S}_{i+j}^{1+q} \mid \tilde{S}_i^q \in \Theta_q / \sim, \tilde{S}_{i+j}^{1+q} \in \Theta_{1+q} / K_1(\pi_1(\text{SO}), \tilde{S}_i^q)\}.$$

Remark 5.3. To compute $\mathcal{S}(S^p \times S^q)$, the latter is superior to the former.

Proof. In case $p+q \geq 6$ and $q \geq p \geq 2$: If we prove that

$$(**) \mathcal{S}(S^p \times S^q) = \{S^p \times \tilde{S}_i^q \# \tilde{S}_{i+j}^{p+q} \mid S^p \times \tilde{S}_i^q \in \mathcal{S}'(S^p \times S^q) \text{ (or } \in \mathcal{S}_0(S^p \times S^q)) \\ \tilde{S}_{i+j}^{p+q} \in \Theta_{p+q} / K_1(\pi_p(\text{SO}_q), \tilde{S}_i^q)\}$$

then two theorems for $p+q \geq 6$ and $q \geq p \geq 2$ follow from Proposition 4.3 and Lemma 4.6 in §4.

Given a diffeomorphism

$$f : S^p \times \tilde{S}_1^q \# \tilde{S}_1^{p+q} \longrightarrow S^p \times \tilde{S}_2^q \# \tilde{S}_2^{p+q} .$$

we have a diffeomorphism

$$f' : S^p \times \tilde{S}_1^q \# \tilde{S}_1^{p+q} \# (-\tilde{S}_2^{p+q}) \longrightarrow S^p \times \tilde{S}_2^q .$$

It follows from Lemma 4.1 that $S^p \times \tilde{S}_1^q$ is diffeomorphic to $S^p \times \tilde{S}_2^q$. Consequently $\tilde{S}_1^{p+q} \# (-\tilde{S}_2^{p+q})$ belongs to the inertia group $I(S^p \times \tilde{S}_1^q) = I(S^p \times \tilde{S}_2^q)$. On the other hand $I(S^p \times \tilde{S}_1^q)$ is equal to

$$K_1(\pi_p(\mathrm{SO}_q), \tilde{S}_1^q)$$

by Lemma 3.1. Hence the natural map H of $\mathcal{S}(S^p \times S^q)$ to the right hand of (**) is well-defined. Obviously H is surjective. If two smooth structures $S^p \times \tilde{S}_1^q \# \tilde{S}_1^{p+q}$ and $S^p \times \tilde{S}_2^q \# \tilde{S}_2^{p+q}$ go into the same element by H , then there exists a diffeomorphism modulo one point

$$f_1 : S^p \times \tilde{S}_1^q - \mathrm{Int} D^{p+q} \longrightarrow S^p \times \tilde{S}_2^q - \mathrm{Int} D^{p+q}$$

i.e., there exists a diffeomorphism

$$f_2 : S^p \times \tilde{S}_1^q \# \tilde{S}^{p+q} \longrightarrow S^p \times \tilde{S}_2^q$$

for some homotopy sphere \tilde{S}^{p+q} . It follows from Lemma 4.1 that there exists actually an orientation preserving diffeomorphism

$$f_3 : S^p \times \tilde{S}_1^q \longrightarrow S^p \times \tilde{S}_2^q .$$

On the other hand, since $\tilde{S}_1^{p+q} \# (-\tilde{S}_2^{p+q})$ belongs to $K_1(\pi_p(\mathrm{SO}), \tilde{S}_1^q)$, $S^p \times \tilde{S}_1^q \# (\tilde{S}_1^{p+q} \# (-\tilde{S}_2^{p+q}))$ is diffeomorphic to $S^p \times \tilde{S}_1^q$ by an orientation preserving diffeomorphism

$$f_4 : S^p \times \tilde{S}_1^q \# (\tilde{S}_1^{p+q} \# (-\tilde{S}_2^{p+q})) \longrightarrow S^p \times \tilde{S}_1^q$$

by Lemma 3.1. Then the composition $f_3 \circ f_4$ gives an orientation preserving diffeomorphism between $S^p \times \tilde{S}_1^q \# (\tilde{S}_1^{p+q} \# (-\tilde{S}_2^{p+q}))$ and $S^p \times \tilde{S}_2^q$. It follows that $S^p \times \tilde{S}_1^q \# \tilde{S}_1^{p+q}$ is diffeomorphic to $S^p \times \tilde{S}_2^q \# \tilde{S}_2^{p+q}$ by an orientation preserving diffeomorphism. This shows that H is injective and completes the proof of two theorems for $p+q \geq 6$ and $q \geq p \geq 2$.

In case $p=1$ and $q \geq 5$: Given a diffeomorphism

$$f : S^1 \times \tilde{S}_1^q \# \tilde{S}_1^{1+q} \longrightarrow S^1 \times \tilde{S}_2^q \# \tilde{S}_2^{1+q} ,$$

we have a diffeomorphism

$$f' : S^1 \times \tilde{S}_1^q \# (\tilde{S}_1^{1+q} \# (-\tilde{S}_2^{1+q})) \longrightarrow S^1 \times \tilde{S}_2^q.$$

Since Lemma 4.1 holds in case where $p=1$ and $q \geq 4$ too, we shall obtain an orientation preserving diffeomorphism

$$g : S^1 \times \tilde{S}_1^q \longrightarrow S^1 \times \tilde{S}_2^q.$$

Hence \tilde{S}_1^q is equal to \tilde{S}_2^q in Θ_q/\sim by Lemma 4.10. On the other hand $\tilde{S}_1^{1+q} \# (-\tilde{S}_2^{1+q})$ belongs to the inertia group $I(S^1 \times \tilde{S}_1^q) (=I(S^1 \times \tilde{S}_2^q))$ and $I(S^1 \times \tilde{S}_1^q)$ is equal to

$$(K_1\pi_1(\text{SO}), \tilde{S}_1^q)$$

by Lemma 3.1. Thus a map H' of $\mathcal{S}(S^1 \times S^q)$ to the right hand of (*) is well defined. Obviously H' is surjective. Suppose that two smooth structures $S^1 \times \tilde{S}_1^q \# \tilde{S}_1^{1+q}$ and $S^1 \times \tilde{S}_2^q \# \tilde{S}_2^{1+q}$ go into the same element by H' . The similar argument as in Lemma 4.10, proves that there exists an orientation preserving diffeomorphism

$$f : S^1 \times \tilde{S}_1^q \longrightarrow S^1 \times \tilde{S}_2^q.$$

Since $\tilde{S}_1^{1+q} \# (-\tilde{S}_2^{1+q})$ belongs to $K_1(S^1 \times \tilde{S}_1^q)$, $S^1 \times \tilde{S}_1^q \# (\tilde{S}_1^{1+q} \# (-\tilde{S}_2^{1+q}))$ is diffeomorphic to $S^1 \times \tilde{S}_1^q$ by Lemma 3.1. Therefore $S^1 \times \tilde{S}_1^q \# (\tilde{S}_1^{1+q} \# (-\tilde{S}_2^{1+q}))$ is diffeomorphic to $S^1 \times \tilde{S}_2^q$. Consequently $S^1 \times \tilde{S}_1^q \# \tilde{S}_1^{1+q}$ is diffeomorphic to $S^1 \times \tilde{S}_2^q \# \tilde{S}_2^{1+q}$ by an orientation preserving diffeomorphism. This shows that H' is injective and finishes the proof of Theorems 5.1 and 5.2.

6. Some computations

In this section we shall show some examples.

Proposition 6.1. *If (p, q) is any of the following: $(2, 7)$, $(2, 8)$, $(6, 8)$, $(2, 14)$, $(3, 13)$, $(3, 15)$ $(6, 10)$, then*

$$\mathcal{S}(S^p \times S^q) = ((G'(q)/\sim) \times \Theta_{p+q}.$$

Proof. Bredon showed in [2] that if (p, q) is any of the set above, then

$$K_1(\pi_p(\text{SO}_q), \Theta_q) = S^{p+q} \text{ (the natural sphere).}$$

Therefore this is an immediate consequence of Theorem 5.2.

Proposition 6.2. $\mathcal{S}(S^3 \times S^{10}) = \{S^3 \times S^{10}, S^3 \times S^{10} \# \tilde{S}^{13}, S^3 \times S^{10} \# 2\tilde{S}^{13}, S^3 \times \tilde{S}^{10}\}$, i.e., $S^3 \times S^{10}$ admits exactly 4-smooth structures, where \tilde{S}^{10} denotes a

generator of the 3-component Z_3 of $\Theta_{10} \cong Z_2 \oplus Z_3$, and \tilde{S}^{13} denotes a generator of $\Theta_{13} \cong Z_3$.

This follows from the following computations.

Kervaire and Milnor showed in [11] that every element of the group

$$G(10) \cong Z_2 \oplus Z_3$$

is represented by the Pontrjagin-Thom map of some framed imbedding

$$\tilde{S}^{10} \times D^N \subset S^{10+N}.$$

Since non-zero elements of the 3-component Z_3 of $G(10) \cong Z_2 \oplus Z_3$ do not come from the unstable group $\pi_{14}(S^4)$ by the suspension homomorphism

$$E : \pi_{14}(S^4) \longrightarrow \pi_{10+N}(S^N) \quad (N : \text{large}),$$

$S^3 \times \tilde{S}^{10}$ is not diffeomorphic to $S^3 \times S^{10}$ modulo one point for a generator \tilde{S}^{10} of $Z_3 \subset Z_2 \oplus Z_3 \cong G'(10)$. In [8], it is shown that

$$I(S^3 \times \tilde{S}^{10}) = K_1(\pi_3(SO), \tilde{S}^{10}) = \Theta_{13}.$$

On the other hand, the 2-component Z_2 of $G(10) \cong Z_2 \oplus Z_3$ comes from the unstable group $\pi_{12}(S^2)$ (see H. Toda [18]). Therefore $S^3 \times \tilde{S}^{10}$ is diffeomorphic to $S^3 \times S^{10}$ modulo a point for the generator \tilde{S}^{10} of the 2-component Z_2 . By Lemma 4.1, we can deduce that $S^3 \times \tilde{S}^{10}$ is actually diffeomorphic to $S^3 \times S^{10}$. Therefore Theorem 5.2 gives the requiring result.

Remark 6.3. Let \tilde{S}^{10} denote a generator of the 3-component $Z_3 \subset G'(10) \cong Z_2 \oplus Z_3$. Since there exist orientation reversing diffeomorphisms $f_1 : S^3 \rightarrow S^3$ and $f_2 : \tilde{S}^{10} \rightarrow 2\tilde{S}^{10}$, we have an orientation preserving diffeomorphism

$$f = f_1 \times f_2 : S^3 \times \tilde{S}^{10} \longrightarrow S^3 \times 2\tilde{S}^{10}.$$

Proposition 6.4. *The order of $\mathcal{S}(S^3 \times S^{14})$ is 24.*

Proof. Since $G'(14) = Z_2 \subset G(14) = Z_2 \oplus Z_2$ (see Kervaire and Milnor [11]), $\mathcal{S}(S^3 \times S^{14})$ is the quotient set of $Z_2 \times \Theta_{14}$. In [8], it is proved that

$$I(S^3 \times \tilde{S}^{14}) = K_1(\pi_3(SO), \tilde{S}^{14}) = Z_2 \neq 0 \text{ modulo } \Theta_{17}(\partial\pi)$$

for the generator \tilde{S}^{14} of $\Theta_{14} \cong Z_2$. Consequently we have

$$\begin{aligned} \mathcal{S}(S^3 \times S^{14}) = & \{S^3 \times S^{14} \# \tilde{S}_i^{17} \mid \tilde{S}_i^{17} \in \Theta_{17}\} \cup \\ & \{S^3 \times \tilde{S}^{14} \# \tilde{S}_j^{17} \mid \tilde{S}^{14} \neq S^{14}, \tilde{S}_j^{17} \in \Theta_{17}/K_1(\pi_3(SO), \tilde{S}^{14})\}, \end{aligned}$$

which proves Proposition 6.4.

Proposition 6.5. *If $p+3 \geq q \geq p$, then $\mathcal{S}(S^p \times S^q)$ is in one-to-one correspondence with Θ_{p+q} by*

$$S^p \times S^q \# \tilde{S}^{p+q} \longrightarrow \tilde{S}^{p+q}.$$

Proof. Hsiang, Levine and Szczarba [7] showed that \tilde{S}^q can be embedded in the $(q+p+1)$ -dimensional euclidean space R^{q+p+1} with a trivial normal bundle for $q-2 \leq p+1$. It follows from Lemma 4.4 that $S^p \times \tilde{S}^q$ is diffeomorphic to $S^p \times S^q$ by an orientation preserving diffeomorphism. Hence the inertia group $I(S^p \times \tilde{S}^q)$ is trivial by Corollary 3 in Kawakubo [8]. Consequently Proposition 6.5 follows from Theorem 5.1.

Proposition 6.6. *If $p \equiv 2, 4, 5, 6 \pmod{8}$, then $\mathcal{S}(S^p \times S^q)$ is in one-to-one correspondence with $(G'(q)/\sim) \times \Theta_{p+q}$.*

Proof. Since $K_1(\pi_p(SO), \tilde{S}^q) = K_1(0, \tilde{S}^q) = S^{p+q}$ for $p \equiv 2, 4, 5, 6 \pmod{8}$, the inertia group of $S^p \times \tilde{S}^q$ is trivial for every $\tilde{S}^q \in \Theta_q$. Therefore this Proposition is an immediate consequence of Theorem 5.2.

Proposition 6.7. $\mathcal{S}(S^1 \times S^7) = (\Theta_7/\sim) \times \Theta_8$, i.e., *the order of $\mathcal{S}(S^1 \times S^7)$ is 30.*

Proof. In general, the following diagram

$$\begin{array}{ccc} \pi_p(SO_q) \times \Theta_q & \xrightarrow{K_1} & \Theta_{p+q} \\ J \times \omega \downarrow & & \downarrow \omega \\ G(p) \times G(q)/\text{Im } J & \xrightarrow{C} & G(p+q)/\text{Im } J \end{array}$$

is commutative up to sign where j denotes the Hopf-Whitehead homomorphism and ω denotes the Kervaire-Milnor homomorphism and C denotes the composition of stable homotopy groups. Since $\omega(\Theta_7) = 0$ and $\omega|_{\Theta_8}$ is injective, K_1 is trivial. Hence $\mathcal{S}(S^1 \times S^7) = (\Theta_7/\sim) \times \Theta_8$ by Theorem 5.2. Since the order of Θ_7/\sim is $\frac{28}{2} + 1 = 15$, the order of $\mathcal{S}(S^1 \times S^7)$ is 30.

Proposition 6.8. $\mathcal{S}(S^1 \times S^8) = \{S^1 \times S^8 \# \tilde{S}_i^9 \mid \tilde{S}_i^9 \in \Theta_9\} \cup \{S^1 \times \tilde{S}^8 \# \tilde{S}_j^9 \mid \tilde{S}^8 \# S^8, \tilde{S}_j^9 \in \Theta_9/K_1(\pi_1(SO), \tilde{S}^8)\}$, i.e., *the order of $\mathcal{S}(S^1 \times S^8)$ is 12.*

Proof. Obviously $\Theta_8/\sim = \Theta_8$. It is shown in Bredon [2] that

$$K_1((\pi_1(SO), \tilde{S}^8) \# S^9 \quad \text{for } \tilde{S}^8 \# S^8.$$

Therefore the order of $K_1(\pi_1(SO), \tilde{S}^8)$ is 2, and hence the order of $\Theta_9/K_1(\pi_1(SO), \tilde{S}^8)$ is 4. The order of $\mathcal{S}(S^1 \times S^8)$ is consequently $8+4=12$.

Proposition 6.9. *The order of $\mathcal{S}(S^1 \times S^{14})$ is 24384.*

Proof. Since $K_1(\pi_1(SO), \tilde{S}^{14}) \cong Z_2 \subset \Theta_{15}$ for $\tilde{S}^{14} \neq S^{14}$ (see Bredon [2]), the order of $\Theta_{15}/K_1(\pi_1(SO), \tilde{S}^{14})$ is 8128. Obviously $\Theta_{14}/\sim = \Theta_{14}$. Hence we have

$$\begin{aligned} \mathcal{S}(S^1 \times S^{14}) = & \{S^1 \times S^{14} \# \tilde{S}_i^{15} \mid \tilde{S}_i^{15} \in \Theta_{15}\} \cup \\ & \{S^1 \times \tilde{S}^{14} \# \tilde{S}_j^{15} \mid \tilde{S}^{14} \neq S^{14}, \tilde{S}_j^{15} \in \Theta_{15}/K_1(\pi_1(SO), \tilde{S}^{14})\}, \end{aligned}$$

i.e., $S^1 \times S^{14}$ admits exactly 24384 smooth structures.

Proposition 6.10. *The order of $\mathcal{S}(S^1 \times S^{16})$ is 24.*

Proof. Since $K_1(\pi_1(SO), \tilde{S}^{16}) \cong Z_2 \subset \Theta_{17}$ for $\tilde{S}^{16} \neq S^{16}$ (see Bredon [2]), the order of $\Theta_{17}/K_1(\pi_1(SO), \tilde{S}^{16})$ is 8. Obviously $\Theta_{16}/\sim = \Theta_{16}$. Hence we have

$$\begin{aligned} \mathcal{S}(S^1 \times S^{16}) = & \{S^1 \times S^{16} \# \tilde{S}_i^{17} \mid \tilde{S}_i^{17} \in \Theta_{17}\} \cup \\ & \{S^1 \times \tilde{S}^{16} \# \tilde{S}_j^{17} \mid \tilde{S}^{16} \neq S^{16}, \tilde{S}_j^{17} \in \Theta_{17}/K_1(\pi_1(SO), \tilde{S}^{16})\}, \end{aligned}$$

i.e., $S^1 \times S^{16}$ admits exactly 24 smooth structures.

7. Smooth structures on a sphere bundle over sphere with a cross section

Let $S : \pi_{q-1}(SO_p) \rightarrow \pi_{q-1}(SO_{p+1})$ be the natural homomorphism induced by the inclusion.

Denote by $M_h(\tilde{S}^q)$ the p -sphere bundle over a homotopy q -sphere \tilde{S}^q with a characteristic map $h \in \pi_{q-1}(SO_{p+1})$. Define a homomorphism

$$K[h, \tilde{S}^q] : \pi_p(SO_q) \longrightarrow \Theta_{p+q}$$

by

$$K[h, \tilde{S}^q](l) = K_1(l, \tilde{S}^q) + K_2(l, h)$$

for $h \in \pi_{q-1}(SO_{p+1})$, $\tilde{S}^q \in \Theta_q$ and $l \in \pi_p(SO_q)$ (see Kawakubo [8]). Obviously the fact that the bundle $M_h(\tilde{S}^q)$ has a cross section is equal to the fact that the characteristic map h belongs to the image S . Then we have

Theorem 7.1.* *For $p+q \geq 6$, $q+2 \geq p \geq 2$ and $h \in S(\pi_{q-1}(SO_p))$,*

$$\mathcal{S}(M_h(S^q)) = \{M_h(\tilde{S}_i^q) \# \tilde{S}_{ij}^{p+q} \mid \tilde{S}_i^q \in G'(q)/\sim, \tilde{S}_{ij}^{p+q} \in \Theta_{p+q}/K[h, \tilde{S}_i^q](\pi_p(SO_q))\}.$$

For $p=1$ and $q \geq 5$,

(*) Added in proof. After the preparation of this paper, Professor R. Schultz kindly sent me a letter which said that similar results were obtained.

$$\mathcal{S}(M_h(S^q)) = \{S^1 \times \tilde{S}^q \# \tilde{S}^{1+q}_i \mid \tilde{S}^q_i \in \Theta_q / \sim, \tilde{S}^{1+q}_i \in \Theta_{1+q} / K_1(\pi_1(SO), \tilde{S}^q_i)\}.$$

Proof. Suppose that $p \neq q$. Let $p : M_h(S^q) \rightarrow S^q$ be the projection of this bundle. Let c denote a cross section in $M_h(S^q)$ and Let U denote a small open tubular neighbourhood of c . According to Serre [17], the following sequence

$$\cdots \longrightarrow H_i(S^p) \xrightarrow{i_*} H_i(M_h(S^q)) \xrightarrow{p_*} H_i(S^q) \xrightarrow{\partial_*} \cdots$$

is exact for $i \leq p+q-1$. Since the bundle $p : M_h(S^q) \rightarrow S^q$ has a cross section c , we have the splitting short exact sequence,

$$0 \longrightarrow H_i(S^p) \xrightarrow{i_*} H_i(M_h(S^q)) \begin{array}{c} \xrightarrow{p_*} \\ \xleftarrow{c_*} \end{array} H_i(S^q) \longrightarrow 0$$

i.e., the homology groups of $M_h(S^q)$ are as follows,

$$H_i(M_h(S^q)) = \begin{cases} \mathbb{Z} & i = 0, p, q, p+q \\ 0 & \text{otherwise.} \end{cases}$$

Let a smooth structure $M_h(S^q)_\alpha$ on $M_h(S^q)$ be given i.e., assume that there is given a piecewise differentiable homeomorphism $f : M_h(S^q) \rightarrow M_h(S^q)_\alpha$. Since $f(U)$ is an open submanifold in $M_h(S^q)$, $f(U)$ has an induced smooth structure $\{f(U)\}_\alpha$. Since $M_h(S^q)$ is a total space of a sphere bundle over sphere with a cross section c associated with a vector bundle, U has a vector bundle structure. It follows from Lashof and Rothenberg [13] that there exists a homotopy sphere \tilde{S}^q such that $\{f(U)\}_\alpha$ is diffeomorphic to a total space V of an open disk bundle over \tilde{S}^q with the characterisic map h' which is the characteristic map of the tubular neighbourhood of c in $M_h(S^q)$. Let

$$d_1 : \{f(U)\}_\alpha \longrightarrow V$$

be this diffeomorphism. Let x_0 denote a point of S^q and let $R_{x_0}^q$ denote a small open neighbourhood of x_0 in S^q which is PL-homeomorphic to the euclidean space R^q . Since $f(p^{-1}(R_{x_0}^q))$ is an open submanifold in $M_h(S^q)_\alpha$, $f(p^{-1}(R_{x_0}^q))$ has an induced smooth structure $\{f(p^{-1}(R_{x_0}^q))\}_\alpha$. Obviously $p^{-1}(R_{x_0}^q)$ is PL-homeomorphic to $S^p \times R^q$. As is shown in §2, $\{f(p^{-1}(R_{x_0}^q))\}_\alpha$ is diffeomorphic to $\tilde{S}^p \times R^q$ for some homotopy sphere \tilde{S}^p . But Hsiang, Levine and Szczarba [7] showed that $\tilde{S}^p \times R^q$ is diffeomorphic to $S^p \times R^q$ for $q+2 \geq p$. Let

$$d_2 : \{f(p^{-1}(R_{x_0}^q))\}_\alpha \longrightarrow S^p \times R^q$$

be this diffeomorphism. Then $d_1^{-1}(\tilde{S}^q)$ (resp. $d_2^{-1}(S^p)$) obviously represents a generator of

$$H_q(M_h(S^q)_\alpha) \cong Z \quad (\text{resp. } H_p(M_h(S^q)_\alpha) \cong Z)$$

where \tilde{S}^q (resp. S^p) denotes the zero cross section of the bundle V (resp. $S^p \times S^q$). We may assume that $d_1^{-1}(\tilde{S}^q)$ and $d_2^{-1}(S^p)$ intersect transversally at one point. It follows that there exists a smooth imbedding

$$g : \bar{V} \sqcup S^p \times D^q \longrightarrow M_h(S^q)_\alpha - \text{Int } D^{p+q}$$

inducing isomorphism of homology groups

$$g_* : H_*(\bar{V} \sqcup S^p \times D^q) \xrightarrow{\cong} H_*(M_h(S^q)_\alpha - \text{Int } D^{p+q})$$

where \bar{V} denotes the closed disk bundle associated with V and \sqcup denotes a generalized plumbing of two manifolds obtained as follows. When we regard

$$\bar{V} \quad \text{as} \quad D^p \times D_+^q \cup_{hr} D^p \times D_-^q$$

and

$$S^p \times D^q \quad \text{as} \quad D_+^p \times D^q \cup_{id} D_-^p \times D^q,$$

$\bar{V} \sqcup S^p \times D^q$ denotes the oriented differentiable $(p+q)$ -manifold formed from the disjoint sum $\bar{V} \cup S^p \times D^q$ by indentifying $D^p \times D_-^q$ with $D_+^p \times D^q$ in such a way that $D^p = D_+^p$ and $D_-^q = D^q$. Applying the similar argument as in §2, we can show that $\bar{V} \sqcup S^p \times D^q$ is diffeomorphic to

$$M_h(S^q)_\alpha - \text{Int } D^{p+q}.$$

Let $M_h(\tilde{S}^q)$ denote the total space of S^p bundle over a homotopy sphere \tilde{S}^q with a characteristic map h . Then, applying the similar argument as above, we have that $\bar{V} \sqcup S^p \times D^q$ is diffeomorphic to

$$M_h(\tilde{S}^q) - \text{Int } D^{p+q}.$$

Thus $M_h(S^q)_\alpha$ is diffeomorphic to

$$M_h(\tilde{S}^q) \# \tilde{S}^{p+q}$$

for some homotopy spheres \tilde{S}^q and \tilde{S}^{p+q} for $p+q \geq 6$. When $p=q$, we can also obtain the similar result. Consequently we have

Lemma 7.2. *If $p+q \geq 6$ and $q+2 \geq p \geq 1$, every smooth structure $M_h(S^q)_\alpha$ is diffeomorphic to $M_h(\tilde{S}^q) \# \tilde{S}^{p+q}$ for some homotopy spheres \tilde{S}^q and \tilde{S}^{p+q} .*

Let $B_h(\tilde{S}^q)$ denote a D^{p+1} bundle over a homotopy sphere \tilde{S}^q with a characteristic map $h \in \pi_{q-1}(SO_{p+1})$. Let B_l denote a D^q bundle over S^{p+1} with a characteristic map $l \in \pi_p(SO_q)$. Let

$$B_l \sqcup B_h(\tilde{S}^q)$$

denote a generalized plumbing of two manifolds obtained as follows. When we regard

$$B_h(\tilde{S}^q) \text{ as } D^{p+1} \times D_+^q \cup_{hr} D^{p+1} \times D_-^q$$

and

$$B_l \text{ as } D_+^{p+1} \times D^q \cup_l D_-^{p+1} \times D^q,$$

$B_l \sqcup B_h(\tilde{S}^q)$ denotes the oriented differentiable $(p+q+1)$ -manifold formed from the disjoint sum $B_l \cup B_h(\tilde{S}^q)$ by identifying $D^{p+1} \times D_-^q$ with $D_+^{p+1} \times D^q$ in such a way that $D^{p+1} = D_+^{p+1}$ and $D_-^q = D^q$. Define the homomorphism

$$E[h, \tilde{S}^q] : \pi_p(SO_q) \longrightarrow \Theta_{p+q}$$

by

$$E[h, \tilde{S}^q](l) = \partial(B_l \sqcup B_h(\tilde{S}^q)) \quad \text{for } l \in \pi_p(SO_q).$$

As is shown in [8], it is easily verified that this is a well-defined homomorphism. Now, we show

Lemma 7.3. $E[h, \tilde{S}^q] = K[h, \tilde{S}^q]$.

Proof. Let B_h be the D^{p+1} bundle over S^q with a characteristic map $h \in \pi_{q-1}(SO_{p+1})$. Let B_l be the D^q bundle over S^{p+1} with a characteristic map $l \in \pi_p(SO_q)$. Then, consider the following manifold,

$$W = (B_l \vee B_h) \natural (B_l \circlearrowleft D^{p+1} \times \tilde{S}^q)$$

where \natural denotes the boundary connected sum and \vee and \circlearrowleft denote the plumbings (see Kawakubo [8]). Since

$$\partial((B_l \vee B_h) \natural (B_l \circlearrowleft D^{p+1} \times \tilde{S}^q)) = \partial(B_l \vee B_h) \# \partial(B_l \circlearrowleft D^{p+1} \times \tilde{S}^q),$$

the boundary ∂W is equal to $K[h, \tilde{S}^q](l)$ by definitions. On the other hand, we can rewrite W as follows. Let

$$\begin{aligned} e_1 &\in H_q(B_l \vee B_h), & e_1' &\in H_p(B_l \vee B_h), \\ e_2 &\in H_q(B_l \circlearrowleft D^{p+1} \times \tilde{S}^q), & e_2' &\in H_p(B_l \circlearrowleft D^{p+1} \times \tilde{S}^q) \end{aligned}$$

be the natural homology basis. Now, introducing a new basis in $H_*(W)$ by the formulas

$$\begin{aligned} f_1 &= e_1, & f_1' &= e_1' - e_2', \\ f_2 &= e_1 + e_2, & f_2' &= e_2', \end{aligned}$$

we obtain

$$W = (B_0 \vee B_h) \natural (B_l \sqcup B_h(\tilde{S}^q))$$

where B_0 denotes a trivial bundle $S^{b+1} \times D^q$. Hence

$$\partial W = \partial(B_0 \vee B_h) \# \partial(B_l \sqcup B_h(\tilde{S}^q)) = \partial(B_l \sqcup B_h(\tilde{S}^q)) = E[h, \tilde{S}^q](l),$$

completing the proof of Lemma.

Similarly to Lemma 3.1 we have the following

Lemma 7.4. $I(M_h(\tilde{S}^q)) = K[h, \tilde{S}^q](\pi_p(SO_q))$.

Proof. First we shall prove that $I(M_h(\tilde{S}^q))$ is contained in $K[h, \tilde{S}^q](\pi_p(SO_q))$. For an element $\alpha \in I(M_h(\tilde{S}^q))$, there exists a diffeomorphism

$$H : M_h(\tilde{S}^q) - \text{Int } D^{b+q} \longrightarrow M_h(\tilde{S}^q) - \text{Int } D^{b+q}$$

such that $H|_{\partial D} \in \Gamma_{p+q}$ represents α . Using this diffeomorphism, we construct a manifold

$$B_h(\tilde{S}^q) \cup_{\mu} B_h(\tilde{S}^q)$$

which is denoted by X . Clearly $\partial X = \alpha$ and similarly to the proof of Theorem A in Kawakubo [8], we can prove that X can be written as

$$B_l \sqcup B_h(\tilde{S}^q)$$

for some characteristic map $l \in \pi_p(SO_q)$. Hence

$$\alpha = \partial X = \partial(B_l \sqcup B_h(\tilde{S}^q)) = E[h, \tilde{S}^q](l) = K[h, \tilde{S}^q](l)$$

by Lemma 7.3.

Conversely, for $\alpha = K[h, \tilde{S}^q](l)$, we can represent α by

$$\partial(B_l \sqcup B_h(\tilde{S}^q))$$

by Lemma 7.3. The similar argument employed in the proof of Lemma in [8] proves that $B_l \sqcup B_h(\tilde{S}^q)$ is diffeomorphic to a manifold

$$B_h(\tilde{S}^q) \cup_{\mu} B_h(\tilde{S}^q)$$

for some diffeomorphism

$$H : \partial B_h(\tilde{S}^q) - \text{Int } D^{b+q} \longrightarrow \partial B_h(\tilde{S}^q) - \text{Int } D^{b+q}.$$

This implies that α belongs to the inertia group $I(M_h(\tilde{S}^q))$, completing the

proof of Lemma 7.4.

We now prove the following Lemma which corresponds to Lemma 4.1.

Lemma 7.5. *If $M_h(\tilde{S}_1^q)$ is diffeomorphic to $M_h(\tilde{S}_2^q)$ modulo a point, then $M_h(\tilde{S}_1^q)$ is actually diffeomorphic to $M_h(\tilde{S}_2^q)$ for $p+q \geq 5$.*

Proof. Let $f : M_h(\tilde{S}_1^q) \# \tilde{S}^{p+q} \rightarrow M_h(\tilde{S}_2^q)$ be a diffeomorphism, and let

$$f' : M_h(\tilde{S}_1^q) - \text{Int } D^{p+q} \longrightarrow M_h(\tilde{S}_2^q) - \text{Int } D^{p+q}$$

be the restriction of f . Let X be the manifold obtained by attaching two manifolds $B_h(\tilde{S}_1^q)$ and $B_h(\tilde{S}_2^q)$ by the diffeomorphism

$$f' : \partial B_h(\tilde{S}_1^q) - \text{Int } D^{p+q} \longrightarrow \partial B_h(\tilde{S}_2^q) - \text{Int } D^{p+q}.$$

Obviously the boundary ∂X is diffeomorphic to the homotopy sphere \tilde{S}^{p+q} . The similar argument as in the proof of Lemma 7.4 shows that X is diffeomorphic to

$$B_l \sqcup B_h(\tilde{S}_1^q)$$

for some characteristic map $l \in \pi_p(SO_q)$, hence

$$\tilde{S}^{p+q} = \partial X = \partial(B_l \sqcup B_h(\tilde{S}_1^q)) = E[h, \tilde{S}_1^q](l) = K[h, \tilde{S}_1^q](l).$$

Since

$$K[h, \tilde{S}_1^q](\pi_p(SO_q))$$

is exactly the inertia group $I(M_h(\tilde{S}_1^q))$ (see Lemma 7.4),

$$M_h(\tilde{S}_1^q) = M_h(\tilde{S}_1^q) \# \tilde{S}^{p+q} = M_h(\tilde{S}_2^q),$$

which completes the proof of Lemma 6.5.

Lemma 7.6. *The set $\mathcal{S}_0(S^p \times S^q)$ ($= S'(S^p \times S^q) = \Theta_q^p / \approx = G'(q) / \sim$) is in one-to-one correspondence with the set $S'(M_h(\tilde{S}^q))$ by*

$$S^p \times \tilde{S}^q \longrightarrow M_h(\tilde{S}^q)$$

for $p+q \geq 6$ and $q+2 \geq p \geq 2$.

Proof. Firstly we shall show that the map

$$A : \mathcal{S}_0(S^p \times S^q) \longrightarrow S'(M_h(S^q))$$

defined by

$$A(S^p \times \tilde{S}^q) = M_h(\tilde{S}^q)$$

is well-defined. Let $f : S^p \times \tilde{S}_1^q \rightarrow S^p \times \tilde{S}_2^q$ be a diffeomorphism. According

to Lemma 4.5, $S^p \times (\tilde{S}_1^q \# \tilde{S}_2^q)$ or $S^p \times (\tilde{S}_1^q \# (-\tilde{S}_2^q))$ is diffeomorphic to $S^p \times S^q$. Suppose $S^p \times (\tilde{S}_1^q \# (-\tilde{S}_2^q))$ is diffeomorphic to $S^p \times S^q$. Let

$$f' : S^p \times (\tilde{S}_1^q \# (-\tilde{S}_2^q)) \longrightarrow S^p \times S^q$$

be this diffeomorphism. Making use of the diffeomorphism S in the proof of Lemma 4.5 (if necessary), we can assume that

$$f' | S^p \times D^q = id.$$

Hence we have a diffeomorphism $f'' : S^p \times \tilde{S}_1^q \rightarrow S^p \times \tilde{S}_2^q$ such that

$$f'' | S^p \times D^q = id.$$

Consequently we have the following orientation preserving diffeomorphism

$$f_0 : (S^p \times \tilde{S}_1^q - \text{Int}(S^p \times D^q)) \cup_h S^p \times D^q \rightarrow (S^p \times \tilde{S}_2^q - \text{Int}(S^p \times D^q)) \cup_h S^p \times D^q$$

and obviously we have that

$$(S^p \times \tilde{S}_1^q - \text{Int}(S^p \times D^q)) \cup_h S^p \times D^q = M_h(\tilde{S}_1^q)$$

and

$$(S^p \times \tilde{S}_2^q - \text{Int}(S^p \times D^q)) \cup_h S^p \times D^q = M_h(\tilde{S}_2^q).$$

Suppose $S^p \times (\tilde{S}_1^q \# \tilde{S}_2^q)$ is diffeomorphic to $S^p \times S^q$, then we have an orientation preserving diffeomorphism

$$f_0 : M_h(\tilde{S}_1^q) \longrightarrow M_h(-\tilde{S}_2^q)$$

by the above argument. Since $M_h(\tilde{S}^q)$ has a cross section, there always exists an orientation reversing diffeomorphism of $M_h(\tilde{S}^q)$ for all \tilde{S}^q . It follows that we always have an orientation preserving diffeomorphism

$$f_1 : M_h(-\tilde{S}_2^q) \longrightarrow M_h(\tilde{S}_2^q).$$

Hence $f_1 \circ f_0$ gives the desired diffeomorphism.

Secondly we shall show that the map A is surjective. But this follows obviously from Lemma 7.2.

Thirdly we shall show that the map A is injective. Given a diffeomorphism modulo a point, then there exists actually a diffeomorphism

$$f : M_h(\tilde{S}_1^q) \longrightarrow M_h(\tilde{S}_2^q)$$

by Lemma 7.5. When we write

$$M_h(\tilde{S}_1^q) \text{ as } S^p \times D_1^q \underset{hr_1}{\cup} S^p \times D_2^q$$

and

$$M_h(\tilde{S}_2^q) \text{ as } S^p \times D_1^q \underset{hr_2}{\cup} S^p \times D_2^q$$

such that $* \times D_1^q \cup * \times D_2^q$ corresponds to a cross section, we may assume

$$f(S^p \times D_1^q) = S^p \times D_1^q.$$

Suppose $f_*[S^p \times 0] = [S^p \times 0]$, then we may assume

$$f|S^p \times 0 = id$$

and that

$$f|D_1^p \times D_1^q = id$$

when we write $S^p \times D_1^q$ as $D_1^p \times D_1^q \underset{id}{\cup} D_2^p \times D_1^q$ such that $* \times D_1^q$ corresponds to $0 \times D_1^q$ (0 denotes the original point of D_1^p). Then we can define a diffeomorphism

$$f' : (M_h(\tilde{S}_1^q) - S^p \times \text{Int } D_1^q) \underset{(-r_1)h^{-1}}{\cup} S^p \times D^q \rightarrow (M_h(\tilde{S}_2^q) - S^p \times \text{Int } D_1^q) \underset{f(-r_1)h^{-1}}{\cup} S^p \times D^q$$

by

$$f'|M_h(\tilde{S}_1^q) - S^p \times \text{Int } D_1^q = f|M_h(\tilde{S}_1^q) - S^p \times \text{Int } D_1^q$$

and

$$f'|S^p \times D^q = id.$$

Clearly

$$(M_h(\tilde{S}_1^q) - S^p \times \text{Int } D_1^q) \underset{(-r_1)h^{-1}}{\cup} S^p \times D^q$$

is diffeomorphic to $S^p \times S^q$. Since $f|D^p \times D^q = id$, $0 \times D^q \underset{r_2-r_1}{\cup} * \times D_2^q$ (where 0 denotes the original point of $D^p = D_1^p$) is imbedded in

$$(M_h(\tilde{S}_2^q) - S^p \times \text{Int } D_1^q) \underset{f(-r_1)h^{-1}}{\cup} S^p \times D^q$$

with a trivial normal bundle, i.e., $\tilde{S}_2^q \# (-\tilde{S}_1^q)$ embeds in $S^p \times S^q$ with a trivial normal bundle. Hence $\tilde{S}_2^q \# (-\tilde{S}_1^q)$ embeds in R^{p+q+1} with a trivial normal bundle. It follows from Lemma 4.4 that $S^p \times (\tilde{S}_2^q \# (-\tilde{S}_1^q))$ is diffeomorphic to $S^p \times S^q$. Consequently we have $S^p \times \tilde{S}_1^q = S^p \times \tilde{S}_2^q$ by Lemma 4.5.

When $f_*[S^p \times 0] = -[S^p \times 0]$, there exists an orientation preserving diffeomorphism

$$f' : M_h(\tilde{S}_1^q) \longrightarrow M_h(-\tilde{S}_2^q)$$

such that

$$f'_*[S^p \times 0] = [S^p \times 0].$$

Making use of the similar argument as above, we can verify that $S^p \times \tilde{S}_1^q$ is diffeomorphic to $S^p \times \tilde{S}_2^q$. This makes the proof of Lemma 7.6 complete.

We can now prove Theorem 7.1.

In case $p=1$ and $q \geq 5$: This is reducible to the proof of Theorem 5.2, since $\pi_{q-1}(SO_2) = \{0\}$ for $q \geq 5$.

In case $p+q \geq 6$ and $q+2 \geq p \geq 2$: Given a diffeomorphism

$$f : M_h(\tilde{S}_1^q) \# \tilde{S}_1^{p+q} \longrightarrow M_h(\tilde{S}_2^q) \# \tilde{S}_2^{p+q},$$

we have a diffeomorphism

$$f' : M_h(\tilde{S}_1^q) \# \tilde{S}_1^{p+q} \# (-\tilde{S}_2^{p+q}) \longrightarrow M_h(\tilde{S}_2^q).$$

It follows from Lemma 7.6 that \tilde{S}_1^q and \tilde{S}_2^q represent the same element of $G'(q)$. On the other hand, $M_h(\tilde{S}_1^q)$ is diffeomorphic to $M_h(\tilde{S}_2^q)$ by Lemma 7.5. Consequently $\tilde{S}_1^{p+q} \# (-\tilde{S}_2^{p+q})$ belongs to the inertia group $I(M_h(\tilde{S}_1^q)) = I(M_h(\tilde{S}_2^q))$. Since $I(M_h(\tilde{S}_1^q))$ is equal to

$$K[h, \tilde{S}_1^q](\pi_p(SO_q))$$

by Lemma 7.4, the natural map

$$H : \mathcal{S}(M_h(S^q)) \rightarrow \{M_h(\tilde{S}_i^q) \# \tilde{S}_i^{p+q} \mid \tilde{S}_i^q \in G'(q)/\sim, \tilde{S}_i^{p+q} \in \Theta_{p+q}/K[h, \tilde{S}_i^q](\pi_p(SO_q))\}$$

is well-defined. Obviously H is surjective. If two smooth structures $M_h(\tilde{S}_1^q) \# \tilde{S}_1^{p+q}$ and $M_h(\tilde{S}_2^q) \# \tilde{S}_2^{p+q}$ go into the same element by H , then $M_h(\tilde{S}_1^q)$ and $M_h(\tilde{S}_2^q)$ represent the same element of $\mathcal{S}'(M_h(S^q))$ by Lemma 7.6. It follows from Lemma 7.5 that there exists actually an orientation preserving diffeomorphism

$$f_1 : M_h(\tilde{S}_1^q) \longrightarrow M_h(\tilde{S}_2^q).$$

On the other hand, since $\tilde{S}_1^{p+q} \# (-\tilde{S}_2^{p+q})$ belongs to $K[h, \tilde{S}_1^q](\pi_p(SO_q))$, $M_h(\tilde{S}_1^q) \# \tilde{S}_1^{p+q} \# (-\tilde{S}_2^{p+q})$ is diffeomorphic to $M_h(\tilde{S}_1^q)$ by an orientation preserving diffeomorphism

$$f_2 : M_h(\tilde{S}_1^q) \# \tilde{S}_1^{p+q} \# (-\tilde{S}_2^{p+q}) \rightarrow M_h(\tilde{S}_1^q)$$

by Lemma 7.4. Then the composition $f_1 \circ f_2$ gives an orientation preserving diffeomorphism between $M_h(\tilde{S}_1^q) \# \tilde{S}_1^{p+q} \# (-\tilde{S}_2^{p+q})$ and $M_h(\tilde{S}_2^q)$. It follows that $M_h(\tilde{S}_1^q) \# \tilde{S}_1^{p+q}$ is diffeomorphic to $M_h(\tilde{S}_2^q) \# \tilde{S}_2^{p+q}$ by an orientation preserving diffeomorphism. This shows that H is injective and finishes the proof of Theorem 7.1.

8. A James problem on H -spaces

Let M_h denote a p -sphere bundle over a q -sphere with a characteristic map $h \in \pi_{q-1}(SO_{p+1})$ as in §7.

When h belongs to the image of

$$S: \pi_6(SO_3) \rightarrow \pi_6(SO_4)$$

and also to the kernel of

$$J: \pi_6(SO_4) \rightarrow \pi_{10}(S^4),$$

M_h is an example of H -spaces (see [12]). Then I. James posed the following problem [12, p. 586]:

Is the H -space M_h homeomorphic to $S^3 \times S^7$ for $h \in \text{Image } S \cap \text{Kernel } J \cong Z_4$?

In this section we shall present the answer.

Theorem 8.1. *M_h is homeomorphic to $S^3 \times S^7$ for $h \in \text{Image } S \cap \text{Kernel } J$.*

Let $S^3 \rightarrow M_h \xrightarrow{p} S^7$ be the bundle. Then the tangent bundle $\tau(M_h)$ of the total space M_h is stably isomorphic to

$$p!(\tau(S^7)) \oplus p!(\bar{M}_h)$$

where \bar{M}_h stands for the vector bundle associated with the bundle $S^3 \rightarrow M_h \xrightarrow{p} S^7$.

Lemma 8.2. *M_h is a π -manifold, more over M_h is actually a parallelizable manifold.*

Proof. As is well-known, the tangent bundle $\tau(S^7)$ of S^7 is trivial. Since $\pi_6(SO_N)$ (N : large) is trivial, the bundle

$$S^3 \rightarrow M_h \xrightarrow{p} S^7$$

is stably trivial. Therefore M_h is a π -manifold. It is obvious that the only obstruction to the triviality of $\tau(M_h)$ is a well defined cohomology class

$$\mathfrak{D}_{10}(M_h) \in H^{10}(M_h; \pi_9(SO_{10})) \cong \pi_9(SO_{10}) \cong Z_2 \oplus Z.$$

Since the Euler number $\chi(M_h)$ of M_h is zero, $\mathfrak{D}_{10}(M_h)$ belongs to the image of the homomorphism

$$\begin{array}{ccc} S_*: H^{10}(M_h; \pi_9(SO_9)) & \longrightarrow & H^{10}(M_h; \pi_9(SO_{10})) \\ \Downarrow & & \Downarrow \\ Z_2 \oplus Z_2 & & Z_2 \oplus Z \end{array}$$

induced by the natural map $S: \pi_9(SO_9) \rightarrow \pi_9(SO_{10})$, i.e., $\mathfrak{D}_{10}(M_h)$ is a torsion element.

The fact that M_h is a π -manifold implies that $\mathfrak{D}_{10}(M_h)$ goes into the zero element by the homomorphism

$$\begin{array}{ccc} S_*: H^{10}(M_h; \pi_9(SO_{10})) & \longrightarrow & H^{10}(M_h; \pi_9(SO_{11})) \\ \parallel & & \parallel \\ Z_2 \oplus Z & & Z_2 \end{array}$$

Since the Kernel S_* is isomorphic to Z , $\mathfrak{D}_{10}(M_h)$ is zero. This completes the proof of Lemma 8.2.

Proof of Theorem 8.1. Since $\tau(M_h)$ is trivial and M_h is homotopy equivalent to $S^3 \times S^7$, M_h is classified by the Novikov's theorem [16]. The similar argument as in §4 proves that M_h is diffeomorphic to $S^3 \times S^7 \# \tilde{S}^{10}$ for some homotopy sphere \tilde{S}^{10} . Hence M_h is homeomorphic to $S^3 \times S^7$, completing the proof of Theorem 8.1. We don't know whether M_h is diffeomorphic to $S^3 \times S^7$ or not.

9. Actions of homotopy spheres which do not bound spin manifolds

Let M'' be a simply connected, spin manifold. Then we have

Lemma 9.1. *The inertia group $I(M'')$ does not contain homotopy spheres not bounding spin-manifolds.*

Proof. Let \tilde{S}'' be a homotopy sphere which does not bound a spin-manifold. Since M'' and \tilde{S}'' are spin-manifolds, $M'' \# \tilde{S}''$ has also a spin structure (see Milnor [14]). It is well known that the number of distinct spin structures on the tangent bundle $\tau(M'')$ (resp. $\tau(M'' \# \tilde{S}'')$) is equal to the number of elements in $H^1(M''; Z_2)$ (resp. $H^1(M'' \# \tilde{S}''; Z_2)$) (see Milnor [14]). Since $\pi_1(M'') \cong \pi_1(M'' \# \tilde{S}'') \cong \{1\}$, we have

$$H^1(M''; Z_2) \cong H^1(M'' \# \tilde{S}''; Z_2) \cong \{0\}$$

and hence M'' (resp. $M'' \# \tilde{S}''$) has a unique spin structure. Therefore, if $M'' \# \tilde{S}''$ is diffeomorphic to M'' by an orientation preserving diffeomorphism, both $M'' \# \tilde{S}''$ and M'' represent the same element of the spin cobordism group Ω_n^{spin} . This contradicts the fact that \tilde{S}'' is not a spin boundary.

Milnor has shown the existence of homotopy spheres in dimensions 9, 10, 17 and 18 not bounding spin-manifolds [15] and Anderson, Brown and Peterson have extended this to dimensions 1 or 2 mod 8 [1].

Proposition 9.2. *There exists a homotopy sphere in dimension $8k+2$ which does not belong to the inertia group $I(\mathbf{CP}^{4k+1})$ of the $(8k+2)$ -dimensional complex projective space \mathbf{CP}^{4k+1} for all $k \geq 1$.*

This Proposition digresses from the line, but it is an interesting application.

Proof. It is well known that $\pi_1(\mathbf{CP}^{4k+1}) \cong \{1\}$ and the total Stiefel-Whitney class $W(\mathbf{CP}^{4k+1})$ is

$$(1+\alpha)^{4k+2}$$

where α is the non zero class in $H^2(\mathbf{CP}^{4k+1}; \mathbb{Z}_2)$. Hence the second Stiefel-Whitney class $W_2(\mathbf{CP}^{4k+1})$ is zero and \mathbf{CP}^{4k+1} is a spin-manifold (see Milnor [15]). Consequently this Proposition follows from Lemma 9.1.

Proposition 9.3. *For all $p (\geq 2)$ and all $k (\geq 1)$, the inertia group $I(S^p \times \tilde{S}^{8k+1-p})$ (resp. $I(S^p \times \tilde{S}^{8k+2-p})$) does not contain the above homotopy sphere \tilde{S}^{8k+1} (resp. \tilde{S}^{8k+2}).*

Proof. Obviously $S^p \times \tilde{S}^{8k+1-p}$ (resp. $S^p \times \tilde{S}^{8k+2-p}$) is a simply connected, spin-manifold, hence this Proposition follows from Lemma 9.1. As a corollary we have

Corollary 9.4. *There exists an element $\mu_{8k+2} \in G(8k+2)$ for all $k \geq 1$ which does not belong to the set of the compositions of Image J_p and $G(8k+2-p)$ for all $p \geq 2$.*

Proof. Since the homotopy sphere \tilde{S}^{8k+2} not bounding a spin-manifold does not belong to the inertia group $I(S^p \times \tilde{S}^{8k+2-p})$, \tilde{S}^{8k+2} does not belong to the group $K_1(\pi_p(SO), \tilde{S}^{8k+2-p})$ by Lemma 3.1. As in the proof of Proposition 6.7, the following diagram

$$\begin{array}{ccc} \pi_p(SO) \times \Theta_{8k+2-p} & \xrightarrow{K_1} & \Theta_{8k+2} \\ J_p \times \omega \downarrow & & \downarrow \omega \\ G(p) \times G(8k+2-p)/\text{Im } J_{8k+2-p} & \xrightarrow{C} & G(8k+2) \end{array}$$

is commutative up to sign, since $J(\pi_{8k+2}(SO)) = J(0) = 0$. According to Kervaire and Milnor [11], $\omega|_{\Theta_{8k+2}}$ is injective. It follows that the group

$$C(J_p(\pi_p(SO)), \omega(\tilde{S}^{8k+2-p}))$$

does not contain $\omega(\tilde{S}^{8k+2})$ for all $\tilde{S}^{8k+2-p} \in \Theta_{8k+2-p}$. Since the homomorphism

$$\omega: \Theta_{8k+2-p} \longrightarrow G'(8k+2-p)/\text{Im } J$$

is surjective by definition, the set

$$C(J_p(\pi_p(SO)), G'(8k+2-p)/\text{Im } J_{8k+2-p})$$

does not contain the element $\omega(\tilde{S}^{8k+2})$. If $p \equiv 2, 4, 5, 6 \pmod{8}$, we have $\pi_p(SO)=0$, and hence we can replace $G'(8k+2-p)$ by $G(8k+2-p)$ trivially. If $p \equiv 1, 3, 7 \pmod{8}$, $G'(8k+2-p)$ is equal to $G(8k+2-p)$ (see Novikov [16]). Brown and Peterson [3] showed that the Kervaire invariant is zero in dimensions $8k+2$.

Hence $G'(8k+2-p)$ is also equal to $G(8k+2-p)$ when $p \equiv 0 \pmod{8}$. Consequently $\mu_{8k+2}=\omega(\tilde{S}^{8k+2})$ satisfies the condition of Corollary 9.4. This makes the proof complete.

On the contrary, we shall show that Lemma 9.1 is false in general when the manifolds is not simply connected.

Theorem 9.5. *The inertia group $I(S^1 \times \tilde{S}^{8k+1})$ contains \tilde{S}^{8k+2} for all $k \geq 1$ where \tilde{S}^{8k+1} and \tilde{S}^{8k+2} are the homotopy spheres not bounding spin-manifolds constructed by Milnor [15] and Anderson, Brown and Peterson [1].*

Proof. It is obvious that $\eta \circ \omega(\tilde{S}^{8k+1}) = \omega(\tilde{S}^{8k+2})$ where η denotes the generator of $J(\pi_1(SO)) \cong \mathbb{Z}_2$ (see Anderson, Brown and Peterson [1]). Hence this Theorem follows from Lemma 3.1.

Remark 9.6. $\omega(\tilde{S}^{8k+2})$ does not belong to the set of the compositions of $J_p(\pi_p(SO))$ and $G(8k+2-p)$ for all $p \geq 2$ by Proposition 9.4, but Theorem 9.5 says that $\omega(\tilde{S}^{8k+2}) = \eta \circ \omega(\tilde{S}^{8k+1}) \in C(\pi_1(SO), G(8k+1))$.

10. A concluding remark

Remark 10.1. Propositions 6.2 and 6.4 in the section 6 show that there exists in general on the set $\mathcal{S}(S^p \times S^q)$ no group structure such that the natural map

$$S: C(S^p \times S^q) \cong \Gamma_p \oplus \Gamma_q \oplus \Gamma_{p+q} \longrightarrow \mathcal{S}(S^p \times S^q)$$

is a group homomorphism.

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