Kumano-go, H.
Osaka Math. J.
14 (1962), 181-212.

ON THE UNIQUENESS OF THE SOLUTION OF THE CAUCHY PROBLEM AND THE UNIQUE CONTINUATION THEOREM FOR ELLIPTIC EQUATION

By

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§ 0. Introduction. We shall consider differential operators with complex valued coefficients in a neighborhood of the origin in the $(\nu+1)$-dimensional Euclidean space whose points are denoted by $(t, x)=(t, x_1, \cdots, x_\nu)$ or $(r, \theta)=(r, \theta_1, \cdots, \theta_\nu)$ or simply $(x)=(x_1, \cdots, x_{\nu+1})$.

The object of this note is to prove the following two theorems by a unified method.

The one is the theorem on the uniqueness of the solution of the Cauchy problem for the differential equation of the form

$$Lu \equiv \sum_{i+|\mu|\leqq m} a_{i,\mu}(t, x)\frac{\partial^{i+|\mu|}}{\partial t^i \partial x^\mu}u(t, x) = f(t, x) \tag{0.1}$$

$(\mu=(\mu_1, \cdots, \mu_\nu),\ |\mu|=\mu_1+\cdots+\mu_\nu\,;\ x=(x_1, \cdots, x_\nu),\ \partial x^\mu=\partial x_1^{\mu_1}\cdots\partial x_\nu^{\mu_\nu})$ under the following conditions: Set $L_m \equiv \sum_{i+|\mu|=m} a_{i,\mu}(t, x)\frac{\partial^m}{\partial t^i \partial x^\mu}$. We assume that the associated characteristic polynomial $L_m(t, x, \lambda, \xi)=\sum_{i+|\mu|=m} a_{i,\mu}(t, x)\lambda^i\xi^\mu$ $(\xi=(\xi_1, \cdots, \xi_\nu),\ \xi^\mu=\xi_1^{\mu_1}\cdots\xi_\nu^{\mu_\nu})$ can be written as

$$L_m(t, x, \lambda, \xi') = \prod_{i=1}^{k}(\lambda-\lambda_i^{(1)}(t, x, \xi'))\prod_{j=1}^{m-k}(\lambda-\lambda_j^{(2)}(t, x, \xi')) \qquad (0\leqq k\leqq m) \tag{0.2}$$

for ξ' in some neighborhood of any ξ'_0 on the unit sphere $S=\{\xi'\,;\,|\xi'|=1\}$ $(|\xi'|=(\sum_{i=1}^{\nu}\xi_i'^2)^{1/2})$ and for (t, x) in some neighborhood of the origin where $\lambda_i^{(1)}=-q_i^{(1)}+ip_i^{(1)}$ $(i=1, \cdots, k)$ and $\lambda_j^{(2)}=-q_j^{(2)}+ip_j^{(2)}$ $(j=1, \cdots, m-k)$ are distinct respectively and infinitely differentiable with respect to (t, x, ξ') ($\lambda_i^{(1)}$ and $\lambda_j^{(2)}$ may coincide at some point for some i and j). Furthermore we assume that $\lambda_i^{(1)}(t, x, \xi)=\lambda_i^{(1)}(t, x, \xi|\xi|^{-1})|\xi|$ $(i=1, \cdots, k)$ satisfy the condition of M. Matsumura [8], that is

$$(0.3)\qquad \frac{\partial}{\partial t}p_i^{(1)}+\sum_{j=1}^{\nu}\left\{\frac{\partial}{\partial x_j}p_i^{(1)}\frac{\partial}{\partial \xi_j}q_i^{(1)}-\frac{\partial}{\partial x_j}q_i^{(1)}\frac{\partial}{\partial \xi_j}p_i^{(1)}\right\}=\gamma_i p_i^{(1)}\quad (i=1,\cdots,k)$$

for some $\gamma_i=\gamma_i(t, x, \xi)\in C^{\infty}_{(t,x,\xi)}$ $(\xi\neq 0)$, and that none of $p_j^{(2)}$ $(j=1,\cdots, m-k)$ vanishes.

The other is the unique continuation theorem for the elliptic differential equation of the form

$$(0.4)\qquad Lu=\sum_{|\mu|\leqq m} r^{-(m-|\mu|)}a_\mu(x)\frac{\partial^{|\mu|}}{\partial x^{\mu}}u(x)=0$$

$(x=(x_1,\cdots,x_{\nu+1}),\ r=(\sum_{i=1}^{\nu+1}x_i^2)^{1/2};\ \mu=(\mu_1,\cdots,\mu_{\nu+1}),\ |\mu|=\mu_1+\cdots+\mu_{\nu+1})$ under an exponential vanishing condition, that is

$$(0.5)\qquad \lim_{r\to 0}\exp\{\alpha r^{-l}\}\frac{\partial^{|\mu|}}{\partial x^{\mu}}u(x)=0\qquad (0\leqq|\mu|\leqq m)$$

for a fixed l depending only on L and for every α.

Here we make the following assumption for the characteristic polynomial $L_m(x,\eta)=\sum_{|\mu|=m}a_\mu(x)\eta^\mu$. After transforming $L_m(x,\eta)$ dy (2.14), it can be expressed as

$$(0.6)\qquad L_m(x,\eta)=a^*(x)\prod_{i=1}^{k}(\lambda-r^{-1}\lambda_i^{(1)}(r,\theta,\xi'))\prod_{j=1}^{m-k}(\lambda-r^{-1}\lambda_j^{(2)}(r,\theta,\xi'))$$
$$(0\leqq k<m)$$

for ξ' in some neighborhood of any ξ'_0 on S and for (r,θ) in some neighborhood of the origin where $\lambda_i^{(1)}$ $(i=1,\cdots,k)$ and λ_j $(j=1,\cdots,m-k)$ are distinct respectively and infinitely differentiable.

Strictly speaking it is sufficient to assume that the smoothness of $\lambda_i^{(1)}$ and $\lambda_j^{(2)}$ with respect to (t, x) in (0.2) or to (r,θ) in (0.6) is sufficiently high depending only on m and ν. Furthermore the constant k may depend on ξ'_0 on S, but it is sufficient to treat only the case when the representation (0.2) or (0.6) holds in the whole of the product set of S and some neighborhood of the origin with a fixed constant k, which will be proved in Theorem 4 of §4. Appendix using the idea of S. Mizohata [11]. In this note for the convinience sake we assume $\lambda_i^{(1)}$ and $\lambda_j^{(2)}$ are infinitely differentiable in ξ' on S and in (t, x) or (r,θ) in a neighborhood of the origin.

We can easily see from the proof of Theorem 4 that we need not impose restriction on the dimension of the space, and also we see that the condition (0.3) corresponds to a sufficient condition obtained by L. Hörmander [7] for the existence of the solution of first order differential equation.

The results of A. P. Calderón [3], S. Mizohata [9] and L. Hörmander [6] are contained in ours for the case of $k=m$, of $m=4$, $k=2$ and of $P_i^{(1)}\neq 0$ $(i=1,\cdots,k)$ in (0.2) respectively if we assume the sufficient differentiability for the leading coefficients $a_{i,\mu}(t,x)$ $(i+|\mu|=m)$ of L.

The result of the second theorem contains that of M. H. Protter [12], and partly I. S. Bernstein [1] that corresponds to the case of $k=0$ in (0.6).

As a consequence of the first theorem we can also prove the local existence theorem for a certain differential equation $Lu=f$ of the form (3.6).

The idea of the proofs is based on the methods of S. Mizohata [9] and M. Yamaguti [13].

We wish to thank Prof. M. Nagumo, Dr. H. Tanabe and my colleague for valuable discussions.

§1. **Preliminary lemmas.** In this chapter we shall consider singular integral operators in the sense of M. Yamaguti [13] in the ν-dimensional Euclidean space.

The singular integral operator of A. P. Calderón and A. Zygmund [2] is an operator in the sense of M. Yamaguti if it is of type C_β^∞ $(\beta=\infty)$.

DEFINITION 0. We call $H=\sum_{r=0}^{\infty} a_r h_r$ a singular integral operator with the symbol $\sigma(H)=\sum_{r=0}^{\infty} a_r(x)\tilde{h}_r(\xi)$ $(\tilde{h}_0(\xi)=1)$ in the sence of M. Yamaguti if the following conditions are satisfied: $a_r(x)\in C^\infty_{(x)}$, $\tilde{h}_r(\xi)\in C^\infty_{(\xi\neq 0)}$ $(r=0,1,\cdots)$, and for every k and l there exists a constant $A_{k,l}$ such that $\left|\frac{\partial^{|\mu|}}{\partial x^\mu}a_0(x)\right|\leqq A_{k,l}$, $\left|\frac{\partial^{|\mu|}}{\partial x^\mu}a_r(x)\right|\leqq A_{k,l}r^{-l}$ for $r\geqq 1$ $(|\mu|\leqq k)$, and for every k there exists constants B_k and l'_k such that $\left|\frac{\partial^{|\mu|}}{\partial \xi^\mu}\tilde{h}_r(\xi)\right|\leqq B_k r^{l'_k}|\xi|^{-|\mu|}$ $(|\mu|\leqq k,\ r=1,2,\cdots)$.

We define for $u\in L^2$ the Fourier transform $\mathfrak{F}$ by $\mathfrak{F}[u]=\tilde{u}(\xi)=\frac{1}{\sqrt{2\pi}^\nu}\int e^{-ix\cdot\xi}u(x)dx$, and convolution operators h_r by $\widetilde{h_r u}=\tilde{h}_r(\xi)\tilde{u}(\xi)$.

Then, Hu is defined by

$$Hu=\sum_{r=0}^{\infty} a_r(x)(h_r u)(x) \quad \text{or} \quad Hu=\frac{1}{\sqrt{2\pi}^\nu}\int e^{ix\cdot\xi}\sigma(H)\tilde{u}(\xi)d\xi\,.$$

DEFINITION 1. A function $u=u(t,x)\in C^m_{(t,x)}$ defined in a neighborhood of the origin is said to be of class $\mathfrak{F}_h^{(m)}=\mathfrak{F}_{h,K}^{(m)}$ if car. u=closure of $\{x\ ;\ u(x)\neq 0\}$ is contained in $\left\{(t,x)\ ;\ 0\leqq t<h<\frac{1}{2},\ |x|<K\right\}$ $(|x|=(\sum_{i=1}^{\nu}x_i^2)^{1/2}$ and $\frac{\partial^{j-1}}{\partial t^{j-1}}u(0,x)=0$ $(j=1,\cdots,m)$.

DEFINITION 2. A function $u=u(r,\theta)\in C^m_{(r,\theta)}$ defined in a neighborhood of the origin is said to be of class $\mathfrak{G}^{(m)}_{r_0,l}=\mathfrak{G}^{(m)}_{r_0,K,l}$ if cas. u is contained in

$$\{(r,\theta);\ 0\leqq r<r_0<1,\ |\theta|<K\}\ \ (|\theta|=(\sum_{i=1}^{\nu}\theta_i^2)^{1/2}) \text{ and}$$

$$\lim_{r\to 0}\exp\{\alpha r^{-l}\}\frac{\partial^{i+|\mu|}}{\partial r^i\partial\theta^\mu}u(r,\theta)=0\ \ (0\leqq i+|\mu|\leqq m) \text{ for every } \alpha.$$

DEFINITION. 3. A function $u=u(x)\in C^m_0(\mathfrak{D})$, $\mathfrak{D}=\{x;\ |x|<r_0<1\}$ is said to be of class $\mathfrak{H}^{(m)}_{r_0,l}$ if $\lim_{r\to 0}\exp\{\alpha r^{-l}\}\frac{\partial^{|\mu|}}{\partial x^\mu}u(x)=0$ $(0\leqq|\mu|\leqq m)$ for every α $(x=(x_1,\cdots,x_{\nu+1}),\ r=|x|=(\sum_{i=1}^{\nu+1}x_i^2)^{1/2})$.

In this note we shall use the next lemma without proof.

Lemma 1. i) *Let P and Q be singular integral operators of type $C^\infty_\beta(\beta>1)$ in the sense of* [2] *with real valued symbols, then the following operator norms*

$$(1.1)\qquad \begin{gathered}\|(Q\Lambda-\Lambda Q^*)\|,\ \|(P\Lambda-\Lambda P^*)\|,\\ \|(P^*Q-Q^*P)\Lambda\|,\ \|\Lambda(P^*Q-Q^*P)\|\end{gathered}$$

where Λ is defined by $\widetilde{\Lambda u}(\xi)=|\xi|\tilde{u}(\xi)$ and P^* means the adjoint operator of P, are all bounded; see [2].

ii) *Let H, H_1 and H_2 be singular integral operators, then we have for any positive integers p and q the next representations*

$$(1.2)\qquad \begin{aligned} H\Lambda^p-\Lambda^pH &= H_{p,q}\Lambda^{p-1}+H'_{p,q}\\ (H_1H_2-H_1\circ H_2)\Lambda &= H_q+H'_q\,,\end{aligned}$$

where $H_{p,q}$ and H_q are singular integral operators, and $H'_{p,q}$ and H'_q are bounded operators together with $\Lambda^iH'_{p,q}\Lambda^j$ and $\Lambda^iH'_q\Lambda^j$ $(0\leqq i+j\leqq q)$ respectively. $H_1\circ H_2$ shows a singular integral operator with the symbol $\sigma(H_1)\sigma(H_2)$; see [13].

iii) *Let H be a singular integral operator such as $|\sigma(H)|\geqq\delta>0$, then there exists a positive constant C such that*

$$(1.3)\qquad \|H\Lambda u\|^2\geqq\frac{\delta^2}{8}\|\Lambda u\|^2-C\|u\|^2;\qquad \textit{see S. Mizohata } [10].$$

REMARK. The sign $\|\ \|$ always shows L^2 norm.

Lemma 2. *Let P and Q be singular integral operators with real valued symbols.*

Then we have the following representation

$$(1.4)\qquad -i(P\Lambda Q-\Lambda Q^*P)\Lambda = (K_1-K_2)\Lambda+K_0P\Lambda+K',$$

where K_1 and K_2 are singular integral operators with

$$(1.5)\quad \sigma(K_1) = \sum_{j=1}^{\nu}\frac{\partial}{\partial x_j}\sigma(P)\frac{\partial}{\partial\xi_j}(\sigma(Q)|\xi|),\ \sigma(K_2) = \sum_{j=1}^{\nu}\frac{\partial}{\partial x_j}\sigma(Q)\frac{\partial}{\partial\xi_j}(\sigma(P)|\xi|)$$

respectively, and K_0 and K' are bounded operators.

Proof. Here we shall prove it roughly, details are easily derived from M. Yamaguti [13]. See also the proof of Lemma 6 in § 4 of this note.

As a simple case we consider $P=ah$ and $Q=bk$ with $\sigma(P)=a(x)\tilde{h}(\xi)$ and $\sigma(Q)=b(x)\tilde{k}(\xi)$ respectively.

Take $\alpha(\xi)\in C^{\infty}_{0(\xi)}(\alpha(\xi)=1$ on $|\xi|\leqq 1)$, we write $P=ah_1+ah_2(\sigma(h_1)=\alpha(\xi)\tilde{h}(\xi)$, $\sigma(h_2)=(1-\alpha(\xi))\tilde{h}(\xi))$, and so $Q=bk_1+bk_2$.

Then, we can write $(P\Lambda Q-\Lambda Q^*P)\Lambda=a(h_2\Lambda)b(k_2\Lambda)-(\Lambda k_2)ba(h_2\Lambda)+$ *a bounded operator*, and $a(h_2\Lambda)b(k_2\Lambda)-(\Lambda k_2)ba(h_2\Lambda)=\{a((h_2\Lambda)b-b(h_2\Lambda))(k_2\Lambda)+abh_2k_2\Lambda^2\}-\{((\Lambda k_2)b-b(\Lambda k_2))ah_2\Lambda+b((\Lambda k_2)a-a(\Lambda k_2))h_2\Lambda+abh_2k_2\Lambda^2\}$. Now, for sufficiently large l we use the following representation for $u\in C^{\infty}_{0(x)}$

$$\begin{aligned}
&((h_2\Lambda)b-b(h_2\Lambda))u(x)\\
&=\int((h_2\Lambda)(x-y)b(y)-b(x)(h_2\Lambda)(h_2\Lambda)(x-y))u(y)dy\\
&\qquad\qquad\qquad\qquad\text{(in the distribution's sense)}\\
&=-\sum_{j=1}^{\nu}\int\frac{\partial}{\partial x_j}b(x)(x_j-y_j)(h_2\Lambda)(x-y)u(y)dy\\
&\quad+\sum_{2\leqq|\mu|\leqq l}(-1)^{|\mu|}\int\frac{\partial^{|\mu|}}{\partial x^{\mu}}b(x)\,\frac{(x-y)^{\mu}}{\mu!}(h_2\Lambda)(x-y)u(y)dy\\
&\quad+\sum_{|\mu|=l+1}\int(x-y)^{\mu}(h_2\Lambda)(x-y)b_{\mu}(x,y)u(y)dy,
\end{aligned}$$

then, the operator for the first term is equal to a singular integral operator with the symbol $-i\sum_{j=1}^{\nu}\frac{\partial}{\partial x_j}b(x)\frac{\partial}{\partial\xi_j}(\tilde{h}_2|\xi|)$, and we can see the operators for remaining term are equal to a bounded operator K together with $K\Lambda$.

Using the above representation, if we set K_2 a singular integral operator with $\sigma(K_2)=\sum_{j=1}^{\nu}\frac{\partial}{\partial x_j}\sigma(Q)\frac{\partial}{\partial\xi_j}(\sigma(P)|\xi|)$, then, we can obtain $-ia((h_2\Lambda)b-b(h_2\Lambda))(k_2\Lambda)=-K_2\Lambda+K_2'$ where K_2' is a bounded operator.

Similarly, if we set K_1 a singular integral operator with $\sigma(K_1)=\sum_{j=1}^{\nu}\frac{\partial}{\partial x_j}\sigma(P)\frac{\partial}{\partial\xi_j}(\sigma(Q)|\xi|)$, we obtain $+ib((\Lambda k_2)a-a(\Lambda k_2))h_2\Lambda=K_1\Lambda+K_1'$ with a bounded operator K_1'. By (1.1), $(\Lambda k_2)b-b(\Lambda k_2)=\Lambda Q^*-Q\Lambda$ is bounded.

Consequently, we get (1. 4) for $P=ah$ and $Q=bk$. For general case, we write $\sigma(P)=\sum_{\mu} a_{\mu}(x)\tilde{h}_{\mu}(\xi)$ and $\sigma(Q)=\sum_{\mu'} b_{\mu'}(x)\tilde{k}_{\mu'}(\xi)$ and we can prove (1. 4) dy the same manner as the above simple case. Q.E.D.

Now we shall prove the next fundamental lemmas 3 and 3'.

Lemma 3. *Let $P(t)$ and $Q(t)$ be singular integral operators with real valued symbols defined in (x)-space with t as a parameter and satisfy the condition of M. Matsumura* [8], *that is*

$$(1.6)\quad \frac{\partial}{\partial t}\sigma(P)+\sum_{j=1}^{\nu}\left\{\frac{\partial}{\partial x_j}\sigma(P)\frac{\partial}{\partial \xi_j}(\sigma(Q)|\xi|)-\frac{\partial}{\partial x_j}\sigma(Q)\frac{\partial}{\partial \xi_j}(\sigma(P)|\xi|)\right\} = \gamma\sigma(P)$$

in a neighborhood of the origin $(t, x)=(0, 0)$ for some $\gamma=\gamma(t, x, \xi)\in C^{\infty}_{(t,x,\xi)}$ $(\xi\neq 0)$.

Then, if we set $J=\frac{\partial}{\partial t}+(P+iQ)\Lambda$, there exists a positive constant h_0 depending only on P and Q such that for $0<h\leqq h_0$, $r=t+h$ and sufficiently large n

$$(1.7)\quad \int_0^h r^{-2n}||Ju||^2dt\geqq\frac{h^{-2}n}{8}\int_0^h r^{-2n}||u||^2dt+\frac{1}{8n}\int_0^h r^{-2n}||P\Lambda u||^2dt \quad \text{for all } u\in\mathfrak{F}_h^{(1)}.$$

Especially, if $|\sigma(P)|\geqq\delta>0$, then we have for a positive constant C'

$$(1.8)\quad \int_0^h r^{-2n}||Ju||^2dt\geqq\frac{h^{-2}n}{9}\int_0^h r^{-2n}||u||^2dt+\frac{C'}{n}\left\{\int_0^h r^{-2n}\left\|\frac{\partial u}{\partial t}\right\|^2dt+\int_0^h r^{-2n}||\Lambda u||^2dt\right\}\qquad u\in\mathfrak{F}_h^{(1)}.$$

REMARK: If $\sigma(P)\equiv 0$ or $|\sigma(P)|\geqq\delta>0$, (1. 6) is satisfied.

Proof. Set $u=r^n v$, then $r^{-n}Ju=\left(\frac{dv}{dt}+iQ\Lambda v\right)+(P\Lambda v+nr^{-1}v)$, so that

$$\begin{aligned}(1.9)\quad \int_0^h r^{-2n}||Ju||^2dt &= \int_0^h\left\|\frac{dv}{dt}+iQ\Lambda v\right\|^2dt+\int_0^h||P\Lambda v+nr^{-1}v||^2dt\\ &+\int_0^h\left\{\left(\frac{dv}{dt}, P\Lambda v\right)+\left(P\Lambda v, \frac{dv}{dt}\right)\right\}dt+n\int_0^h r^{-1}\frac{d}{dt}||v||^2dt\\ &+i\int_0^h\{(Q\Lambda v, P\Lambda v)-(P\Lambda v, Q\Lambda v)\}dt+in\int_0^h r^{-1}\{(Q\Lambda v, v)-(v, Q\Lambda v)\}dt\\ &\equiv\sum_{i=1}^6 I_i .\end{aligned}$$

Integrating by part, $I_4=n\int_0^h r^{-2}||v||^2dt$ and applying Schwarz's inequality we have

$$(1.10)\qquad I_2+I_4 \geqq \int_0^h \{||P\Lambda v||^2-2nr^{-1}||P\Lambda v||\,||v||+n(n+1)r^{-2}||v||^2\}\,dt$$
$$\geqq \frac{2}{3}n\int_0^h r^{-2}||v||^2dt+\frac{1}{4n}\int_0^h ||P\Lambda v||^2dt.$$

By (1.1) we have for a positive constant C_1

$$(1.11)\qquad I_6 = in\int_0^h r^{-1}((Q\Lambda-\Lambda Q^*)v,\, v)dt \geqq -C_1hn\int_0^h r^{-2}||v||^2dt.$$

For I_3, we use the method of S. Mizohata [9], and consider it together with I_5, then integrating by parts and using (1.1) we get for a constant $C_2(>0)$

$$I_3 = -\int_0^h (v,\, P'\Lambda v)\,dt+\int_0^h \Big((P\Lambda-\Lambda P^*)v,\, \frac{dv}{dt}+iQ\Lambda v\Big)dt$$
$$-\int_0^h (v,\, i(\Lambda P^*-P\Lambda)Q\Lambda v)dt \geqq -\int_0^h (v,\, (P'+i(\Lambda P^*-P\Lambda)Q)\Lambda v)dt$$
$$-I_1-C_2h^2\int_0^h r^{-2}||v||^2dt, \text{ and } I_5 = -\int_0^h (v,\, i\Lambda(Q^*P-P^*Q)\Lambda v)dt.$$

Consequently we get

$$I_3+I_5 \geqq -\int_0^h (v,\, (P'-i(P\Lambda Q-\Lambda Q^*P))\Lambda v)dt-I_1-C_2h^2\int_0^h r^{-2}||v||^2dt,$$

and by Lemma 2, we have

$$-i(P\Lambda Q-\Lambda Q^*P)\Lambda = (K_1-K_2)\Lambda+K_0P\Lambda+K',$$

where K_1 and K_2 are singular integral operators with

$$\sigma(K_1-K_2) = \sum_{j=1}^{\nu}\Big\{\frac{\partial}{\partial x_j}\sigma(P)\frac{\partial}{\partial \xi_j}(\sigma(Q)\,|\xi|)-\frac{\partial}{\partial x_j}\sigma(P)\frac{\partial}{\partial \xi_j}(\sigma(Q)\,|\xi|)\Big\},$$

and K_0 and K' are bounded operators, on the other hand P' is a singular integral operator with $\sigma(P')=\frac{\partial}{\partial t}\sigma(P)$. Hence, by the condition (1.6) we get $\sigma(P'+(K_1-K_2))=\gamma\sigma(P)$, then using (1.2) and Schwarz's inequality, we have for a constant $C_3(>0)$

$$(1.12)\qquad I_1+I_3+I_5 \geqq -\frac{1}{8n}\int_0^h ||P\Lambda v||^2dt-C_3h^2n\int_0^h r^{-2}||v||^2dt.$$

From (1.9)–(1.12), we have

$$(1.13)\qquad \int_0^h r^{-2n}||Ju||^2dt \geqq \Big(\frac{2}{3}n-C_1h^2n\Big)\int_0^h r^{-2}||v||^2dt+\frac{1}{8n}\int_0^h ||P\Lambda v||^2dt.$$

Remarking $v=r^{-n}u$, we get (1.7) for a sufficiently small h because of $r^{-2}\geqq\frac{1}{4}h^{-2}$ for $0\leqq t\leqq h$.

In order to prove (1.8) we use (1.3) by $|\sigma(P)|\geqq\delta>0$, and remarking $\left\|\frac{\partial u}{\partial t}\right\|^2\leqq 2\|Ju\|^2+C_4\|\Lambda u\|^2$ $(C_4>0)$, we have (1.8). Q.E.D.

Lemma 3′. *Let $P(r)$ and $Q(r)$ be singular integral operators defined in a neighborhood of the origin in (θ)-space with r as a parameter and have real valued symbols.*

Suppose $|\sigma(P)|\geqq\delta>0$, then for the operator $J=\frac{\partial}{\partial r}+r^{-1}(P+iQ)\Lambda$, there exist positive constants l_0 and C depending only on P and Q such that for every l $(\geqq l_0)$ and sufficiently larg α

$$(1.14)\quad \int_0^{r_0} r^{2\beta}\exp\{2\alpha r^{-l}\}\|Ju\|^2dr$$
$$\geqq C\left\{\alpha l^2\int_0^{r_0} r^{2\beta-l-2}\exp\{2\alpha r^{-l}\}\|u\|^2dr\right.$$
$$\left.+\frac{1}{\alpha}\int_0^{r_0} r^{2\beta+l}\exp\{2\alpha r^{-l}\}\left(\left\|\frac{\partial u}{\partial r}\right\|^2+r^{-2}\|\Lambda u\|^2\right)dr\right\}\qquad u\in\mathfrak{G}^{(1)}_{r_0,l}\,.$$

Proof. Set $u=\exp\{-\alpha r^{-l}\}v$, then, $\exp\{\alpha r^{-l}\}Ju=\left(\frac{dv}{dr}+ir^{-1}Q\Lambda v\right)+(r^{-1}P\Lambda v+\alpha lr^{-l-1}v)$. Hence,

$$(1.15)\quad \int_0^{r_0}\exp\{2\alpha r^{-l}\}\|Ju\|^2dr=\int_0^{r_0}\left\|\frac{dv}{dr}+ir^{-1}Q\Lambda v\right\|^2dr$$
$$+\int_0^{r_0}\|r^{-1}P\Lambda v+\alpha lr^{-l-1}v\|^2dr+\int_0^{r_0}\left\{\left(\frac{dv}{dr},\,r^{-1}P\Lambda v\right)+\left(r^{-1}P\Lambda v,\,\frac{dv}{dr}\right)\right\}dr$$
$$+\alpha l\int_0^{r_0} r^{-l-1}\frac{d}{dr}\|v\|^2dr+i\int_0^{r_0}\{(r^{-1}Q\Lambda v,\,r^{-1}P\Lambda v)-(r^{-1}P\Lambda v,\,r^{-1}Q\Lambda v)\}dr$$
$$+i\alpha l\int_0^{r_0} r^{-l-2}\{(Q\Lambda v,\,v)-(v,\,Q\Lambda v)\}dr$$
$$\equiv\sum_{i=1}^{6}I'_i\,.$$

We shall estimate each term parallel to the proof of Lemma 3.

Integrating by part, we have $I'_4=\alpha l(l+1)\int_0^{r_0} r^{-l-2}\|v\|^2dr$, hence, using Schwarz's inequality

$$I'_2+I'_4\geqq\int_0^{r_0} r^{-2}\{\|P\Lambda v\|^2-2\alpha lr^{-l}\|P\Lambda v\|\,\|v\|+\alpha l^2(\alpha r^{-l}+1)r^{-l}\|v\|^2\}dr$$
$$\geqq\frac{1}{2}\alpha l^2\int_0^{r_0} r^{-l-2}\|v\|^2dr+\frac{1}{4\alpha}\int_0^{r_0} r^{l-2}\|P\Lambda v\|^2dr\,.$$

By the assumption of the lemma we can apply (1.3) to the above inequality and we get for a positive constant C_1 and sufficiently large α

$$(1.16) \qquad I_2'+I_4' \geqq \frac{1}{3}\alpha l^2\int_0^{r_0} r^{-l-2}||v||^2dr+\frac{C_1}{\alpha}\int_0^{r_0} r^{l-2}||\Lambda v||^2dr\,.$$

Integrating by parts and using (1.1) we get

$$(1.17) \qquad I_3' \geqq -\frac{C_1}{4\alpha}\int_0^{r_0} r^{l-2}||\Lambda v||^2dr-C_2\alpha\int_0^{r_0} r^{-l-2}||v||^2dr-I_1' \ (C_2>0)$$

and

$$(1.18) \qquad I_5'+I_6' \geqq -\frac{C_1}{4\alpha}\int_0^{r_0} r^{l-2}||\Lambda v||^2dr-C_3\alpha l\int_0^{r_0} r^{-l-2}||v||^2dr \ (C_3>0)\,.$$

From (1.15)-(1.18), there exists a positive constant l_0 such that

$$(1.19) \qquad \int_0^{r_0}\exp\{2\alpha r^{-l}\}||Ju||^2dr \geqq \frac{1}{4}\alpha l^2\int_0^{r_0} r^{-l-2}||u||^2dr+\frac{C_1}{2\alpha}\int_0^{r_0} r^{l-2}||\Lambda v||^2dr$$

for every l $(\geqq l_0)$ and sufficiently large α.

Remarking $v=\exp\{\alpha r^{-l}\}u$ and $\left\|\frac{du}{dr}\right\|^2 \leqq 2||Ju||^2+C_4r^{-2}||\Lambda u||^2$ $(C_4>0)$ we obtain (1.14) for $\beta=0$, and replacing u by $r^\beta u$ we get (1.14) for sufficiently large α. Q.E.D.

Lemma 4. *Let $H_i(t)(i=1,\cdots,k$ for $k\geqq 2)$ be singular integral operators defined in (x)-space with t as a parameter such that $|\sigma(H_i-H_j)|\geqq\delta>0$ $(i\neq j)$.*

We set $J_i=\frac{\partial}{\partial t}+H_i\Lambda(i=1,\cdots,k)$, and $J_{i_1}\cdot J_{i_2}\cdot\cdots\cdot J_{i_{k-1}}$ $(i_\nu\neq i_\mu$ for $\nu\neq\mu)$ are the product operators for the permutations from $J_1, J_2,\cdots,$ and J_k.

Then, we have for positive constants C and C',

$$(1.20) \qquad \sum_{i_1,i_2,\cdots,i_{k-1}}||J_{i_1}\cdot J_{i_2}\cdot\cdots\cdot J_{i_{k-1}}u||^2 \geqq C\sum_{i+j=k-1}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2-C'\sum_{0\leqq i+j\leqq k-2}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2.$$

Proof. For the case $k=2$, $J_1-J_2=(H_1-H_2)\Lambda$. From the assumption $|\sigma(H_1-H_2)|\geqq\delta>0$, if we apply (1.3) of Lemma 1, we get

$$\frac{\delta^2}{8}||\Lambda u||^2-C_1||u||^2 \leqq ||(H_1-H_2)\Lambda u||^2 \leqq 2(||J_1u||^2+||J_2u||^2) \ (C_1>0)\,,$$

and $\left\|\frac{\partial u}{\partial t}\right\|^2 \leqq 2(||J_1u||^2+||H_1\Lambda u||^2)$, hence we get (1.20) for $k=2$.

For the general case $k\geqq 3$, using (1.3) we have for $2\leqq i_\nu\leqq k$ and $i_\nu\neq i_\mu$ for $\nu\neq\mu$,

$$(1.21)\quad ||(J_1-J_{i_1})J_{i_2}\cdot\cdots\cdot J_{i_{k-1}}u||^2 = ||(H_1-H_{i_1})\Lambda J_{i_2}\cdot\cdots\cdot J_{i_{k-1}}u||^2$$
$$\geqq \frac{\delta^2}{8}||\Lambda J_{i_2}\cdot\cdots\cdot J_{i_{k-1}}u||^2 - C_2\sum_{0\leqq i+j\leqq k-2}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2 \qquad (C^2>0)$$

and because of $\frac{\partial}{\partial t}=J_1-H_1\Lambda$

$$(1.22)\quad \left\|\frac{\partial}{\partial t}J_{i_2}\cdot\cdots\cdot J_{i_{k-1}}u\right\|^2$$
$$\leqq 2(||J_1\cdot J_{i_2}\cdot\cdots\cdot J_{i_{k-1}}u||^2+||H_1\Lambda J_{i_2}\cdot\cdots\cdot J_{i_{k-1}}u||^2)\,.$$

On the other hand, using (1.2) we have for constant $C_3(>0)$,

$$(1.23)\quad A\equiv ||J_{i_2}\cdot\cdots\cdot J_{i_{k-1}}\Lambda u||^2+\left\|J_{i_2}\cdot\cdots\cdot J_{i_{k-1}}\frac{\partial u}{\partial t}\right\|^2$$
$$\leqq C_3\left\{||\Lambda J_{i_2}\cdot\cdots\cdot J_{i_{k-1}}u||^2+\left\|\frac{\partial}{\partial t}J_{i_2}\cdot\cdots\cdot J_{i_{k-1}}u\right\|^2+\sum_{0\leqq i+j\leqq k-2}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2\right\}.$$

Since $J_{i_2}\cdot\cdots\cdot J_{i_{k-1}}$ are the permutation from $J_2,\cdots,J_k$, we can apply the assumption of the induction to A and get for positive constant C_4 and C_5

$$(1.24)\qquad A\geqq C_4\sum_{i+j=k-1}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2 - C_5\sum_{0\leqq i+j\leqq k-2}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2.$$

Combining (1.21)-(1.24) we can prove (1.20) for the general case. Q.E.D.

Lemma 4′. *Let $H(r)$ ($i=1,\cdots,k$ for $k\geqq 2$) be singular integral operators defined in (θ)-space with r as a parameter and satisfy the assumption of Lemma 4.*

We set $J_i=\frac{\partial}{\partial r}+r^{-1}H_i\Lambda$ ($i=1,\cdots,k$) and $J_{i_1}\cdot J_{i_2}\cdot\cdots\cdot J_{i_{k-1}}$ ($i_\nu\neq i_\mu$ for $\nu\neq\mu$) are the product operators for the permutations from $J_1, J_2,\cdots$, and J_k.

Then, we have for positive constants C and C'

$$(1.25)\quad \sum_{i_1,i_2,\cdots,i_{k-1}}||J_{i_1}\cdot J_{i_2}\cdot\cdots\cdot J_{i_{k-1}}u||^2$$
$$\geqq C\sum_{i+j=k-1}r^{-2(k-1-i)}\left\|\frac{\partial^i}{\partial r^i}\Lambda^j u\right\|^2 - C'\sum_{0\leqq i+j\leqq k-2}r^{-2(k-1-i)}\left\|\frac{\partial^i}{dr^i}\Lambda^j u\right\|^2.$$

Proof. We can prove it by the method parallel to that of Lemma 4, but we must remark the fact that $\frac{\partial}{\partial r}r^{-1}H\Lambda u-r^{-1}H\Lambda\frac{\partial}{\partial r}u=\left(\frac{\partial}{\partial r}(r^{-1}H)\right)\Lambda u$ and $(\Lambda r^{-1}H\Lambda-r^{-1}H\Lambda^2)u=r^{-1}(\Lambda H-H\Lambda)\Lambda u$, then using (1.2) we get (1.25). Q.E.D.

Lemma 5. *Let $H_i(t)=P_i(t)+iQ_i(t)$ $(i=1, \cdots, k)$ be singular integral operators defined in (x)-space with t as a parameter, and assume each of P_i and Q_i $(i=1, \cdots, k)$ satisfies the condition* (1.6) *of M. Matsumura* [8].

Set $J_i=\frac{\partial}{\partial t}+H_i\Lambda$ $(i=1, \cdots, k)$, *then we have for the operator* $A=J_1\cdot\cdots\cdot J_k$, *and a positive constant* C

$$(1.26)\qquad \int_0^h r^{-2}\|Au\|^2 dt \geqq C \sum_{0\leqq i+j=\tau\leqq k-1} (h^{-2}n)^{k-\tau}\int_0^h r^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2 dt \qquad u\in\mathfrak{F}_h^{(k)},$$

where $r=t+h$ *and* h *is a sufficiently small constant depending only on* P_i *and* Q_i.

Especially, if $|\sigma(P_i)|\geqq\delta>0$, *then we have for a positive constant* C',

$$(1.27)\qquad \int_0^h r^{-2n}\|Au\|^2 dt \geqq C'\frac{1}{n}\sum_{0\leqq i+j=\tau\leqq k} (h^{-2}n)^{k-\tau}\int_0^h r^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2 dt \qquad u\in\mathfrak{F}_h^{(k)}.$$

Proof. (a) The proof of (1.26). For the case $k=1$, the proof is trivial from (1.7) of Lemma 3.

For the general case $k\geqq 2$, we use for example the equality $J_1J_2-J_2J_1 =\left(\frac{\partial}{\partial t}(H_1-H_2)\right)\Lambda+(H_1\Lambda H_2\Lambda-H_2\Lambda H_1\Lambda)=\left(\frac{\partial}{\partial t}(H_1-H_2)\right)\Lambda-\{H_1(\Lambda H_2-H_2\Lambda)+(H_1H_2-H_1\circ H_2)\Lambda-(H_2\circ H_1-H_2H_1)\Lambda-H_2(H_1\Lambda-\Lambda H_1)\}\Lambda$. Then, applying (1.2) to the above equality we can write with a singular integral operator H' and a operator H'' which for every q has a singular integral operator H_q such as $\Lambda^i(H''-H_q)\Lambda^j$ $(0\leqq i+j\leqq q)$ bounded,

$$(1.28)\qquad J_1\cdot J_2-J_2J_1=H'\Lambda+H''.$$

If we use (1.28) for any $J_iJ_j-J_jJ_i$, we get for a constant $C_1(>0)$

$$(1.29)\qquad \|(J_1\cdot\cdots\cdot J_k-J_{i_1}\cdot\cdots\cdot J_{i_k})u\|^2\leqq C_1\sum_{0\leqq i+j\leqq k-1}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2 \qquad (i_\nu\neq i_\mu \text{ for } \nu\neq\mu),$$

hence for constants C_2 and $C_3(>0)$, we get

$$(1.30)\qquad \|Au\|^2\geqq C_2\sum_{i_1,\cdots,i_k}\|J_{i_1}\cdot\cdots\cdot J_{i_k}u\|^2-C_3\sum_{0\leqq i+j\leqq k-1}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2.$$

Now, we apply (1.7) to the operators $J_{i_1}\cdot\cdots\cdot J_{i_k}$ and use (1.30), then we get for constants C_4 and C_5 (>0)

$$(1.31)\quad \int_0^h r^{-2n}\|Au\|^2dt \geq C^4h^{-2}n\sum_{i_2,\cdots,i_k}\int_0^h r^{-2n}\|J_{i_2}\cdot\cdots\cdot J_{i_k}u\|^2dt - C_5\sum_{0\leq i+j\leq k-1}\int_0^h r^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2 dt.$$

By the assumption of the induction,

$$(1.32)\quad \varepsilon h^{-2}n\int_0^h r^{-2n}\|J_1\cdot\cdots\cdot J_{k-1}u\|^2dt \geq \varepsilon C\sum_{0\leq i+j=\tau\leq k-2}(h^{-2}n)^{k-\tau}\int_0^h r^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2 dt \quad (\varepsilon>0).$$

Then, if we apply Lemma 4 to the first term of the right hand side of (1.31), and use (1.32) for sufficiently small ε, we get (1.26) for sufficiently large n.

(b) The proof of (1.27). By the assumption we can apply (1.8) of Lemma 3 to $J_{i_1}\cdot\cdots\cdot J_{i_k}$ ($i_\nu\neq i_\mu$ for $\nu\neq\mu$), and using (1.30) we obtain for constants C_6 and C_7 (>0),

$$\int_0^h r^{-2n}\|Au\|^2dt \geq C_6\frac{1}{n}\sum_{i_2,\cdots,i_k}\int_0^h r^{-2n}\left(\left\|\frac{\partial}{\partial t}J_{i_2}\cdot\cdots\cdot J_{i_k}u\right\|^2+\|\Lambda J_{i_2}\cdot\cdots\cdot J_{i_k}u\|^2dt\right.$$
$$+\frac{1}{2}\int_0^h r^{-2n}\|Au\|^2dt - C_7\sum_{0\leq i+j\leq k-1}\int_0^h r^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2 dt.$$

In the first term of the right hand side in the above inequality we estimate the commutators $\left(\frac{\partial}{\partial t}J_{i_2}\cdot\cdots\cdot J_{i_k}-J_{i_2}\cdot\cdots\cdot J_{i_k}\frac{\partial}{\partial t}\right)u$ and $(\Lambda J_{i_2}\cdot\cdots\cdot J_{i_k}-J_{i_2}\cdot\cdots\cdot J_{i_k}\Lambda)u$ by (1.2) and apply Lemma 4, and we apply (1.26) to the second term, then we have for constants C_8 and C_9(>0)

$$\int_0^h r^{-2n}\|Au\|^2dt \geq C_8\frac{1}{n}\sum_{i+j=k}\int_0^h r^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2 dt - C_9\sum_{0\leq i+j\leq k-1}\int_0^h r^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2 dt$$
$$+C\sum_{0\leq i+j=\tau\leq k-1}(h^{-2}n)^{k-\tau}\int_0^h r^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2 dt.$$

Then, for sufficiently large n we get (1.27). Q.E.D.

Lemma 5′. *Let $H_i(r)=P_i(r)+iQ_i(r)$ ($i=1,\cdots,k$) be singular integral operators defined in (θ)-space with r as a parameter, and assume $|\sigma(P_i)|\geq\delta>0$ ($i=1,\cdots,k$).*

Set $J_i=\frac{\partial}{\partial r}+r^{-1}(P_i+iQ_i)\Lambda$ ($i=1,\cdots,k$), then we have for the operator $A=J_1\cdot\cdots\cdot J_k$ and a positive constant C

$$(1.33)\quad \int_0^{r_0} r^{2\beta}\exp\{2\alpha r^{-l}\}\|Au\|^2dr$$
$$\geqq C\alpha\sum_{0\leqq i+j=\tau\leqq k-1} l^{2(k-\tau)}\int_0^{r_0} r^{2\beta-l-2(k-i)}\exp\{2\alpha r^{-l}\}\left\|\frac{\partial^i}{\partial r^i}\Lambda^j u\right\|^2dr$$
$$u\in\mathfrak{G}^{(k)}_{r_0,l},$$

and for another positive constant C'

$$(1.34)\quad \int_0^{r_0} r^{2\beta}\exp\{2\alpha r^{-l}\}\|Au\|^2dr$$
$$\geqq C'\frac{1}{\alpha}\sum_{0\leqq i+j=\tau\leqq k} l^{2(k-\tau)}\int_0^{r_0} r^{2\beta+l-2(k-i)}\exp\{2\alpha r^{-l}\}\left\|\frac{\partial^i}{\partial r^i}\Lambda^j u\right\|^2dr$$
$$u\in\mathfrak{G}^{(k)}_{r_0,l}.$$

Proof. The proofs are played by the same process with that of Lemma 5. Corresponding to (1.30) we have

$$\|Au\|^2\geqq C_1\sum_{i_1,\cdots,i_k}\|J_{i_1}\cdot\cdots\cdot J_{i_k}u\|^2-C_2\sum_{0\leqq i+j\leqq k-1} r^{-2(k-i)}\left\|\frac{\partial^i}{\partial r^i}\Lambda^j u\right\|^2,$$

and

$$\left\|\frac{\partial}{\partial r}J_{i_1}\cdot\cdots\cdot J_{i_{k-1}}u\right\|^2+r^{-2}\|\Lambda J_{i_1}\cdot\cdots\cdot J_{i_{k-1}}u\|^2$$
$$\geqq C_3\left\{\left\|J_{i_1}\cdot\cdots\cdot J_{i_{k-1}}\frac{\partial u}{\partial r}\right\|^2+r^{-2}\|J_{i_1}\cdot\cdots\cdot J_{i_{k-1}}\Lambda u\|^2\right\}$$
$$-C_4\sum_{0\leqq i+j\leqq k-1} r^{-2(k-i)}\left\|\frac{\partial^i}{\partial r^i}\Lambda^j u\right\|^2$$

where C_1, C_2, C_3 and C_4 are positive constants. Remarking the above inequality, if we apply (1.14) of Lemma 3′ according to the proofs of (1.26) and (1.27), we get for positive constants C_5 and C_6

$$(1.35)\quad \int_0^{r_0} r^{2\beta}\exp\{2\alpha r^{-l}\}\|Au\|^2dr$$
$$\geqq C_5\sum_{0\leqq i+j=\tau\leqq k-1}(\alpha l^2)^{k-\tau}\int_0^{r_0} r^{2\beta-l(k-\tau)-2(k-i)}\exp\{2\alpha r^{-l}\}\left\|\frac{\partial^i}{\partial r^i}\Lambda^j u\right\|^2dr$$

and

$$(1.36)\quad \int_0^{r_0} r^{2\beta}\exp\{2\alpha r^{-l}\}\|Au\|^2dr$$
$$\geqq C_6\frac{1}{\alpha}\sum_{0\leqq i+j=\tau\leqq k}(\alpha l^2)^{k-\tau}\int_0^{r_0} r^{2\beta-l(k-1-\tau)-2(k-i)}\exp\{2\alpha r^{-l}\}\left\|\frac{\partial^i}{\partial r^i}\Lambda^j u\right\|^2dr$$

respectively.

Hence, if we note $r^{-l(k-\tau)}\geqq r^{-l}$ for $\tau\leqq k-1$ and $r^{-l(k-1-\tau)}\geqq r^{l}$ for $\tau\leqq k$

because of $0 \leqq r \leqq r_0 < 1$, and $(\alpha l^2)^{k-\tau} \geqq \alpha l^{2(k-\tau)}$ for $\tau \leqq k-1$ and $(\alpha l^2)^{k-\tau} \geqq l^{2(k-\tau)}$ for $\tau \leqq k$, then from (1.35) and (1.36) we can easily obtain (1.33) and (1.34) respectively. Q.E.D.

§2. **Main theorems.** First we shall prove a theorem which will be used for the uniqueness of the Cauchy problem.

Let $L_m(t, x, \lambda, \xi) = \sum_{j=0}^{m} H_j(t, x, \xi)\lambda^{m-j}$ be a homogeneous differential polynomial where $H_j(t, x, \xi) = \sum_{|\mu|=j} a_\mu(t, x)\xi^\mu$ $(H_0 = 1)$ are differential polynomials with respect to ξ with complex valued infinitely differentiable caefficients $a_\mu(t, x)$ defined in a neighborhood of the origin.

Now we resolve L_m into the form

$$(2.1)\quad L_m(t, x, \lambda, \xi) = \prod_{i=1}^{k} (\lambda - \lambda_i^{(1)}(t, x, \xi)) \prod_{j=1}^{m-k} (\lambda - \lambda_j^{(2)}(t, x, \xi)) \qquad (0 \leqq k \leqq m),$$

and we write

$$(2.2)\quad \begin{aligned} \lambda_i^{(1)}(t, x, \xi) &= -q_i^{(1)}(t, x, \xi) + ip_i^{(1)}(t, x, \xi) \qquad (i = 1, \cdots, k), \\ \lambda_j^{(2)}(t, x, \xi) &= -q_j^{(2)}(t, x, \xi) + ip_j^{(2)}(t, x, \xi) \qquad (j = 1, \cdots, m-k). \end{aligned}$$

Theorem 1. *Let* $L = L(t, x, \lambda, \xi) = L_m(t, x, \lambda, \xi) + \sum_{0 \leqq i+|\mu| \leqq m-1} b_{i,\mu}(t, x)\lambda^i\xi^\mu$ *be a differential polynomial of order m with bounded measurable coefficients* $b_{i,\mu}(t, x)$.

Suppose $\lambda_i^{(1)} (i = 1, \cdots, k)$ *and* $\lambda_j^{(2)} (j = 1, \cdots, m-k)$ *in* (2.1) *are distinct for* $\xi \neq 0$ *respectively and infinitely differentiable, and* $p_i^{(1)}$ *and* $q_i^{(1)}$ $(i = 1, \cdots, k)$ *in* (2.2) *satisfy the condition of M. Matsumura* [8], *that is*

$$(2.3)\quad \frac{\partial}{\partial t} p_i^{(1)} + \sum_{j=1}^{\nu} \left\{ \frac{\partial}{\partial x_j} p_i^{(1)} \frac{\partial}{\partial \xi_j} q_i^{(1)} - \frac{\partial}{\partial x_j} q_i^{(1)} \frac{\partial}{\partial \xi_j} p_i^{(1)} \right\} = \nu_i p_i^{(1)} \qquad (i = 1, \cdots, k)$$

in a neighborhood of the origin for some $\nu_i = \nu_i(t, x, \xi) \in C^\infty_{(t,x,\xi)}$ $(\xi \neq 0)$, *and* $p_j^{(2)}$ $(j = 1, \cdots, m-k)$ *in* (2.2) *do not vanish for* $\xi \neq 0$.

Then, there exist positive constants C and h such that

$$(2.4)\quad \int_0^h r^{-2n} \|Lu\|^2 dt \geqq C \sum_{0 \leqq i+|\mu|=\tau \leqq m-1} h^{-2(m-\tau)} \int_0^h r^{-2n} \left\| \frac{\partial^\tau}{\partial t^i \partial x^\mu} u \right\|^2 dt$$
$$(r = t+h, \quad u \in \mathfrak{F}_h^{(m)})$$

for sufficiently large n.

Proof. By Theorem 4 we may consider that (2.1) and (2.3) hold for every (t, x). Let $P_i^{(1)} + iQ_i^{(1)}$ $(i = 1, \cdots, k)$ and $P_j^{(2)} + iQ_j^{(2)}$ $(j = 1, \cdots, m-k)$ be singular integral operators with $\sigma(P_i^{(1)} + iQ_i^{(1)}) = -i\lambda_i^{(1)} |\xi|^{-1}$ and $\sigma(P_j^{(2)} + iQ_j^{(2)}) =$

$-i\lambda_j^{(2)}|\xi|^{-1}$ respectively, then they are of type C_β^∞ ($\beta=\infty$) in the sense of [2].

Set $A_1=\prod_{i=1}^{k}\left(\frac{\partial}{\partial t}+(P_i^{(1)}+Q_i^{(1)})\Lambda\right)$ and $A_2=\prod_{j=1}^{m-k}\left(\frac{\partial}{\partial t}+(P_j^{(2)}+iQ_j^{(2)})\Lambda\right)$. Then, using (1.2) of Lemma 1, we have for a positive constant C_1,

$$\|(A_1\cdot A_2-L)u\|^2\leqq C_1\sum_{0\leqq i+j\leqq m-1}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2. \tag{2.5}$$

By the assumptions of the theorem, we can apply (1.26) and (1.27) of Lemma 5 to A_1 and A_2 respectively. Hence, first using (1.26)

$$\int_0^h r^{-2n}\|A_1A_2u\|^2dt\geqq C\sum_{0\leqq i+j=\tau\leqq k-1}(h^{-2}n)^{k-\tau}\int_0^h r^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^jA_2u\right\|^2dt \tag{2.6}$$

and using (1.2) we get for positive constants C_2 and C_3

$$\begin{aligned}&\sum_{0\leqq i+j=\tau\leqq k-1}\left\|\frac{\partial^i}{\partial t^i}\Lambda^jA_2u\right\|^2\\ &\geqq C_2\sum_{0\leqq i+j=\tau\leqq k-1}\left\|A_2\frac{\partial^i}{\partial t^i}\Lambda^ju\right\|^2-C_3\sum_{0\leqq i'+j'=\tau'\leqq \tau+(m-k)-1}\left\|\frac{\partial^{i'}}{\partial t^{i'}}\Lambda^{j'}u\right\|^2.\end{aligned} \tag{2.7}$$

Now, by (1.27) for a positive constants C_4

$$\begin{aligned}&\sum_{0\leqq i+j=\tau\leqq k-1}(h^{-2}n)^{k-\tau}\int_0^h r^{-2n}\left\|A_2\frac{\partial^i}{\partial t^i}\Lambda^ju\right\|^2dt\\ &\geqq C_4\frac{1}{n}\sum_{0\leqq i+j=\tau\leqq m-1}(h^{-2}n)^{m-\tau}\int_0^h r^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^ju\right\|^2dt.\end{aligned} \tag{2.8}$$

From the second term of the right hand side of (2.7) we get $k-\tau\leqq m-1-\tau'$, hence combining (2.6)-(2.8) we have for positive constants C_5 and C_6

$$\begin{aligned}\int_0^h r^{-2n}\|A_1A_2u\|^2dt&\geqq C_5\frac{1}{n}\sum_{0\leqq i+j=\tau\leqq m-1}(h^{-2}n)^{m-\tau}\int_0^h r^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^ju\right\|^2dt\\ &-C_6\sum_{0\leqq i'+j'=\tau'\leqq m-2}(h^{-2}n)^{m-1-\tau'}\int_0^h r^{-2n}\left\|\frac{\partial^{i'}}{\partial t^{i'}}\Lambda^{j'}u\right\|^2dt.\end{aligned}$$

Then, if we use (2.5) and $\left\|\frac{\partial^{i+|\mu|}}{\partial t^i\partial x^\mu}u\right\|\leqq\left\|\frac{\partial^i}{\partial t^i}\Lambda^{|\mu|}u\right\|$, and note $m-1-\tau\geqq0$ for $\tau\leqq m-1$, we can get (2.4) for sufficiently small h. Q.E.D.

Corollary 1. *Let L_i ($i=1,\cdots,s$) be differential polynomials of order m_i, and assume each of them satisfies the conditions of Theorem* 1.

Then, there exist positive constants C' and h such that

$$\int_0^h r^{-2n}\|L_1\cdot\cdots\cdot L_s u\|^2 dt \geq C' \sum_{0\leq i+|\mu|=\tau\leq M-s} h^{-2(M-\tau)} \int_0^h r^{-2n} \left\| \frac{\partial^\tau}{\partial t^i \partial x^\mu} u \right\|^2 dt \tag{2.9}$$

$$\left(M = \sum_{i=1}^s m_i, \quad u \in \mathfrak{F}_h^{(M)}\right)$$

for sufficiently large n.

Proof. If we consider $L_1\cdot\cdots\cdot L_s u$ as $L_1\cdot\cdots\cdot L_{s-1}(L_s u)$, and apply the assumption of the induction, then by using the inequality for $M_s = M - m_s$ and sufficiently small h

$$\sum_{0\leq i+|\mu|=\tau\leq M_s-(s-1)} h^{-2(M_s-\tau)} \left\| \frac{\partial^\tau}{\partial t^i \partial x^\mu} L_s u \right\|^2$$

$$\geq C_1 \sum_{0\leq i+|\mu|=\tau\leq M_s-(s-1)} h^{-2(M_s-\tau)} \left\| L_s \frac{\partial^\tau}{\partial t^i \partial x^\mu} u \right\|^2$$

$$-C_2 h^2 \sum_{0\leq i+|\mu|=\tau\leq M-s} h^{-2(M-\tau)} \left\| \frac{\partial^\tau}{\partial t^i \partial x^\mu} u \right\|^2 \qquad (C_1, C_2 > 0)$$

we can easily prove (2.9). Q.E.D.

Next we shall prove the theorem concerning the unique continuation for elliptic differential operator.

Let $L = L(x, \eta) = \sum_{|\mu|\leq m} a_\mu(x)\eta^\mu$ be an elliptic differential polynomial with complex valued bounded coefficients defined in a neighborhood of the origin in the $(\nu+1)$-dimensional Euclidean space, and assume for constants δ_1 and δ_2 (>0)

$$\delta_1 \geq \left|\sum_{|\mu|=m} a_\mu(x)\eta^\mu\right| \geq \delta_2 > 0 \qquad (|\eta| = 1). \tag{2.10}$$

Now we transform the coordinates (x) to polar coordinates (r, θ), for example

$$\begin{aligned} &x = (x_1, \cdots, x_\nu, x_{\nu+1}) = r\phi(\theta) = r(\theta_1, \cdots, \theta_\nu, \sqrt{1-|\theta|^2}) \\ &\qquad\qquad\qquad (|\theta| = \{\sum_{i=1}^\nu \theta\}^{1/2} < 1), \\ &r = \sqrt{\sum_{i=1}^{\nu+1} x_i^2}, \quad \theta_i = \frac{x_i}{\sqrt{\sum_{i=1}^{\nu+1} x_i^2}} \quad (i = 1, \cdots, \nu) \quad (x_{\nu+1} > 0). \end{aligned} \tag{2.11}$$

Then,

$$\begin{aligned} &\frac{\partial}{\partial x_i} = \theta_i \frac{\partial}{\partial r} + r^{-1} \sum_{j=1}^\nu (\delta_{ij} - \theta_i\theta_j) \frac{\partial}{\partial \theta_j} \qquad (i = 1, \cdots, \nu), \\ &\frac{\partial}{\partial x_{\nu+1}} = \sqrt{1-|\theta|^2}\left(\frac{\partial}{\partial r} - r^{-1}\sum_{j=1}^\nu \theta_j \frac{\partial}{\partial \theta_j}\right). \end{aligned} \tag{2.12}$$

Hence, if we define a matrix D by

$$(2.13)\qquad D = D(\theta) = \begin{pmatrix} 1-\theta_1^2, & -\theta_1\theta_2, \cdots, & -\theta_1\theta_\nu, & \theta_1 \\ \vdots & & & \\ -\theta_\nu\theta_1, & -\theta_\nu\theta_2, \cdots, & 1-\theta_\nu^2, & \theta_\nu \\ -\theta_1\sqrt{1-|\theta|^2}, \cdots, & & -\theta_\nu\sqrt{1-|\theta|^2}, & \sqrt{1-|\theta|^2} \end{pmatrix}$$

then, the principal part $L_m = L_m(r, \theta, \lambda, \xi)$ of the above differential polynomial L as the operator with respect to (r, θ), is obtained in $\sum_{|\mu|=m} a_\mu(x)\eta^\mu$ by replacing $a_\mu(x)$ by $a_\mu(r\phi(\theta))$ and transforming η by

$$(2.14)\qquad \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_\nu \\ \eta_{\nu+1} \end{pmatrix} = D \begin{pmatrix} r^{-1}\xi_1 \\ \vdots \\ r^{-1}\xi_\nu \\ \lambda \end{pmatrix}$$

respectively.

We write L_m

$$(2.15)\qquad L_m \equiv a^*(x)\{\lambda^m + \sum_{i=1}^m r^{-i} H_i(r, \theta, \xi)\lambda^{m-i}\},$$

where $H_i(r, \theta, \xi) = \sum_{|\mu|=i} b_\mu(r, \theta)\xi^\mu$, $a^*(x) = \sum_{|\mu|=m} a_\mu(x)\left(\frac{x}{r}\right)^\mu$ and by (2.10) and $\left|\frac{x}{r}\right| = 1$ we have

$$(2.16)\qquad \delta_1 \geqq |a^*(x)| \geqq \delta_2 > 0.$$

REMARK 1. Since the elements of the matrix D is analytic, $b_\mu(r, \theta)$ are infinitely differentiable with respect to (r, θ) if $a_\mu(x)$ ($|\mu| = m$) are infinitely differentiable with respect to (x).

2. Since $D(0)$ = unit matrix, for the associated differential polynomial

$$(2.17)\qquad L_m^*(r, \theta, \lambda, \xi) \equiv \lambda^m + \sum_{i=1}^m r^{-i} H_i(r, \theta, \xi)\lambda^{m-i} = \prod_{i=1}^m (\lambda - r^{-1}\lambda_i(r, \theta, \xi)),$$

$\lambda_i(r, \theta, \xi)$ $(i = 1, \cdots, m)$ are distinct if the equation $\sum_{|\mu|=m} a_\mu(x)\eta^\mu = 0$ has distinct roots as the polynomial with respect to $\eta_{\nu+1}$.

Theorem 1'. *Let $L(x, \eta) = \sum_{|\mu|\leqq m} a_\mu(x)\eta^\mu$ be an elliptic differential polynomial of order m defined in a neighborhood of the origin which satisfies* (2.10), *and leading coefficients are infinitely differentiable and remaining coefficients bounded measurable.*

Suppose for any representation of polar coordinates we can write L_m^* of (2.17) *such as*

$$(2.18)\qquad L_m^*(r,\theta,\lambda,\xi)=\prod_{i=1}^{k}(\lambda-r^{-1}\lambda_i^{(1)}(r,\theta,\xi))\prod_{j=1}^{m-k}(\lambda-r^{-1}\lambda_j^{(2)}(r,\theta,\xi))\qquad (0\leqq k<m),$$

where $\lambda_i^{(1)}(i=1,\cdots,k)$ *and* $\lambda_j^{(2)}(j=1,\cdots,m-k)$ *are distinct respectively, and infinitely differentiable for* $\xi\neq 0$.

Then, there exist positive constants C *and* l_0 *depending only on* L *such that*

$$(2.19)\qquad \int_{|x|<r_0} r^{2\beta}\exp\{2\alpha r^{-l}\}|Lu|^2dx \geqq C\sum_{0\leqq|\mu|\leqq m-1} l^{2(m-|\mu|)}\int_{|x|<r_0} r^{2\beta-2(m-|\mu|)}\exp\{2\alpha r^{-l}\}\left|\frac{\partial^{|\mu|}}{\partial x^\mu}u\right|^2dx \qquad u\in\mathfrak{H}_{r_0,l}^{(m)}$$

for every l $(\geqq l_0)$ *and sufficiently large* α.

Proof. For L_m^* of (2.18), we define $A_1=\prod_{i=1}^{k}\left(\frac{\partial}{\partial r}+r^{-1}(P_i^{(1)}+iQ_i^{(1)})\Lambda\right)$ and $A_2=\prod_{j=1}^{m-k}\left(\frac{\partial}{\partial r}+r^{-1}(P_j^{(2)}+iQ_j^{(2)})\Lambda\right)$ where $P_i^{(1)}+iQ_i^{(1)}$ $(i=1,\cdots,k)$ and $P_j^{(2)}+iQ_j^{(2)}(j=1,\cdots,m-k)$ are singular integral operators with symbols $-i\lambda_i^{(1)}|\xi|^{-1}$ and $-i\lambda_j^{(2)}|\xi|^{-1}$ respectively.

Then, the assumptions of the theorem it is easy A_1 and A_2 satisfy the conditions of Lemma 5′.

We remark here by estimating commutators using (1.2)

$$(2.20)\qquad \left\|\left(L_m^*\left(r,\theta,\frac{\partial}{\partial r},\frac{\partial}{\partial\theta}\right)-A_1A_2\right)u\right\|^2\leqq C_1\sum_{0\leqq i+j\leqq m-1} r^{-2(m-i)}\left\|\frac{\partial^i}{\partial r^i}\Lambda^ju\right\|^2$$

and considering L as a operators with respect to (r,θ)

$$(2.21)\qquad \|(L-a^*L_m^*)u\|^2\leqq C_2\sum_{0\leqq i+j\leqq m-1} r^{-2(m-i)}\left\|\frac{\partial^i}{\partial r^i}\Lambda^ju\right\|^2$$

for $u\in C_{(r_0,\theta)}^{(m)}$ and positive constants C_1 and C_2.

Now, if we apply (1.34) to A_1, we get

$$(2.22)\qquad \int_0^{r_0} r^{2\beta}\exp\{2\alpha r^{-l}\}\|A_1A_2u\|^2dr \geqq C'\frac{1}{\alpha}\sum_{0\leqq i+j=\tau\leqq k} l^{2(k-\tau)}\int_0^{r_0} r^{2\beta+l-2(k-i)}\exp\{2\alpha r^{-l}\}\left\|\frac{\partial^i}{\partial r^i}\Lambda^jA_2u\right\|^2dr \qquad u\in\mathfrak{G}_{r_0,l}^{(m)},$$

and if we estimate the commutators by (1.2) we get

$$(2.23)\quad \sum_{0\leqq i+j=\tau\leqq k} l^{2(k-\tau)}r^{-2(k-i)}\left\|\frac{\partial^i}{\partial r^i}\Lambda^j A_2 u\right\|^2$$
$$\geqq C_3\sum_{0\leqq i+j=\tau\leqq k} l^{2(k-\tau)}r^{-2(k-i)}\left\|A_2\frac{\partial^i}{\partial r^i}\Lambda^j u\right\|^2$$
$$-C_4\sum_{0\leqq i'+j'=\tau'\leqq\tau+(m-k)-1} l^{2(k-\tau)}r^{-2(m-i')}\left\|\frac{\partial^{i'}}{dr^{i'}}\Lambda^{j'}u\right\|^2 \qquad (C_3,\,C_4>0)\,.$$

Noting $k-\tau\leqq m-1-\tau'$ and $\tau'\leqq m-1$, and replacing i', j' and τ' by i, j and τ respectively, we can see that the second term of the right hand side in (2.23) is not larger than $C_5 l^{-2}\sum_{0\leqq i+j=\tau\leqq m-1} l^{2(m-\tau)}r^{-2(m-i)}\left\|\frac{\partial^i}{\partial r^i}\Lambda^j u\right\|^2$ ($C_5>0$). Hence, if we replace the right hand side of (2.22) by that of (2.23) and apply (1.33) to the terms $\int_0^{r_0} r^{2\beta+l-2(k-i)}\exp\{2\alpha r^{-l}\}\left\|A_2\frac{\partial^i}{\partial r^i}\Lambda^j u\right\|^2 dr$ then we get

$$(2.24)\quad \int_0^{r_0} r^{2\beta}\exp\{2\alpha r^{-l}\}\|A_1A_2u\|^2dr$$
$$\geqq C_6\sum_{0\leqq i+j=\tau\leqq m-1} l^{2(m-\tau)}\int_0^{r_0} r^{2\beta-2(m-i)}\exp\{2\alpha r^{-l}\}\left\|\frac{\partial^i}{\partial r^i}\Lambda^j u\right\|^2dr$$
$$-C_7\frac{l^{-2}}{\alpha}\sum_{0\leqq i+j=\tau\leqq m-1} r_0^l\, l^{2(m-\tau)}\int_0^{r_0} r^{2\beta-2(m-i)}\exp\{2\alpha r^{-l}\}\left\|\frac{\partial^i}{\partial r^i}\Lambda^j u\right\|^2dr$$
$$(C_6,\,C_7>0)\,.$$

By (2.20), (2.21) and (2.24), if we consider L as

$$L=(L-a^*L_m^*)+a^*(L_m^*-A_1A_2)+a^*A_1A_2\,,$$

then, by (2.16) we have the following important inequality for positive constants l_0 and C_8

$$(2.25)\quad \int_0^{r_0} r^{2\beta}\exp\{2\alpha r^{-l}\}\|Lu\|^2dr$$
$$\geqq C_8\sum_{0\leqq i+j=\tau\leqq m-1} l^{2(m-\tau)}\int_0^{r_0} r^{2\beta-2(m-i)}\exp\{2\alpha r^{-l}\}\left\|\frac{\partial^i}{\partial r^i}\Lambda^j u\right\|^2dr$$
$$u\in\mathfrak{G}_{r_0,l}^{(m)}$$

for every l ($\geqq l_0$) and sufficiently large α.

Now we use the partition of the unity such that

$$(2.26)\qquad \Theta_i\left(\frac{x}{|x|}\right)\in C^\infty_{(|x|>0)}\ (i=1,\cdots,s)\,,\quad \sum_{i=1}^s\Theta_i^2=1\,,$$

for any $u(x)\in\mathfrak{H}_{r_0,l}^{(m)}$, $u_i=(\Theta_i u)(r\phi(\theta))$ belong to $\mathfrak{G}_{r_0,l}^{(m)}$ and we can apply the inequality (2.25) to each u_i. It is easy that such partition of the unity exists from the assumption of Theorem 1′.

We have for such u_i the following inequality

$$(2.27)\quad \left|\frac{\partial^{|\mu|}}{\partial x^{\mu}}u\right|^2 \leqq C_9 \sum_{i=1}^{s}\left|\frac{\partial^{|\mu|}}{\partial x^{\mu}}u_i\right|^2,$$

$$\sum_{i=1}^{s}|Lu_i|^2 \leqq 2|Lu|^2 + C_9 \sum_{0\leqq|\mu|\leqq m-1} r^{-2(m-|\mu|)}\left|\frac{\partial^{|\mu|}}{\partial x^{\mu}}u\right|^2 \quad (C_9>0).$$

On the other hand by (2.12) and (2.14), if we set $r^{\nu}drd\theta=\psi(x)dx$, then $\frac{1}{2}\leqq\psi(x)\leqq 2$ for sufficiently small θ. Hence, we have for any $v(x)=v(r,\theta)\in\mathfrak{G}_{r_0,l}^{(0)}$

$$(2.28)\quad 2\int_{|x|<r_0} r^{2\beta-\nu}\exp\{2\alpha r^{-l}\}|v|^2dx \geqq \int_0^{r_0} r^{2\beta}\exp\{2\alpha r^{-l}\}\|v\|^2dr \geqq \frac{1}{2}\int_{|x|<r_0} r^{2\beta-\nu}\exp\{2\alpha r^{-l}\}|v|^2dx,$$

and for any $v\in\mathfrak{G}_{r_0,l}^{(|\mu|)}$ we have

$$(2.29)\quad r^{-2(m-|\mu|)}\int\left|\frac{\partial^{|\mu|}}{\partial x^{\mu}}v\right|^2 d\theta \leqq C_{10}\sum_{0\leqq i+j\leqq|\mu|} r^{-2(m-i)}\left\|\frac{\partial^{i}}{\partial r^{i}}\Lambda^{j}v\right\|^2 \quad (C_{10}>0).$$

From (2.25), (2.28) and (2.29), we get

$$(2.30)\quad \int_{|x|<r_0} r^{2\beta-\nu}\exp\{2\alpha r^{-l}\}|Lu_i|^2dx \geqq C_{11}\sum_{0\leqq|\mu|\leqq m-1} l^{2(m-|\mu|)} \int_{|x|<r_0} r^{2\beta-\nu-2(m-|\mu|)}\exp\{2\alpha r^{-l}\}\left|\frac{\partial^{|\mu|}}{\partial x^{\mu}}u_i\right|^2dx \quad (C_{11}>0).$$

In the above inequality we replace $2\beta-\nu$ by 2β and using (2.27) we get (2.19) for sufficiently large l. Q.E.D.

Corollary 1′. *Let L_i ($i=1,\cdots,s$) be elliptic differential polynomials of order m_i, and assume each of them satisfies the conditions of Theorem* 1′.

Then, there exist positive constants C' and l' such that

$$(2.31)\quad \int_{|x|<r_0} r^{2\beta}\exp\{2\alpha r^{-l}\}|L_1\cdot\cdots\cdot L_s u|^2dx$$
$$\geqq C'\sum_{0\leqq|\mu|\leqq M-s} l^{2(M-|\mu|)}\int_{|x|<r_0} r^{2\beta-2(m-|\mu|)}\exp\{2\alpha r^{-l}\}\left|\frac{\partial^{|\mu|}}{\partial x^{\mu}}u\right|^2dx$$
$$\left(M=\sum_{i=1}^{s}m_i,\ u\in\mathfrak{H}_{r_0,l}^{(M)}\right)$$

for every l ($\geqq l_0$) and sufficiently large α.

Proof. We can easily prove it by the method of the induction. Q.E.D.

§ 3. Uniqueness and unique continuation.

First we shall state the uniqueness of the Cauchy problem. Let $L(y, \eta)=\sum_{|\mu|\leqq m} a_\mu(y)\eta^\mu$ be a differential polynomial defined in a neighborhood of the origin in the $(\nu+1)$-dimensional Euclidean space.

We take Holmgren's transformation to $y=(y_1, \cdots, y_{\nu+1})$

$$t = y_1+\sum_{j=1}^{\nu} y_{j+1}^2,\ x_i = y_{i+1} \qquad (i=1, \cdots, \nu), \tag{3.1}$$

and we consider only the operator L such that after that transformation the principal polynomial of L is of the form a^*L_m ($|a^*|\geqq\delta>0$), where

$$L_m = L_m(t, x, \lambda, \xi) = \prod_{i=1}^{k} (\lambda-\lambda_i^{(1)}(t, x, \xi)) \prod_{j=1}^{m-k} (\lambda-\lambda_j^{(2)}(t, x, \xi)). \tag{3.2}$$
$$(0\leqq k\leqq m)$$

Theorem 2. *Let $L=L(y, \eta)=\sum_{|\mu|\leqq m} a_\mu(y)\eta^\mu$ be a differential polynomial of order m defined in a neighborhood of the origin of which leading coefficients are infinitely differentiable and remaining coefficients bounded measurable, and let $u=u(y)\in C_{(y)}^m$ defined in a neighborhood of the origin satisfy the differential equation $L\left(y, \frac{\partial}{\partial y}\right)u(y)=0$ and the initial conditions*

$$\frac{\partial^{j-1}}{\partial y_1^{j-1}}u(0, y_2, \cdots, y_{\nu+1}) = 0 \qquad (j=1, \cdots, m). \tag{3.3}$$

Suppose after the transformation (3.1) *the roots $\lambda_i^{(1)}=-q_i^{(1)}+ip_i^{(1)}$ $(i=1, \cdots, k)$ and $\lambda_j^{(2)}=-q_j^{(2)}+ip_j^{(2)}$ $(j=1, \cdots, m-k)$ of the associated polynomial L_m in* (3.2) *are distinct respectively and infinitely differentiable, and $p_i^{(1)}$ and $q_i^{(1)}$ $(i=1, \cdots, k)$ satisfy the condition* (2.3) *of M. Matsumura* [8], *and $p_j^{(2)}$ $(j=1, \cdots, m-k)$ do not vanish for $\xi\neq 0$.*

Then, $u(y)=u(t, x)$ vanishes identically in a neighborhood of the origin.

Proof. From the assumption of Theorem 2 $a^{*^{-1}}L$ as the operator with respect to (t, x) satisfies the assumptions of Theorem 1.

Now we take a function $\varphi(t)\in C_{(t)}^\infty$ such that

$$\varphi(t) = 1 \text{ on } \left[0, \frac{h}{2}\right],\quad \varphi(t) = 0 \text{ for } t\geqq\frac{2}{3}h, \tag{3.4}$$

then by (3.1) and (3.3) $w(t, x)=\varphi(t)u(t, x)$ belongs to $\mathfrak{F}_h^{(m)}$.

Applying (2.4) of Theorem 1 to $a^{*^{-1}}L$ and w and remarking $|a^*|\geqq\delta>0$ we get

$$(3.5)\qquad \int_0^h r^{-2n}\|Lw\|^2dt \geqq C_1 \sum_{0\leqq i+|\mu|=\tau\leqq m-1} h^{-2(m-\tau)}\int_0^h r^{-2n}\left\|\frac{\partial^\tau}{\partial t^i\partial x^\mu}w\right\|^2dt$$
$$(r=t+h)$$

for sufficiently large n and $C_1=\delta^{-2}C$.

By (3.4) $Lw=Lu=0$ for $t\in\left[0, \frac{h}{2}\right]$ and because of $h\leqq r\leqq 2h<1$ for $0\leqq t\leqq h$ we get

$$\int_{h/2}^h r^{-2n}\|Lw\|^2dt \geqq C_1\int_0^{h/2} r^{-2n}\|u\|^2dt.$$

Hence, noting $0<r^{-1}\leqq\left(\frac{h}{2}+h\right)^{-1}=\frac{2}{3}h^{-1}$ for $\frac{h}{2}\leqq r\leqq h$ and $r^{-1}\geqq\left(h+\frac{h}{3}\right)^{-1}$ $=\frac{3}{4}h^{-1}$ for $0\leqq r\leqq\frac{h}{3}$, we have

$$C_1^{-1}\left(\frac{8}{9}\right)^{2n}\int_{h/2}^h \|Lw\|^2dt \geqq \int_0^{h/3}\|u\|^2dt$$

and letting $n\to\infty$ we get u vanishes identically in $0\leqq t\leqq\frac{h}{3}$.

This completes the proof. Q.E.D.

EXAMPLE 1. $L_m(t, x, \lambda, \xi)=\lambda^8+2(\sum_{i=1}^{\nu}\xi_i^2)^2\lambda^4+(\sum_{i=1}^{\nu}\xi_i^2)^4-a(t, x)^2\sum_{i=1}^{\nu}\xi_i^8$, where $a(t, x)\in C^\infty(t, x)$ in a neighborhood of the origin and $a(0,0)=0$ but $a(t, x)\not\equiv 0$ in any neighborhood of the origin. We can write this operator

$$\begin{aligned} L_m &= \{\lambda^4+((\sum_i\xi_i^2)^2+a(t, x)(\sum_i\xi_i^8)^{1/2})\}\{\lambda^4+((\sum_i\xi_i^2)^2-a(t, x)(\sum_i\xi_i^8)^{1/2})\} \\ &= \prod_{i=1}^4(\lambda-\lambda_i^{(1)})\prod_{j=1}^4(\lambda-\lambda_j^{(2)})\equiv A_1A_2 \end{aligned}$$

where $\lambda_i^{(1)}=e^{\pi/4(2i-1)\sqrt{-1}}b_1$ $(i=1, \cdots, 4)$ and $\lambda_j^{(2)}=e^{\pi/4(2i-1)\sqrt{-1}}b_2$ $(i=1, \cdots, 4)$ with $b_1=((\sum_i\xi_i^2)^2+a(t, x)(\sum_i\xi_i^8)^{1/2})^{1/4}$ and $b_2=((\sum_i\xi_i^2)^2-a(t, x)(\sum\xi_i^8)^{1/2})^{1/4}$ respectively. Then, A_1 and A_2 have distinct roots respectively and infinitely differentiable, but at the origin $\lambda_i^{(1)}=\lambda_i^{(2)}$ $(i=1, \cdots, 4)$.

Hence, for the operator $L=L_m+\sum_{0\leqq i+|\mu|\leqq m-1} b_{i,\mu}(t, x)\lambda^i\xi^\mu$ the uniqueness of the Cauchy problem holds. We must note that we can not write L_m as the product of two differential operators; see L. Hörmander [6].

Corollary 2. *Let $L_i (i=1, \cdots, s)$ be differential polynomials of order m_i and each of them satisfy the conditions of Theorem 2.*

Then, if $u=u(y)$ satisfies the differential equation $L_1\cdots L_su$ $=\sum_{|\mu|\leqq M-s} a_\mu(y)\frac{\partial^{|\mu|}}{\partial y^\mu}u$ $(M=\sum_{i=1}^s m_i)$ in a neighborhood of the origin, and satisfies the initial conditions

$$\frac{\partial^{j-1}}{\partial y_1^{j-1}} u\ (0, y_2, \cdots, y_{\nu+1}) = 0 \qquad (j=1, \cdots, M),$$

then $u(y)$ vanishes identically in a neighborhood of the origin.

Next we shall prove the unique continuation theorem.

Theorem 2′. *Let $L=L(x, \eta)=\sum_{|\mu|\leqq m} a_\mu(x)\eta^\mu$ be an elliptic differential polynomial of order m which satisfies the conditions of Theorem 1′.*

Suppose $u=u(x)\in C^m_{(x)}$ satisfies the differential equation $Lu=0$ in a neighborhood of the origin, and

$$\lim_{r\to 0} \exp\{\alpha r^{-l}\} \frac{\partial^{|\mu|}}{\partial x^\mu} u(x) = 0 \quad \textit{for every } \alpha \ \ (|\mu|\leqq m,\ r=\{\sum_{i=1}^{\nu+1} x_i^2\}^{1/2})$$

for sufficiently large l for which we can apply Theorem 1′.

Then, $u=u(x)$ vanishes identically in a neighborhood of the origin.

Proof. We take a function $\varphi(x)\in C^\infty_{0(|x|<r_0)}$ such that $\varphi(x)=1$ on $\left\{x\,;|x|<\frac{r_0}{2}\right\}$, then $w(x)=(\varphi u)(x)$ belongs to $\mathfrak{H}^{(m)}_{r_0, l}$.

Hence by the same process with the proof of Theorem 2 we can derive an inequality

$$\int_{r_0/2\leqq |x|<r_0} \exp\{2\alpha r^{-l}\}\,|Lw|^2 dx \geqq C_1 \int_{|x|\leqq r_0/3} \exp\{2\alpha r^{-l}\}\,|u|^2 dx \quad (C_1>0)$$

and letting $\alpha\to\infty$ we have u vanishes identically in $\left\{x\,;|x|\leqq\frac{r_0}{3}\right\}$. Q.E.D.

EXAMPLE 2. a) $A(x, \eta)=\prod_{i=1}^{s}(\eta_1^2+a_i(x)\eta_2^2)$ $(a_i(x)>0\,;\ i=1, \cdots, s)$ where $a_i(x)\in C^\infty_{(x)}$ and $a_i(x)\neq a_j(x)$ for $i\neq j$ in a neighborhood of the origin in $(x)=(x_1, x_2)$-space. Then, the associated operator A^*_m in (2. 17) for A has distinct roots in any representation of polar coordinates, hence for the operator $L=A^2+\sum_{|\mu|\leqq 4s-1} b_\mu(x)\eta^\mu$ the unique cotinuation theorem holds.

b) $$\begin{aligned} L &\equiv \Delta_1^2+\varepsilon^2(\Delta_2^2+\Delta_3^2)-2\varepsilon(\Delta_1\Delta_2+\Delta_2\Delta_3+\Delta_3\Delta_1) \\ &= \{\Delta_1-\varepsilon(\sqrt{\Delta_2}+\sqrt{\Delta_3})^2\}\{\Delta_1-\varepsilon(\sqrt{\Delta_2}-\sqrt{\Delta_3})^2\} \equiv A_1A_2 \\ &\quad (\Delta_j=\eta_1^2+j\eta_2^2\,;\ j=1, 2, 3 \ \text{ and } \ \varepsilon=\varepsilon(x_1, x_2)\in C^\infty_{(x)}). \end{aligned}$$

By the remark of a), after any orthogonal transformation $\frac{\partial}{\partial\eta_1}\sqrt{\Delta_j} = \frac{1}{2\sqrt{\Delta_j}}\frac{\partial}{\partial\eta_1}\Delta_j$ $(j=2, 3)$ are bounded in a neighborhood of $(\eta_1, \eta_2)=(\pm i, \pm 1)$, so that for sufficiently small ε the roots of $A_j=0$ $(j=1, 2)$ are distinc and belong to $C^\infty_{(x)}$ because of $\frac{\partial}{\partial\eta_1}A_j\neq 0$ at $A_j=0$ respectively.

Hence, for L Theorem 2′ holds, but we can not represent L as the product of two second order elliptic polynomials.

Corollary 2′. *Let L_i $(i=1, \cdots, s)$ be elliptic differential polynomials of order m_i which satisfy the conditions of Theorem* 1′.

Suppose $u=u(x)$ satisfies a differential equation $L_1\cdot\cdots\cdot L_s u = \sum_{|\mu|\leqq M-s} b_\mu(x)\frac{\partial^{|\mu|}}{\partial x^\mu}u$ $(M=\sum_{i=1}^{s} m_i)$ in a neighborhood of the origin, and satisfies $\lim_{r\to 0}\exp\{\alpha r^{-l}\}\frac{\partial^{|\mu|}}{\partial x^\mu}u(x)=0$ $(|\mu|\leqq M)$ for every α and sufficiently large l for which we can apply Theorem 1′ for each L_i $(i=1, \cdots, s)$.

Then, $u=u(x)$ vanishes identically in a neighborhood of the origin.

EXAMPLE 3. Let L_i $(i=1, \cdots, s)$ be elliptic differential polynomials of order 2 with real valued leading coefficients and sufficiently smooth remaining ones.

In this case the principal parts of L_i have distinct roots for every direction respectively.

Then, by the remark 1 in the chapter 2, each pair $L_{2j-1}L_{2j}$ $\left(1\leqq j\leqq\left[\frac{s}{2}\right]\right)$ satisfies the conditions of Theorem 1′, consequently for the operator $L=L_1\cdot\cdots\cdot L_s+\sum_{|\mu|\leqq[3/2s]} b_\mu(x)\eta^\mu$ the unique continuation theorem holds; see [9] and [12].

Finary we shall state the local existence theorem for the operator concerning Theorem 1.

Theorem 3. *Let $L^{(1)}=L^{(1)}(t, x, \lambda, \xi)$ be an elliptic differential polynomial of order m and $L_i^{(2)}=L_i^{(2)}(t, x, \lambda, \xi)$ $(i=1, \cdots, s)$ be differential polynomials of order m_i which satisfy the conditions of Theorem* 1.

Set $L^{(2)}=L_1^{(2)}\cdot\cdots\cdot L_s^{(2)}+\sum_{i+|\mu|\leqq M-s} b_{i,\mu}(t, x)\lambda^i\xi^\mu$ $(M=\sum_{i=1}^{s} m_i)$ and $L=L^{(1)}L^{(2)}+\sum_{i+|\mu|\leqq M+m-s} a_{i,\mu}(t, x)\lambda^i\xi^\mu$, and suppose the coefficients are sufficiently smooth.

Then, the equation $L\left(t, x, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)u=f$ has, for any $f\in L^2(\Omega)$ (Ω is a sufficiently small neighborhood of the origin) at least one maximal solution u in the sense of L. Hörmander [5], *that is $u\in L^2[\Omega]$ and*

$$(f, v)=(u, L^*v) \quad \text{for any} \quad v\in C_0^\infty(\Omega). \tag{3.6}$$

Proof. The conditions of Theorem 1 are determined by the principal parts of $L_i^{(2)}$ $(i=1, \cdots, s)$, so that the formal adjoint polynomials $L_i^{(2)*}$ of $L_i^{(2)}$ satisfy the conditions of Theorem 1 respectively. Hence we can apply Corollary 1 to $(L_1^{(2)}\cdot\cdots\cdot L_s^{(2)})^*=L_s^{(2)*}\cdot\cdots\cdot L_1^{(2)*}$.

Remarking the condition $u\in\mathfrak{F}_h^{(M)}$ is required so that the boundary value may vanish together with its derivatives in integrating by parts, we get for sufficiently small domain $\Omega_h(\subset\{(t,x)\,;\ t^2+|x|^2<h^2/4\})$,

$$\int_{\Omega_h} r^{-2n}|(L_1^{(2)}\cdot\cdots\cdot L_s^{(2)})^* L^{(1)*}v|^2dtdx\geqq C_1\sum_{i+|\mu|=\tau\leqq M-s}h^{-2(M-\tau)}$$
$$\int_{\Omega_h} r^{-2n}\left|\frac{\partial^\tau}{\partial t^i\partial x^\mu}L^{(1)*}v\right|^2dtdx \quad (C_1>0,\ v\in C_0^\infty(\Omega_h)).$$

Remarking $|(L^{(2)*}-(L_1^{(2)}\cdot\cdots\cdot L_s^{(2)})^*)L^{(1)*}v|^2\leqq C_2\sum_{i+|\mu|=\tau\leqq M-s}\left|\frac{\partial^\tau}{\partial t^i\partial x^\mu}L^{(1)*}v\right|^2$, if we take domain $\Omega_{h,n}$ such as $\left(\frac{h+t_1}{h+t_2}\right)^{2n}\geqq\frac{1}{2}$ for $(t_i,\ x)\in\Omega_{h,n}$ $(i=1,2)$, then

$$(3.7)\quad \int_{\Omega_{h,n}}|L^{(2)*}L^{(1)*}v|^2dtdx\geqq\frac{1}{3}C_1\sum_{i+|\mu|=\tau\leqq M-s}h^{-2(M-\tau)}\int_{\Omega_{h,n}}\left|\frac{\partial^\tau}{\partial t^i\partial x^\mu}L^{(1)*}v\right|^2dtdx$$
$$\geqq C_3\sum_{i+|\mu|=\tau\leqq M-s}h^{-2(M-\tau)}\int_{\Omega_{h,n}}\left|L^{(1)*}\frac{\partial^\tau}{\partial t^i\partial x^\mu}v\right|^2dtdx$$
$$-C_4\sum_{i'+|\mu'|=\tau'\leqq m+\tau-1}h^{-2(M-\tau)}\int_{\Omega_{h,n}}\left|\frac{\partial^{\tau'}}{\partial t^{i'}\partial x^{\mu'}}v\right|^2dtdx$$
$$\equiv I_1-I_2\quad (C_3,\ C_4>0).$$

By Gålding's inequality [4] and (1.3) of L. Hörmander [7] we get

$$(3.8)\quad I_1\geqq C_5\sum_{i+|\mu|=\tau\leqq M-s}h^{-2(M-\tau)}\sum_{i'+|\mu'|=\tau'\leqq m}h^{-2(m-\tau')}\int_{\Omega_{h,n}}\left|\frac{\partial^{\tau+\tau'}}{\partial t^{i+i'}\partial x^{\mu+\mu'}}v\right|^2dtdx$$
$$\geqq C_6\sum_{i+|\mu|=\tau\leqq M+m-s}h^{-2(M+m-\tau)}\int_{\Omega_{h,n}}\left|\frac{\partial^\tau}{\partial t^i\partial x^\mu}v\right|^2dtdx\quad (C_5,\ C_6>0),$$

and for I_2, remarking $M-\tau\leqq M+m-\tau'-1$ we get

$$(3.9)\quad I_2\leqq C_7h^2\sum_{i'+|\mu'|=\tau'\leqq M+m-s}h^{-2(M+m-\tau')}\int_{\Omega_{h,n}}\left|\frac{\partial^{\tau'}}{\partial t^{i'}dx^{\mu'}}v\right|^2dtdx.$$

Hence, from (3.7)-(3.9) and $|(L^*-L^{(2)*}L^{(1)*})v|^2\leqq C_8\sum_{i+|\mu|\leqq M+m-s}\left|\frac{\partial^{i+|\mu|}}{\partial t^i\partial x^\mu}v\right|^2$ we get for sufficiently small $h(>0)$

$$\int_{\Omega_{h,n}}|L^*v|^2dtdx\geqq C_9\sum_{i+|\mu|=\tau\leqq M+m-s}h^{-2(M+m-\tau)}\int_{\Omega_{h,n}}\left|\frac{\partial^\tau}{\partial t^i\partial x^\mu}v\right|^2dtdx$$
$$\geqq C_9h^{-2(M+m)}\int_{\Omega_{h,n}}|v|^2dtdx\quad (C_9>0,\ v\in C_0^\infty(\Omega_{h,n})).$$

This shows $L^{*^{-1}}$ is bounded, and by Lemma 1.7 of L. Hörmander [5] proves the existence theorem of maximal solutions for $Lu=f$ in $\Omega_{h,n}$ (h, n ; fixed). Q.E.D.

§4. Appendix. Let $H=\sum_{r=0}^{\infty} a_r h_r$ be a singular integral operator in the sense of M. Yamaguti such that for every μ $(0 \leqq |\mu| \leqq k)$

$$\left|\frac{\partial^{|\mu|}}{\partial x^{\mu}} a_0(x)\right| \leqq A_{k,l}, \quad \left|\frac{\partial^{|\mu|}}{\partial x^{\mu}} a_r(x)\right| \leqq A_{k,l} r^{-l} \quad (r=1, 2, \cdots);$$
$$\tilde{h}_0(\xi) = 1, \quad \left|\frac{\partial^{|\mu|}}{\partial \xi^{\mu}} \tilde{h}_r(\xi)\right| \leqq B_k r^{l'_k} |\xi|^{-|\mu|} \quad (r=1, 2, \cdots) \tag{4.1}$$

whose meaning is stated in Definition 0 of §1.

We consider a convolution operator α defined by $\widetilde{\alpha u} = \tilde{\alpha}(\xi)\tilde{u}(\xi)$ $(u \in L^2)$ where $\tilde{\alpha}(\xi)$ is an infinitely differentiable function such that

$$\tilde{\alpha}(\xi) = 0 \quad \text{on} \quad \{\xi ; |\xi| \leqq 1\}, \tag{4.2}$$

and for every k there exists a constant B'_k such that

$$\left|\frac{\partial^{|\mu|}}{\partial \xi^{\mu}} \tilde{\alpha}(\xi)\right| \leqq B'_k |\xi|^{-|\mu|} \quad (0 \leqq |\mu| \leqq k). \tag{4.3}$$

Tnen, setting $\Xi_\delta = \{x ; |x| < \delta\}$ $(\delta > 0)$ we have the next

Lemma 6. *Let H be a singular integral operator in the sense of M. Yamaguti and α is a convolution operator which satisfies* (4.2) *and* (4.3).

Suppose $\sigma(H) = \sum_{r=0}^{\infty} a_r(x) \tilde{h}_r(\xi) = 0$ for $x \in \Xi_{2\delta}$ and $\xi \in car.\ \tilde{\alpha}(\xi)$. Then, for every non-negative integer p there exists a constant C depending only on H, α, p, ν and δ such that

$$\|H \Lambda^p \alpha u\|_{L^2} \leqq C \|u\|_{L^2} \quad \text{for } u \in C_0^p(\Xi_\delta). \tag{4.4}$$

Proof. Take a function $\varphi(x) \in C_0^\infty(\Xi_{2\delta})$ such that $\varphi(x) = 1$ for $x \in \Xi_\delta$. Then, for $u \in C_0^\infty(\Xi_\delta)$ we have

$$H\Lambda^p \alpha u = \sum_{r=0}^{\infty} a_r((h_r \Lambda^p \alpha)\varphi - \varphi(h_r \Lambda^p \alpha))u + \sum_{r=0}^{\infty} a_r \varphi(h_r \Lambda^p \alpha) u$$
$$= \sum_{r=0}^{\infty} a_r(x) \int (h_r \Lambda^p \alpha)(x-y)(\varphi(y) - \varphi(x)) u(y)\, dy + \varphi H \alpha \Lambda^p u$$
$$\text{(in the distribution's sense)}$$
$$= \sum_{r=0}^{\infty} a_r(x) \Big\{ \sum_{1 \leqq |\mu| \leqq k-1} (-1)^{|\mu|} \frac{\partial^{|\mu|}}{\partial x^{\mu}} \varphi(x) \int \frac{(x-y)^{\mu}}{\mu!} (h_r \alpha \Lambda^p)(x-y) u(y)\, dy$$
$$+ \sum_{|\mu|=k} \int (x-y)^{\mu} (h_r \Lambda^p \alpha)(x-y) \varphi_\mu(x, y) u(y)\, dy \Big\} + \varphi H \alpha \Lambda^p u$$

$$= \sum_{0\leq|\mu|\leq k-1} C_\mu \int e^{ix\cdot\xi}\frac{\partial^{|\mu|}}{\partial\xi^\mu}\Big(\frac{\partial^{|\mu|}}{\partial x^\mu}\varphi(x)\sigma(H)\tilde{\alpha}(\xi)|\xi|^p\Big)\tilde{u}(\xi)\,d\xi$$
$$+\sum_{r=0}^{\infty} a_r(x)\sum_{|\mu|=k}\int(x-y)^\mu(h_r\alpha\Lambda^p)(x-y)\varphi_\mu(x,y)u(y)\,dy\,.$$

From the assumption of $\sigma(H)$ and $\varphi\in C_0^\infty(\Xi_{2\delta})$ we have

$$\frac{\partial^{|\mu|}}{\partial x^\mu}\varphi(x)\sigma(H)\tilde{\alpha}(\xi)=0\,,$$

hence the first term vanishes, and by an well known theorem for the convolution operator, i.e. $\|v*u\|_{L^p}\leq\|v\|_{L^1}\cdot\|u\|_{L^p}$ for $v\in L^1$ and $u\in L^p$ ($p\geq 1$), we have

$$(4.5)\quad \|H\Lambda^p\alpha u\|_{L^2}\leq\sum_{r=0}^{\infty}\operatorname{Max}_x|a_r(x)|\sum_{|\mu|=k}\operatorname{Max}_{x,y}|\varphi_\mu(x,y)|\,\|x^\mu(h_r\alpha\Lambda^p)(x)\|_{L^1}\cdot\|u\|_{L^2}.$$

Now we consider $x^\mu(h_r\alpha\Lambda^p)(x)$ ($|\mu|=k$).
Since $\mathfrak{F}[x^\mu(h_r\alpha\Lambda^p)(x)](\xi)=i^k\frac{\partial^k}{\partial\xi^\mu}(\tilde{h}_r(\xi)\tilde{\alpha}(\xi)|\xi|^p)$,
we have by (4.1)-(4.3)

$$\mathfrak{F}[x^\mu(h_r\alpha\Lambda^p)](\xi)=0\quad\text{on}\quad\{\xi\,;|\xi|\leq 1\}$$

and
$$|\mathfrak{F}[x^\mu(h_r\alpha\Lambda^p)(x)](\xi)|\leq C_{p,k}r^{l'_k}B_kB'_k|\xi|^{p-k}\,.$$

We take $k=p+\nu+1$, then for every x

$$|x^\mu(h_r\alpha\Lambda^p)(x)|\leq\frac{1}{\sqrt{2\pi}^\nu}\Big|\int_{|\xi|\geq1}e^{ix\cdot\xi}\mathfrak{F}[x^\mu(h_r\alpha\Lambda^p)(x)](\xi)\,d\xi\Big|\leq C_{p,k,\nu,\alpha}r^{l'_k}B_k\,,$$

and for x ($|x|\geq 1$)

$$|x^\mu(h_r\alpha\Lambda^p)(x)| = |x|^{-2([\nu/2]+1)}|\,|x|^{2([\nu/2]+1)}(h_r\alpha\Lambda^p)(x)|$$
$$\leq|x|^{-2([\nu/2]+1)}\frac{1}{\sqrt{2\pi}^\nu}\int_{|\xi|\geq1}|\Delta_\xi^{([\nu/2]+1)}\frac{\partial^k}{\partial\xi^\mu}(\tilde{h}_r(\xi)\tilde{\alpha}(\xi)|\xi|^p)|\,d\xi$$
$$\leq C_{p,k',\nu,\alpha}r^{l'_k}B_{k'}|x|^{-2([\nu/2]+1)}\qquad\Big(|\mu|=k,\ k'=k+2\Big(\Big[\frac{\nu}{2}\Big]+1\Big)\Big),$$

so that we have

$$(4.6)\qquad \|x^\mu(h_r\alpha\Lambda^p)(x)\|_{L^1}\leq C_{p,k',\nu,\alpha}r^{l'_{k'}}B_{k'}\,.$$

In (4.1) we take $l=l'_{k'}+2$ then by (4.5) and (4.6)

$$\|H\Lambda^p\alpha u\|_{L^2}\leq C_{p,k',\nu,\alpha}A_{0,l'_{k'}}B_{k'}(1+\sum_{r=1}^{\infty}r^{-2})\|u\|_{L^2}\leq C\|u\|_{L^2}\,.\quad\text{Q.E.D.}$$

Set $\Omega_{r_0}=\{(t,x)\,;\,t^2+|x|^2<r_0^2\}$ and $S_{(s)}=S_{(s)}^{(\delta)}=\{\xi'\,;\,|\xi'-\xi'_{(s)}|<\delta\}$. Then, by the compactness of $S=\{\xi' ; |\xi'|=1\}$ there exist positive constants r_0 and δ such that we have the representation (0. 2) in each $S_{(s)}=S_{(s)}^{(\delta)}$ $(s=1, \cdots, p)$ and in Ω_{3r_0}, and $S\subset\sum_{s=1}^{p} S_{(s)}$.

Now we take $\psi(t, x)\in C_0^\infty(\Omega_{3r_0})$ such that

$$(4.7)\qquad 1\geqq\psi(t, x)\geqq 0,\ \psi(t, x)=1 \quad \text{for} \quad (t, x)\in\Omega_{2r_0},$$

and for $a_\mu^*(t, x)=\psi(t, x)a_{i,\mu}(t, x)+(1-\psi(t, x))a_{i,\mu}(0, 0)$ $(i+|\mu|=m)$ consider the associated polynomial $L_m^*(t, x, \lambda, \xi)=\sum_{i+|\mu|=m} a_\mu^*(t, x)\xi^\mu\lambda^i$.

Then, we have

$$(4.8)\qquad L_m\Big(t, x, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}\Big)u=L_m^*\Big(t, x, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}\Big)u \quad \text{for} \quad u\in C_0^m(\Omega_{2r_0}),$$

and we can represent L_m^* as the form

$$(4.9)\qquad L_m^*=\sum_{j=0}^{m} H_j^*\Lambda^j\frac{\partial^{m-j}}{\partial t^{m-j}}$$

where H_j^* are singular integral operators of type C_β^∞ $(\beta=\infty)$ with $\sigma(H_j^*)=i^j\sum_{|\mu|=j} a_\mu^*(t, x)\xi^\mu|\xi|^{-j}$ in the sense of [2].

According to $S_{(s)}$ $(s=1, \cdots, p)$ we take the following real valued functions $\alpha_s'(\xi')$ $(s=1, \cdots, p)$ and $\beta(\xi')$ such that

$$(4.10)\qquad \begin{gathered}\alpha_s'(\xi')\in C_0^\infty(S_{(s)})\ (s=1, \cdots, p),\ \sum_{s=1}^{p}\alpha_{(s)}'^2(\xi')=1;\\ \beta(\xi)\in C_{(\xi)}^\infty,\ \begin{cases}\beta(\xi)=0 & \text{for } \xi\ (|\xi|\leqq 1)\\ 0<\beta(\xi)<1 & \text{for } \xi\ (1<|\xi|<2)\\ \beta(\xi)=1 & \text{for } \xi\ (|\xi|\geqq 2).\end{cases}\end{gathered}$$

Setting

$$(4.11)\qquad \begin{aligned}\tilde\alpha_0(\xi)&=(1-\beta(\xi)^2)^{1/2},\\ \tilde\alpha_s(\xi)&=\beta(\xi)\alpha_s'(\xi|\xi|^{-1})\quad(s=1, \cdots, p)\end{aligned}$$

we consider the convolution operators α_s defined by

$$(4.12)\qquad \alpha_s;\ \widetilde{\alpha_s u}=\tilde\alpha_s(\xi)\tilde u(\xi)\ (s=0, \cdots, p) \quad \text{for} \quad u\in L^2,$$

then α_s $(s=1, \cdots, p)$ satisfy the conditions (4. 2) and (4. 3), and

$$(4.13)\qquad ||u||^2=\sum_{s=0}^{p}||\alpha_s u||^2 \quad \text{for} \quad u\in L^2.$$

For each α_s' $(s=1, \cdots, p)$ we take $\gamma_s'(\xi')\in C_0^\infty(S_{(s)})$ such that $\gamma_s'(\xi')=1$ on car. $\alpha'_s(\xi')$, and set $\gamma_s(\xi)=\gamma'_s(\xi|\xi|^{-1})$. Now we write $L_m(t, x, \lambda, \xi)$ simply $L_m=\prod_{j=1}^{m}(\lambda-\lambda_j(t, x, \xi))$. We define

$$\lambda_j^*(t, x, \xi) = \psi(t, x)\lambda_j(t, x, \xi)+(1-\psi(t, x))\lambda_j^*(0, 0, \xi)\,,$$
$$\lambda_{j,s}^*(t, x, \xi) = \gamma_s(\xi)\lambda_j^*(t, x, \xi)+(1-\gamma_s(\xi))\lambda_j^*(t, x, \xi'_{(s)}|\xi|) \qquad (s=1, \cdots, p)\,,$$

then $\lambda_{j,s}^* \in C^{\infty}_{(t,x,\xi)}$ for $\xi \neq 0$ and are homogeneous of order 1 with respect to ξ.

Set $L_s^*(t, x, \lambda, \xi)=\prod_{j=1}^{m}(\lambda-\lambda_{j,s}^*) = \sum_{j=0}^{m} h_{j,s}^*(t, x, \xi)|\xi|^j\lambda^{m-j}$ and define the associated operator $L_{m,s}^*$ by

$$L_{m,s}^* = \sum_{j=0}^{m} H_{j,s}^*\Lambda^j \frac{\partial^{m-j}}{\partial t^{m-j}} \qquad (s=1, \cdots, p) \tag{4.14}$$

where $H_{j,s}^*$ are singular integral operators with $\sigma(H_{j,s}^*)=i^j h_{j,s}^*$ which are of type C_β^∞ $(\beta=\infty)$ in the sense of A. P. Calderón and A. Zygmund [2]. Then, by the definition it follows that

$$\text{(4.15)}\qquad \begin{aligned} &H_{0,s}^* = H_0^* = 1\,, \\ &\sigma(H_{j,s}^*) = \sigma(H_j^*) \quad \text{for} \quad (t, x) \in \Omega_{2r_0},\ \xi \in \text{car. } \tilde{\alpha}_s(\xi) \qquad (j=1, \cdots, p)\,. \end{aligned}$$

Taking the number p sufficiently large we may assume $L_s^*(t, x, \lambda, \xi)$ have the form (0.2) on the whole unit sphere and for every (t, x), and the condition (0.3) of M. Matsumura is satisfied for $(t, x) \in \Omega_{2r_0}$ and $\xi \in$ car. $\tilde{\alpha}_s(\xi)$.

Theorem 4. *Let differential operators in* (0.1) *and* (0.4) *satisfy the condition stated in* § 0. *Introduction respectively. Then, the inequalities* (2.4) *of Theorem* 1 *and* (2.9) *of Theorem* 1′ *hold respectively.*

Proof. We shall prove the theorem only for the operator in (0.1), the proof for the operator in (0.4) is played quite similarly.

Let a function $u=u(t, x)$ be of class $\mathfrak{F}_{h,K}^{(m)}$ $(h^2+K^2<r_0^2)$. We consider $\alpha_s u$ $(s=1, \cdots, p)$ defined by (4.12) and for each $\alpha_s u$ we operate $L_{m,s}^*$ defined by (4.14).

Considering the process of the construction of $L_{m,s}^*$ we can write the associated polynomials $L_{m,s}^*(t, x, \lambda, \xi)$ as

$$L_{m,s}^*(t, x, \lambda, \xi) = \prod_{i=1}^{k}(\lambda-\lambda_{i,s}^{(1)}(t, x, \xi)) \prod_{j=1}^{m-k}(\lambda-\lambda_{j,s}^{(2)}(t, x, \xi))$$

so that $\lambda_{i,s}^{(1)}$ and $\lambda_{j,s}^{(2)}$ may satisfy the conditions of Theorem 1 for every

(t, x, ξ) $(\xi \neq 0)$, but the condition (0.3) or (2.3) of M. Matsumura is satisfied only for $(t, x) \in \Omega_{2r_0}$ and $\xi \in \text{car. } \tilde{\alpha}_s(\xi)$.

Now, we consider the operators $J_{i,s}^{(1)} = \frac{\partial}{\partial t} + (P_{i,s}^{(1)} + iQ_{i,s}^{(1)})\Lambda$ $(i=1, \cdots, k)$ and $J_{j,s}^{(2)} = \frac{\partial}{\partial t} + (P_{j,s}^{(2)} + iQ_{j,s}^{(2)})\Lambda$ $(j=1, \cdots, m-k)$ where $P_{i,s}^{(1)} + iQ_{i,s}^{(1)}$ and $P_{j,s}^{(2)} + iQ_{j,s}^{(2)}$ are singular integral operators with the symbols $-i\lambda_{i,s}^{(1)}|\xi|^{-1}$ and $-i\lambda_{j,s}^{(2)}|\xi|^{-1}$ respectively.

Then, by Lemma 3 and Lemma 6 we get for $u \in \mathfrak{F}_{h,K}^{(1)}$

$$\int_0^h r^{-2n}\|J_{j,s}^{(1)}\alpha_s u\|^2 dt \geqq \frac{1}{8} h^{-2} n \int_0^h r^{-2n}\{\|\alpha_s u\|^2 - C_1 h^2 \|u\|^2\}\, dt$$
$$(s=1, \cdots, p;\ i=1, \cdots, k_s)$$

and for a positive constant C_2

$$\int_0^h r^{-2n}\|J_{j,s}^{(2)}\alpha_s u\|^2 dt \geqq C_2\{h^{-2}n\int_0^h r^{-2n}\|\alpha_s u\|^2 dt + \frac{1}{n}\int_0^h r^{-2n}\left\{\left\|\frac{\partial}{\partial t}\alpha_s u\right\|^2 + \|\Lambda \alpha_s u\|^2\right\} dt$$
$$(s=1, \cdots, p;\ j=1, \cdots, m-k_s).$$

Using the above inequalities we proceed the same step with the proofs of Lemma 5 and Theorem 1, then we get

$$\int_0^h r^{-2n}\|L_{m,s}^* \alpha_s u\|^2 dt \geqq C_3 \sum_{i+|\mu|=\tau \leqq m-1} h^{-2(m-\tau)}$$
$$\int_0^h r^{-2n}\left\{\left\|\frac{\partial^\tau}{\partial t^i \partial x^\mu}\alpha_s u\right\|^2 - C_4 h^2 \left\|\frac{\partial^\tau}{\partial t^i \partial x^\mu} u\right\|^2\right\} dt$$
$$(s=1, \cdots, p;\ C_3, C_4 > 0;\ u \in \mathfrak{F}_{h,K}^{(m)}).$$

We write $\alpha_s L_m u$ $(s=1, \cdots, p)$ as

$$\alpha_s L_m u = \alpha_s L_m^* u = (\alpha_s L_m^* - L_m^* \alpha_s) u + (L_m^* - L_{m,s}^*)\alpha_s u + L_{m,s}^* \alpha_s u,$$

then estimating $(\alpha_s L_m^* u - L_m^* \alpha_s) u$ by (1.2) and $(L_m^* - L_{m,s}^*)\alpha_s u$ by Lemma 6 we get important inequalities

$$\int_0^h r^{-2n}\|\alpha_s L_m u\|^2 dt \geqq C_5 \sum_{i+|\mu|=\tau \leqq m-1} h^{-2(m-\tau)}$$
$$\int_0^h r^{-2n}\left\{\left\|\frac{\partial^\tau}{\partial t^i \partial x^\mu}\alpha_s u\right\|^2 - C_6 h^2 \left\|\frac{\partial^\tau}{\partial t^i \partial x^\mu} u\right\|^2\right\} dt \tag{4.16}$$
$$(s=1, \cdots, p;\ C_5,\ C_6 > 0;\ u \in \mathfrak{F}_{h,K}^{(m)}).$$

On the other hand we have for $\alpha_0 L_m$ and $u \in \mathfrak{F}_{h,K}^{(m)}$

$$\alpha_0 L_m u = \alpha_0 L_m^* u = \alpha_0 \frac{\partial^m}{\partial t^m} u + \alpha_0 \sum_{j=1}^m H_j^* \Lambda^j \frac{\partial^{m-j}}{\partial t^{m-j}} u$$

and

$$\alpha_0 \sum_{j=1}^{m} H_j^* \Lambda^j \frac{\partial^{m-j}}{\partial t^{m-j}} u = \sum_{j=1}^{m} \alpha_0 (H_j^* \Lambda - \Lambda H_j^*) \Lambda^{j-1} \frac{\partial^{m-j}}{\partial t^{m-j}} u + \alpha_0 \Lambda \sum_{j=1}^{m} \Lambda^{j-1} \frac{\partial^{m-j}}{\partial t^{m-j}} u \, .$$

Since $\alpha_0(H_j^*\Lambda - \Lambda H_j^*)$ and $\alpha_0\Lambda$ are bounded operators we have for a constant C_7

$$\left\| \alpha_0 \sum_{j=1}^{m} H_j^* \Lambda^j \frac{\partial^{m-j}}{\partial t^{m-j}} u \right\|^2 \leqq C_7 \sum_{i+|\mu|=m-1}^{m} \left\| \frac{\partial^{m-1}}{\partial t^i \partial x^\mu} u \right\|^2 .$$

As a special case of Lemma 3 ($P=Q=0$) we get

$$\int_0^h r^{-2n} \left\| \alpha_0 \frac{\partial^m}{\partial t^m} u \right\|^2 dt = \int_0^h r^{-2n} \left\| \frac{\partial}{\partial t} \left(\frac{\partial^{m-1}}{\partial t^{m-1}} \alpha_0 u \right) \right\|^2 dt \geqq C_8 n h^{-2}$$

$$\int_0^h r^{-2n} \left\| \frac{\partial^{m-1}}{\partial t^{m-1}} \alpha_0 u \right\|^2 dt \quad (C_8 > 0)$$

and so on we get

$$(4.17) \qquad \int_0^h r^{-2n} \| \alpha_0 L_m u \|^2 dt \geqq C_9 \sum_{i=0}^{m-1} h^{-2(m-i)} \int_0^h r^{-2n} \left\| \frac{\partial^i}{\partial t^i} \alpha_0 u \right\|^2 dt$$

$$- C_{10} \sum_{i+|\mu|=m-1} \int_0^h r^{-2n} \left\| \frac{\partial^{m-1}}{\partial t^i \partial x^\mu} u \right\|^2 dt \quad (C_9, \ C_{10} > 0) \, .$$

By (4.13) we get $\|L_m u\|^2 = \sum_{s=0}^{n} \|\alpha_s L_m u\|^2$, and since $\left\| \frac{\partial^{i+|\mu|}}{\partial t^i \partial x^\mu} \alpha_0 u \right\|^2 = \left\| \tilde{\alpha}_0(\xi) \xi^\mu \frac{\partial^i}{\partial t^i} \tilde{u}(t, \xi) \right\|^2 \leqq C_\mu \left\| \frac{\partial^i}{\partial t^i} u \right\|^2$ we get for i and μ ($i + |\mu| = \tau$)

$$\left\| \frac{\partial \tau}{\partial t^i \partial x^\mu} u \right\|^2 = \sum_{s=0}^{p} \left\| \alpha_s \frac{\partial \tau}{\partial t^i \partial x^\mu} u \right\|^2$$

$$= \sum_{s=0}^{n} \left\| \frac{\partial \tau}{\partial t^i \partial x^\mu} \alpha_s u \right\|^2 \leqq \sum_{s=1}^{p} \left\| \frac{\partial \tau}{\partial t^i \partial x^\mu} \alpha_s u \right\|^2 + \left\| \frac{\partial \tau}{\partial t^\tau} \alpha_0 u \right\|^2 + C_\tau \sum_{0 \leqq j < \tau} \left\| \frac{\partial^j}{\partial t^j} u \right\|^2 .$$

Hence, combining (4.16) and (4.17), and remarking $\|(L - L_m)u\|^2 \leqq C_{12} \sum_{i+|\mu| \leqq m-1} \left\| \frac{\partial^{i+|\mu|}}{\partial t^i \partial x^\mu} u \right\|^2$ we get

$$(4.18) \quad \int_0^h r^{-2n} \|Lu\|^2 dt \geqq C_{13} \sum_{0 \leqq i+|\mu|=\tau \leqq m-1} h^{-2(m-\tau)} \int_0^h r^{-2n} (1 - C_{14} h^2) \left\| \frac{\partial \tau}{\partial t^i \partial x^\mu} u \right\|^2 dt$$

$$(r = t + h \, ; \ C_{13}, \ C_{14} > 0 \, ; \ u \in \mathfrak{F}_{h,K}^{(m)}) \, ,$$

so that we get (2.4) of Theorem 1 for sufficiently small fixed h. Q.E.D.

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(Received March 5, 1962)

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