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ON FIXED POINT FREE INVOLUTIONS OF $S^1 \times S^2$

Dedicated to Professor K. Shoda on his sixtieth birthday

By

YOKO TAO

Introduction

In 1958, J. H. C. Whitehead [10] generalized the sphere theorem by C. D. Papakyriakopoulos in the following way:

WHITEHEAD'S SPHERE THEOREM. *Let M be an orientable 3-manifold, compact or not, with boundary which may be empty, such that* $\pi_2(M) \neq 0$. *Then there exists a 2-sphere S semi-linearly embedded in M, such that* $S \not\simeq 0$[1] *in M.*

As the example $S^1 \times P^2$[2] (S^k means k-sphere) shows, the above sphere theorem does not hold generally for non-orientable 3-manifolds. Therefore it remains as a question that for what 3-manifolds the sphere theorem does not hold? This problem naturally leads to the fixed point free involution (homeomorphism on itself of order 2) of $S^1 \times S^2$ as Theorem 2 of § 3 in this paper shows.

The main purpose of this paper is to prove the following

Theorem 1. *If T is a fixed point free involution of* $S^1 \times S^2$, *and if M is the 3-manifold obtained by identifying x and Tx in* $S^1 \times S^2$, *then M is either homeomorphic to* (1) $S^1 \times S^2$, *or* (2) *3-dimensional Klein Bottle*[3] (*we denote it by* K^3), *or* (3) $S^1 \times P^2$, *or* (4) $P^3 \#$[4] P^3.

This theorem may be regarded as an analogy of the following

Theorem (G. R. LIVESAY [4]). *If T is a fixed point free involution*

1) $\not\simeq 0$ means not homotopic to a constant.

2) P^2 is the real projective plane.

3) 3-dimensional Klein Bottle is defined as follows: let S_0, S_1 be the boundaries of $S^2 \times [0, 1]$. Then S_0 S_1 have the orientations induced from the orientation of $S^2 \times [0, 1]$. Let f be an orientation preseving homeorphirm from S_0 to S_1. Identifying S_0 with S_1 by f in $S^2 \times [0. 1]$, we obtain a non-orientable closed 3-manifold which we call 3-dim. Klein Bottle.

4) P^3 is the projective space. $P^3 \# P^3$ is defined as follows: Let E', E'' be two open 3-cells in P^3, P^3 respectively. Matching the boundaries of $P^3 - E'$ and $P^3 - E''$, we obtain a new closed 3-manifold which we denote $P^3 \# P^3$.

of S^3, *then the space obtained by identifying* x *and* Tx *in* S^3 *is the projective space.*

Theorem 2 follows almost immediately from Therem 1.

§ 1

According to E. E. Moise [7], we may suppose that $S^1\times S^2$ and M have fixed triangulations and that T is simplicial on some subdivision of the triangulation of $S^1\times S^2$ (See Chap. 1. of [4]). Therefore we stand throughout this paper on the semi-linear point of view : i.e., a 2-sphere will be considered as a 2-sphere semi-linearly embedded in M and any curve will be considered as polygonal, any homeomorphism as a semi-linear homeomorphism and so on.

Lemma 1. *Let* E, E_1, E_2 *be disks in a connected closed 3-manifold* M, *such that they have a common boundary* c *and* $E_1\cap E_2=E\cap E_1=E\cap E_2=c$. *If any two of 2-spheres* $S=E_1\cup E_2$, $S_1=E\cup E_1$ *and* $S_2=E\cup E_2$ *separate* M, *then the other one also separates* M.

Proof. Suppose S and S_1 separate M. Let A, B be two components of $M-S$, and let A_1, B_1 be two components of $M-S_1$. Since $E\cap S=\partial E=c$, $\operatorname{Int} E\subset A$ or $\operatorname{Int} E\subset B$. Here we suppose $\operatorname{Int} E\subset A$. In the same way, we may suppose $\operatorname{Int} E_2\subset A_1$. Take a point P on $\operatorname{Int} E_2$. Then, there exist two points P_1, P_2 sufficiently close to P, such that $P_1\in A_1\cap A$ and $P_2\in A_1\cap B$.

Suppose S_2 does not separate M. Then we can take a simple arc w in M which starts from P_1 and ends in P_2, such that $w\cap S_2=\phi$, $w\cap \operatorname{Int} E_1$ consists of an even number of points, Q_1, Q_2, $\cdots$, Q_{2n-1}, Q_{2n}. Let w_i $(i=1, 2, \cdots, n)$ be the subarcs of w from Q_{2i-1} to Q_{2i}. Then we replace w_i by w'_i, such that w'_i is an arc from Q_{2i-1} to Q_{2i} on $\operatorname{Int} E_1$, and $w'_i\cap w'_j=\phi$, if $i\neq j$. For convenience, we denote w by the same letter w after the deformation. Then shifting each w'_i slightly into A_1, we can delete the intersection $w\cap E_1$, keeping $w\cap S_2=\phi$ and getting any new intersections of w and E_1. Hence P_1 is joined with P_2 in $M-S$ by an arc, which contradicts that S separates M. Therefore S_2 must separate M.

Thus Lemma 1 is proved.

Lemma 2. *Let* $S^1\times S^2$ *be obtained from* $I\times S^2$, *where* I *is the closed interval* [0, 1], *by identifying its boundaries* $0\times S^2$ *and* $1\times S^2$. *Let* S *be a 2-sphere semi-linearly embedded in* $S^1\times S^2$, *such that* $S\cap(0\times S^2)=\phi$ *and* S *does not separate* $S^1\times S^2$. *Then* S *is isotopic to* $0\times S^2$ *in* $S^1\times S^2$.

Proof. Let S^3 be a 3-sphere obtained from $I\times S^2$ by filling in the boundaries $0\times S^2$ and $I\times S^2$ with two 3-cells e_1^3, e_2^3. Since S is semi-linearly embedded in S^3, by Alexander's theorem[5] ([1], [2], [6]), S divides S^3 into two 3-cells E_1^3, E_2^3 such that $S^3=E_1^3\cup E_2^3$ and $E_1^3\cap E_2^3=\partial E_1^3=\partial E_2^3=S$. Since S does not separate $S^1\times S^2$, we may suppose that $\mathrm{Int}\,E_1^3\supset 0\times S^2$ and $\mathrm{Int}\,E_2^3\supset 1\times S^2$. Therefore there exists a homeomorphism $h: I\times S^2\to E_1^3-\mathrm{Int}\,e_1^3$ by Alexander's theorem.

Thus Lemma 2 is proved.

Hereafter we suppose throughout this paper that $S^1\times S^2$ is obtained from $I\times S^2$ by identifying its boundaries $0\times S^2$ and $1\times S^2$.

Lemma 3. *There exists a 2-sphere S^* in $S^1\times S^2$ which is isotopic to $0\times S^2$, such that $S^*\cap TS^*=\phi$ or $TS^*=S^*$.*

Proof. Let $S=0\times S^2$. If $TS\neq S$, nor $S\cap TS\neq\phi$, then we may suppose $S\cap TS$ consists of a finite number of simple closed curves c_1, c_2, $\cdots$, c_n. If otherwise, by a small isotopic simplicial deformation of S, we obtain $S\cap TS$ in such a form. Let c be one of the innermost intersection curves on TS: i.e., there exists a disk E on TS, such that $c=\partial E$ and $\mathrm{Int}E\cap S=\phi$. c divides S into two disks E_1, E_2 such that $E_1\cup E_2=S$ and $E_1\cap E_2=\partial E_1=\partial E_2=c$. Since there is no intersection curves on $\mathrm{Int}\,TE$, we may suppose, without loss of generality, $TE\subseteqq E_1$ (equality holds, if and only if $c=Tc$). Let $S_1=E\cup E_1$, $S_2=E\cup E_2$. Then one of S_1 or S_2 does not separate $S^1\times S^2$. For, if both S_1 and S_2 separate $S^1\times S^2$, then S, S_1 and S_2 satisfy the conditions of Lemma 1. Therefore S separates $S^1\times S^2$ by the conclusion of Lemma 1, which contradicts the first assumption.

(1) S_1 does not separate $S^1\times S^2$.

If $c=Tc$, then $TS_1=T(E\cup E_1)=TE\cup TE_1=E_1\cup E=S_1$. Hence S_1 is an invariant 2-sphere under T.

If $c\neq Tc$, then $TE\subsetneqq E_1$. Take a simple closed curve c' on E_1 so close to c that the ring domain R bounded by c and c' on E_1 has no intersection with TS except c (Fig. 1). Then span a disk E' on c' so close to E that $S_1'=(E_1-R)\cup E'$ does not separate $S^1\times S^2$, $E'\cap TE'=\phi$, $E'\cap TS=\phi$ and $E'\cap S=\partial E'=c'$. From the way of construction of S_1', $S_1'\cap TS_1'$ consists of a subset of $\{c_1, c_2, \cdots, c_n\}$. we shall denote by $n(S\cap TS)$ the numbar of intersection curves of $S\cap TS$. Then it follows that $n(S_1'\cap TS_1')<n(S\cap TS)$, because the former is diminished at least by 2 (c and Tc) from the latter.

5) Alexander's theorem: Let S be a polygonal 2-sphere in the 3-sphere S^3. Then $S^3=e_1\cup e_2$ and $e_1\cap e_2=\partial e_1=\partial e_2=S$ where e_1, e_2 are topological 3-cells.

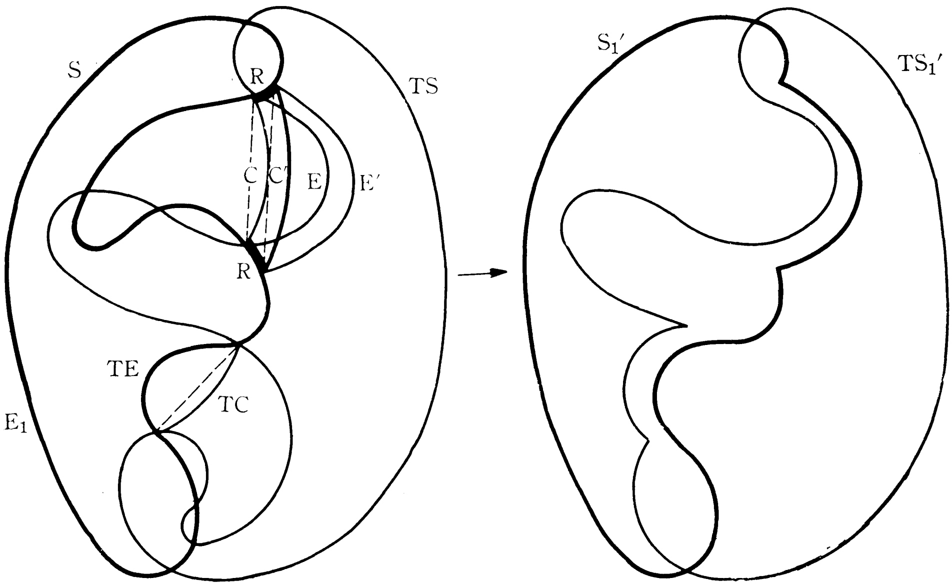

Fig. 1

(2) S_2 does not separate $S^1 \times S^2$.

If $c = Tc$, then we can take a simple closed curve c' and a disk E' so close to c and E that they satisfy the following conditions: (i) $c' \subset E_2$ and the domain R bounded by c and c' on E_2 has no intersection with TS, (ii) $E' \cap TS = \phi$ and $E' \cap S = \partial E' = c'$, (iii) $S_2' = (E_2 - R) \cup E'$ does not separate $S^1 \times S^2$. Furthermore we can take E' such that $E' \cap TE' = \phi$, because $TE' \cap S = \phi$, $TE' \cap TS = Tc'$ (Fig. 2). Then $n(S_2' \cap TS_2') <$

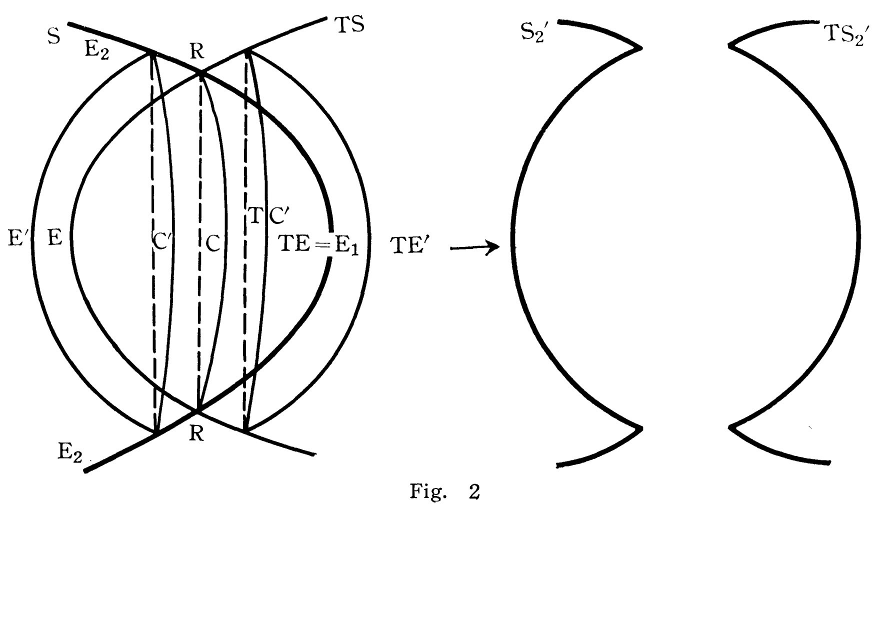

Fig. 2

$n(S\cap TS)$, because c is deleted and there arise no new intersection curves.

If $c\neq Tc$, then $n(S_2\cap TS_2)=n(E_2\cap TE_2)<n(S\cap TS)$, because c and Tc are not contained in $S_2\cap TS_2$.

As has been shown there exists in both cases (1), (2), a 2-sphere S' which does not separate $S^1\times S^2$ such that $n(S'\cap TS')<n(S\cap TS)$ or $TS'=S'$. Furthermore, from the way of our construction of S', we have by a small deformation of S' $S'\cap S=\phi$ without changing any other situations. Therefore it follows from Lemma 2 that S' is isotopic to S. Since $n(S\cap TS)$ is a non negative integer, we can find by proceeding with the above procedure a 2-sphere S^* which is isotopic to S and $S^*\cap TS^*=\phi$ or $TS^*=S^*$, in a finite step.

Thus Lemma 3 is proved.

§ 2

Proof of Theorem 1. By Lemma 3, there exists a 2-sphere S which is semi-linearly embeded in $S^1\times S^2$ and is isotopic to $0\times S^2$, such that $S\cap TS=\phi$ or $S=TS$. we divide our proof into the following two cases:

(1) $S\cap TS=\phi$,

(2) $S=TS$ and there is no 2-sphere S' which is isotopic to $0\times S^2$ and $S'\cap TS'=\phi$.

(1) $S\cap TS=\phi$. Since S is isotopic to $0\times S^2$, we may suppose $S=0\times S^2$. Then $S^1\times S^2-(S\cup TS)$ consists of two components A, B. Here A and B are homeomorphic to $I\times S^2$ by Lemma 2. Then the following two cases are possible:

(a) $TA=A$, (b) $TA=B$.

Case (a). Let p be a map from $S^1\times S^2$ onto M defined by $px=pTx$. Then $p: S^1\times S^2\to M$ is a double covering. Let M_A, M_B be closed 3-manifolds obtained from pA, pB by filling in the boundary 2-sphere $S'=p(S\cup TS)$ with 3-cells respectively. Filling in $\partial A=S\cup TS$ with two 3-cells, we obtain from A a 3-sphere S^3. Since T is a fixed point free involution of A, T is extended naturally to a fixed point free involution T' of S^3. Then, by Theorem 3 of [4], T' is equivalent to the antipodal map: i.e. there exists a homeomorphism $h: S^3\to S^3$ such that $hT'h^{-1}$ is an antipodal map. Hence $M_A=P^3$. In the same way, it follows that $M_B=P^3$. Therefore $M=P^3\# P^3$ (For example Fig. 3).

Case (b). In this case, M is homeomorphic to the manifold obtained from A by matching S and TS by the homeomorphism T. Therefore M is either homeomorphic to $S^1\times S^2$ or K^3.

(2) By Lemma 2, we may suppose $S=\frac{1}{2}\times S^2$. There are the following two cases to be considered:

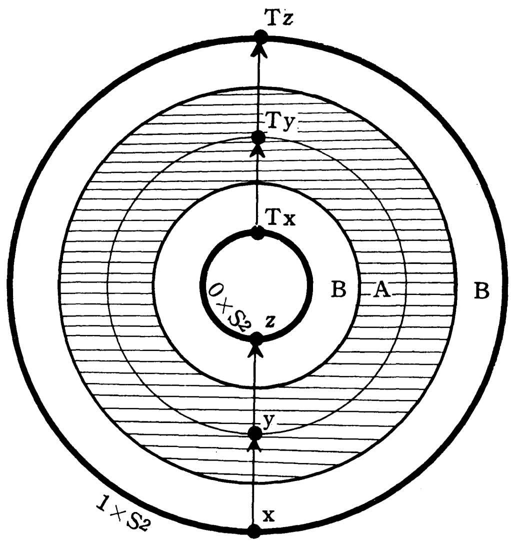

Fig. 3

(c). For all number ε, where $0<\varepsilon<\frac{1}{2}$, there exists a point Q such that $Q\in\left[\frac{1}{2}, \frac{1}{2}+\varepsilon\right]\times S^2$, $TQ\notin\left[\frac{1}{2}, 1\right)\times S^2$.

(d). There exists a number ε such that $0<\varepsilon<\frac{1}{2}$ and $T\left(\left[\frac{1}{2}, \frac{1}{2}+\varepsilon\right]\times S^2\right)\subset\left[\frac{1}{2}, 1\right)\times S^2$.

Case (c). Since $\frac{1}{2}\times S^2$ is invariant under T, there exists two numbers α, β such that $0<\beta<\alpha<\frac{1}{2}$, and $T\left(\left[\frac{1}{2}-\beta, \frac{1}{2}+\beta\right]\times S^2\right)\subset\left[\frac{1}{2}-\alpha, \frac{1}{2}+\alpha\right]$. Then by the assumption, there exists $\gamma\times S^2$ with $0<\gamma<\beta$, such that $T(\gamma\times S^2)\cap(\gamma\times S^2)=\phi$. Hence this case does not actually occur.

Case (d). Cut $S^1\times S^2$ by $\frac{1}{2}\times S^2$. Then it is homeomorphic to $I\times S^2$ and we may suppose that T is a fixed point free involution of $I\times S^2$. Then T restricted to the boundary $i\times S^2$ $(i=0,1)$ is an antipodal map A on 2-sphere S^2. Hence, by Lemma 3.1 of [3] T is equivalent to $e\times A$: $I\times S^2\to I\times S^2$, where $e: I\to I$ is the identity. Matching again the boundary of $I\times S^2$, we obtain that M is homeomorphic to $S^1\times P^2$.

Thus Theorem 1 is proved.

§ 3

A connected closed 3-manifold M is said to be *irreducible* if every 2-sphere which is semi-linearly embedded in M and separates M bounds a 3-cell in M.

Using Theorem 1, we obtain the following

Theorem 2. *Under Poincaré Hypothesis*[6], *the following two propositions are equivalent*:

(I) *Let $\tilde{M}$ be the orientable double covering of a connected closed 3-manifold M. If M is irreducible, then $\tilde{M}$ is irreducible.*

(II) $S^1 \times P^2$ *is the only one connected closed 3-manifold for which the sphere theorem does not hold.*

Proof. First suppose that (I) is true. Let M be a connnected closed 3-manifold for which the sphere theorem does not hold: i.e., $\pi_2(M) \neq 0$ but every 2-sphere semi-linearly embedded in M is homotopic to a constant in M. Then M is irreducible. For, if M is reducible, then $\pi_1(M)$ is a free product of two non trivial groups. Then it follows from Whitehead's theorem (Theorem 1.1 of [9]) that there exists a 2-sphere S semi-linearly embedded in M, such that $S \not\simeq 0$ in M, which contradicts the assumption. Therefore by (I), $\tilde{M}$ is irreducible. From Milnor's result ([5], [8]) and Poincaré Hypothesis, $\tilde{M}$ is homeomorphic to (1) $S^1 \times S^2$, or (2) is aspherical, or (3) has a non trivial finite fundamental group. Case (2) or (3) does not occur. For, if $\tilde{M}$ is aspherical, then $\pi_2(M) \approx \pi_2(\tilde{M}) = 0$, which contradicts the assumption. If $\pi_1(\tilde{M})$ is finite, then the universal covering $\tilde{\tilde{M}}$ of $\tilde{M}$ is S^3. Hence $\pi_2(M) \approx \pi_2(\tilde{M}) \approx \pi_2(\tilde{\tilde{M}}) = 0$, which contradicts the assumption. Therefore $\tilde{M}$ is homeomorphic to $S^1 \times S^2$. Since M is irreducible and non orientable, it follows from Theorem 1 that M is homeomorphic to $S^1 \times P^2$.

Next, suppose that (II) is true. Suppose $\tilde{M}$ is redudible. Then by Whitehead's theorem, $\pi_2(\tilde{M}) \neq 0$. Therefore $\pi_2(M) \neq 0$. If there exists a 2-sphere semi-linearly embedded in M, such that $S^2 \not\simeq 0$, then by Whitehead's theorem, M is reducible or $M = S^1 \times S^2$ or $M = K^3$. If there is no 2-sphere semi-linearly embeddee in M, such that $S^2 \not\simeq 0$, then it follows from (II) that $M = S^1 \times P^2$. On the other hand, by Theorem 1, if $M = S^1 \times S^2$, or $= K^3$, or $= S^1 \times P^2$, then $\tilde{M} = S^1 \times S^2$, which contradicts the first assumption. Hence M is reducible. Therefore (I) is true under the assumption of (II).

Thus Theorem 2 is proved.

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6) Poincaré Hypothesis: Every simply connected closed 3-manifold is the 3-sphere.

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