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Decompositions of a Completely Simple Semigroup

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§1. Introduction.

In this paper we shall study the method of finding all the decompositions of a completely simple semigroup and shall apply the result to the two special cases: an indecomposable completely simple semigroup [2] and a $\mathfrak{H}$-semigroup [4]. By a decomposition of a semigroup S we mean a classification of the elements of S due to a congruence relation in S. Let S be a completely simple semigroup throughout this paper. According to Rees [1], it is faithfully represented as a regular matrix semigroup whose ground group is G and whose defining matrix semigroup is $P=(p_{\mu\lambda})$, $\mu\in L$, $\lambda\in M$, that is, either $S=\{(x;\lambda\mu)\,|\,x\in G,\ \mu\in L,\ \lambda\in M\}$ or S with a two-sided zero 0, where the multiplication is defined as

$$(x;\lambda\mu)(y;\xi\eta)=\begin{cases}(xp_{\mu\xi}y;\lambda\eta) & \text{if } p_{\mu\xi}\neq 0\\ 0 & \text{if } p_{\mu\xi}=0 \text{ and hence } S \text{ has } 0.\end{cases}$$

For the sake of simplicity S is denoted as

$$Simp.(G,\ 0;\ P) \quad \text{or} \quad Simp.(G;\ P)$$

according as S has 0 or not. L and M may be considered as a right-singular semigroup and a left-singular semigroup respectively [5].

§2. Normal Form of Defining Matrix.

We define two equivalence relations $\overset{0}{\underset{M}{\sim}}$ and $\overset{0}{\underset{L}{\sim}}$ in M and L respectively: we mean by $\lambda\overset{0}{\underset{M}{\sim}}\sigma$ that $p_{\eta\lambda}\neq 0$ if and only if $p_{\eta\sigma}\neq 0$ for all $\eta\in L$; by $\mu\overset{0}{\underset{L}{\sim}}\tau$ that $p_{\mu\xi}\neq 0$ if and only if $p_{\tau\xi}\neq 0$ for all $\xi\in M$. Let $L=\sum_{\mathfrak{l}} L_{\mathfrak{l}}$, and $M=\sum_{\mathfrak{m}} M_{\mathfrak{m}}$ be the classifications of the elements of L and M due to the relations $\overset{0}{\underset{L}{\sim}}$ and $\overset{0}{\underset{M}{\sim}}$ respectively.

Lemma 1. *A defining matrix is equivalent to one which satisfies the following two conditions. Let e be a unit of G.*

(1) *For any $\mathfrak{m}$, there is $\alpha(\mathfrak{m})\in L$ such that $p_{\alpha(\mathfrak{m}),\xi}=e$ for all $\xi\in M_{\mathfrak{m}}$.*

(2) *For any $\mathfrak{l}$, there is $\beta(\mathfrak{l})\in M$ such that $p_{\eta,\beta(\mathfrak{l})}=e$ for all $\eta\in L_{\mathfrak{l}}$.*

Proof. First, for any $\mathfrak{m}$, we can easily choose $\alpha(\mathfrak{m}) \in L$ such that

(3) $p_{\alpha(\mathfrak{m}),\xi} \neq 0$ for all $\xi \in M_{\mathfrak{m}}$,

(4) If $\alpha(\mathfrak{m}_1) \overset{0}{\underset{L}{-}} \alpha(\mathfrak{m}_2)$, then $\alpha(\mathfrak{m}_1) = \alpha(\mathfrak{m}_2)$.

Next, for a mapping $\mathfrak{m} \to \alpha(\mathfrak{m})$, $\beta(\mathfrak{l}) \in M$ is determined such that the following conditions are satisfied:

(5) $p_{\eta,\beta(\mathfrak{l})} \neq 0$ for all $\eta \in L_{\mathfrak{l}}$

(6) if there is $\mathfrak{m}$ such that $\alpha(\mathfrak{m}) \in L_{\mathfrak{l}}$, then we let $\beta(\mathfrak{l}) \in M_{\mathfrak{m}_1}$ and $\alpha(\mathfrak{m}_1) \in L_{\mathfrak{l}}$ for one $\mathfrak{m}_1$ among $\mathfrak{m}$.

Consider the matrices

$$Q = (q_{\lambda_1\lambda_2}) \qquad \lambda_1, \lambda_2 \in M$$

and

$$R = (r_{\mu_1\mu_2}) \qquad \mu_1, \mu_2 \in L$$

where

$$q_{\lambda_1\lambda_2} = \begin{cases} p^{-1}_{\alpha(\mathfrak{m}),\xi} & \text{if } \lambda_1 = \lambda_2 = \xi \in M_{\mathfrak{m}} \\ 0 & \text{if } \lambda_1 \neq \lambda_2 \end{cases}$$

$$r_{\mu_1\mu_2} = \begin{cases} e & \text{if } \mu_1 = \mu_2 = \alpha(\mathfrak{m}) \text{ for some } \mathfrak{m} \\ p_{\alpha(\mathfrak{m}'),\beta(\mathfrak{l})}\, p^{-1}_{\eta,\beta(\mathfrak{l})} & \text{if } \alpha(\mathfrak{m}) \neq \mu_1 = \mu_2 = \eta \text{ for all } \mathfrak{m}, \text{ and we} \\ & \quad \text{let } \eta \in L_{\mathfrak{l}} \text{ and } \beta(\mathfrak{l}) \in M_{\mathfrak{m}'} \\ 0 & \text{if } \mu_1 \neq \mu_2 \end{cases}$$

Then, setting $R(PQ) = (t_{\mu\lambda})$, we have

$$t_{\mu\lambda} = \begin{cases} p_{\mu\lambda}\, p^{-1}_{\alpha(\mathfrak{m}),\lambda} & \text{if } \mu = \alpha(\mathfrak{m}'') \text{ for some } \mathfrak{m}'', \text{ and } \lambda \in M_{\mathfrak{m}} \\ p_{\alpha(\mathfrak{m}'),\beta(\mathfrak{l})}\, p^{-1}_{\mu,\beta(\mathfrak{l})}\, p_{\mu\lambda}\, p^{-1}_{\alpha(\mathfrak{m}),\lambda} & \text{if } \mu \neq \alpha(\mathfrak{m}'') \text{ for all } \mathfrak{m}'', \text{ we let } \lambda \in M_{\mathfrak{m}}, \\ & \quad \mu \in L_{\mathfrak{l}},\ \beta(\mathfrak{l}) \in M_{\mathfrak{m}'}, \end{cases}$$

and it is easily shown that RPQ satisfies (1) and (2). The conditions (4) and (6) are available for the proof of (2) in the case that $\eta = \alpha(\mathfrak{m}) \in L_{\mathfrak{l}}$ for some $\mathfrak{m}$. Thus the proof of the Lemma is completed.

The form, RPQ, which satisfies (1) and (2), is called a normal form of P.

§ 3. Decompositions.

Hereafter we shall assume that S has a matrix of normal form as the defining matrix. Let $\sim$ denote a congruence relation in S. $\sim$ is said to be trivial if either $x \sim y$ for all x, y or $x \sim y$ for only $x = y$.

Lemma 2. *Let $\sim$ be a non-trivial congruence relation.* $(x\,;\,\lambda\mu) \sim (y\,;\,\sigma\tau)$ *implies* $\lambda \overset{0}{\underset{M}{-}} \sigma$, $\mu \overset{0}{\underset{L}{-}} \tau$ *and hence there are* α *and* β *such that*

$p_{\alpha\lambda}=p_{\alpha\sigma}=e$, $p_{\mu\beta}=p_{\tau\beta}=e$ where e is a unit of G.

Proof. Suppose $p_{\eta_0\lambda}\neq 0$ as well as $p_{\eta_0\sigma}=0$ for some η_0, and take any element $(u\,;\,\xi\eta)$, then, for certain $p_{\mu\xi_0}\neq 0$,

$$\begin{aligned}(u\,;\,\xi\eta) &= (ux^{-1}p_{\eta_0\lambda}^{-1}\,;\,\xi\eta_0)(x\,;\,\lambda\mu)(p_{\mu\xi_0}^{-1}\,;\,\xi_0\eta)\\ &\sim (ux^{-1}p_{\eta_0\lambda}^{-1}\,;\,\xi\eta_0)(y\,;\,\sigma\tau)(p_{\mu\xi_0}^{-1}\,;\,\xi_0\eta) = 0\,.\end{aligned}$$

This shows that the relation $\sim$ is a trivial congruence relation, contradicting the assumption. The remaining part is similarly proved. The existence of α and β is clear by a normal form of the defining matrix, q. e. d.

Now we derive the relations $\approx$ in G, $\underset{M}{\sim}$ in M, and $\underset{L}{\sim}$ in L from the congruence relation $\sim$ in S as defined in the following way.

$x\approx y$ if there are $\lambda,\sigma\in M$, $\mu,\tau\in L$ such that $(x\,;\,\lambda\mu)\sim(y\,;\,\sigma\tau)$,

$\lambda\underset{M}{\sim}\sigma$ if there are $x,y\in G$, $\mu,\tau\in L$ such that $(x\,;\,\lambda\mu)\sim(y\,;\,\sigma\tau)$,

$\mu\underset{L}{\sim}\tau$ if there are $x,y\in G$, $\lambda,\sigma\in M$ such that $(x\,;\,\lambda\mu)\sim(y\,;\,\sigma\tau)$.

Lemma 3. *The relations $\approx$, $\underset{M}{\sim}$ and $\underset{L}{\sim}$ are all congruence relations.*

Proof. Reflexivity and symmetry are evident. Let us prove transitivity. By $x\approx y$ and $y\approx z$ there are λ, μ, σ, τ, σ', τ', κ and ν such that

$$(x,\,\lambda\mu)\sim(y\,;\,\sigma\tau)\,,\quad (y\,;\,\sigma'\tau')\sim(z\,;\,\kappa\nu)\,.$$

According to Lemma 2,

$$p_{\alpha\lambda}=p_{\alpha\sigma}=e\,,\quad p_{\mu\beta}=p_{\tau\beta}=e \quad \text{for certain } \alpha \text{ and } \beta,$$

so that we get

$$(e\,;\,\sigma'\alpha)(x\,;\,\lambda\mu)(e\,;\,\beta\tau')\sim(e\,;\,\sigma'\alpha)(y\,;\,\sigma\tau)(e\,;\,\beta\tau')$$

and hence
$$(x\,;\,\sigma'\tau')\sim(z\,;\,\kappa\nu)\,.$$

Thus we have proved $x\approx z$.

Transitivity of $\underset{M}{\sim}$ is proved from $(x\,;\,\lambda\mu)\sim(y\,;\,\sigma\tau)$, $(y'\,;\,\sigma\tau')\sim(z\,;\,\kappa\nu)$ and $(x\,;\,\lambda\mu)(y^{-1}y'\,;\,\beta\tau')\sim(y\,;\,\sigma\tau)(y^{-1}y'\,;\,\beta\tau')$ where $p_{\mu\beta}=p_{\tau\beta}=e$. We get transitivity of $\underset{L}{\sim}$ analogously.

Next, $x\approx y$ implies $xz\approx yz$ and $zx\approx zy$ because

$$\begin{gathered}(x\,;\,\lambda\mu)(z\,;\,\beta\mu)\sim(y\,;\,\sigma\tau)(z\,;\,\beta\mu)\\ (z\,;\,\lambda\alpha)(x\,;\,\lambda\mu)\sim(z\,;\,\lambda\alpha)(y\,;\,\sigma\tau)\end{gathered}$$

under the assumption $(x\,;\,\lambda\mu)\sim(y\,;\,\sigma\tau)$ where $p_{\mu\beta}=p_{\tau\beta}=p_{\alpha\lambda}=p_{\alpha\sigma}=e$. The proof for $\underset{M}{\sim}$ and $\underset{L}{\sim}$ is clear.

Lemma 4. *If* $\lambda \underset{M}{\sim} \sigma$ *and* $p_{\eta\lambda} \neq 0$, *then* $p_{\eta\lambda} \approx p_{\eta\sigma} \neq 0$ *for all* η. *If* $\mu \underset{L}{\sim} \tau$ *and* $p_{\mu\xi} \neq 0$, *then* $p_{\mu\xi} \approx p_{\tau\xi} \neq 0$ *for all* ξ.

Proof. By Lemma 2, it is evident that $p_{\eta\sigma} \neq 0$, $p_{\tau\xi} \neq 0$. Find β such that $p_{\mu\beta} = p_{\tau\beta} = e$. Multiplying each of $(x ; \lambda\mu)$ and $(y ; \sigma\tau)$ by $(x^{-1} ; \beta\mu)$ from right, we get

$$(e ; \lambda\mu) \sim (yx^{-1} ; \sigma\mu) \quad \text{whence} \quad e \approx yx^{-1}.$$

Moreover, from $(e ; \lambda\eta)(e ; \lambda\mu) \sim (e ;\lambda\, \eta)(yx^{-1} ; \sigma\mu)$

we have $$p_{\eta\lambda} \approx p_{\eta\sigma} yx^{-1} \approx p_{\eta\sigma}$$

completing the proof. Similarly $p_{\mu\xi} \approx p_{\tau\xi}$ is proved, q. e. d.

Conversely, consider congruence relations $\underset{M}{\sim}$, $\underset{L}{\sim}$, $\approx$ in M, L, G respectively such that

$$\begin{aligned} \lambda \underset{M}{\sim} \sigma \quad &\text{implies} \quad \lambda \overset{0}{\underset{M}{\sim}} \sigma, \\ \mu \underset{L}{\sim} \tau \quad &\text{implies} \quad \mu \overset{0}{\underset{L}{\sim}} \tau, \\ \approx \quad &\text{makes Lemma 4 hold.} \end{aligned}$$

For these congruence relations, a relation $\sim$ in S is defined as

$$(x ; \lambda\mu) \sim (y ; \sigma\tau) \quad \text{if } x \approx y,\ \lambda \underset{M}{\sim} \sigma, \text{ and } \mu \underset{L}{\sim} \tau.$$

Then it is easily shown that the relation is a congruence relation.

Theorem 1. *We obtain, as follows, every congruence relation in a completely simple semigroup* S *with a ground group* G *and with a defining matrix* $P = (p_{\mu\lambda})$, $\lambda \in M$, $\mu \in L$. *First, for a pair of the congruence relations* $\underset{M}{\sim}$ *and* $\underset{L}{\sim}$ *taken arbitrarily, independently each other, there is at least one congruence relation* $\approx$ *in* G *which satisfies*

$$\begin{aligned} \lambda \underset{M}{\sim} \sigma \quad &\textit{implies} \quad p_{\eta\lambda} \approx p_{\eta\sigma} \quad \textit{for all } \eta, \\ \mu \underset{L}{\sim} \tau \quad &\textit{implies} \quad p_{\mu\xi} \approx p_{\tau\xi} \quad \textit{for all } \xi. \end{aligned}$$

By a triplet of the three congruence relations $\underset{M}{\sim}$, $\underset{L}{\sim}$, $\approx$, *a congruence relation* $\sim$ *in* S *is determined as*

$$(x ; \lambda\mu) \sim (y ; \sigma\tau) \quad \textit{means that} \quad x \approx y,\ \lambda \underset{M}{\sim} \sigma, \textit{ and } \mu \underset{L}{\sim} \tau.$$

§ 4. Examples.

We shall arrange a few examples which follow from Theorem 1.

First, we can easily determine the structure of an indecomposable completely simple semigroup, which was obtained in [2].

Example 1. A completely simple semigroup S is indecomposable if and only if the following three conditions are satisfied.

(7) The ground group is $G=\{e\}$.

(8) $\lambda \overset{0}{\underset{M}{\sim}} \sigma$ if and only if $\lambda=\sigma$.

(9) $\mu \overset{0}{\underset{L}{\sim}} \tau$ if and only if $\mu=\tau$.

Example 2. Consider a finite simple semigroup S with a ground group G and with the defining matrix $\begin{pmatrix} e \\ e \end{pmatrix}$ or $(e\ e)$.

Let $x \to f(x)$ be a homomorphism of G to certain group $G' : G'=f(G)$, $e'=f(e)$. Then any homomorphism of S is given as either (10) or (11).

(10) $(x\,;\ \lambda\mu) \to (f(x)\,;\ \lambda\mu)$

where the homomorphic image S' of S is also a completely simple semigroup in which G' is the ground group and $P'=(f(p_{\mu\lambda}))$ is the defining matrix.

(11) $(x\,;\ \lambda\mu) \to f(x)$ where $S'=G'$.

Example 3. A finite simple semigroup S with a ground group G and with the defining matrix $\begin{pmatrix} e & e \\ e & a \end{pmatrix}$ where $a \neq 0$. Any homomorphic image of S is given as one of

$(x\,;\ \lambda\mu) \to (f(x)\,;\ \lambda\mu)$ where $S'=Simp.\left(f(G)\,;\ \begin{pmatrix} e' & e' \\ e' & f(a) \end{pmatrix}\right)$,

$(x\,;\ \lambda\mu) \to (f(x)\,;\ \lambda 1)$ where $S'=Simp.\,(f(G)\,;\ (e'\ e'))$ and $f(e)=f(a)=e'$,

$(x\,;\ \lambda\mu) \to (f(x)\,;\ 1\mu)$ where $S'=Simp.\left(f(G)\,;\ \begin{pmatrix} e' \\ e' \end{pmatrix}\right)$ and $f(e)=f(a)=e'$.

$(x\,;\ \lambda\mu) \to f(x)$ where $S'=f(G)$.

§ 5. $\mathfrak{H}$-Semigroups.

In this paragraph S denotes a finite simple semigroup. If a decomposition of S classifies the elements into some classes composed of equal number of elements, then the decomposition is called homogeneous. We term by $\mathfrak{H}$-semigroup a finite semigroup S with $\mathfrak{H}$-property, i.e., the property that every decomposition of S is homogeneous [4]. It goes without saying that any semigroup of order 2 and any indecomposable finite semigroup are $\mathfrak{H}$-semigroups. We assume that the order of S is >2.

Lemma 5. *A $\mathfrak{H}$-semigroup is simple.*

Proof. If a $\mathfrak{H}$-semigroup S is not simple, a proper ideal I exists so that the difference semigroup $(S:I)$ of S modulo I would result in a non-homogeneous decomposition of S, q. e.d.

Lemma 6. *If a $\mathfrak{H}$-semigroup S has zero* **0**, *then S is indecomposable.*

Proof. Let $\sim$ be a congruence relation in S. From Lemma 5 follows that there is nothing but the trivial decompositions, i.e., either $0\sim x$ for all $x\in S$ or $0\sim x$ for only $x=0$. In the latter case, by homogeneity, $x\neq y$ implies $x\nsim y$ for every x, y, q. e. d.

Corollary 1. *If a $\mathfrak{H}$-semigroup S has a non-trivial decomposition, then S is a simple semigroup without zero.*

Accordingly a $\mathfrak{H}$-semigroup S may be considered as a semigroup $S=\text{Simp.}(G;(p_{ji}))$ where let G be a group of order g, let $P=(p_{ji})$ be a matrix of (l, m) type i.e. $i=1,\cdots,m$; $j=1,\cdots,l$.

Lemma 7. *If S is a $\mathfrak{H}$-semigroup which has no zero, then $m\leqq 2$, $l\leqq 2$.*

Proof. Suppose, for example, $m\geqq 3$. Consider a congruence relation $\sim$ in S as follows.

$(x;kj)\sim(y;kj')$ for any $k>2$, any j, and any j',
$(x;kj)\nsim(y;k'j')$ for any $k>2$, $k'>2$, $k\neq k'$ any j, and any j',
$(x;1j)\sim(y;2j')$ for any j, and any j',

where x and y run independently throughout G.

Then we have a non-homogeneous decomposition

$$S=\bar{S}\cup S_3\cup S_4\cup\cdots$$

where

$$\bar{S}=\{(x;ij)\,|\,x\in G,\ i=1,2;\ 1\leqq j\leqq l\},$$
$$S_k=\{(x;kj)\,|\,x\in G,\ 1\leqq j\leqq l\},\quad k=3,4,\cdots$$

and the order of $\bar{S}$ is $2gl$, that of S_k is gl. This contradicts the assumption of $\mathfrak{H}$. Hence $m\leqq 2$. Similarly $l\leqq 2$ is proved.

Therefore a $\mathfrak{H}$-semigroup which has no zero must have a structure of the following four.

$$Simp.\left(G;\begin{pmatrix}e\\e\end{pmatrix}\right),$$
$$Simp.\,(G;(e\ e)),$$
$$Simp.\left(G;\begin{pmatrix}e&e\\e&a\end{pmatrix}\right),$$

Group.

On the other hand, Examples 2, 2′ and 3 show that every decomposition of them is homogeneous. At last we have arrived at

Theorem 2. *A finite semigroup is a $\mathfrak{H}$-semigroup of order $\geqq 2$ if and only if it is one of the following six cases.*

(C_1) *a z-semigroup of order* 2 *or a semilattice of order* 2

(C_2) *a finite group of order* $\geqq 2$

(C_3) *an indecomposable finite semigroup of order* >1

$$\left.\begin{array}{ll} (C_4) & Simp.\left(G\,;\,\begin{pmatrix} e \\ e \end{pmatrix}\right) \\ (C_5) & Simp.\,(G\,;\,(e\ e)) \\ (C_6) & Simp.\left(G\,;\,\begin{pmatrix} e & e \\ e & a \end{pmatrix}\right) \end{array}\right\} \text{ where } G \text{ is a finite group of order } \geqq 1,\ a \neq 0$$

§ 6. Relations between $\mathfrak{S}$-property and $\mathfrak{H}$-property.

In the paper [3, 4] we defined $\mathfrak{S}$-property of a finite semigroup and proved that an $\mathfrak{S}$-semigroup is one of the above cases except (C_3).

Immediately we have

Theorem 3. *$\mathfrak{S}$-property implies $\mathfrak{H}$-property. Though the converse is not true, it is true that a $\mathfrak{H}$-semigroup which has a proper decomposition is an $\mathfrak{S}$-semigroup.*

By the way we give a few theorems.

Theorem 4. *A unipotent $\mathfrak{S}$-semigroup of order >2 is a group. A unipotent $\mathfrak{H}$-semigroup of order >2 is so also.*

Theorem 5. *A subsemigroup of an $\mathfrak{S}$-semigroup is an $\mathfrak{S}$-semigroup. A subsemigroup of an indecomposable semigroup is not always a $\mathfrak{H}$-semigroup, but $\mathfrak{H}$-property, in the other cases, is preserved in a subsemigroup. $\mathfrak{S}$-property and $\mathfrak{H}$-property are both preserved in a homomorphic image.*

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[2] T. Tamura: Indecomposable completely simple semigroups except groups, Osaka Math. J. **8**, 35–42 (1956).

[3] ――――: Finite semigroups in which Lagrange's theorem holds, Jour. of Gakugei, Tokushima Univ. **10**, 33–38 (1959).

[4] ――――: Note on finite semigroups which satisfy certain group-like condition, Proc. of Jap. Acad. **36**, 62–64 (1960).

[5] I have called L a right-singular semigroup if $xy=y$ for all $x, y \in L$. In p. 62, [4], I find a misprint: for "left-singular", read "right-singular". But this is not essential for discussion.