Kinoshita, Shin'ichi
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On Wendt's Theorem of Knots, II

By Shin'ichi KINOSHITA

1. Recently R. H. Fox introduced in his paper [1] an operation τ called a single twist. Using this operation τ, we introduce now a numerical knot[1] invariant $\bar{\bar{s}}(k)$ defined as the minimal number of τ^n which change the given knot k to the trivial one, where the natural number n is not fixed. By definition of $\bar{s}(k)$ and $s(k)$[2]

$$\bar{\bar{s}}(k) \leqq \bar{s}(k) \leqq s(k)\,.$$

Then the purpose of this note is to prove

$$(*) \qquad e_g \leqq (g-1)\bar{\bar{s}}(k)\,,$$

where e_g is the minimal number of essential generators of the 1-dimensional homology group of the g-fold cyclic covering space of S, branched along k. From the above inequality $(*)$ it follows that

$$e_g \leqq (g-1)\bar{s}(k)\,, \qquad e_g \leqq (g-1)s(k)\,,$$

where the former is proved in [2] and the latter is due to H. Wendt [3].

2. Now we prove our inequality $(*)$. Let k be a knot. Suppose that k is deformed into k' by τ^n. Then we are only to prove that

$$e_g(k') \leqq e_g(k) + (g-1)\,.$$

Let $F(S-k)$ be the fundamental group of $S-k$. By [1] we may assume that

$$\begin{aligned} F(S-k) &= (a,\ b,\ A,\ B,\ x_1,\ x_2, \cdots : \\ &\qquad a=A,\ b=B,\ r_1=1,\ r_2=1, \cdots), \\ F(S-k') &= (a,\ b,\ A,\ B,\ x_1,\ x_2, \cdots : \\ &\qquad a=A,\ b=A^nB,\ r_1=1,\ r_2=1, \cdots). \end{aligned}$$

The 1-dimensional homology groups of $S-k$ and $S-k'$ are infinite cyclic. We denote by t a generator of either group. Then abelianization of $F(S-k)$ or $F(S-k')$ maps A into t^q and B into 1, where q is an integer. By usual methods the presentations of $F(S-k)$ and $F(S-k')$ can be transfomed to the following one:

---

1) A knot is a polygonal simple closed curve in the 3-sphere S.
2) $\bar{s}(k)$ and $s(k)$ are defined in [2].

$$
\begin{aligned}
F(S-k) &= (t,\ \bar{a},\ \bar{b},\ \bar{A},\ \bar{B},\ \bar{x}_1,\ \bar{x}_2,\cdots :\\
&\qquad \bar{a}=\bar{A},\ \bar{b}=\bar{B},\ \bar{r}_1=1,\ \bar{r}_2=1,\cdots,\ tf_1^{-1}=1),\\
F(S-k') &= (t,\ \bar{a},\ \bar{b},\ \bar{A},\ \bar{B},\ \bar{x}_1,\ \bar{x}_2,\cdots :\\
&\qquad \bar{a}=\bar{A},\ t^{nq}\bar{b}=(t^q\bar{A})^n\bar{B},\ \bar{r}_1=1,\ \bar{r}_2=1,\cdots,\ tf_2^{-1}=1).
\end{aligned}
$$

Furthermore we may suppose that $f_1=f_2$. Then the 1-dimensional homology group of the g-fold cyclic covering space of S, branched along k, is given by the matrix:

$$
(1)\qquad
\begin{array}{c}
\begin{array}{ccccc} \bar{B} & \bar{b} & \bar{A} & \bar{a} & \bar{x}_i \end{array}\\
\left(\begin{array}{c|c|c}
\begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & 0 \\ \hline
0 & \multicolumn{2}{c}{*}
\end{array}\right)
\end{array}
\begin{array}{l}\left.\begin{matrix}\\ \\ \\ \end{matrix}\right\} g \text{ rows}\\ \\ \\ \end{array}
$$

$$\underbrace{\qquad\quad}_{g\text{ columns}}$$

and that of S, branched along k', is given by the following one:

$$
(1')\qquad
\begin{array}{c}
\begin{array}{ccccc} \bar{B} & \bar{b} & \bar{A} & \bar{a} & \bar{x}_i \end{array}\\
\left(\begin{array}{c|c|c|c}
\begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}-1&&\\&\ddots&\\&&-1\end{matrix} & \begin{matrix}a_0\cdots\cdots a_{g-1}\\ a_{g-1}\cdots a_{g-2}\\ \cdots\cdots\cdots\\ a_1\cdots\cdots a_0\end{matrix} & 0 \\ \hline
0 & \multicolumn{3}{c}{*}
\end{array}\right).
\end{array}
$$

Putting $\sum_{i=0}^{g-1} a_i=\alpha$, we can transform (1) and (1′) to the following one, respectively:

$$
(2)\qquad
\left(\begin{array}{c|c}
\begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & 0 \\ \hline
0 & *
\end{array}\right),
$$

$$
(2') \qquad \left(\begin{array}{c|c|c|c}
\begin{matrix} 11\cdots 1 \\ 1\cdots 1 \\ \cdots \\ \cdots 1 \end{matrix} & 0 & \begin{matrix} \alpha\cdots\cdots\alpha \\ a_{g-1}\cdots a_{g-2} \\ \cdots\cdots \\ a_1\cdots\cdots a_0 \end{matrix} & 0 \\
\hline
0 & \multicolumn{3}{c}{*}
\end{array}\right).
$$

(2′) is equivalent to

$$
(3) \qquad \left(\begin{array}{c|c|c|c}
\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 & \begin{matrix} 0\cdots\cdots 0 \\ a_{g-1}\cdots a_{g-2} \\ \cdots\cdots \\ a_1\cdots\cdots a_0 \end{matrix} & 0 \\
\hline
0 & \multicolumn{3}{c}{*}
\end{array}\right).
$$

From (2) and (3) it is easy to see that

$$e_g(k') \leqq e_g(k)+(g-1)\,.$$

Thus our proof is complete.

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References

[1] R. H. Fox: Congruence classes of knots, Osaka Math. J. **10**, 37–41 (1958).
[2] S. Kinoshita: On Wendt's theorem of knots, Osaka Math. J. **9**, 61–66 (1957).
[3] H. Wendt: Die gordische Auflösung von Knoten, Math. Z. **42**, 680–696 (1937).