On Wendt's Theorem of Knots

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1. By the *unknotting number*[1] $s(k)$ of a knot[2] k we will mean the minimal number of cuts (See Fig. 1) in order to change the given knot k to an unknotted one.   Let $M_g(k)$ be the g–fold branched cyclic covering
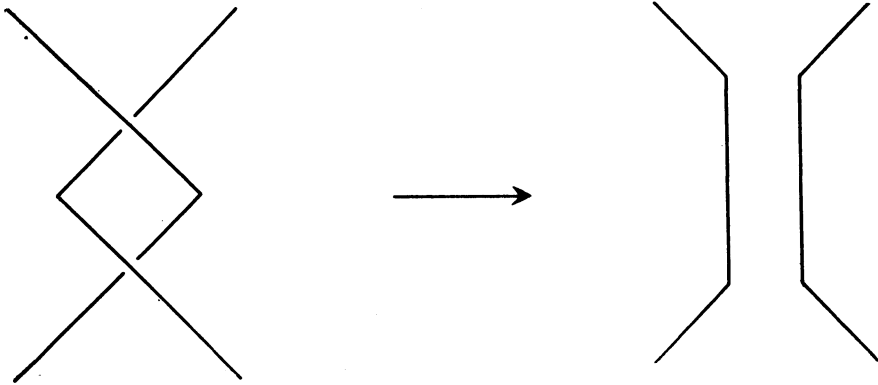

Fig. 1.

space of k and $e_g(k)$ the minimal number of essential generators of the 1–dimensional homology group $H_1(M_g(k))$ of $M_g(k)$. Then in 1937 H. Wendt [3] proved that

$$e_g \leqq s(g-1) .$$

Now we introduce the *weak unknotting number* $\bar{s}(k)$ defined very similarly to the above one as follows: $\bar{s}(k)$ is the minimal number of cuts as shown in Fig. 2, which change the given knot k to an unknotted one. The purpose of this note is to prove the following inequality

(∗) $$e_g \leqq \bar{s}(g-1) .$$

Since $\bar{s} \leqq s$ by the definition, Wendt's inequality follows immediately from ours.

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1) Überschneidungszahl in H. Wendt [3], see K. Reidemeister [2], p. 17.
2) A knot is a polygonal simple closed curve in the 3-sphere S^3.
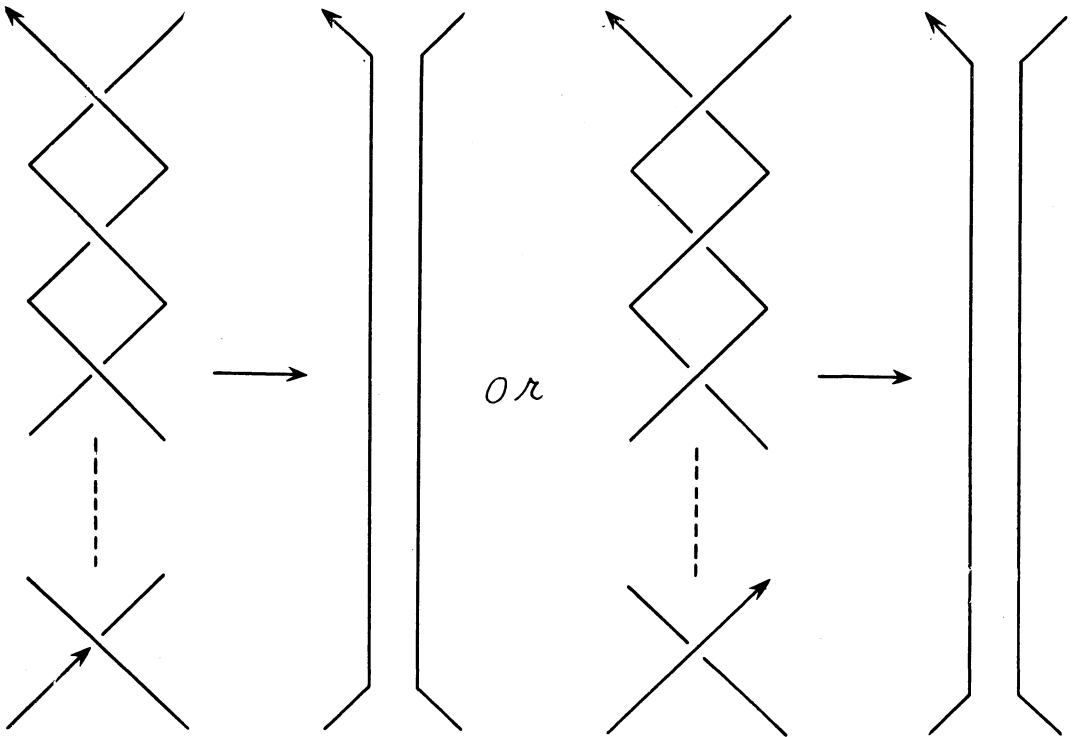
Fig. 2.

2.   Now we prove our inequality (∗).   Let k be a knot.   Suppose that k is deformed into k' by one cut, where we assume that k and k' remain the same except for the neighbourhood of the cut.   Then we are only to prove that

$$e_g(k) \leqq e_g(k') + (g-1)^{3)} .$$

Suppose that k is oriented.   Then there are two different cases[4] as shown in Fig. 3 and Fig. 4 respectively.

Firstly we prove for the case 1.   Suppose that the crossing number of k' is equal to $m+2$.   Then the crossing number of k is $m+2n$, where $2n$ is the crossing number of two sub-polygons of k as shown in Fig. 3. The regions on the plane decomposed by the projection of k will be denoted by letters as in Fig. 3.   Then the Alexander presentation[5] of the group $F(S^3 - k)^{6)}$ is as follows:

3)   See H. Wendt [3].
4)   Other cases can be proved by the similar method to one of these two cases.
5)   See J. W. Alexander [1].   See also R. H. Fox, Ann. of Math. 59 (1954) p. 196.
6) $F(S^3 - k)$ is the fundamental group of $S^3 - k$.
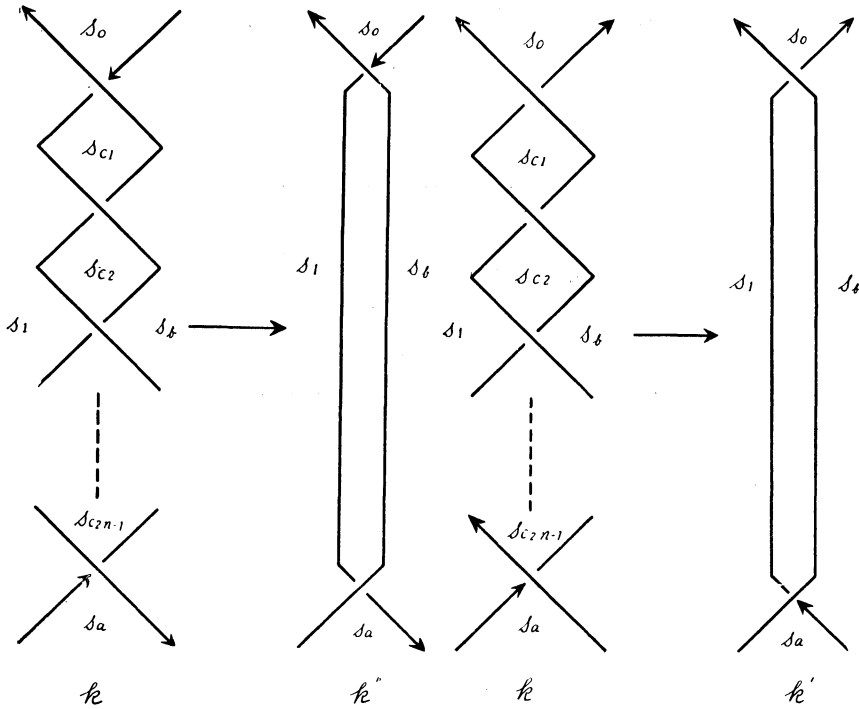
Fig. 3.　　　　　　　　　　　　　　　　Fig. 4.

Generators: $\begin{cases} x,\ s_0,\ s_1,\ \cdots,\ s_m,\ s_a,\ s_b, \\ s_{c_1},\ s_{c_2},\ \cdots,\ s_{c_{2n-1}}. \end{cases}$

Relations: $\left.\begin{cases} s_0 = s_1 = 1, \\ x s_{c_1} s_b^{-1} x^{-1} s_0 s_1^{-1} = 1, \\ x s_{c_1} s_1^{-1} x^{-1} s_{c_2} s_b^{-1} = 1, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ x s_{c_{2n-1}} s_1^{-1} x^{-1} s_a s_b^{-1} = 1, \end{cases} \right\} 2n$

$\left.\begin{cases} x s_t s_u^{-1} x^{-1} s_v s_w^{-1} = 1, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots. \end{cases}\right\} m$

This presentation can be transformed to the following one.

Generators: $x,\ s_0,\ s_1,\ \cdots,\ s_m,\ s_a,\ s_b.$

Relations: $\left.\begin{cases} s_0 = s_1 = 1, \\ x s_b^{-n} x^{-1} s_b^n s_a^{-1} = 1, \\ x s_t s_u^{-1} x^{-1} s_v s_w^{-1} = 1, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots. \end{cases}\right\} m$

Similarly a presentation of $F(S^3 - k')$ is as follows:

Generators: $x, s_0, s_1, \cdots, s_m, s_a, s_b$.

Relations: $\begin{cases} s_0 = s_1 = s_a = 1, \\ x s_t s_u^{-1} x^{-1} s_v s_w^{-1} = 1, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots . \end{cases} \Bigg\} m$

Putting

$$s_{i,j} = x^j s_i x^{-j}, \qquad \begin{cases} i = 0, 1, \cdots, m, a, b \\ j = 0, \pm 1, \pm 2, \cdots \end{cases}$$

we know that $H_1(M_g(k))$ is given by the following square matrix:

$s_{a,0}, \cdots, s_{a,g-1}, \quad s_{b,0}, \cdots, s_{b,g-1}, \quad \cdots\cdots\cdots\cdots\cdots$

$$\left(\begin{array}{ccc|ccc|c} -1 & & 0 & n & -n & 0 & \\ & \ddots & & & \ddots & & \\ & & \ddots & & & \ddots & 0 \\ 0 & & -1 & -n & 0 & n & \\ \hline & & & & & & \\ & * & & & d_{ij} & & \\ & & & & & & \end{array} \right).$$

Adding each $(1+i)$-th row to the first row $(i = 1, 2, \cdots, g-1)$, we have

$s_{a,0}, \cdots, s_{a,g-1}, \quad s_{b,0}, \cdots, s_{b,g-1}, \quad \cdots\cdots\cdots\cdots\cdots$

$$\left(\begin{array}{cccc|cccc|c} -1 & -1 & \cdots & -1 & 0 & 0 & 0\cdots & 0 & \\ 0 & -1 & & 0 & 0 & n & -n & 0 & \\ \vdots & & \ddots & & & & \ddots & & 0 \\ 0 & .0 & & -1 & -n & 0 & 0 & n & \\ \hline & & & & & & & & \\ & & * & & & & d_{ij} & & \\ & & & & & & & & \end{array} \right),$$

which can be deforemed to

$$\left(\begin{array}{cccc|cccc|c} -1 & 0 & \cdots\cdots & 0 & 0 & 0 & 0\cdots & 0 & \\ 0 & -1 & & 0 & 0 & n & -n & 0 & \\ \vdots & & \ddots & & & & \ddots & & 0 \\ 0 & 0 & & -1 & -n & 0 & 0 & n & \\ \hline & & & & & & & & \\ & & *' & & & & d_{ij} & & \\ & & & & & & & & \end{array} \right).$$

On the other hand $H_1(M_g(k'))$ is given by (d_{ij}). From this it is easy to see that

$$e_g(k) \leq e_g(k') + (g-1) ,$$

and the proof for the case 1 is complete.

3. Now we prove for the case 2. The same notation as Nr. 2 will be used. Then the Alexander presentation of $F(S^3 - k)$ is as follows:

Generators: $\begin{cases} x, \ s_0, \ s_1, \ \cdots, s_m, \ s_a, \ s_b, \\ s_{c_1}, \ s_{c_2}, \ \cdots, s_{2n-1} . \end{cases}$

Relations: $\begin{cases} s_0 = s_1 = 1 , \\ xs_0s_1^{-1}x^{-1}s_{c_1}s_b^{-1} = 1 , \\ xs_{c_1}s_1^{-1}x^{-1}s_{c_2}s_b^{-1} = 1 , \\ \cdots\cdots\cdots\cdots\cdots\cdots , \\ xs_{c_{2n-1}}s_1^{-1}x^{-1}s_as_b^{-1} = 1 , \\ xs_ts_u^{-1}x^{-1}s_vs_w^{-1} = 1 , \\ \cdots\cdots\cdots\cdots\cdots\cdots . \end{cases} \begin{matrix} \Big\} 2n \\ \\ \\ \Big\} m \end{matrix}$

This presentation can be transformed to the following one:

Generators: $x, s_0, s_1, \cdots, s_m, s_a, s_b .$

Relations: $\begin{cases} s_0 = s_1 = 1 , \\ s_a^{-1}(xs_b^{-1}x)^{2n} (x^{-2}s_b)^{2n} = 1 , \\ xs_ts_u^{-1}x^{-1}s_vs_w^{-1} = 1 , \\ \cdots\cdots\cdots\cdots\cdots\cdots . \end{cases} \Big\} m$

Put

$$-(1 - x + x^2 - \cdots - x^{2n-1}) = b_0 + b_1x + \cdots + b_{g-1}x^{g-1} \qquad (\text{mod } x^g - 1) .$$

Then $\sum_{i=0}^{q-1} b_i = 0$. It can easily be seen that $H_1(M_g(k))$ is given by the following square matrix:

$$s_{a,0}, \ \cdots, \ s_{a,g-1}, \ s_{b,0}, \ \cdots, \ s_{b,g-1}, \ \cdots\cdots\cdots\cdots\cdots$$

$$\begin{pmatrix} \begin{array}{ccccc|cccc} 1 & 0 & \cdots\cdots & 0 & b_0 & b_1 & \cdots & b_{g-1} \\ 0 & 1 & & 0 & b_{g-1} & b_0 & & b_{g-2} \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \\ 0 & 0 & & 1 & b_1 & b_2 & & b_0 \\ \hline \\ & & * & & & & d_{ij} & \\ \\ \end{array} & & 0 \\ \end{pmatrix} .$$

Adding each $(1+i)$-th row to the first row $(i=1, 2, \cdots, g-1)$, we have

$$s_{a,0}, \cdots, s_{a,g-1}, s_{b,0}, \cdots, s_{b,g-1}, \cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left(\begin{array}{ccccc|ccccc|c}
1 & 1 & \cdots\cdots & 1 & 0 & 0 & \cdots\cdots & 0 & & \\
0 & 1 & & 0 & b_{g-1} & b_0 & & b_{g-2} & & \\
\vdots & & \ddots & & \vdots & & \ddots & & & 0 \\
0 & 0 & & 1 & b_1 & b_2 & & b_0 & & \\
\hline
& & * & & & & d_{ij} & & & \\
\end{array}\right),$$

which can be deformed to

$$\left(\begin{array}{ccccc|ccccc|c}
1 & 0 & \cdots\cdots & 0 & 0 & 0 & \cdots\cdots & 0 & & \\
0 & 1 & & 0 & b_{g-1} & b_0 & & b_{g-2} & & \\
\vdots & & \ddots & & \vdots & & \ddots & & & 0 \\
0 & 0 & & 1 & b_1 & b_2 & & b_0 & & \\
\hline
& & *' & & & & d_{ij} & & & \\
\end{array}\right).$$

On the other hand $H_1(M_g(k'))$ is given by (d_{ij}). Thus

$$e_g(k) \leq e_g(k') + (g-1) ,$$

which completes the proof of our inequality $(*)$.

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References

[1] J. W. Alexander: Topological invariants of knots and links, Trans. Amer. Math Soc. **30**, 275–306 (1928).

[2] K. Reidemeister: Knotentheorie, Berlin (1932).

[3] H. Wendt: Die gordische Auflösung von Knoten, Math. Z. **42**, 680–696 (1937).