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Mass Distributions on the Ideal Boundaries of Abstract Riemann Surfaces. I[1]

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We shall extend some theorems of potential theory in space to abstract Riemann surfaces. In the present article we shall be concerned with Evans-Selberg's theorem on Riemann surfaces with null-boundary.

G. C. Evans and H. Selberg[2] proved the following theorem. *Given a closed set F of capacity zero in space, then there exists a positive mass distribution on F whose potential is positively infinite at every point of F.* We shall extend this theorem to abstract Riemann surfaces with null-boundary.

Let R^* be a Riemann surface with null-boundary and $\{R_n\}$ $(n=0, 1, 2, \cdots)$ be its exhaustion with compact relative boundaries $\{\partial R_n\}$. Put $R=R^*-R_0$. Let $G_n(z, p)$ be the Green's function of R_n-R_0 with pole at p. Clearly, $G_n(z, p)\uparrow G(z, p)$ as $n\to\infty$. Since $\int_{\partial R_0}\frac{\partial G_n(z, p)}{\partial n}ds\leqq 2\pi$ for every n, $G(z, p)$ is not constant infinity and harmonic in R except at p where $G(z, p)$ has a logarithmic singularity.

Take M large so that the set $V_M(p)=E[z\in R: G(z, p)\geqq M]$ is compact in R. Let $\omega_n(z)$ be a harmonic function in $R_n-R_0-V_M(p)$ such that $\omega_n(z)=0$ on $\partial R_0+\partial V_M(p)$ and $\omega_n(z)=M$ on ∂R_n. Then since R^* is a Riemann surface with null-boundary, $\lim_{n=\infty}\omega_n(z)=0$. Let $\bar{G}_n(z, p)$, $G_n'(z, p)$ and $\underline{G}_n(z, p)$ be harmonic functions in $R_n-R_0-V_M(p)$ such that $\bar{G}_n(z, p)=G_n'(z, p)=\underline{G}_n(z, p)=M$ on $\partial V_M(p)$, $\bar{G}_n(z, p)=G_n'(z, p)=\underline{G}_n(z, p)=0$ on ∂R_0 and $\bar{G}_n(z, p)=M$, $\frac{\partial G_n'(z, p)}{\partial n}=0$ and $\underline{G}_n(z, p)=0$ on ∂R_n respectively. Since $0<G_n'(z, p)<M$ on ∂R_n, we have by the maximum principle

$$\underline{G}_n(z, p)<G_n'(z, p)<\bar{G}_n(z, p),\quad \underline{G}_n(z, p)<G(z, p)<\bar{G}_n(z, p)$$

and

$$0<\bar{G}_n(z, p)-\underline{G}_n(z, p)=M\omega_n(z).$$

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1) Resumé of this part is reported in Proc. Japan Acad. 32, 1956.

2) G. C. Evans: Potential and positively infinite singularities of harmonic functions. Monatsch. f. Math. u. Phys. 43, 1936, 419–424.

H. Selberg: Über die ebenen Punktmengen von der Kapazität Null. Avh. Norske Vid-akad, Oslo, 1, Nr. 10, 1937, 1–10.

Hence

$$\lim_{n=\infty} \bar{G}(z, p) = \lim_{n=\infty} G_n'(z, p) = \lim_{n=\infty} \underline{G}_n(z, p) = G(z, p).$$

Then by Green's formula and by the compactness of $V_M(p)$

$$\int_{\partial R_0} \frac{\partial G(z, p)}{\partial n} ds = \int_{\partial R_0} \lim_{n=\infty} \frac{\partial G_n'(z, p)}{\partial n} ds = - \int_{\partial V_M(p)} \lim_{n=\infty} \frac{\partial G_n'(z, p)}{\partial n} ds$$
$$= - \int_{\partial V_M(p)} \frac{\partial G(z, p)}{\partial n} ds = 2\pi.$$

$G(z, p)$ is called the Green's function of R with pole at p.

After R. S. Martin[3] we shall define the ideal boundary points as follows: let $G(z, p)$ be the Green's function of R with pole at p. Then by definition, the flux of $G(z, p)$ along ∂R_0 is 2π and $G(z, p)$ is positive. Consider now a sequence of points $\{p_i\}$ of R having no point of accumulation in $R+\partial R_0$. In any compact part of R, the corresponding functions $G(z, p_i)$ $(i=1, 2, \cdots)$ form, from some i on, a bounded sequence of harmonic functions—thus a normal family. A sequence of these functions, therefore, is convergent in every compact part of R to a positive harmonic function. A sequence $\{p_i\}$ of R having no point of accumulation in $R+\partial R_0$, for which the corresponding $G(z, p_i)_s$' have the property just mentioned, that is, converges to a harmonic function—will be called fundamental. Two fundamental sequences are called equivalent if their corresponding $G(z, p_i)_s$' have the same limit. The class of all fundamental sequences equivalent to a given one determines an *ideal boundary point* of R. The set of all the ideal boundary points of R will be denoted by B and the set $R+B$, by $\bar{R}$. The domain of definition of $G(z, p)$ may now be extended by writing $G(z, p) = \lim_{i=\infty} G(z, p_i)$ $(z \in R,\ p \in B)$, where $\{p_i\}$ is any fundamental sequence determining p. For p in B, $G(z, p)$ is positive, harmonic and $\int_{\partial R_0} \frac{\partial G(z, p)}{\partial n} ds = 2\pi$ and further $G(z, p)$ is unbounded in R, because if $G(z, p)$ is bounded in R, $G(z, p) \equiv 0$ by the maximum principle. This contradicts $\int_{\partial R_0} \frac{\partial G(z, p)}{\partial n} ds = 2\pi$. Evidently, the function $G(z, p)$ is characteristic of the point p in the sense that the identity of two points of $\bar{R}$ is equivalent to the equality of their corresponding $G(z, p)_s$' as a function of z. The function $\delta(p_1, p_2)$ of two points p_1 and p_2 in $\bar{R}$ is defined by

$$\delta(p_1, p_2) = \sup_{z \in R_1 - R_0} \left| \frac{G(z, p_1)}{1+G(z, p_1)} - \frac{G(z, p_2)}{1+G(z, p_2)} \right|.$$

3) R. S. Martin: Minimal positive harmonic functions. Trans. Amer. Math. Soc. **39**, 1941.

Evidently, $\delta(p_1, p_2)=0$ is equivalent to $G(z, p_1)=G(z, p_2)$ for all points z in R_1-R_0. Therefore we have $G(z, p_1)=G(z, p_2)$ for all points in R, that is $\delta(p_1, p_2)=0$ implies $p_1=p_2$ and it is clear that $\delta(p_1, p_2)$ satisfies the axioms of distance. Therefore $\delta(p_1, p_2)$ can be considered as the distance between two points p_1 and p_2 of $\overline{R}$. The topology induced by this metric is homeomorphic to the original topology when it is restricted in R. Since $G(z, p_i)(p_i\in\overline{R})$ is also a normal family, both $(R-R_1)+\partial R_1+B$ and B are closed and compact. For fixed z, $G(z, p)$ is continuous with respect to this metric (we denote shortly it by δ-continuous) as a function of p in $\overline{R}$ except at $z=p$.

First we shall prove the following

Lemma 1. *Let G_i be a compact or non-compact domain with an analytic relative boundary ∂G_i $(i=1, 2, \cdots, k)$. Let $U_i(z)$ $(i=1, 2, \cdots, k)$ be a function which is harmonic in $R-G_i$ and on ∂G_i, such that the Dirichlet integral of $U_i(z)$ taken over $R-G_i$ is finite. Then there exists a sequence of compact curves $\{\gamma_n\}$ such that γ_n separates B from ∂R_0, $\{\gamma_n\}$ clusters at B and that $\int_{\gamma_n-G_i}\left|\frac{\partial U_i(z)}{\partial n}\right|ds$ tends to zero as $n\to\infty$, for every i.*

Proof. Let $\omega_n'(z)$ be a harmonic function in R_n-R_0 such that $\omega_n'(z)=1$ on ∂R_n and $\omega_n'(z)=0$ on ∂R_0. Then $\lim_{n=\infty}\omega_n'(z)=0$, since R^* is a Riemann surface with null-boundary. Hence, for any given number n' there exists a number n_0 such that $\omega_n'(z)<\frac{1}{2}$ in $R_{n'}-R_0$, for any $n\geqq n_0$. We denote by $\omega_n(z)$ a harmonic function in R_n-R_0 which vanishes on ∂R_0 and assumes a constant value M_n on ∂R_n and whose flux along ∂R_0 is 2π. It is evident that $\omega_n(z)=M_n\omega_n'(z)$ and $\lim_{n=\infty}M_n=\infty$. Then for a number n' chosen in the manner above stated, the niveau curve with height $\geqq\frac{M_n}{2}$ is contained in $R_n-R_{n'}$.

Put
$$e^{\omega_n(z)+i\bar{\omega}_n(z)}=re^{i\theta},$$

where $\bar{\omega}_n(z)$ is the conjugate harmonic function of $\omega_n(z)$.

Let $U(z)$ be one of $U_i(z)$ and put

$$L(r)=\int_{\underline{C}_r}\left|\frac{\partial U(z)}{\partial r}\right|rd\theta=\int_{\underline{C}_r}\left|\frac{\partial U(z)}{\partial n}\right|ds,$$

where $\underline{C}_r$ is the part of the niveau curve C_r of $\omega_n(z)$ with height r contained in $R-G$.

Suppose that there exist two positive constants η and δ and infinitely many numbers n with the property as follows: there exists a closed set F_n in the interval $(e^{M_n}, e^{M_n/2})$ such that $\frac{\text{mes } F_n}{(e^{M_n}-e^{M_n/2})} \geqq \eta$ and that $L(r) \geqq \delta$ for any $r \in F_n$. Since $\int_{C_r} d\theta = 2\pi$, $\int_{C_r} d\theta \leqq 2\pi$. Then by Schwarz's inequality, we have

$$D_{R-G}(U(z)) = \iint_{R-G} \left\{ \left(\frac{\partial U(z)}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial U(z)}{\partial \theta} \right)^2 \right\} r dr d\theta \geqq \frac{1}{2\pi} \int_1^{e^{M_n}} \frac{L^2(r)}{r} dr$$

$$> \frac{1}{2\pi} \int_{e^{M_n/2}}^{e^{M_n}} \frac{L^2(r)}{r} dr > \frac{1}{2\pi} \int_{e^{M_n - \eta(M_n - M_n/2)}}^{e^{M_n}} \frac{\delta^2}{r} dr = \frac{M_n}{4\pi} \eta \delta^2 .$$

Let $n \to \infty$. Then the right hand side diverges. This contradicts the finiteness of $D(U(z))$. Hence there exists a sequence of exceptional sets $\{E_n\}$ in the intervals $\{(e^{M_n}, e^{M_n/2})\}$ such that $\lim_{n=\infty} \frac{\text{mes } E_n}{(e^{M_n}-e^{M_n/2})} = 0$ and that $r \notin E_n$ implies $L(r) < \delta_n$, where $\lim_{n=\infty} \delta_n = 0$.

Returning to case of $U_i(z)$, let $\{E_{i,n}\}$ be a sequence of exceptional sets corresponding to $U_i(z)$ and $\{\delta_{i,n}\}$ be the corresponding quantities of $\{E_{i,n}\}$. Then we see that $\frac{\sum_{i=1}^{k} \text{mes } E_{i,n}}{(e^{M_n}-e^{M_n/2})}$ and $\max_i \delta_{i,n}$ tend to zero as $n \to \infty$. On the other hand, the niveau curves with height $\geqq \frac{M_n}{2}$ are are contained in $R - R_{n'}$, since $\omega_n(z) < \frac{M_n}{2}$ in $R_{n'} - R_0$. It follows that every C_r with $r \in (e^{M_n}, e^{M_n/2}) - \sum_{i=1}^{k} E_{i,n}$ clusters at B as $n \to \infty$ and that $\int_{C_r \cap (R-G_i)} \left| \frac{\partial U_i(z)}{\partial n} \right| ds \leqq \max_i \delta_{i,n}$. Consider a niveau curve C_r above mentioned as γ_n. Then we have the lemma.

Next, we shall consider the behaviour of $G(z, p)$ $(p \in \bar{R})$.

Lemma 2. *Put* $V_m(p) = E[z \in R: G(z, p) \geqq m]$. *Then* $\int_{\partial V_m(p)} \frac{\partial G(z, p)}{\partial n} ds$[4] $= 2\pi$ *and the Dirichlet integral* $D_{R-V_m(p)}(G(z, p)) \leqq 2\pi m$, *where* $p \in \bar{R}$ *and* $m \geqq 0$.

Proof. We shall prove the lemma in three cases:

Case 1. $p \in R$ *and* $V_m(p)$ *is compact.*

Case 2. $p \in R$ *and* $V_m(p)$ *is non-compact.*

Case 3. $p \in B$.

Case 1. $p \in R$ *and* $V_m(p)$ *is compact.* Let $\omega_n(z)$ be a harmonic function in $R_n - R_0 - V_m(p)$ such that $\omega_n(z) = 1$ on ∂R_n and $\omega_n(z) = 0$ on

4) In the sequel, $\frac{\partial}{\partial n}$ means derivative with respect to inner normal with the exception that $\frac{\partial G(z, p)}{\partial n}$ on the niveau curves of $G(z, p)$ means derivative with respect to inner or outer normal so that $\frac{\partial G(z, p)}{\partial n} \geqq 0$.

$\partial R_0 + \partial V_m(p)$. Since R^* is a Riemann surface with null-boundary, $\lim_{n=\infty} \omega_n(z) = 0$. Let $\bar{G}_n(z, p)$ and $\underline{G}_n(z, p)$ be harmonic functions in $R_n - R_0 - V_m(p)$ such that $\bar{G}_n(z, p) = \underline{G}_n(z, p) = m$ on $\partial V_m(p)$, $\bar{G}(z, p) = \underline{G}_n(z, p) = 0$ on ∂R_0 and $\bar{G}_n(z, p) = m$ on ∂R_n and $\underline{G}_n(z, p) = 0$ an ∂R_n respectively.

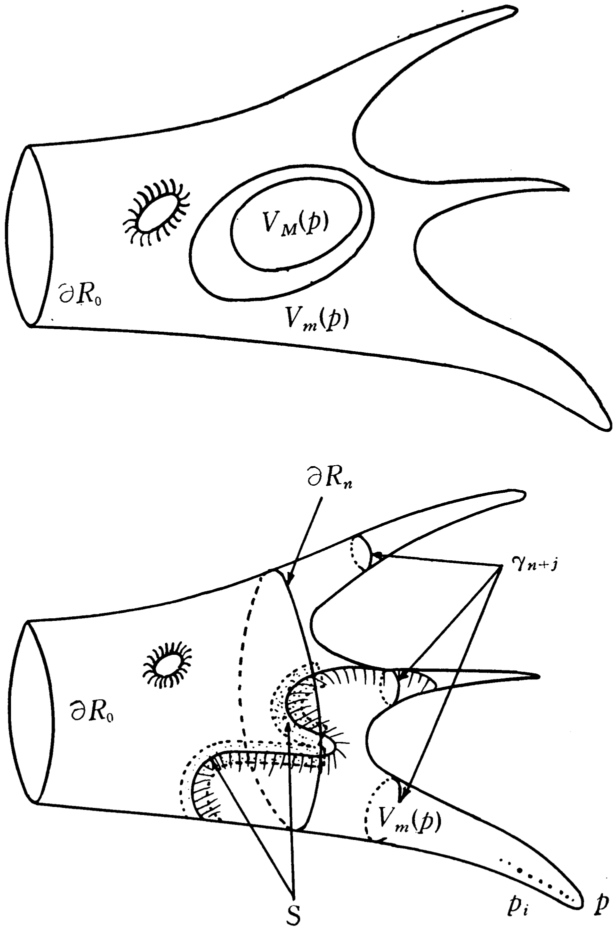

Fig. 1.

Then

$$\bar{G}_n(z, p) > G(z, p) > \underline{G}_n(z, p) \quad \text{and} \quad 0 < \bar{G}_n(z, p) - \underline{G}_n(z, p) = m\omega_n(z).$$

Hence $\lim_{n=\infty} \bar{G}_n(z, p) = G(z, p) = \lim_{n=\infty} \underline{G}_n(z, p)$.

The Dirichlet integral of $G_n(z, p)$ taken over $R_n - R_0 - V_m(p)$ is $m \int_{\partial V_m(p)} \frac{\partial G_n(z, p)}{\partial n} ds$. Therefore, we have by Fatou's lemma

$$D_{R-V_m(p)}(G(z, p)) \leqq \varliminf_{n=\infty} D_{R_n-R_0-V_m(p)}(G(z, p)) = \lim_{n=\infty} m \int_{\partial V_m(p)} \frac{\partial G_n(z, p)}{\partial n} ds$$
$$= m \int_{\partial V_m(p)} \frac{\partial G(z, p)}{\partial n} ds = 2\pi m,$$

because $\int_{\partial V_m(p)} \frac{\partial G(z, p)}{\partial n} ds = 2\pi$ is clear by the compactness of $V_m(p)$.

Case 2. *$p \in R$ and $V_m(p)$ is non-compact.* Take M large enough so that $V_M(p)$ is compact. Then by the results of the case 1, $2\pi M \geqq D_{R-V_M(p)}(G(z, p)) > D_{R-V_m(p)}(G(z, p))$. Consider $G(z, p)$ as $U(z)$ in lemma 1. Then there exists a sequence of compact curves $\{\gamma_n\}$ clustering at B such that γ_n separates B from ∂R_0 and $\lim_{n=\infty} \int_{\gamma_n - V_m(p)} \left| \frac{\partial G(z, p)}{\partial n} \right| ds = 0$. Denote by R_n' the compact component of R bounded by γ_n and ∂R_0. On the other hand, it is obvious that

$$\int_{\partial R_0} \frac{\partial G(z, p)}{\partial n} ds - \int_{\partial V_m(p) \cap R_n'} \frac{\partial G(z, p)}{\partial n} ds + \int_{\gamma_n - V_m(p)} \frac{\partial G(z, p)}{\partial n} ds = 0.$$

Since $\{\gamma_n\}$ clusters at B and $\frac{\partial G(z, p)}{\partial n} \geqq 0$ on $\partial V_m(p)$, by mentioning to the above equality, we have

$$\int_{\partial V_m(p)} \frac{\partial G(z, p)}{\partial n} ds = 2\pi.$$

The Dirichlet integral of $G(z, p)$ is

$$D_{R_n'-V_m(p)}(G(z, p)) = \int_{\partial V_m(p) \cap R_n'} G(z, p) \frac{\partial G(z, p)}{\partial n} ds + \int_{\gamma_n - V_m(p)} G(z, p) \frac{\partial G(z, p)}{\partial n} ds.$$

Since $\{\gamma_n\}$ clusters at B and the second term on the right hand side tends to zero as $n \to \infty$, we have

$$D_{R-V_m((p)}(G(z, p)) = 2\pi m.$$

Case 3. $p \in B$. Let $\{p_i\}$ be a fundamental sequence determining p. Consider the Dirichlet integral $D_{R_n-R_0-V_m(p)}(G(z, p))$. For any given positive number ε, we can find a narrow strip S such that the interior of S contains $\partial V_m(p) \cap (R_n - R_0)$, $D_{R_n-R_0-S-V_m(p)}(G(z, p)) \geqq D_{R_n-R_0-V_m(p)}(G(z, p)) - \varepsilon$ and that $R - V_m(p_i) \supset R_n - R_0 - S - V_m(p)$ for any $i \geqq i_0(S, \varepsilon)$, where $i.(S, \varepsilon)$ is a suitable number depending on S and ε, because $G(z, p_i)$ converges to $G(z, p)$ uniformly in $R_n - R_0$ and hence the niveau curves $\partial V_m(p_i)$ tend to $\partial V_m(p)$ as $i \to \infty$, (Fig. 1). Since the derivatives of $G(z, p_i)$ converge uniformly to those of $G(z, p)$ as $i \to \infty$, we have

$$D_{R_n-R_0-S-V_m(p)}(G(z,\ p))\leqq\varliminf_{i=\infty} D_{R-V_m(p_i)}(G(z,\ p_i))\leqq 2\pi m.$$

By letting $\varepsilon\to 0$ and then $n\to\infty$.

$$D_{R-V_m(p)}(G(z,\ p))\leqq 2\pi m.$$

Hence, by lemma 1, we can prove the existence of a sequence of compact curves $\{\gamma_n\}$ such that γ_n separates B from ∂R_0 and $\{\gamma_n\}$ clusters at B and that $\lim_{n=\infty}\int_{\gamma_n-V_m(p)}\left|\frac{\partial G(z,\ p)}{\partial n}\right|ds=0$. Therefore we have

$$\int_{\partial V_m(p)}\frac{\partial G(z,\ p)}{\partial n}ds=\int_{\partial R_0}\frac{\partial G(z,\ p)}{\partial n}ds=2\pi.$$

Thus we have the lemma.

Lemma 3. (*Extension of Green's formula*). *Let q be a point in $R-V_m(p)$. Then for every point $p\in\overline{R}$,*

$$\frac{1}{2\pi}\int_{\partial V_m(p)}G(z,\ q)\frac{\partial G(z,\ p)}{\partial n}ds=G(q,\ p). \qquad (1)$$

Proof. Since $q\in R$, there exists a number n' such that $R_{n'}-R_0\ni q$, whence there exists a constant L such that $G(z,\ q)\leqq L$ in $R-R_{n'}$. Hence by lemma 2, $D_{R-R_{n'}}(G(z,\ q))\leqq D_{R-V_L(p)}(G(z,\ q))\leqq 2\pi L$ and $D_{R-V_m(p)}(G(z,\ p))\leqq 2\pi m$. Therefore by lemma 1, there exists a sequence of compact curves $\{\gamma_n\}$ such that γ_n separates B from ∂R_0, $\{\gamma_n\}$ clusters at B and that both $\int_{\gamma_n}\left|\frac{\partial G(z,\ q)}{\partial n}\right|ds$ and $\int_{\gamma_n-V_m(p)}\left|\frac{\partial G(z,\ p)}{\partial n}\right|ds$ tend to zero as $n\to\infty$. Denote by R'_n the component bounded by γ_n and ∂R_0. Suppose $R'_n\subset R_{n'}$. Apply the Green's formula to $G(z,\ p)$ and $G(z,\ q)$ in $R'_n-V_m(p)$. Then

$$\int_{\partial V_m(p)\cap R'_n}G(z,\ q)\frac{\partial G(z,\ p)}{\partial n}ds=2\pi G(q,\ p)+\int_{\partial V_m(p)\cap R'_n}G(z,\ p)\frac{\partial G(z,\ q)}{\partial n}ds$$
$$+\int_{\gamma_n-V_m(p)}G(z,\ p)\frac{\partial G(z,\ q)}{\partial n}ds-\int_{\gamma_n-V_m(p)}G(z,\ q)\frac{\partial G(z,\ p)}{\partial n}ds.$$

We shall see that every term, except the first, on the right hand side tends to zero as $n\to\infty$. In fact, $\left|\int_{\partial V_m(p)\cap R'_n}G(z,\ p)\frac{\partial G(z,\ q)}{\partial n}ds\right|$ $\leqq G(z,\ p)\left|\int_{\partial V_m(p)\cap R'_n}G(z,\ q)\,ds\right|\leqq m\int_{\gamma_n\cap V_m(p)}\left|\frac{\partial G(z,\ q)}{\partial n}\right|ds\leqq m\int_{\gamma_n}\left|\frac{\partial G(z,\ q)}{\partial n}\right|ds$, $\left|\int_{\gamma_n-V_m(p)}G(z,\ p)\frac{\partial G(z,\ q)}{\partial n}ds\right|\leqq m\int_{\gamma_n-V_m(p)}\left|\frac{\partial G(z,\ q)}{\partial n}\right|ds$ and $\left|\int_{\gamma_n-V_m(p)}G(z,\ q)\frac{\partial G(z,\ p)}{\partial n}ds\right|\leqq L\int_{\gamma_n-V_m(p)}\left|\frac{\partial G(z,\ p)}{\partial n}\right|ds$. On the other hand, $G(z,\ q)\frac{\partial G(z,\ p)}{\partial n}\geqq 0$ on

$\partial V_m(p)$. Therefore we have the lemma.

We shall consider the behaviour of the topology induced by δ-metric.

Corollary. *Let $v_n(p)$ be a δ-neighbourhood of $p \in \overline{R}$, that is $v_n(p) = E\left[z \in R:\ \delta(z, p) < \frac{1}{n}\right]$. Then for any given $V_m(p)$, there exists a neighbourhood $v_n(p)$ such that*

$$V_m(p) \supset (v_n(p) \cap R).$$

Proof. The assertion is evident for $p \in R$, because our topology is homemorphic to the original one in R. Hence it is sufficient to prove the corollary for $p \in B$. Suppose that the assertion is false. Then there exists a number m_0 such that $V_{m_0}(p) \not\supset (v_{n'}(p) \cap R)$ for infinitely many numbers n'. Hence we can find a sequence of points $\{q_i\}$ in $R - V_{m_0}(p)$, tending to p with respect to δ-metric. Let $m \geqq 3m_0$. Then we can find a number n_0 by lemma 2, such that

$$\int_{\partial V_m(p) \cap (R_{n_0} - R_0)} \frac{\partial G(z, p)}{\partial n} ds \geqq \pi.$$

Since $q_i \in R - V_{m_0}(p)$, we have by (1),

$$\int_{\partial V_m(p) \cap (R_n - R_0)} G(z, q_i) \frac{\partial G(z, p)}{\partial n} ds < \int_{\partial V_m(p)} G(z, q_i) \frac{\partial G(z, p)}{\partial n} ds = 2\pi G(q_i, p) \leqq 2\pi m_0.$$

Since $\frac{\partial G(z, p)}{\partial n} \geqq 0$ on $\partial V_m(p)$, there exists one point z_i on $\partial V_m(p) \cap (R_{n_0} - R_0)$ such that $G(z_i, q_i) \leqq 2m_0$. Let i tend to ∞. They by the compactness of $\partial V_m(p) \cap (R_{n_0} - R_0)$, we have $G(z_0, p) \leqq 2m_0$, where z_0 is one of limiting points of $\{z_i\}$. This contradicts $G(z_0, p) = m \geqq 3m_0$. Therefore we have the corollary.

If two points p and q are contained in R, we have, by definition $G_n(p, q) = G_n(q, p)$, where $G_n(z, p)$ and $G_n(z, q)$ are Green's functions of $R_n - R_0$ with pole p and q respectively. Hence, by letting $n \to \infty$, we have $G(p, q) = G(q, p)$. Next, suppose $p \in B$ and $q \in R$. Let $\{p_i\}$ be one of fundamental sequences determining p. Then, since $G(p_i, q) = G(q, p_i)$ and since $G(z, p_i)$ converges to $G(z, p)$ uniformly in every compact set of R, $G(p_i, q)$ has a limit denoted by $G(p, q)$ as $p_i \to p$. More generally, suppose that a sequence $\{p_i\}$ of $\overline{R}$ tends to p with respect to δ-metric and that q belongs to R. Then we have

$$G(q, p) = \lim_{i=\infty} G(q, p_i) = \lim_{i=\infty} G(p_i, q).$$

Hence $G(z, q) (q \in R)$ has a limit when z tends to $p \in \overline{R}$ with respect to

δ-metric. In this case we define the value of $G(z, q)$ at p as this limit denoted by $G(p, q)$. Thus we have the following

Lemma 4. *If at least one of two points p and q is contained in R, then*

$$G(p, q) = G(q, p). \tag{2}$$

$G(z, q)$ is defined in $\overline{R}$ for $q \in R$ but $G(z, q)$ has been defined only in R for $q \in B$. In what follows, we shall define $G(z, q)$ in $\overline{R}$, even in case $q \in B$. For this purpose, we shall prove the following

Lemma 5. *Suppose that p and q are contained in $\overline{R}$. Let $V_m(p) = E[z \in R: G(z, p) \geqq m]$ and $V_{n'}(p) = E[z \in R: G(z, p) \geqq m']$, where $m < m'$,* i. e. *$V_m(p) \supset V_{m'}(p)$. Then*

$$2\pi\, G_{V_{m'}(p)}(p, q) = \int_{\partial V_{m'}(p)} G(z, q) \frac{\partial G(z, p)}{\partial n} ds \geqq \int_{\partial V_m(p)} G(z, q) \frac{\partial G(z, p)}{\partial n} ds$$
$$= 2\pi\, G_{V_m(p)}(p, q).$$

Proof. At first, if $p \in R$, since $G(z, q)$ is harmonic in $\overline{R}$ for $q \in \overline{R}$, $2\pi\, G(p, q) = \int_{\partial V_m(p)} G(z, q) \frac{\partial G(z, p)}{\partial n} ds$ for every $V_m(p)$ such that $V_m(p) \not\ni q$. Next, if $p \in B$ and $q \in R$, we have also by (1), $2\pi\, G(p, q) = 2\pi G(q, p) = \int_{\partial V_m(p)} G(z, q) \frac{\partial G(z, p)}{\partial n} ds$ for $V_m(p) \not\ni q$. Hence our assertion

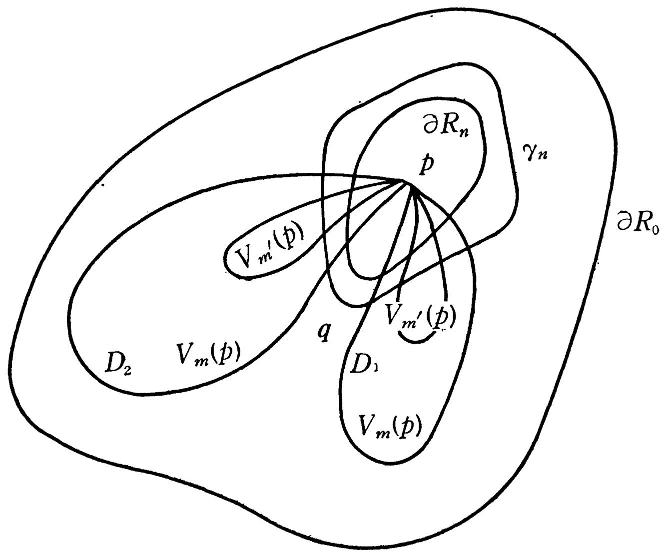

Fig. 2.

is clear if either p or q, at least belongs to R. Therefore it is sufficient to prove the lemma when both p and q belong to B. Let $\{q_j\}$ be a fundamental sequence determining q. $V_m(p)$ may consist of at most a enumerably infinite number of domains D_l ($l=1, 2, \cdots$), (Fig. 2).

Let D be one of them. Let $G_{D,n}(z, q_j)$ be a harmonic function in $D \cap (R_n - R_0)$ such that $G_{D,n}(z, q_j) = G(z, q_j)$ on $\partial D \cap (R_n - R_0)$ and $G_{D,n}(z, q_j) = 0$ on $\partial R_n \cap D$. Then we have by Green's formula

$$G(z, q_j) > G_{D,n}(z, q_j) = \frac{1}{2\pi} \int_{\partial D \cap (R_n - R_0)} G(\xi, q_j) \frac{\partial G_n^D(\xi, z)}{\partial n} ds,$$

where $G_n^D(\xi, z)$ is the Green's function of $D \cap (R_n - R_0)$ with pole at z.

Since $G_n^D(\xi, z)$ is increasing with respect to n, $\frac{\partial G_n^D(\xi, z)}{\partial n} \uparrow \frac{\partial G^D(\xi, z)}{\partial n}$ at every point ξ on ∂D, where $G^D(\xi, z)$ is the Green's function of D. Hence

$$G(z, q_j) \geqq G_D(z, q_j) = \lim_{n=\infty} G_{D,n}(z, q_j) = \frac{1}{2\pi} \int_{\partial D} G(\xi, q_j) \frac{\partial G^D(\xi, z)}{\partial n} ds.$$

We call $G_D(z, q_j)$ the solution of Dirichlet problem in D with boundary value $G(z, q_j)$ on ∂D. Let q_j tend to q. Then, since $G(\xi, q_j)$ tends to $G(\xi, q)$ at every point ξ on ∂D, we have by Fatou's lemma

$$G(z, q) = \lim_{j=\infty} G(z, q_j) \geqq \lim_{j=\infty} G_D(z, q_j) \geqq \frac{1}{2\pi} \int_{\partial D} \lim_{j=\infty} G(\xi, q_j) \frac{\partial G^D(\xi, z)}{\partial n} ds$$
$$= \frac{1}{2\pi} \int_{\partial D} G(\xi, q) \frac{\partial G^D(\xi, z)}{\partial n} ds = G_D(z, q), \qquad (3)$$

where $G_D(z, q)$ is the solution of Dirichlet problem in D with the boundary value $G(z, q)$.

Put $G^M(z, q) = \min[M, G(z, q)]$. Then $G^M(z, q)$ is superharmonic in R. Let $\bar{G}_n^M(z, q)$, $G_n^M(z, q)$ and $\underline{G}_n^M(z, q)$ be harmonic functions in $D \cap (R_n - R_0)$ such that $\bar{G}_n^M(z, q) = G_n^M(z, q) = \underline{G}_n^M(z, q) = G^M(z, q)$ on $\partial D \cap (R_n - R_0)$ and $\bar{G}_n^M(z, q) = M$, $G_n^M(z, q) = G^M(z, q)$ and $\underline{G}_n^M(z, q) = 0$ on $\partial R_n \cap D$ respectively. Then $\bar{G}_n^M(z, q) > G_n^M(z, q) > \underline{G}_n^M(z, q)$ and $\bar{G}_n^M(z, q) - \underline{G}_n^M(z, q) \leqq M\omega_n(z)$, where $\omega_n(z)$ is a harmonic function in $R_n - R_0$ such that $\omega_n(z) = 0$ on ∂R_0 and $\omega_n(z) = 1$ on ∂R_n, whence

$$G_D^M(z, q) = \lim_{n=\infty} \bar{G}_n^M(z, q) = \lim_{n=\infty} G_n^M(z, q) = \lim_{n=\infty} \underline{G}_n^M(z, q).$$

Evidently, $G_D^M(z, q)$ is the solution of Dirichlet problem in D with the boundary value $G^M(z, q)$ on ∂D and $G_D^M(z, q) = \frac{1}{2\pi} \int_{\partial D} G^M(\xi, q) \frac{\partial G^D(\xi, z)}{\partial n} ds.$

Therefore

$$\lim_{M=\infty} G_D^M(z, q) = G_D(z, q).$$

In the sequel, we denote briefly by $G_{V_m(p)}(z, q)$ the function which is equal to $G_{D_l}(z, q)$ which is the solution of Dirichlet problem in D_l with boundary value $G(z, q)$, in every domain D_l $(l=1, 2, \cdots)$.

Consider the Dirichlet integral of $G^M_{V_m(p)}(z, q)$ which is equal to the solution of Dirichlet problem $G^M_{D_l}(z, q)$ with the boundary value $G^M(z, q)$, in every domain D_l. Then by Dirichlet principle

$$\sum_l D_{D_l \cap (R_n - R_0)}(G_n^M(z, q)) \leqq \sum_l D_{D_l \cap R}(G^M(z, q)) = D_{V_m(q)}(G^M(z, q)) \leqq 2\pi M,$$

because the Dirichlet integral of $G^M(z, q)$ over R equals $D_{R-V_M(q)}(G(z, q)) \leqq 2\pi M$. Let $n \to \infty$. Then

$$D_{V_m(p)}(G^M_{V_m(q)}(z, q)) \leqq \lim_{n=\infty} \sum_l D_{D_l}(G^M_{D_l, n}(z, q)) \leqq 2\pi M.$$

Since $D_{V_m(p)}(G^M_{V_m(p)}(z, q))$ and $D_{R-V_{m'}(p)}(G(z, p))(\leqq 2\pi m')$ are bounded, there exists, by lemma 1, a sequence of compact curves $\{\gamma_n\}$ separating B from ∂R_0 such that $\{\gamma_n\}$ clusters at B and that both $L_1(\gamma_n) = \int_{\gamma_n - V_{m'}(p)} \left| \frac{\partial G(z, p)}{\partial n} \right| ds$ and $L_2(\gamma_n) = \int_{\gamma_n \cap V_m(p)} \left| \frac{\partial G^M_{V_m(p)}(z, q)}{\partial n} \right| ds$ tend to zero as $n \to \infty$. Denoting by R'_n the compact component of R bounded by γ_n and ∂R_0, apply the Green's formula to $G^M_{V_m(p)}(z, q)$ and $G(z, p)$ in $(V_m(p) - V_{m'}(p)) \cap R'_n$. Then

$$\begin{aligned}
&\int_{\partial V_m(p) \cap R'_n} G^M_{V_m(p)}(z, q) \frac{\partial G(z, p)}{\partial n} ds - \int_{\partial V_{m'}(p) \cap R'_n} G^M_{V_m(p)}(z, q) \frac{\partial G(z, p)}{\partial n} ds \\
&= \int_{\partial V_m(p) \cap R'_n} G(z, p) \frac{\partial G^M_{V_m(p)}(z, q)}{\partial n} ds - \int_{\gamma_n \cap (V_m(p) - V_{m'}(p))} G^M_{V_m(p)}(z, q) \frac{\partial G(z, p)}{\partial n} ds \\
&+ \int_{\partial V_{m'}(p) \cap R'_n} G(z, p) \frac{\partial G^M_{V_m(p)}(z, q)}{\partial n} ds + \int_{\gamma_n \cap (V_m(p) - V_{m'}(p))} G(z, p) \frac{\partial G^M_{V_m(p)}(z, q)}{\partial n} ds.
\end{aligned}$$

It can be proved, as in lemma 3, that every term on the right hand side tends to zero as $n \to \infty$, by the fact that $L_i(\gamma_n)$ $(i=1, 2)$ tends to zero. Now $G^M_{V_m(p)}(z, q) \frac{\partial G(z, p)}{\partial n} \geqq 0$ on $\partial V_m(p) + \partial V_{m'}(p)$. Hence

$$\int_{\partial V_m(p)} G^M_{V_m(p)}(z, q) \frac{\partial G(z, p)}{\partial n} ds = \int_{\partial V_{m'}(p)} G^M_{V_m(p)}(z, q) \frac{\partial G(z, p)}{\partial n} ds.$$

By letting $M \to \infty$ and by (3)

$$\int_{\partial V_m(p)} G(z, q) \frac{\partial G(z, p)}{\partial n} ds = \int_{\partial V_m(p)} G_{V_m(p)}(z, q) \frac{\partial G(z, p)}{\partial n} ds$$
$$= \int_{\partial V_{m'}(p)} G_{V_m(p)}(z, q) \frac{\partial G(z, p)}{\partial n} ds \leq \int_{\partial V_{m'}(p)} G(z, q) \frac{\partial G(z, p)}{\partial n} ds.$$

Thus we have the lemma.

Definition of $G(z, q)$ for z and q belonging to $\bar{R}$.

Since $G_{V_m(p)}(p, q) = \frac{1}{2\pi} \int_{\partial V_m(p)} G(z, q) \frac{\partial G(z, p)}{\partial n} ds$ is increasing with respect to m, $G_{V_m(p)}(p, q)$ has a limit as $m \to \infty$ which we denote by $G(p, q)$. We define the value of $G(z, q) (q \in \bar{R})$ at $p \in \bar{R}$ by this limit. It is easily seen that this definition of $G(p, q)$ coincides with what was given previously in case either p or q is contained in R. In fact, it is evident that $G_{V_m(p)}(p, q) = \frac{1}{2\pi} \int_{\partial V_m(p)} G(z, q) \frac{\partial G(z, p)}{\partial n} ds = G(p, q)$ for $p \in R$ and $V_m(p) \not\ni q$ and that, by (1) $G_{V_m(p)}(p, q) = \frac{1}{2\pi} \int_{\partial V_m(p)} G(z, q) \frac{\partial G(z, p)}{\partial n} ds$ $= G(q, p) = \lim_{i=\infty} G(q, p_i) = \lim_{i=\infty} G(p_i, q) = G(p, q)$ for $p \in B$ and $q \in R$, where $\{p_i\}$ is a fundamental sequence determining p.

Definition of Superharmonicity at a point $p \in \bar{R}$.

Suppose a function $U(z)$ in $\bar{R}$. If $U(p) \geq \frac{1}{2\pi} \int_{\partial V_m(p)} U(z) \frac{\partial G(z, p)}{\partial n} ds$ holds for the niveau curves of $G(z, p)$, we say that $U(z)$ is *superharmonic in the weak sense* at a point p.

In what follows, we shall show that $G(z, q)$ (z and $q \in \bar{R}$) defined as above, has the essential properties of the logarithmic potential in the plane. Now we have the following

Theorem 1. *The Green's function in $\bar{R}$ has the following properties:*
1) $G(p, p) = \infty$.
2) *$G(z, q)$ is lower semicontinuous in $\bar{R}$ with respect to δ-metric.*
3) *$G(z, q)$ is superharmonic in the weak sense at every point of $\bar{R}$.*
4) $G(p, q) = G(q, p)$.

Proof. 1) and 3) are clear by the definition of $G(z, q)$.

Proof of 2). Suppose that $\{p_i\}$ tends to p with respect to δ-metric. Since $G_{V_m(p)}(p, q) = \frac{1}{2\pi} \int_{\partial V_m(p)} G(z, q) \frac{\partial G(z, p)}{\partial n} ds$, there exists a number n_0 for any given positive number ε such that

$$G_{V_m(p)}(p, q) \leq \frac{1}{2\pi} \int_{\partial V_m(p) \cap (R_n - R_0)} G(z, q) \frac{\partial G(z, p)}{\partial n} ds + \varepsilon, \quad \text{for} \quad n \geq n_0.$$

Here $(R_{n_0} - R_0) \cap \partial V_m(p)$ is composed of at most a finite number of analytic curves. We make a narrow strip S in $R_{n_0+1}-R_0$ such that the interior of S containes $\partial V_m(p)\cap(R_{n_0}-R_0)$ and ∂S cuts $\partial V_m(p)$ orthogonally at the end points of $\partial V_m(p)\cap(R_{n_0}-R_0)$. We divide S into a finite number of narrow strips S_l $(l=1,2,\cdots,k)$ so that ∂S_l intersects $\partial V_m(p)$ with angles being not equal to 0 or π and map S_l onto a rectangle: $0\leqq \mathrm{Im}\,\zeta\leqq\delta$ (δ is sufficiently small), $-1\leqq Re\zeta\leqq 1$, on the ζ-plane so that every vertical line: $Re\zeta=s$ $(-1\leqq s\leqq 1)$ intersects only once $\partial V_m(p_i)$ for $j\geqq j_0$, where j_0 is a suitable number. This is possible, because $G(z,p_j)$ tend to $G(z,p)$ that is, $\partial V_m(p_j)$ tends to $\partial V_m(p)$ and

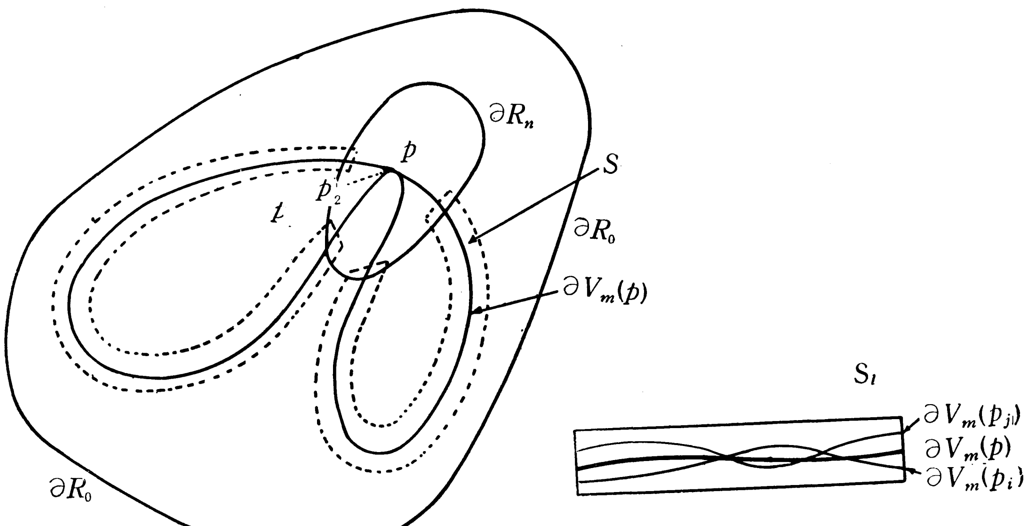

Fig. 3.

the derivatives of $G(z,p_j)$ tend to those of $G(z,p)$ on $R_{n_0}-R_0$. We make a point α_j of $\partial V_m(p_j)$ correspond to a point α of $\partial V_m(p)$ so that $Re\alpha_j=Re\alpha$. Then we have

$$\lim_{j=\infty}\int_{S\cap\partial V_m(p_j)} G(z,q)\frac{\partial G(z,p_j)}{\partial n}ds=\int_{S\cap\partial V_m(p)} G(z,q)\frac{\partial G(z,p)}{\partial n}ds,$$

because $\frac{\partial G(\alpha_j,p_j)}{\partial n}ds\geqq 0$ and uniformly bounded in S, $\frac{\partial G(\alpha_j,p_j)}{\partial n}ds$ $\to\frac{\partial G(\alpha,p)}{\partial n}ds$ and $G(\alpha_j,p_j)\to G(\alpha,p)$. Hence

$$\varliminf_{j=\infty} 2\pi G_{V_m(p_j)}(p_j,q)=\varliminf_{j=\infty}\int_{\partial V_m(p_j)} G(z,q)\frac{\partial G(z,p_j)}{\partial n}ds$$

$$\geqq\varliminf_{j=\infty}\int_{\partial V_m(p_j)\cap(R_{n_0}-R_0)} G(z,q)\frac{\partial G(z,p)}{\partial n}ds\geqq\int_{\partial V_m(p)} G(z,q)\frac{\partial G(z,p)}{\partial n}ds$$

$$-\varepsilon=2\pi G_{V_m(p)}p,q)-\varepsilon,$$

whence by letting $\varepsilon \to 0$,

$$\varliminf_{j=\infty} G_{V_m(p_j)}(p_j, q) \geqq G_{V_m(p)}(p, q).$$

Hence $G_{V_m(p)}(p, q)$ is lower semicontinuous at p for fixed m. Since $G_{V_m(p)}(p, q) \uparrow G(p, q)$, $G(p, q)$ is also lower semicontinuous at p. Therefore $G(z, q)$ is lower semicontinuous in $\overline{R}$.

Proof of 4). If p or q belongs to R, 4) is clear by (2). We suppose that both p and q belong to B. Let ξ and η be points in R. Then by (1) and (2) we have the following

$$G(p, \eta) = G(\eta, p) = \frac{1}{2\pi} \int_{\partial V_m(p)} G(z, \eta) \frac{\partial G(z, p)}{\partial n} ds \quad \text{for} \quad \eta \notin V_m(p), \tag{4}$$

$$G(p, \eta) = G(\eta, p) \geqq \frac{1}{2\pi} \int_{\partial V_m(p)} G(z, \eta) \frac{\partial G(z, p)}{\partial n} ds \quad \text{for} \quad \eta \in V_m(p). \tag{5}$$

Since $G_{V_m(p)}(p, q) = \frac{1}{2\pi} \int_{\partial V_m(p)} G(\xi, q) \frac{\partial G(\xi, p)}{\partial n} ds$ and since $\{V_m(q)\}$ clusters at B as $n \to \infty$, there exists a number n for any given positive number ε, such that

$$G_{V_m(p)}(p, q) - \varepsilon \leqq \frac{1}{2\pi} \int_{\partial \underline{V}_m(p)} G(\xi, q) \frac{\partial G(\xi, p)}{\partial n} ds,$$

where $\partial \underline{V}_m(p)$ is the part of $\partial V_m(p)$ outside of $V_n(q)$.

Suppose that ξ is on $\partial V_m(p)$, then $\xi \notin V_n(q)$, whence

$$G(\xi, q) = G(q, \xi) = \frac{1}{2\pi} \int_{\partial V_n(q)} G(\eta, \xi) \frac{\partial G(\eta, q)}{\partial n} ds.$$

Accordingly we have

$$\begin{aligned} G_{V_m(p)}(p, q) - \varepsilon &\leqq \frac{1}{4\pi^2} \int_{\partial \underline{V}_m(p)} \left(\int_{\partial V_n(q)} G(\eta, \xi) \frac{\partial G(\eta, q)}{\partial n} ds \right) \frac{\partial G(\xi, p)}{\partial n} ds \\ &= \frac{1}{4\pi^2} \int_{\partial V_n(q)} \left(\int_{\partial \underline{V}_m(p)} G(\xi, \eta) \frac{\partial G(\xi, p)}{\partial n} ds \right) \frac{\partial G(\eta, q)}{\partial n} ds. \end{aligned}$$

Now by (4) and (5)

$$\begin{aligned} \frac{1}{2\pi} \int_{\partial \underline{V}_m(p)} G(\xi, \eta) \frac{\partial G(\xi, p)}{\partial n} ds &\leqq \frac{1}{2\pi} \int_{\partial V_m(p)} G(\xi, \eta) \frac{\partial G(\xi, p)}{\partial n} ds \\ &= G(\eta, p) = G(p, \eta) \quad \text{for} \quad \eta \notin V_m(p). \end{aligned}$$

$$\begin{aligned} \frac{1}{2} \int_{\partial \underline{V}_m(p)} G(\xi, \eta) \frac{\partial G(\xi, p)}{\partial n} ds &\leqq \frac{1}{2\pi} \int_{\partial V_m(p)} G(\xi, \eta) \frac{\partial G(\xi, p)}{\partial n} ds \\ &\leqq G(\eta, p) = G(p, \eta) \quad \text{for} \quad \eta \in V_m(p). \end{aligned}$$

On the other hand,

$$G_{V_n(q)}(q, p)=\frac{1}{2\pi}\int_{\partial V_n(q)} G(p, \eta)\frac{\partial G(\eta, q)}{\partial n}ds.$$

Hence

$$G_{V_m(p)}(p, q)-\varepsilon\leqq\frac{1}{4\pi^2}\int_{\partial V_n(q)}\left(\int_{\partial V_m(p)} G(\xi, \eta)\frac{\partial G(\xi, p)}{\partial n}ds\right)\frac{\partial G(\eta, q)}{\partial n}ds$$

$$\leqq\frac{1}{2\pi}\int_{\partial V_n(q)} G(\eta, p)\frac{\partial G(\eta, q)}{\partial n}ds=G_{V_n(q)}(q, p).$$

Thus by letting $\varepsilon\to 0$,

$$G_{V_m(p)}(p, q)\leqq G_{V_n(s)}(q, p).$$

Since the inverse inequality holds for the other pair of $V_{m'}(p)$ and $V_{n'}(q)$ and since $G_{V_m(p)}(p, q)\uparrow G(p, q)$ and $G_{V_n(q)}(q, p)\uparrow G(q\ p,)$, we have 4).

Transfinite Diameter. Let A be a δ-closed subset of B (closed with respect to δ-metric). We define the transfinite diameter of A of order n as follows:

$$1/{}_AD_n=\frac{1}{2\pi{}_nC_2}\Big(\inf\sum_{\substack{p_s,\ p_t\in A\\ s<t,\ s=1,\ t=1}}^{n,\ n} G(p_s, p_t)\Big).$$

Then we have the following:

a) *From the definition, it is clear that* $A_1\supseteqq A_2$ *implies* ${}_{A_1}D_n\geqq{}_{A_2}D_n$.

b) *Put* $\bar{\Omega}_m=\bar{R}-R_m+\partial\Omega_m$ *and let* $1/{}_{\bar{\Omega}_m}D_n=\frac{1}{2\pi{}_nC_2}\Big(\inf\sum_{p_s,\ p_t\in\bar{\Omega}_m}^{n,n} G(p_s, p_t)\Big)$.

Then every p_t *is situated on* $\partial\Omega_m$.

In fact,

$$\sum_{s<t}^{n,n} G(p_s, p_t)=\sum_{\substack{i,\ j\neq s\\ i\neq j}}^{n} G(p_i, p_j)+\sum_{i\neq s}^{n} G(p_s, p_i).$$

The sum of the first term does not depend on p_s and by 2) of Theorem 1, $\sum_i G(p_s, p_i)=U(p_s)$ is superharmonic at every point p_s of $\bar{R}$ for fixed p_i. We make $V_M(p_i)$ correspond to every point p_i $(i\neq s)$ such that $U(p_s)\geqq M$ in $V_M(p_i)$, where $M\geqq\min_{p_s\in\partial\Omega_m} U(p_s)+1$. Since $U(p_s)$ is δ-lower semicontinuous, $U(p_s)$ attains its the minimum m_0 at z_0 on a δ-closed set $\bar{\Omega}_m$ ($\bar{\Omega}_m$ is the closure of Ω_m). We show that $z_0\in\partial\Omega_m$. $U(p_s)$ does not attain its minimum in $(\bar{\Omega}_m-\sum_i V_M(p_i))\cap R$ by the minimum principle, because $U(p_s)$ is harmonic and bounded in $(\bar{\Omega}_m-\sum_i V_M(p_i))\cap R$ and R^* is a Riemann surface with null-boundary. Next, suppose, $U(z_0)\leqq m_0$ $=\min_{p_s\in\partial\Omega_m} U(p_s)$ $(z_0\in B)$. Then by 3) of Theorem 1, $U(z_0)\geqq\frac{1}{2\pi}\int_{\partial V_{M'}(z_0)} U(z)$

$\times \frac{\partial G(z, z_0)}{\partial n} ds$, where M' is large so that $V_{M'}(z_0) = E[z \in R: G(z, z_0) \geqq M']$ is contained in Ω_m, whence there exists at least one point z' in $\Omega_m \cap R$ such that $U(z') \leqq m_0$. This contradicts the minimum principle. Hence $U(p_s)$ attains its minimum on $\partial\Omega_m$. Therefore every p_t is on $\partial\Omega_m$.

We can discuss mass distributions on $\bar{R}$ by $G(z, p)$, that is, the potential of an unit mass at p is given by $G(z, p)$ and we can define also the energy integral of mass distributions as in space. In our case, since $\partial\Omega_m$ is compact, it is easily proved that there exists the unique unit mass distribution μ on $\bar{\Omega}_m$ called the equilibrium distribution, whose energy $I(\mu)$ is minimal and that whose potential $U(z) = \int G(z, p) \, d\mu(p)$ is a constant on $\partial\Omega_m$, that is, $U(z) = \omega_m(z)$, where $\omega_m(z)$ is a harmonic function in $R_m - R_0$ such that $\omega_m(z) = 0$ on ∂R_0, $\omega_m(z) = M_m$ on ∂R_m and $\int_{\partial R_0} \frac{\partial \omega_m(z)}{\partial n} ds = 2\pi$. Moreover, it is easily proved by (b) as in space that the transfinite diameter ${}_{\bar{\Omega}_m}D = \lim_{n=\infty} {}_{\bar{\Omega}_m}D_n$ is equal to $1/I(\mu) = 1/2\pi M_m$.

Given a system of n points $p_1, p_2, \cdots, p_n$ on A, we can choose an $(n+1)_{st}$ point p ($p = p(p_1, p_2, \cdots, p_n)$) on A such that

$$V(p) = (\sum_{i=1}^{n} G(p, p_i))/2\pi n$$

is minimal, because the above function is δ-lower semicontinuous on A. Let ${}_AV_n$ be the least upper bound of the minimum above defined as $p_1, p_2, \cdots, p_n$ vary on A. Then there exists a system $(p_1^*, p_2^*, \cdots, p_n^*)$ such that

$$V(p, p_1^*, p_2^*, \cdots, p_n^*) \geqq {}_AV_n - \frac{1}{2\pi n} \qquad \text{for} \quad p \text{ on } A.$$

Denote by $V(z)$ the potential

$$V(z) = \frac{1}{2\pi n} (\sum_{i=1}^{n} G(z, p_i^*)) .$$

This is the potential of a certain distribution of equal point mass on A of total mass unity and it is clear that $V(z) \geqq {}_AV_n - \frac{1}{2\pi n}$ for all points of A admitting ∞ as a possible value of either member. Furthermore, since $V(z)$ is δ-lower semicontinuous, $\varliminf_{z_j \to q \in A} V(z_j) \geqq {}_AV_n - \frac{1}{2\pi n}$ for every sequence $\{z_j\}$ tending to A with respect to δ-metric.

Now, since $G(p_i, p_j) = G(p_j, p_i)$,

$$\binom{n+1}{2} / {}_AD_{n+1} = \frac{1}{2\pi} \min_{p_\kappa, p_i \in A} \Big(\sum_{i<\kappa}^{1, 2, \cdots, n+1} G(p_i, p_\kappa) \Big) \leqq \frac{1}{2} \cdot \frac{1}{2\pi} \sum_{\substack{\kappa=1 \\ \kappa \neq i}}^{n+1} \Big(\sum_{i=1}^{n+1} G(p_i, p_n) \Big) .$$

Hence ${}_AV_n \geqq 1/{}_AD_{n+1}$, whence

$$V(z) \geqq 1/{}_AD_{n+1} - \frac{1}{2\pi n} \qquad \text{on } A.$$

Since $A \subset \bar{\Omega}_m$ for every m and $\lim_{m=\infty} M_m = \infty$.

$$\infty = 1/{}_AD = \lim_{n=\infty} 1/{}_AD_n = \lim_{n=\infty} \Big(\sum_{p_s,\, p_t \in A}^{n,\, n} G(p_s,\, p_t)/{}_nC_2 \Big).$$

Therefore, for any given large number M, we can find a system of $n(M)$ points $p_1, p_2, \cdots, p_n$ such that the function

$$V(z) = \frac{1}{2\pi n} \Big(\sum_{i=1}^{n} G(z,\, p_i) \Big) \geqq M \qquad \text{on } A.$$

Theorem 2. *Let A be a δ-closed subset of B. Then there exists a potential $U(z)$ such that* 1°. *$U(z)$ is harmonic in R.* 2°. *$U(z) = 0$ on ∂R_0.* 3°. *The flux of $U(z)$ along ∂R_0 is 2π.* 4°. $\lim_{z \to A} U(z) = \infty$.

Proof. Let N be an integer larger than 3. Then since $\lim_{n=\infty} {}_AD_n = 0$, there exists, for any positive integer m, $n(N, m)$ number of points $p_1, p_2, \cdots, p_n$ such that

$$V^m(z) = \frac{1}{2\pi n} \Big(\sum_{i=1}^{n} G(z,\, p_i) \Big) \geqq N^m \qquad \text{on } A.$$

Put $\sum_{m=1}^{\infty} V^m(z)/2^m = U(z)$. Then, clearly $U(z)$ is the function required.

For an F_σ set of R, the capacity of F_σ is defined usually. Let A be an F_σ subset of $\bar{R}$ of capacity zero. Then both $A \cap R$ (R is open) and $A \cap B$ are F_σ sets. Hence we have at once the following

Corollary. *Let A be an F_σ subset of $\bar{R}$ of capacity zero. Then there exists a potential $U(z)$ satisfying the four conditions of Theorem 2.*

Let $\{G_n\}$ be a decreasing sequence of non compact subsurfaces of R with compact relative boundaries $\{\partial G_n\}$ such that $\bigcap_{n>1} G_n = 0$. Two such sequences $\{G_n\}$ and $\{G_n'\}$ are called equivalent if for given m, there exists a number n such that $G_m \supset G_n'$ and $G_m' \supset G_n$. We consider that any equivalent sequences determine an unique ideal boundary component. Denote the set of all the ideal components by $\underline{B}$. A topology is introduced on $R + \underline{B} + \partial R_0$ by the usual manner and it is easily seen that $R + \underline{B} + \partial R_0$ and $\underline{B}$ are closed and compact. Let $\underline{A}$ be a closed subset of $\underline{B}$ and let A be the set of ideal boundary points on $\underline{A}$. Then since $\{G(z,\, p_i)\}$ for $p_i \in A$ is a normal family, A is also a δ-closed set. Hence we have

Theorem 3. *Let A be the subset of B on a closed subset $\underline{A}$ of $\underline{B}$.*

Then there exists a harmonic function $U(z)$ *satisfying the conditions of Theorem* 2 *and moreover* 5°. $\overline{\lim_{z\to q\notin A}} U(z)<\infty$.

It is sufficient to prove that the condition 5° is satisfied, since the other four conditions are clearly satisfied. Let q be a point of the complementary set of A. Then there exists a component $G(q)$ of $R-R_m$ (m is a suitable number with a compact relative boundary $\partial G(q)$ such that $G(q)\ni q$ and $G(q)\cap A=0$. Then $\max_{z\in\partial G(q)} U(z)\leqq M$, which implies $\sup_{z\in G(q)} U(z)\leqq M$, by the maximum principle, because $U(z)$ is harmonic and bounded in $G(q)$ and R^* is a Riemann surface with a null-boundary.

Corollary. *Let A be the subset of* $\overline{R}$ *on an* F_σ *subset of* $R+\underline{B}$ *of capacity zero. Then there exists a harmonic function* $U(z)$ *satisfying the conditions of Theorem* 3.

R. S. Martin defined the ideal boundary points by the use of the function $K(z, p)=\dfrac{G(z, p)}{G(0, p)}$, where 0 is a fixed point of R. However, in case R^* is a Riemann surface with null-boundary, since $G(z, 0)\geqq\delta>0$ in $R-R_{n'}$, $K(z, p)$ is a multiple of $G(z, p)$, where $R_{n'}\in 0$. $G(z, p)$ plays consequently the same role as $K(z, p)$. Hence Martin's assertions hold even in our case.

Let $U(z)$ be a positive harmonic function in R vanishing on ∂R_0. If $U(z)\geqq V(z)>0$ implies $V(z)=KU(z)$ for any harmonic function $V(z)$ in R, $U(z)$ is called a minimal function. Martin proved that every minimal function is a multiple of some $G(z, p)$ $(p\in B)$ and that every positive harmonic function vanishing on ∂R_0 is represented uniquely by an integral form of minimal functions.

The condition 5° *of Theorem* 3 *is not always satisfied under the assumptions of Theorem* 2, *that is,a positive harmonic function* $U(z)$ *such that* $U(z)=\infty$ *on a* δ*–closed set A and* $U(z)<\infty$ *except on A does not always exist.*

Example. Suppose that there exist n minimal function $G(z, p_i)$ $(i=1, 2, \cdots, n)$ with pole p_i on a boundary component p. Then every Green's function $G(z, p^*)$[6] with pole p^* on p, being not minimal, must be a linear from $G(z, p^*)=\sum_{i=1}^{n} c_i G(z, p_i)$ $(c_i\geqq 0,\ \sum_{i=1}^{n} c_i=1)$. Put $A=\bigvee_i^n p_i$. Then clearly A is a δ–closed set and $\delta(p^*, A)>0$. Denote by $U(z)$ a positive harmonic function in Theorem 2, that is, $U(z)=0$ on ∂R_0, $\int_{\partial R_0}\frac{\partial U(z)}{\partial n}ds=2\pi$ and $U(z)=\infty$ at every point of A. Then

5) See 4).
6) Clearly, there exists a fundamental sequence $\}p_i^*\}$ determining p^*.

$$U(z) = \int G(z, q_\alpha)\, d\mu(q_\alpha) \quad (q_\alpha \in B)\,.$$

By the symmetry of the Green's function,

$$U(p^*) = \int G(p^*, q_\alpha)\, d\mu(q_\alpha)^{7)} = \int G(q_\alpha, p^*)\, d\mu(q_\alpha) = \int \sum_{i=1}^{n} c_i G(q_\alpha, p_i)\, d\mu(q_\alpha)$$
$$= \sum_{i=1}^{n} c_i \int G(p_i, q_\alpha)\, d\mu(q_\alpha) = \sum_{i=1}^{n} c_i U(p_i)\,.$$

Hence $U(z) = \infty$ on A implies $U(p^*) = \infty$. Therefore any positive harmonic function that is infinite at every point of A must be infinite at any point of B lying on p. Thus there exists no positive harmonic function infinite only on A.

As an application to classification of types of Riemann surfaces, we have

Theorem 4. *R^* is a Riemann surface with null-boundary, if and only if there exists a harmonic function $U(z)$ with one negative logarithmic singularity at a point of R^* such that $U(z)$ has limit ∞ as z tends to B.*

Proof. If the function above stated exists, R^* is clearly a Riemann surface with null-boundary and it is easy to construct the function in this theorem from the function in Theorem 3, by putting $A=B$ and by the alternating process of Schwarz.

Many other applications, for instance, to Nevanlinna's first and second fundamental theorems, will be omitted here.

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7) Since $G_{V_m(p^*)}(p^*, q_\alpha)$ is measurable for fixed p^* and since $G_{V_m(p^*)}(p^*, q_\alpha) \uparrow G(p^*, q_\alpha)$ for $q_\alpha \in B$, $\lim_{m=\infty} \int G_{V_m(p^*)}(p^*, q_\alpha) d\mu(q_\alpha) = \int \lim_{m=\infty} G_{V_m(p^*)}(p^*, q_\alpha) d\mu(q_\alpha)$.

Hence $U(p^*) = \lim_{m=\infty} U_{V_m(p^*)}(p^*) = \lim_{m=\infty} \int_{\partial V_m(p^*)} (\int G(z, q_\alpha) d\mu(q_\alpha)) \frac{\partial G(z, p^*)}{\partial n} ds =$ $\lim_{m=\infty} \int G_{V_m(p^*)}(p^*, q_\alpha) d\mu(q_\alpha) = \int (\lim_{m=\infty} G_{V_m(p^*)}(p^*, q_\alpha)) d\mu(q_\alpha) = \int G(p^*, q_\alpha) d\mu(q_\alpha)$.