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Indecomposable Completely Simple Semigroups Except Groups

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§1. If a semigroup S is homomorphic onto a semigroup T, a factor semigroup of S is obtained, namely, S is decomposed into a union of subsets by gathering elements of S mapped into the same element of T. Among all homomorphisms of S, there are two kinds of special cases: isomorphisms and a mapping of all elements of S to the one-element semigroup, which are called trivial homomorphisms. By an indecomposable semigroup we mean a semigroup without non-trivial homomorphism. As is well known a group is indecomposable if and only if it is simple. Of course finite semigroups of order at most 2 are indecomposable, and we shall call them as trivial cases. It is clear that an indecomposable semigroup has no proper ideal[1]. Otherwise we could consider Rees' difference semigroup of it modulo the proper ideal so that it would have a non-trivial homomorphism. In this paper we shall investigate a structure of indecomposable completely simple semigroups except groups [1].

According to Rees [1], a completely simple semigroup is represented as a regular matrix semigroup. In this paper, we shall use without special explanation the same terminology and notations as Rees'. Let G' denote a group G with zero 0 adjoined. Let P be an (M, L)-matrix, $(p_{\mu\lambda})$, $\mu \in M$, $\lambda \in L$, elements of which belong to G', satisfying the conditions that for any suffix $\mu \in M$ at least one $p_{\mu\lambda} \neq 0$, and that for any suffix $\lambda \in L$ at least one $p_{\mu\lambda} \neq 0$. Then a regular matrix semigroup S with a defining matrix P is defined to be a semigroup whose non-zero elements are all (L, M)-matrices $(x)_{\alpha\beta}$,[2] x varying over G, α over L, β over M, and the multiplication in S is defined as

$$(x)_{\alpha\beta}(y)_{\gamma\delta} = (x p_{\beta\gamma} y)_{\alpha\delta} .$$

In some cases S may contain a zero-matrix **0**, elements of which are all

1) By a proper ideal of S we mean a two-sided ideal distinct from S itself and from a set of only zero.

2) Denote by $(x)_{\alpha\beta}$ a matrix $X=(z_{\lambda\mu})$ where $z_{\lambda\mu}=x$ if $(\lambda, \mu)=(\alpha, \beta)$, and $z_{\lambda\mu}=0$ if $(\lambda, \mu)\neq(\alpha, \beta)$.

zero 0, and to which $(0)_{\lambda\mu}$ shall be equal, if necessary, and $X\mathbf{0}=\mathbf{0}X=\mathbf{0}$ for all $X\in S$. In detail, if P has at least one $p_{\mu\lambda}=0$, S must contain naturally the zero-matrix $\mathbf{0}$, but if P has no $p_{\mu\lambda}=0$, there are two cases of S: the one where $\mathbf{0}$ is contained and the other where not so. It follows that the former[3] S is indecomposable if and only if the number of non-zero elements of S is one, i.e., $S=\{\mathbf{0}, A\}$ where $\mathbf{0}A=A\mathbf{0}=\mathbf{0}$, $A^2=A$, being a trivial case. Hence the former case is out of consideration in the present paper.

§ 2. Now let D be the set of all ordered pairs (λ, μ) where $\lambda\in L$, $\mu\in M$, written $D=L\times M$ and we distinguish it from $D'=M\times L$. Noticing suffixes of elements of a defining matrix P, let $E=\{(\mu, \lambda)\,;\, p_{\mu\lambda}=0\}$. E satisfies the following conditions due to the conditions of P.

(C_1) For any $\lambda\in L$, there is μ such that $(\mu, \lambda)\,\bar{\in}\, E$.

(C_2) For any $\mu\in M$, there is λ such that $(\mu, \lambda)\,\bar{\in}\, E$.

These conditions imply directly that E is a proper subset of D'. For this E, we define a multiplication system $\bar{D}$ as following:

(1) If E is empty, $\bar{D}$ consists of all elements of D and the multiplication is $(\lambda_1, \mu_1)(\lambda_2, \mu_2)=(\lambda_1, \mu_2)$.

(2) If E is not empty, $\bar{D}$ is D adjoined with zero 0_D,

and
$$0_D(\lambda, \mu)=(\lambda, \mu)0_D=0_D^2=0_D,$$

$$(\lambda_1, \mu_1)(\lambda_2, \mu_2)=\begin{cases} 0_D & \text{if } (\mu_1\ \lambda_2)\in E, \\ (\lambda_1, \mu_2) & \text{otherwise.}\end{cases}$$

Then it is easily shown that the correspondence of S to $\bar{D}$ defined in the following manner is a homomorphism.

$$(x)_{\alpha\beta}\to(\alpha, \beta), \quad \mathbf{0}\to 0_D \qquad \text{if } S \text{ has } \mathbf{0}.$$

Lemma 1. *S is homomorphic to $\bar{D}$.*

We can assume that at least one of L and M contains distinct elements. Because, if not so, $p_{11}\neq 0$ and the mapping $(xp_{11}^{-1})_{11}\to x$ is an isomorphism between S and the group G; this case is excluded in the initial assumption of this paper.

Suppose that G' contains at least two elements different from 0, then the homomorphism is non-trivial by Lemma 1. Accordingly we have

Lemma 2. *If S is indecomposable, then G is composed of only one*

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3) This means S which contains $\mathbf{0}$, but P of which has no $p_{\mu\lambda}=0$.

element, i.e., $G' = \{0, g\}$, $0g = g0 = 0$, $g^2 = g$; *in other words, if S is indecomposable, the homomorphism mentioned in Lemma* 1 *must be an isomorphism.*

From now on, we shall find what is a necessary condition for $\overline{D}$ with ($\mathbf{C}_1$) and ($\mathbf{C}_2$) to be indecomposable. For simplicity, $\overline{D}_0$ denotes $\overline{D}$ whose E is empty and $\overline{D}_E$ denotes $\overline{D}$ whose E is not empty.

Let us give the set L and M the multiplications $xy = x$ and $xy = y$ respectively; and the former is defined to be a left singular semigroup and the latter a right singular semigroup, and "singular" means "left singular or right singular." Then $\overline{D}_0$ is a direct product of the left singular semigroup L and the right singular semigroup M. Consequently $\overline{D}_0$ is homomorphic to L and M. It follows that $\overline{D}_0$ has a non-trivial homomorphism if both L and M consist of elements more than one.

Lemma 3. *If* $\overline{D}_0$ *is indecomposable, it is isomorphic to L or M, that is to say, it is singular.*

On the other hand, we can easily show that any classification of elements of a left (right) singular semigroup determines a factor semigroup which appears singular. Consequently

Lemma 4. *A singular semigroup of order more than* 2 *has at least one non-trivial homomorphism.*

Thus we have

Theorem 1. *There is no non-trivial indecomposable* $\overline{D}_0$.

§ **3.** Next, let us call $\overline{D}_E$ into question. First suppose that L consists of only one element. (We can treat the case of M similarly.) Denote by E_M the set of $\mu \in M$ such that $(\mu, \lambda) \in E \neq \phi$. On the other hand, it must hold that $(\mu, \lambda) \bar{\in} E$ for any $\mu \in E_M$ due to ($\mathbf{C}_2$). By this contradiction, we get

Lemma 5. *For* $\overline{D}_E$, *both L and M contain at least two elements.*

Now let $L_\mu = \{\lambda;\ (\mu, \lambda) \in E\}$ and $M_\lambda = \{\mu;\ (\mu, \lambda) \in E\}$, where we permit L_μ and M_λ to be empty. Suppose that there are distinct μ_1 and μ_2 such that $L_{\mu_1} = L_{\mu_2}$. Among elements of $\overline{D}_E$ we introduce a relation

$$0_D \sim 0_D, \quad (\lambda, \mu_1) \sim (\lambda, \mu_2) \quad \text{if} \quad L_{\mu_1} = L_{\mu_2}.$$

Of course it is not only an equivalence relation, but a congruence relation. In fact, when $(\lambda, \mu_1) \sim (\lambda, \mu_2)$, we have

$$(\lambda_2, \mu_3)(\lambda, \mu_1) = 0_D = (\lambda_2, \mu_3)(\lambda, \mu_2) \quad \text{for} \quad (\mu_3, \lambda) \in E,$$
$$(\lambda_2, \mu_3)(\lambda, \mu_1) = (\lambda_2, \mu_1) \sim (\lambda_2, \mu_2) = (\lambda_2, \mu_3)(\lambda, \mu_2) \quad \text{for} \quad (\mu_3, \lambda) \bar{\in} E.$$

$L_{\mu_1} = L_{\mu_2}$ implies that $(\mu_1, \lambda_2) \in E$ if and only if $(\mu_2, \lambda_2) \in E$.

$$\text{If} \quad (\mu_1, \lambda_2) \in E, \qquad (\lambda, \mu_1)(\lambda_2, \mu_3) = 0_D = (\lambda, \mu_2)(\lambda_2, \mu_3),$$
$$\text{if} \quad (\mu_1, \lambda_2) \bar{\in} E, \qquad (\lambda, \mu_1)(\lambda_2, \mu_3) = (\lambda, \mu_3) = (\lambda, \mu_2)(\lambda_2, \mu_3).$$

Thus it follows that the equivalence relation $(\lambda, \mu_1) \sim (\lambda, \mu_2)$ gives a non-trivial homomorphism of $\bar{D}_E$. Similarly, assuming that $M_{\lambda_1} = M_{\lambda_2}$ for some $\lambda_1 \neq \lambda_2$, the equivalence relation, $(\lambda_1, \mu) \sim (\lambda_2, \mu)$ for any $\mu \in M$, determines a non-trivial homomorphism of $\bar{D}_E$. Thus we have

Lemma 6. *If $\bar{D}_E$ is indecomposable, then $\mu_1 \neq \mu_2$ implies $L_{\mu_1} \neq L_{\mu_2}$ and $\lambda_1 \neq \lambda_2$ implies $M_{\lambda_1} \neq M_{\lambda_2}$.*

These necessary conditions are proved to be sufficient. In its proof, we should be reminded of the conditions (C_1) and (C_2) as to E.

Now let $\sim$ be any congruence relation of $\bar{D}_E$.

Lemma 7. *If there is $(\lambda_1, \mu_1) \in \bar{D} \subset D_E$ such that $0_D \sim (\lambda_1, \mu_1)$, then $0_D \sim (\lambda, \mu)$ for all $(\lambda, \mu) \in D$.*

Proof. Choose (λ, μ) arbitrarily. According to (C_1) and (C_2), there are μ_2 and λ_2 such that $(\mu_2, \lambda_1) \bar{\in} E$ and $(\mu_1, \lambda_2) \bar{\in} E$. Since $\sim$ is a congruence relation, $0_D \sim (\lambda_1, \mu_1)$ implies that

$$0_D = (\lambda, \mu_2) 0_D \sim (\lambda, \mu_2) \lambda_1, \mu_1) = (\lambda, \mu_1),$$

and further $\quad 0_D = 0_D(\lambda_2, \mu) \sim (\lambda, \mu_1)(\lambda_2, \mu) = (\lambda, \mu).$

Thus the proof of the lemma is completed.

Lemma 8. *If there are distinct (λ_1, μ_1), $(\lambda_2, \mu_2) \in D$ such that $(\lambda_1, \mu_1) \sim (\lambda_2, \mu_2)$, then $0_D \sim (\lambda, \mu)$ for all $(\lambda, \mu) \in D$, in other words, $(\lambda_1, \mu_1) \sim (\lambda', \mu')$ for all $(\lambda, \mu) \neq (\lambda', \mu')$.*

Proof. Suppose $\mu_1 \neq \mu_2$. By Lemma 6, $L_{\mu_1} \neq L_{\mu_2}$ and there is $\bar{\lambda}$ such that one of the two $(\mu_1, \bar{\lambda})$ and $(\mu_2, \bar{\lambda})$ belongs to E and the other does not to E, say, $(\mu_1, \bar{\lambda}) \in E$ and $(\mu_2, \bar{\lambda}) \bar{\in} E$. Then $(\lambda_1, \mu_1)(\bar{\lambda}, \mu) = 0_D$ and $(\lambda_2, \mu_2)(\bar{\lambda}, \mu) = (\lambda_2, \mu)$ whence $0_D \sim (\lambda_2, \mu)$. By Lemma 7, we have $0_D \sim (\lambda, \mu)$ for all $(\lambda, \mu) \in D$. Even if $\lambda_1 \neq \lambda_2$, the proof is similar. Thus we have the following theorems.

Theorem 2. *$\bar{D}_E$ is indecomposable if and only if E satisfies the following conditions:*

(1) $\mu_1 \neq \mu_2$ *implies* $L_{\mu_1} \neq L_{\mu_2}$.
(2) $\lambda_1 \neq \lambda_2$ *implies* $M_{\lambda_1} \neq M_{\lambda_2}$.

Putting the already obtained lemmas and theorems into together, we have

Theorem 3. *A regular matrix semigroup S with a defining (M, L)-matrix P is indecomposable and non-trivial if and only if*

(1) *a non-zero element of a matrix of S is none but* 1, *being an element of a semigroup* $G' = \{0, 1\}$, $01 = 10 = 0^2 = 0$, $1^2 = 1$, *and*

(2) *the defining (M, L)-matrix P fulfils the following conditions.*

(2_1) *Each of L and M has a cardinal more than* 1.

(2_2) *Each element of P is either* 0 *or* 1, *and* 0 *is certainly contained.*

(2_3) *Every two columns differ and every two rows differ.*

Thus S is completely determined by a definining matrix P.

§ **4.** As far as the isomorphism problem is concerned, Rees' theory is applicable and we have

Theorem 4. *Two indecomposable regular matrix semigroups* S, S^* *over* G' *with defining materices* P, P^* *are isomorphic if and only if* P^* *is a permutated matrix from* P. (*See foot* 4))

Moreover we add the following two theorems.

Theorem 5. *Two indecomposable regular matrix semigroups* S, S^* *over* G' *with defining matrices* P, P^* *are anti-isomorphic if and only if* P^* *is a transposed and permutated matrix from* P.

In order to prove that S and S^* are anti-isomorphic if P^* is transposed and permutated from P, it is sufficient to assume P^* to be transposed from P.

Proof of Theorem 5. Let P be an (M, L)-matrix, (p_{ji}), $j \in M$, $i \in L$. P^* is a transposed matrix from P, $P^* = (p^*_{ji})$ where $p^*_{ji} = p_{ij}$; S is composed of (L, M)-matrices $(x)_{ij}$, and S^* is composed of (M, L)-matrices $(x)_{ji}$. The mapping defined as

$$(x)_{ij} \to (x)_{ji} \quad \text{and} \quad 0_S \to 0_{S^*}$$

is clearly anti-isomorphic. Because, we have

$$(x)_{ij}(y)_{kn} = (xp_{jk}y)_{in},$$
$$(y)_{nk}(x)_{ji} = (yp^*_{kj}x)_{ni} = (yp_{jk}x)_{ni}.$$

4) P^* is a matrix gained by permutating columns and rows of P.

On the other hand, since G' is commutative, $xp_{jk}y=yp_{jk}x$, where $(x)_{ij}(y)_{kn}$ corresponds to $(y)_{nk}(x)_{ji}$. Thus it has been proved that S and S^* are anti-isomorphic.

Conversely suppose that S and S^* are anti-isomorphic. Then, according to the former half of this theorem, S is anti-isomorphic to a semigroup S' whose defining matrix P' is a transposed matrix from P. Since S^* and S' are isomorphic, Theorem 4 makes it hold that P^* is permutated from P'; hence P^* is a transposed and permutated matrix from P. The proof of the theorem has been completed.

The proof of the following theorem is very easy.

Theorem 6. *An indecomposable regular matrix semigroup S over G' with a defining matrix P has an anti-automorphism if and only if P is a square and permutated matrix from the transposed matrix P' of P.*

§ 5. Here we shall discuss finite indecomposable semigroups. The structure theory of them is included in the above theorems, for simpleness of finite semigroup in the Rees' sense implies complete simpleness. However we shall be able to investigate the type of a defining matrix more precisely.

Denote by m and l numbers of elements of M and L respectively. We shall call a matrix (a_{ji}), $j=1, \cdots, m$, $i=1, \cdots, l$, an (m, l)-matrix instead of an (M, L)-matrix. Without loss of generality, we can restrict ourselves to $2 \leqq l \leqq m$. The reason is due to Lemma 5 and Theorem 5.

If an (m, l)-matrix, $l \leqq m$, can be a defining matrix of a non-trivial indecomposable regular matrix semigroup, it is not without saying that $2 \leqq l \leqq m < 2^l$. Conversely, however, for any l, m such that $2 \leqq l \leqq m < 2^l$, there exists certainly at least an (m, l)-matrix to be a defining matrix. We can have, for example, an (m, l)-matrix P in the following manner.

(1) Every row contains 1.

(2) Every two rows differ.

(3) $p_{ji}=1$, $1 \leqq j=i \leqq l$; $p_{ji}=0$ $i \neq j$, $1 \leqq j \leqq l$, $1 \leqq i \leqq l$.

Then it is natural that every column contains 1 and every two columns differ. Accordingly we have

Theorem 7. *In order that there exists at least one (m, l)-matrix, $2 \leqq l \leqq m$, which satisfies the conditions stated in Theorem 3, it is necessary and sufficient that $l \leqq m < 2^l$.*

Immediately from the above theorems we get the following theorem.

Theorem 8. *There is a non-trivial indecomposable semigroup of order n except groups if and only if $n-1$ is decomposed into a product* *of two factors lm such that* $2 \leqq l \leqq m < 2^l$. *Then the semigroup is isomorphic or anti-isomorphic to the regular matrix semigroup composed of* $(1)_{ij}$, $i=1, \cdots, l$, $j=1, \cdots, m$, *with a zero-matrix adjoined.*

Let $f(n)$ be the number of non-isomorphic and not-anti-isomorphic indecomposable semigroups S of order n except groups, and let $g(n)$ be the number of non-isomorphic ones. Of course $f(n)$, $n \geqq 3$, is equal to the number of defining matrices giving S. We do not wish to find the form of $f(n)$, but can persist at least the following corollary.

Corollary. *$f(n)$ is unbounded.*

Proof. Let $n_p = p^2+1$ where p is a prime number. For a defining (p, p)-matrix P_p, the two defining $(p+k, p+k)$-matrices P^*_{p+k}, $\bar{P}_{p+k}$ are constructed in the following manner.

$$P^*_{p+k} = \left|\begin{array}{c|ccc} & 0 & \vdots & 0 \\ P_p & \vdots & \vdots & \vdots \\ & 0 & \vdots & 0 \\ \hline 0 \cdots 0 & 1 & & 0 \\ \cdots\cdots\cdots & & \ddots & \\ 0 \cdots 0 & 0 & & 1 \end{array}\right| \quad \bar{P}_{p+k} = \left|\begin{array}{c|ccc} & 0 & \vdots & 0 \\ P_p & \vdots & \vdots & \vdots \\ & 0 & \vdots & 0 \\ \hline 1 \cdots 1 & 1 & & 0 \\ \cdots\cdots\cdots & & \ddots & \\ 1 \cdots 1 & 0 & & 1 \end{array}\right| \text{ where } \left|\begin{array}{ccc} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{array}\right| \text{ is a unit matrix.}$$

Then it is seen that the correspondence $P_p \to P^*_{p+k}$, $P_p \to \bar{P}_{p+k}$ are one to one and any $\bar{P}_{p+k}$ can be neither permutated nor transposed from any P^*_{p+k}. From this it follows that $f(n_p) < f(n_q)$ for any prime numbers $p < q$. Hence the sequence $\{f(n)\}$ contains a monotonly increasing subsequence $\{f(n_p)\}$, completing the proof.

§ **6.** Finally we give, for simple example, all distinct types of defining matrices P for $l=2, 3$, and give the values of $f(n)$, $n \leqq 16$.

Example 1.

$l=m=2$ $\quad \overset{*}{\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}} \; \overset{*}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} \qquad l=2,\ m=3 \quad \begin{vmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{vmatrix}$

$l=m=3$ $\quad \overset{*}{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}} \; \overset{*}{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}} \; \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} \; \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \; \overset{*}{\begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}} \; \overset{*}{\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}}$

$\overset{*}{\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}} \; \overset{*}{\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}}$ $\qquad$ We mark self-dual ones with * .

$l=3,\ m=4$

$$\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\\1&1&0\end{vmatrix}\quad \begin{vmatrix}1&0&0\\0&1&0\\0&0&1\\1&1&1\end{vmatrix}\quad \begin{vmatrix}1&0&0\\0&1&0\\1&1&0\\1&0&1\end{vmatrix}\quad \begin{vmatrix}0&1&0\\0&0&1\\1&1&0\\1&0&1\end{vmatrix}\quad \begin{vmatrix}1&0&0\\0&1&0\\1&1&0\\1&1&1\end{vmatrix}\quad \begin{vmatrix}1&0&0\\0&1&0\\1&0&1\\1&1&1\end{vmatrix}\quad \begin{vmatrix}1&0&0\\1&1&0\\1&0&1\\1&1&1\end{vmatrix}\quad \begin{vmatrix}0&1&0\\1&1&0\\1&0&1\\1&1&1\end{vmatrix}\quad \begin{vmatrix}1&0&0\\1&1&0\\1&0&1\\0&1&1\end{vmatrix}\quad \begin{vmatrix}1&1&0\\1&0&1\\0&1&1\\1&1&1\end{vmatrix}$$

$l=3,\ m=5$

$$\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\\1&1&0\\1&0&1\end{vmatrix}\quad \begin{vmatrix}1&0&0\\0&1&0\\0&0&1\\1&1&0\\1&1&1\end{vmatrix}\quad \begin{vmatrix}1&0&0\\0&1&0\\1&1&0\\1&0&1\\0&1&1\end{vmatrix}\quad \begin{vmatrix}1&0&0\\0&1&0\\1&1&0\\1&0&1\\1&1&1\end{vmatrix}\quad \begin{vmatrix}0&1&0\\0&0&1\\1&1&0\\1&0&1\\1&1&1\end{vmatrix}\quad \begin{vmatrix}1&0&0\\1&1&0\\1&0&1\\0&1&1\\1&1&1\end{vmatrix}$$

$l=3,\ m=6$

$$\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\\1&1&0\\1&0&1\\0&1&1\end{vmatrix}\quad \begin{vmatrix}1&0&0\\0&1&0\\1&1&0\\1&0&1\\0&1&1\\1&1&1\end{vmatrix}\quad \begin{vmatrix}1&0&0\\0&1&0\\0&0&1\\1&1&0\\1&0&1\\1&1&1\end{vmatrix}$$

$l=3,\ m=7$

$$\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\\1&1&0\\1&0&1\\0&1&1\\1&1&1\end{vmatrix}$$

Example 2. The values of $f(n)$ and $g(n)$, $n \leqq 16$.

n	Type of P	$f(n)$	$g(n)$
1	—	1	1
2	—	4	5
3	none	0	0
4	none	0	0
5	2×2	2	2
6	none	0	0
7	2×3	1	2
8	none	0	0
9	none	0	0
10	3×3	8	10
11	none	0	0
12	none	0	0
13	3×4	10	20
14	none	0	0
15	none	0	0
16	3×5	6	12

Remarks. After I had written this paper last autumn, I heard from Dr. Theodore S. Motzkin that he obtained the same results independently of me.

(Received March 25, 1956)

Reference

[1] D. Rees: On semigroups, Proc. Cambridge Philos. Soc. **36**, 387–400 (1940).