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On Perron's Method for the Semi-Linear Hyperbolic System of Partial Differential Equations in Two Independent Variables

By Mitio NAGUMO and Yukio ANASAKO

We use the notation $\partial_x u$ for $\frac{\partial u}{\partial x}$, $\partial^2_{xy}u$ for $\frac{\partial^2 u}{\partial x \partial y}$, and write u for u_1, u_2 $\cdots$, u_k, $f(x, y, u)$ for $f(x, y, u_1, u_2, \cdots, u_k)$. $f(x, y)$ is said to be of C^1 class in a region D if $f(x, y)$ and all its first partial derivatives are continuous in D. In this note we shall consider the system of partial differential equations

$$(1) \qquad \partial_x u_i = \lambda_i(x, y)\partial_y u_i + f_i(x, y, u) \quad (i = 1, 2, \cdots, k)$$

where variables and functions are all real valued.

O. Perron[1] had discussed the Cauchy problem for the system of equations (1) under the condition that $\lambda_i, f_i, \partial_y\lambda_i, \partial_y f_i, \partial_{u_\mu} f_i, \partial^2_{yu_\mu} f_i$, $\partial^2_{u_\mu u_\nu} f_i (i, \mu, \nu = 1, \cdots, k)$ exist and are continuous in some region respectively.

The purpose of this note is to give such an elementary proof for the existence of solution of (1) as by Perron but under weaker assumption. We assume only the continuity of the first derivatives of λ_i, f_i except for $\partial_x\lambda_i$, $\partial_x f_i$ while the proof goes merely by a modification of Perron's method.

Recently A. Douglis[2] proved the existence of the solution of equations ot much more general type where is assumed only the continuity of the first derivatives of the functions in the form of equations. Our result is only a special case of Douglis' theorem, but it may be not insignificant to give an essentially simpler proof for this case.

1. As in Perron's theorem the proof of our theorem is also based on the following

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1) "Über Existenz und Nichtexistenz von Integralen partieller Differentialgleichungssysteme in reellen Gebieten". Math. Zeit. **27** 549-564 (1928).

2) "Some existence theorems for hyperbolic systems of partial differential equations in two independent variables". Commun. on Pure & Appl. Math. **5** (1952), 119-154. See also: K. O. Friedrichs: "Nonlinear hyperbolic differential equations for functions of two independent variables". Amer. Jour. of Math. **70** (1948), 558-589.

Lemma: *Let* $\lambda(x, y)$, $f(x, y)$, $\partial_y\lambda(x, y)$, $\partial_y f(x, y)$ *be continuous in*

$$B_0:\quad 0\leqq x\leqq a,\ |y|+Kx\leqq b$$

and suppose

$$|\lambda(x, y)|\leqq K,\ |\partial_y\lambda(x, y)|\leqq L,\ |f(x, y)|\leqq g(x),\ |\partial_y f(x, y)|\leqq h(x),$$

where a, b, K, L *are positive constants and* $g(x)$, $h(x)$ *are integrable in* $0\leqq x\leqq a$. *Let* $\varphi(y)$ *be of class* C^1 *in* $|y|\leqq b$.

Then there exists one and only one function $u(x, y)$ *such that*

(i) $u(x, y)$ *is of class* C^1 *in* B_0 *and* $u=u(x, y)$ *is a solution of the equation*

$$\partial_x u=\lambda(x, y)\partial_y u+f(x, y) \qquad \text{in } B_0$$

(ii) $u(0,y)=\varphi(y)$ *for* $|y|\leqq b$.

(iii) $|u(x, y)-\Phi(x, y)|\leqq\int_0^x g(\xi)d\xi$,

$$|\partial_y u(x, y)-\partial_y\Phi(x, y)|\leqq e^{Lx}\int_0^x h(\xi)d\xi$$

where $\Phi(x, y)$ *is the solution of* $\partial_x\Phi=\lambda(x, y)\partial_y\Phi$ *with the initial condition* $\Phi(0, y)=\varphi(y)$.[3)]

The proof remains essentially the same as in Perron's work[4)].

2. Our object is to prove

Theorem 1. *Let* $\varphi_i(y)$ *be of class* C^1 *in* $|y|\leqq b$, *and* $\Phi_i(x, y)$ *be the solution of* $\partial_x\Phi=\lambda_i(x, y)\partial_y\Phi$ *such that* $\Phi_i(0, y)=\varphi_i(y)$. *Let* $\lambda_i(x, y)$ *and* $f_i(x, y, u)$ $(i=1, \cdots, k)$ *be continuous in*

$$B_0:\quad 0\leqq x\leqq a,\ |y|+Kx\leqq b \quad \text{and}$$
$$B_1:\quad 0\leqq x\leqq a,\ |y|+Kx\leqq b,\ |u_i-\Phi_i|\leqq C,\ (i=1, \cdots, k)$$

respectively, and

$$|\lambda_i(x, y)|\leqq K,\ |f_i(x, y, u)|\leqq M,$$

where a, b, c, K *and* M *are positive constants.*
Let $\partial_y\lambda_i(x, y)$, $\partial_y f_i(x, y, u)$, $\partial_{u_j} f_i(x, y, u)$ $(i, j=1, \cdots, k)$ *exist and be continuous in* B_0, B_1, B_1 *respectively.*

Then there exists exactly one set of functions $u_i(x, y)$ $(i=1, \cdots, k)$ *such that*

(i) $u_i(x, y)$ *is of class* C^1 *in*

$$B_0':\quad 0\leqq x\leqq l,\ |y|+K\leqq b \ \text{ where } l=\mathrm{Min}\,(a, \frac{c}{M}).$$

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3) $\Phi(x, y)$ is of class C^1 in B_0, the existence of $\Phi(x, y)$ is also clear.

4) ibid. (1)

And $|u_i(x, y)-\Phi_i(x, y)|\leqq c$.

(ii) $u_i=u_i(x, y)$ *is a solution of the system of differential equations*

$$(1) \qquad \partial_x u_i=\lambda_i(x, y)\partial_y u_i+f_i(x, y, u) \quad (i=1, \cdots, k).$$

(iii) $u_i(0, y)=\varphi_i(y)$ *for* $|y|\leqq b$.

3. We prove the theorem by method of successive approximations. Set $u_{i,0}=\Phi_i(x, y)$ $(i=, \cdots, k)$ and define $u_{i,n+1}(x, y)$ by the recursion's formula

$$(2) \qquad \partial_x u_{i,n+1}=\lambda_i(x, y)\partial_y u_{i,n+1}+f_i(x, y, u_n)$$

$$\text{with } u_{i,n+1}(0, y)=\varphi_i(y) \quad (i=1, \cdots, k).$$

$u_{i,n+1}(x, y)$ exist for all n and are of class $C^1[B_0']$. This is proved by Lemma using the inequality

$$(3) \qquad |u_{i,n+1}(x, y)-\Phi_i(x, y)|\leqq Mx\leqq c \quad (i=, 1, \cdots, k).$$

There exist constants L and M' such that $|\partial_y\lambda_i|\leqq L$, $|\partial_y\Phi_i|\leqq M'$ in B_0, $|\partial_y f_i|\leqq M'$, $|\partial_{u_j} f_i|\leqq M'$ in B_1 for all i, j. We shall prove

$$(4) \qquad |\partial_y u_{i,n}(x, y)-\partial_y\Phi_i(x, y)|\leqq (M'+kM'^2)e^{\mu_0 x} \qquad \text{in } B_1 \text{ for all } n, i$$

where

$$(5) \qquad \mu_0=(1+kM')e^{La} \ (>0) .$$

Evidently (4) holds for $n=0$. If (4) is true for some n then

$$\begin{aligned}|\partial_y f_i(x, y, u_n(x, y))| &= |\partial_y f_i+\sum_{j=1}^{k}\partial_{u_j} f_i\cdot\partial_y\Phi_j+\sum_{j=1}^{k}\partial_{u_j} f_i(\partial_y u_{j,n}-\partial_y\Phi_j)| \\ &\leqq M'+kM'^2+kM'(M'+kM'^2)e^{\mu_0 x} \\ &\leqq (M'+kM'^2)(1+kM')e^{\mu_0 x} .\end{aligned}$$

Hence by Lemma and (5)

$$\begin{aligned}|\partial_y u_{i,n+1}-\partial_y\Phi_i| &\leqq e^{La}(M'+kM'^2)\frac{1+kM'}{\mu_0}e^{\mu_0 x} \\ &=(M'+kM'^2)e^{\mu_0 x}\end{aligned}$$

then (4) holds for all n.

Thus there exists a constant G such that

$$(6) \qquad |\partial_y u_{i,n}|\leqq G \qquad \text{in } B_0 \text{ for all } n \text{ and } i.$$

4. Next we shall prove that the sequence $\{u_{i,n}\}$ converges uniformly for any i. There hold next equations

$$\partial_x(u_{i,m+1}-u_{i,n+1})=\lambda_i(x, y)\partial_y(u_{i,m+1}-u_{i,n+1})+f_i(x, y, u_m)-f_i(x, y, u_n).$$

Then from $|\partial_{u_j} f_i| \leqq M'$

$$|f_i(x, y, u_m)-f_i(x, y, u_n)| \leqq kM' e^{\mu_0 x} ||| u_m - u_n |||,$$

where $$||| u_m - u_n ||| \equiv \underset{\substack{1\leqq i\leqq k\\(x,y)\in B_0'}}{\mathrm{Max}} | e^{-\mu_0 x}\{u_{i,m}(x, y) - u_{i,n}(x, y)\} |$$

and μ_0 is defined by (5). Then we get from Lemma

$$|u_{i,m+1} - u_{i,n+1}| \leqq \frac{kM'}{\mu_0} e^{\mu_0 x} ||| u_m - u_n |||.$$

Thus

$$(7) \qquad ||| u_{m+1} - u_{n+1} ||| \leqq \frac{kM'}{\mu_0} ||| u_m - u_n |||.$$

Set

$$\alpha = \limsup_{N\to\infty} \underset{m,n\geqq N}{\mathrm{l.\,u.\,b.}} ||| u_m - m_n |||.$$

Then $0 \leqq \alpha \leqq 2c < +\infty$, and from (7)

$$\alpha \leqq \frac{kM'}{\mu_0}\alpha, \quad \text{where} \quad 0 \leqq \frac{kM'}{\mu_0} = \frac{kM'}{(1+kM')e^{La}} < 1.$$

Hence $\alpha = 0$, namely

$$||| u_m - u_n ||| \to 0 \quad \text{as} \quad m, n \to \infty.$$

Consequently $\| u_m - u_n \| \ (\leqq e^{\mu_0 a} ||| u_m - u_n |||) \to 0$ as $m, n \to \infty$

where $$\| u_m - u_n \| \equiv \underset{\substack{1\leqq i\leqq k\\(x,y)\in B_0'}}{\mathrm{Max}} | u_{i,m}(x, y) - u_{i,n}(x, y) |,$$

i. e. $\{u_{i,n}\}$ converges uniformly in B_0'. We set then

$$(8) \qquad \lim_{n\to\infty} u_{i,n} = u_i \qquad \text{(uniformly)} \quad (i = 1, \cdots, k).$$

5. Now we shall prove the uniform convergence of $\{\partial_y u_{i,n}(x, y)\}$. Since $\partial_y f_i$, $\partial_{u_j} f_i$ are continuous in B_1 and $\{u_n\}$ converges uniformly in B_0', for arbitrary $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for $m, n \geqq N(\varepsilon)$

$$(9) \qquad \begin{array}{l} |\partial_y f_i(x, y, u_m) - \partial_y f_i(x, y, u_n)| < \varepsilon \\ |\partial_{u_j} f_i(x, y, u_m) - \partial_{u_j} f_i(x, y, u_n)| < \varepsilon \end{array} \quad (i, j = 1, \cdots, k) \text{ in } B_0'.$$

Now set $$H_{i,m,n}(x, y) = f_i(x, y, u_m(x,y)) - f_i(x, y, u_n(x, y)),$$

then from (9) and (6)

$$\begin{aligned} |\partial_y H_{i,m,n}| \leqq\ & |\partial_y f_i(x, y, u_m) - \partial_y f_i(x, y, u_n)| \\ & + \sum_{j=1}^{k} \{ |\partial_{u_j} f(x, y, u_m) - \partial_{u_j} f_i(x, y, u_n)| \, |\partial_y u_{j,m}| \\ & + |\partial_y u_{j,m} - \partial_y u_{j,n}| \, |\partial_{u_j} f_i(x, y, u_n)| \} \end{aligned}$$

$$|\partial_y H_{i,m,n}| < \varepsilon + \varepsilon kG + M' e^{\mu_0 x} \sum_{j=1}^{k} |e^{-\mu_0 x}(\partial_y u_{j,m} - \partial_y u_{j,n})|$$
$$\leqq \varepsilon(1+kG)e^{\mu_0 x} + kM' \, ||| \partial_y u_m - \partial_y u_n ||| \, e^{\mu_0 x} .$$

Hence we get from Lemma for $m, n \geqq N(\varepsilon)$ as $\partial_y(u_{i,m+1} - u_{i,n+1}) = \lambda_i(x, y) \times \partial_y(u_{i,m+1} - u_{i,n+1}) + H_{i,m,n}$

$$|\partial_y u_{,m+1} - \partial_y u_{,n+1}| < e^{La}\left\{\frac{\varepsilon(1+kG)}{\mu_0} + \frac{kM'}{\mu_0} \, ||| \partial_y u_m - \partial_y u_n ||| \right\} e^{\mu_0 x} ,$$

then

$$(10) \qquad ||| \partial_y u_{i,m+1} - \partial_y u_{i,n+1} ||| < e^{La}\left\{\frac{\varepsilon(1+kG)}{\mu_0} + \frac{kM'}{\mu_0} \, ||| \partial_y u_m - \partial_y u_n ||| \right\} .$$

Now we set

$$\beta = \limsup_{\nu \to \infty} \underset{m, n \geqq \nu}{\text{l. u. b.}} \, ||| \partial_y u_m - \partial_y u_n ||| .$$

Then $0 \leqq \beta \leqq 2G < \infty$, and from (10) $\beta \leqq \varepsilon(1+kG)e^{La}\mu_0^{-1} + kM'e^{La}\beta\mu_0^{-1}$ or

$$\left(1 - \frac{kM'e^{La}}{\mu_0}\right)\beta \leqq \frac{\varepsilon(1+kG)e^{La}}{\mu_0} .$$

As $0 \leqq kM'e^{La}\mu_0^{-1} < 1$ from (5) and $\varepsilon > 0$ is arbitrarily small we have

$$\beta = 0 . \quad \text{i. d.} \quad ||| \partial_y u_m - \partial_y u_n ||| \to 0 \quad \text{as } m, n \to \infty .$$

The uniform convergence of $\{\partial_y u_{i,n}\}$ in B_0' is thus proved. Hence we have from (8)

$$(11) \qquad \partial_y u_{i,n} \to \partial_y u_i \quad \text{(uniformly)} \quad (i = 1, \cdots, k) .$$

From (2), (8) and (11) we obtain

$$(12) \qquad \partial_x u_{i,n} \to \partial_x u_i \quad \text{(uniformly)} \quad (i = 1, \cdots, k),$$

and from (2), (8), (11) and (12)

$$\partial_x u_i = \lambda_i(x, y)\partial_y u_i + f_i(x, y, u) \quad (i = 1, \cdots, k) .$$

The existence of a desired system of solutions for (1) is thus proved. As the proof of the uniquness of the system of solutions is easy, we shall omit it.

6. From Theorem 1 immediately follows next

Theorem 2: *Let $\lambda_{ij}(x, y)$ be of class C^2 (except that $\partial_{xx}^2 \lambda_{ij}(x, y)$ need not exist) in*

$$B_0: \ 0 \leqq x \leqq a, \ |y| \leqq b,$$

where a, b are positive constants.

Let the characteristic equation:

$$\begin{vmatrix} \lambda_{11}-F & \lambda_{12} & \lambda_{13} \cdots\cdots\cdots & \lambda_{1k} \\ \lambda_{21} & \lambda_{22}-F & \lambda_{23} \cdots\cdots\cdots & \lambda_{2k} \\ \cdot & \cdot & \cdot \cdots\cdots\cdots & \cdot \\ \lambda_{k1} & \cdot & \cdot \cdots\cdots\cdots & \lambda_{kk}-F \end{vmatrix} = 0$$

have real distinct roots and let these roots $F_i(x, y)$ $(i=1, \cdots, k)$ *satisfy*

$$|F_i(x, y)| \leqq K.$$

Let $\varphi_i(y)$ *be of class* $\mathbf{C}^1$ *in* $|y| \leqq b$. *Let* $f_i(x, y, u)$ *be of class* $\mathbf{C}^1$ *(except that* $\partial_x f_i$ *need not exist) in*

$$B_1: \quad 0 \leqq x \leqq a, \quad |y| \leqq b, \quad |u_i - \varphi_i| \leqq c, \quad (i=1, \cdots, k)$$

where c is a positive constant.

Then there exists exactly one system of functions $u(x, y)$ *such that*

(i) $u_i(x, y)$ *is of class* $\mathbf{C}^1$ *in*

$$B_2: \qquad 0 \leqq x \leqq a' \leqq a, \quad |y| + Kx \leqq b$$

where a' *is a positive constant which is determined by* $\lambda_{i,j}$, φ_i *and* f.

And $$|u_i(x, y) - \varphi_i(x, y)| \leqq c.$$

(ii) $u_i = u_i(x, y)$ *satisfies the system*

$$\partial_x u_i = \sum_{j=1}^{k} \lambda_{ij}(x, y) \partial_y u_j + f_i(x, y, u) \quad (i=1, \cdots, k).$$

(iii) $u_i(0, y) = \varphi_i(y)$ in $|y| \leqq b$.

Proof: See Perron's work[5].

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5) ibid. (1)