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An Example of a Null-Boundary Riemann Surface

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We have proved that the Green's function is not[1] uniquely determined, when its pole is at an ideal boundary point of a null-boundary Riemann surface. M. Heins introduced[2] the notion of the minimal function due to R. S. Martin[3] and constructed a boundary point of dimension of preassigned number and conjectured that there would exist a boundary point of dimension infinity. We show by an example that his conjecture holds good.

1) Example. We denote by G the domain bounded by straight lines L_1, L_2 and the semi-circle C such that

$$L_1: 1 \leqq |z| < \infty, \quad \arg z = 0, \quad L_2: 1 \leqq |z| < \infty, \quad \arg z = \pi$$
$$C: |z| = 1, \quad 0 \leqq \arg z \leqq \pi.$$

On G we define a sequence of slits such that

$$I_1^i: 2^{i-1} \leqq |z| \leqq 2^i - \frac{1}{2^{i4}}, \quad \arg z = \frac{\pi}{2} : i = 2, 3, 4, \ldots$$

$$I_2^i: 2^{i-1} \leqq |z| \leqq 2^i - \frac{1}{2^{2i4}}, \quad \arg = \frac{\pi}{4} : i = 3, 4, 5, \ldots$$

....

$$I_n^i: 2^{i-1} \leqq |z| \leqq 2^i - \frac{1}{2^{ni4}}, \quad \arg z = \frac{\pi}{2^n} : i = n+1, n+2, \ldots$$

....

$$n = 1, 2, 3, \ldots$$

Let G^1 and G^2 be the same examplars with the same boundary and connect G^1 with G^2 by identifying L_1, L_2 and $\{I_j^i\}$ of them, to con-

1) Z. Kuramochi: Potential theory and its applications, I, Osaka Math. J. **3** (1951), 123–174.

2) M. Heins: Riemann surfaces of infinite genus, Annals of Math. **55** (1952), 296–317.

3) R. S. Martin: Minimal positive harmonic functions, Trans. Amer. Math. Soc. **19** (1941), 137–172.

struct the symmetric surface with respect to $L_1+L_2+\{I^i_j\}$. We denote such a Riemann surface by F; then F has only one compact relative bonudary lying on C and is of infinite genus and further has it one ideal boundary point at $z=\infty$, and it is clear that F has a null ideal boundary.

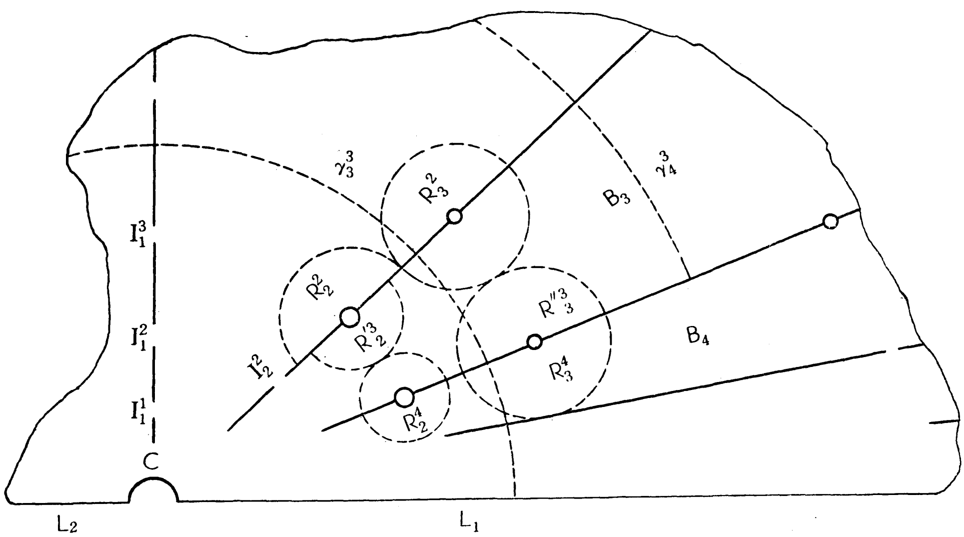

Fig. 1.

2) Let B_n be the subsurface of F with projection on the part $\frac{\pi}{2^n} \leqq \arg z \leqq \frac{\pi}{2^{n-1}}$. Then B_n has boundary on $|z|=1$, $\frac{\pi}{2^n} \leqq \arg z \leqq \frac{\pi}{2^{n-1}}$ and

$$\bar{J}^1_n:\ 1 \leqq |z| \leqq 2^{n-1},\ \arg z = \frac{\pi}{2^{n-1}},\ \overset{+}{J}{}^n_1:\ 1 \leqq |z| \leqq 2^n,\ \arg\ z = \frac{\pi}{2^n}$$

and

$$\bar{J}^i_n:\ 2^i \geqq |z| \geqq 2^i - \delta^i_n,\ \arg z = \frac{\pi}{2^{n-1}}:\ \delta^i_n = \frac{1}{2^{ni4}}:\ i = n+1,\ n+2, \dots$$

$$\overset{+}{J}{}^i_{n+1}:\ 2^i \geqq |z| \geqq 2 - \delta^i_{n+1},\ \arg z = \frac{\pi}{2^n}:\ \delta^i_{n+1} = \frac{2}{2^{(n+1)i4}}:\ i = n+2,\ n+3, \dots$$

We transform B_n by the mapping $w = \pm(ze^{-\frac{\pi}{2^n}})^{2^n}$, where $\pm$ corresponds to the mapping of upper or lower exemplars respectively, then B_n is mapped onto the w-plane slits ${}^+J^i_w$, ${}^-J^i_w$ lying on $\arg w=0$, or $\arg w=\pi$ and having the boundary on $|w|=1$, and ${}^-J^1_n$, ${}^+J^1_n$,

$${}^+J^i_w:\quad (2^i - \delta^i_n)^\alpha \leqq |w| \leqq 2^{i\alpha}:\quad \alpha = 2^n,$$
$${}^-J^i_w:\quad (2^i - \delta^i_{n+1})^\alpha \leqq |w| \leqq 2^{i\alpha}.$$

Then we have

$$^{+}J^{i}_{w}:\ 2^{i\alpha}-\overset{w}{\delta}{}^{i}_{n}\leqq|w|\leqq 2^{i\alpha},\quad \frac{2^{n}}{2^{i+1}}\delta^{n}_{i}\leqq\overset{w}{\delta}{}^{i}_{n}\leqq\frac{2^{n}}{2^{i}}\delta^{n}_{i}$$

$$^{-}J^{i}_{w}:\ 2^{i\alpha}-\overset{w}{\delta}{}^{i}_{n+1}\leqq|w|\leqq 2^{i\alpha},\quad \frac{2^{n}}{2^{i+1}}\delta^{n+1}_{i}\leqq\overset{w}{\delta}{}^{i}_{n+1}\leqq\frac{2^{n}}{2^{i}}\delta^{n+1}_{i}.$$

Denote by $\omega^{+i}(w)$ the harmonic measure of $^{+}J^{i}_{w}$ with respect to the domain $|w|>1$. Then we have by elementary calculation the following inequality

$$\omega^{+}_{i}(w)\leqq\frac{\log\left|\dfrac{1-\bar{a}w}{a-w}\right|}{\log\left|\dfrac{a^{2}-1}{\delta^{w}_{i}}-a\right|}:\quad a=(2^{i})^{\alpha}.$$

On the other hand, denote by $U^{j}_{n}(p)$ the harmonic function on F in the part $|z|<\gamma^{n}_{j}:\ \gamma^{n}_{j}=\left(\dfrac{2^{j\alpha}+2^{(j+1)\alpha}}{2}\right)^{\frac{1}{\alpha}}$, such that $U^{j}_{n}(p)=\log|z|$, when $|z|=\gamma^{n}_{j}$, $\frac{\pi}{2^{n}}\leqq\arg z\leqq\frac{\pi}{2^{n+1}}$ and $U^{j}_{n}(p)=0$, when $|z|=\gamma^{n}_{j}$, $\pi\geqq\arg z\geqq\frac{\pi}{2^{n-1}}$ or $\frac{\pi}{2^{n}}\geqq\arg z\geqq 0$ and further $U^{j}_{n}(p)=0$, when $|z|=1$. Since $U^{j}_{n}(p)\geqq 0$, we define $U_{n}(p)$ by a uniformly convergent subsequence $\{U^{j}_{n}(p)\}$; then it is clear $U_{n}(p)\leqq\log|z|$. On the other hand, let $V^{j}_{n}(p)$ be a harmonic function such that $V^{j}_{n}(p)$ is harmonic in $B_{n}\cap\{|z|<\gamma^{n}_{j}\}$, $V^{n}_{j}(p)=\log|z|$ on $\gamma^{n}_{j}\cap B_{m}$, $V^{j}_{n}(p)=0$ on $2^{i}-\delta^{i}_{n}\leqq|z|\leqq 2^{i}$, $\arg z=\frac{\pi}{2^{n-1}}$ or and $2^{i}-\delta^{i}_{n+1}\leqq|z|\leqq 2^{i}$, $\arg z=\frac{\pi}{2^{n}}$ and on $\bar{J}^{1}_{n}$, $^{+}J^{1}_{n}$ and consider $V^{*}(z)=\log|z|-\sum_{i=n}^{\infty}\log 2^{i}\omega^{+i}(z)-\sum_{i=n}^{\infty}\log 2^{i}\omega^{-i}(z)$, where $\omega^{+i}(z)$ $(\omega^{-i}(z))$ is the harmonic measure of the boundary of B_{n} lying on $\{J^{i}_{n}\}$, $\{J^{i}_{n+1}\}$. Then we have $V^{*}(z)\leqq 0$, $z\in\sum_{i=n}^{\infty}\{I^{i}_{n},I^{i}_{n+1}\}$. Consider $\omega^{+i}(z)$: $|z^{j}|=\gamma^{n}_{j}$, $\arg z_{j}=\frac{3\pi}{2^{n+1}}$, i.e. the value of $\omega^{+i}(w)$ at $w=e^{\pm\frac{\pi}{2}}\,r$: $r=\dfrac{2^{j\alpha}+2^{(j+1)\alpha}}{2}$. Then we have

$$\omega^{+}_{i}(z_{j})\leqq\frac{\log\sqrt{\dfrac{1+a^{2}r^{2}}{a^{2}+r^{2}}}}{\log\left|\left(\dfrac{a^{2}-1}{{}^{w}\delta^{n}_{i}}-a\right)\right|}$$

$$\leqq\frac{\frac{1}{2}\log 2a^{2}}{\log\left(\dfrac{a^{2}}{2\delta_{i}}\right)}\leqq\frac{\frac{1}{2}(\log 2)(1+i\alpha)}{i\alpha\log 2-\log 2+i^{4}n\log 2}\leqq\frac{1+i\alpha}{2i^{4}n},\quad a=(2^{i})^{\alpha}.$$

Thus

$$\sum_{i=n}^{\infty}\log 2^{i}\,\omega^{+i}_{n}(z)+\sum_{i=n}\log 2^{i}\,\omega^{-i}_{n}(z)\quad\text{at}\quad z_{j}\ (j=n,\ n+1,\ \ldots)$$

$$\leqq \log 2\left(\sum_{i=n}^{\infty}\frac{i(1+i\alpha)}{2i^4 n}+\sum_{i=n}^{\infty}\frac{i(1+i\alpha)}{2i^4(n+1)}\right)+k_1<k_2<+\infty ,$$

where k_i is a finite constant, from which follows the unboundedness of $V^*(z)$ at z_j $(j=1, 2, \dots)$, and hence $U_n(p)\geqq V_n^j(p)\geqq V^*(p)$ yields the non-constancy of $U_n(p)$.

3) Next we consider the Dirichlet integral of $U_n(p)$ on F. In B_{n-1} and B_{n+1} and we denote by R_j^{n-1} and R_j^{n+1}, the ring-domains contained in $F-B_n$ with projection such that

$$R_j^{n-1}:\ \frac{\delta_j^{n-1}}{2}\leqq|z-p_j^{n-1}|\leqq 2^j\ \sin\frac{\pi}{2^{n+2}}:\ 0\leqq\arg z-p_j^{n-1}\leqq\pi$$

$$R_j^{n+1}:\ \frac{\delta_j^{n}}{2}\leqq|z-p_j^{n+1}|\leqq 2^j\ \sin\frac{\pi}{2^{n+3}}:\ 0\leqq\arg z-p_j^{n+1}\leqq\pi$$

respectively where

$$p_j^{n-1}:\ \arg z=\frac{\pi}{2^{n-1}},\quad |z|=\left(2^j-\frac{\delta_j^n}{2}\right)$$

$$p_j^{n+1}:\ \arg z=\frac{\pi}{2^{n}},\quad |z|=\left(2-\frac{\delta_j^{n+1}}{2}\right).$$

Then we have

$$\mathfrak{M}_j^{n-1}=\text{module of } R_j^{n-1}=\log\frac{\dfrac{2^j\sin\frac{\pi}{2^{n+2}}}{2}}{\dfrac{2}{2^{nj^4}}}\geqq(j^4n+j-n-2)\log 2+\log\pi\geqq$$

$$(j^4n+j+n-2)\log 2 \text{ and}$$

$$\mathfrak{M}_j^{n+1}=\text{module of } R_j^{n+1}\geqq(j^4(n+1)+j-n-3)\log 2.$$

We denote by $w_n^i(p)$ the harmonic function such that

$$w_n^i(p)\text{ is harmonic in }(F-B_n)\frown\{|z|<\frac{1}{2}(2^i+2^{i+1})\}$$

$$w_n^i(p)=0:\ |z|=\frac{1}{2}(2^i+2^{i+1}),\ z\in\{C_i\frown(F-B_n)\}:\ C_r=|z|=\frac{2^i+2^{i+1}}{2}$$

$$w_n^i(p)=\log 2^j:\ 2^j-\delta_n^j\leqq|z|\leqq 2^j,\quad \arg z=\frac{\pi}{2^{n-1}}:\ j=n,\ n+1,\ \dots,$$

$$w_n^i(p)=\log 2:\ \ 2^j-\delta_{n+1}^j|z|\leqq 2^j,\quad \arg z=\frac{\pi}{2^n}:\ j=n+1;\ n+2,\ \dots.$$

Then

$$D_{F-B_n}(U_n^i(p))\leqq D_{F-B_n}(w_n^i(p)),$$

and further $\tilde{w}_n^i(p)$ is a continuous function such that

$$\tilde{w}_n^i(p)=\log 2^j:\quad |z-p_n^j|\leqq\frac{\delta_n^i}{2},\qquad z\in B_{n-1}: j=n,\ \dots$$

$$\tilde{w}_n^i(p)=\log 2^j:\quad |z-p_{n+1}^j|\leqq\frac{\delta_{n+1}^i}{2},\quad z\in B_{n+1}: j=n+1,\ \dots.$$

In R_j^{n-1} and R_j^{n+1} and $\tilde{w}_n^i(p)$ is harmonic and

$$\tilde{w}_n^i(p)=\log 2^j:\quad |z-p|=\frac{\delta_j^n}{2},$$

$$\tilde{w}_n^i(p)=0\qquad :\quad |z-p|=\sin\frac{\pi}{2^n},$$

$$\tilde{w}_n^i(p)=\log 2^j:\quad |z-p_j^{n+1}|=\frac{\delta_j^{n+1}}{2},$$

$$\tilde{w}_n^i(p)=0\qquad :\quad |z-p_j^{n+1}|=2^j\sin\frac{\pi}{2^{n+1}},$$

$$\tilde{w}_n^i(p)=0\qquad :\quad p\in F-B_n-(\sum_j R_j^{n-1}+\sum_j R_j^{n+1}).$$

Then by Dirichlet principle

$$D_{F-B_n}(w_n^i(p))\leqq \underset{F-B_n}{D}(\tilde{w}_n^i(p))\leqq\sum_j\frac{(\log 2^j)^2}{\mathfrak{M}_j^{n-1}}+\sum_j\frac{(\log 2^j)^2}{\mathfrak{M}_j^{n+1}}+A,$$

for every i, where $A<\infty$.

Thus

$$D_{F-B_n}(U_n(p))\leqq \underset{\substack{i=\infty\\ F-B_n}}{D}(U_n^i(p))\leqq \underset{\substack{i=\infty\\ F-B_n}}{D}(\tilde{w}_n^i(p)))<+\infty.$$

4) Since F has a null-boundary, $D_F(U_n(p))=\infty$, because if $D_F(U_n(p))<\infty$, it follows $U_n(p)=0$, whence

$$D_{B_n}(U_n(p))=\infty.$$

Since $$D_{B_n}(U_m(p))<\infty,\quad\text{if}\quad m\neq n,$$

all $U_n(p)$ are linearly independent.

We show in reality that 1°) $U_n(p)$ are all minimal functions, and 2°) each B_n has only one minimal function.

We denote by B_j^n the ring-domain $2^j\leqq|z|\leqq 2^{j+1}-\delta_n^{j+1}$, $\frac{\pi}{2^n}\leqq\arg z\leqq\frac{\pi}{2^{n-1}}$ contained in B_n, and denote by $\max U_{p\in\gamma}(p)$ the maximum when p is on $B_n\cap C_j$: $C_j=\{|z|=\gamma_n^j\}$. If there exists at least a Jordan curve J in B_j^n starting from p_0, which is on C_j, and reaching at least one boundary component of the ring $2^j\leqq|z|\leqq 2^{j+1}-\delta_n^{j+1}$, $\frac{\pi}{2^n}\leqq\arg z\leqq\frac{\pi}{2^{n-1}}$, and if we denote by $\omega(p)$ the harmonic measure of J with respect to this ring, then there exists a constant K depending only on the module of this ring such that

$$\underset{p\in c_j}{\text{Min}}\, U(p) \geqq K\omega(p)\ \underset{p\in c_j}{\text{Max}}\, U(p) \geqq K\ \text{Max}\, U_{p\in c_j}(p)\,.$$

5) According to R. S. Martin's theorem any positive harmonic function can be expressed uniquely by a linear form of minimal functions, thus

$$U=\int V d\mu\,,$$

therefore there exists at least a function V such that $KU_n \geqq V$: $K<\infty$.

a) *V is unbounded on $|z|=\gamma^j$ in B_n.*

Proof. If $V(p)\leqq M<\infty$, we define $V_j^1(V_j^2(p))$ such that $V_j^1(p)(V_j^2(p))$ is harmonic in $|z|<\gamma^j$, $V_j^1=0$ $(V_j^2=M)$: $p\in C_j\cap B_n$ and $V_j^1=V_j^2=V$: $p\in C_j\cap(F-B_n)$, and take $V^1(p)$, $V^2(p)$ from the uniformly convergent sequences $\{V_j^1\}$, $\{V_j^2\}$. Then $0\leqq V_j^2-V_j^1\leqq M\omega_j(p)$, where $\omega_j(p)$ is the harmonic measure of $C_j\cap B_n$ with respect to domain of F contained in $|z|\leqq\gamma^j$. Since F has a null-boundary, we have $V^1(p)=V^2(p)$. On the other hand, let $U_j(p)$ be the harmonic function in $|z|\leqq\gamma^j$ such that $U_j(p)=0: p\in C_j\cap B_n$, $U_j(p)=U(p): p\in C_j\cap(F-B_n)$, and let $U^*(p)$ be a harmonic function obtained by taking a uniformly convergent subsequence from $\{U_j(p)\}$. We construct ring-domains contained in B_n such that

$$R_i'^{n}\ :\ \delta_i^n\leqq|z-p_i^n|\leqq 2^i\sin\frac{\pi}{2^{n+2}}\,,\qquad 0\leqq(\arg z-p_i^n)\leqq\pi$$

Fig. 1.

$$R_i'^{n+1}:\ \delta_i^{n+1}\leqq|z-p_i^{n+1}|\leqq 2^i\sin\frac{\pi}{2^{n+2}}\,,\quad 0\leqq(\arg z-p_i^{n+1})\leqq\pi$$

where

$$|p^n|=\frac{1}{2}\left(2^i-\frac{1}{2^{(n+1)i4}}\right),\ \arg p_i^n=\frac{\pi}{2^{n-1}}\,,$$

$$|p_i^{n+1}|=\frac{1}{2}\left(2^i-\frac{2}{2^{(n+1)i4}}\right),\ \arg p_i^{n+1}=\frac{\pi}{2^n}\,,$$

and define a continuous function as in the case (3). Then we have

$$D_{B_n}(U_n^*(p))<\infty\,,\quad\text{and}\quad D_F(U_n^*(p))<\infty\,.$$

This implies that $U_n^*(p)=0$. By assumption $V\leqq U$, $V\leqq M$ in B_n,

we have
$V = V^1 = V^2$, $V^1_j \leqq U_j(p)$, and it follow that $V^1 \equiv U^*(p) \equiv V \equiv 0$, therefore V is not bounded on $C^j \cap B_n$ and by (4) $V(z)$ is not bounded on the sequence $\{z_i\} : |z_i| = \gamma^i$, $\arg z_i = \dfrac{3\pi}{2^n}$.

b) *$V(p)$ is invariant by generalized extremisation.*[4]

Let $V_j(p)$ be harmonic in $F \cap \{|z| < \gamma^j\}$, $V_j(p) = V(p) : p \in C_j \cap B_n$ $V_j(p) = 0 : p \in C_j \cap (F - B_n)$: From the unboundedness of $V(p)$ on $\{z_i\}$ and from $V(p) \leqq U(p)$, we can prove as (2) and (3), that there exists a harmonic function $V^*(p)$ from $\{V_i(p)\}$, such that

$$D_{F-B_n}(V^*(p)) < \infty, \quad V^*(p) \not\equiv \text{const.}$$

Since $|V_j(p) - V(p)| \leqq 2U(p)$ on $\arg z = \dfrac{\pi}{2^n}$ or $\arg z = \dfrac{\pi}{2^{n-1}}$, $V(p) = V(p) : p \in C_j \cap B_n$, and hence we have by the same manner used in (2) (3), $D_F(V^*(p) - V(p)) < \infty$, therefore $V^*(p) = V(p)\{V^*(p)$ is obtained by generalized extremisation from $V(p)\}$ and thus, since $U(p) \geqq V(p) \geqq 0.$, it follows that $V(p)$ is invariant by generalized exteremisation with respect to B_n.

c) *There is only one minimal function smaller than $U(p)$.*

Since $V^*(p) \neq 0$, if there are two functions $V_1(p) \geqq V_2(p)$ such that $V_i(p) \leqq U(p)$, then there are two constants K_1, K_2 such that

$$\overline{\lim_j} \operatorname{Max} V_{i\ p \in c_j}(p) = K_i \log |z| : \quad i = 1, 2,$$

but from (4) there exist constants K_3, K_4, such that

$$V_1(p) \leqq K_3 V_2(p) \qquad V_1(p) > K_4 V_2(p)\ ,\ p \in C_j\,.$$

Put $\operatorname{Min} \dfrac{V_1(z)}{V_2(z)} = K_i\,, \quad z \in C_i \cap B_n\,.$

Then

$$\overline{\lim_j} K_i = K\,, \quad K < \infty\,,$$

because

$$\max V_1(z) = K_1' \log |z| : \quad z \in B_n\,,$$
$$\min V_2(z) = K' \log |z| : \quad z \in B_n\,,$$

therefore there exists a subsequence

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4) We such operation generalized extremisation for convenience. This is certainly different from the extremisation.

$$\lim_{i} K'_n = K$$

and

$$0 \leqq V_1(z) - K'_i V_2(z) = \varepsilon'_i V_2(z): \quad \lim_{i} \varepsilon'_i = 0: \quad z \in C_i \cap B_n ,$$

thus

$$V_{1,2}(z) - K_i V_{2,i}(z) \leqq \varepsilon_i V_{i2}(z) \leqq \varepsilon_i S \log |z|: \quad S < \infty$$
$$\lim_{i=\infty} V_{1,i}(z) - K'_n V_{2,i}(z) = 0 ,$$

which implies that

$$V_1(z) = K V_2(z) ,$$

thus $V_{1,i}(p)$ is a minimal function, and if we compare $U(p)$ with $V_1(p)$ by $\{U_i(p), V_{1,i}(p)\}$ we have $U(p) \leqq K'' V_1(p)$, K'' being constant, thus $U(p)$ is a minimal function and we see that on our example there exist exactly enumrable infinity of minimal functions.

6) Positive harmonic function in the neighbourhood of an ideal boundary point.

Let F be a null-boundary Riemann surface with a compact relative boundary Γ_0 and p^∞ be an ideal boundary point.

We denote by $G^i(p, p^\infty)$ $(i = 1, 2, \ldots)$, the positive minimal function with a pole at p^∞ and denote by G_i^N the domain $E[G^i \geqq N]$ and by C_i^{*N} the niveau curve $E[G^i = N]$. Then $\sum_i G_i^N$ is an open set with a compact boundary.

Proof. If $G_i^N \in p_j : \lim_j p_j = p^\infty$ and if $G(p, p_j)$ is the Green's function with its pole at p_j, then since $\frac{\partial G^i(p, p^\infty)}{\partial n} \geqq 0$, $p \in C_i^{*N}$, we have by Green's formula

$$N \geqq G^i(p_j, p^\infty) = \frac{1}{2\pi} \int_{C_i^{*M}} G(p, p_j) \frac{\partial G^i(p. p^\infty)}{\partial n} ds.^{5)}$$

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5)
$$2\pi = \int_{\Gamma_0} \frac{\partial G(z, p_i)}{\partial n} ds = \int_{\Gamma_0} \frac{\partial G(z, p^\infty)}{\partial n} ds$$

and

$$D_{F_n \cap (F_n - D_N)}(G(z, p^\infty)) \leqq D_{F \cap (F - D_N)} (G(z, p_m) \leqq 2\pi ,$$

for sufficiently larrge $m(n)$. Let m and $n \to \infty$. We have

$$D_{F \cup (F - D_N)}(G(z, p^\infty)) \leqq 2\pi N.$$

It follows that

$$\lim_{n=\infty} \int_{\Gamma_n \cap (F - D_N)} \frac{\partial G(z, p^\infty)}{\partial n} ds = 0, \quad \text{and} \quad \int_{\Gamma_n} \frac{\partial G(z, p^\infty)}{\partial n} ds = 2\pi .$$

Let $V_M(p^\infty)$ be a neighbourhood with compact boundaries such that $\int_{\overset{*}{C}_i^M \cap V_M} \frac{\partial G_1^i}{\partial n} ds = \pi$. Then since $N \geqq \frac{2}{2\pi} \int_{\overset{*}{C}_i^M \cap V_M} G(p, p_j) \frac{\partial G^i(p, p^\infty)}{\partial n} ds$, there exists at least one point q_j on $\overset{*}{C}_i^M - (C_i^M \cap V_M(p))$ for every $p_j \overline{\in} G_i^N$ such that $\underline{\lim}_{q_j \in C_i^M} G(q_j, p^\infty) \leqq 2N$ for every M, whence $\lim_{j=\infty} G(p, p_j)$ $(\lim G'(p, p^\infty))$ is free from the minimal functions $G^i(p, p^\infty)$. If $G^N = \sum_i G_i^N$ is not compact, there exists a sequence $r_1, r_2, \ldots, r_i \overline{\in} G^N$, and there exists at least one general Green's function $G(p, r^\infty)$, but $G(p, r^\infty)$ must be expressed by a linear form of $G^1(p, p^\infty)$, $G^2(p, p^\infty) \cdots$, which contradicts the preceding assertion.

Since at any point p there exists a constant $k(p)$ such that $U(p) \leqq k(p)$ for any positive harmonic function $U(p)$ satisfying $\frac{1}{2\pi} \int_{\Gamma_0} \frac{\partial U}{\partial n} ds = 1$, $\underset{P \in \Gamma_0}{U(p)} = 0$, we have $G_N \supset G_{2N} \cdots$, $\prod_{i=1}^{\infty} \overline{G}_{Ni} = p^\infty$.

If $\omega_{Nn}(p)$ denotes the harmonic measure of the boundary G_{Nn} with respect to the domain $F - G_{Nn}$, then $Nn\omega_{nN}(p)$ is a monotonously increasing function. Hence if $Nn \int_{\Gamma_0} \frac{\partial \omega_{nN}(p)}{\partial n} ds < \infty$, then $\lim_{n=\infty} \omega_{nN}(p) = \omega^*(p)$ is harmonic and $\lim_{p \to p_\infty} \omega^*(p) = \infty$. Then as a special case we have

Corollary. *If* p_∞[6] *is finite dimensional, then the solution of Evans's*[7] *problem exists.*

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6) See 2).
7) See 1) and 2).