Osaka Mathematical Journal
Vol. 4, No. 2, December, 1952

# *A Remark on the Bounded Analytic Function*

By Zenjiro KURAMOCHI

Z. Nehari[1] solved many extremal problems in the case when the domain is planer. We shall show that his method is also applicable to Riemann surfaces.

$F$ denotes a compact Riemann surface bounded by $n$ closed analytic curves $\Gamma_1, \Gamma_2, \dots, \Gamma_n$; $\Gamma = \Gamma_1 + \Gamma_2 + \dots \Gamma_n$, and $g$ genus denoted by $\alpha_1 \dots \alpha_g$. $\beta_1, \beta_2 \dots \beta_g$, its corresponding canonical cuts.

$g(z, \zeta)$ denotes the Green's function of $F$ with respect to the point $\zeta$.

$w_i(z)$ denotes the harmonic measure of $\Gamma_i$ with respect to $F$.

${}^{\alpha}w_i(z)$ denotes the first kind harmonic function which is harmonic in $F$ and has period 1 corresponding to the cut $\alpha_i$ and has no period and moreover has the boundary value 0 on $\Gamma$. Such harmonic function can be defined in the following.

If $g(z, \zeta)$ is the Green's function of $F$ with respect to $\zeta$, then

$$w(z; \alpha_i) = \int_{\alpha_1} dg(z, \zeta)\, d\zeta$$

where the integration is performed along $\alpha_i$, the function vanishes for $z \in \Gamma$, since $g(z, \zeta) = 0$ for $z \in \Gamma$, and further has a period about the cut $\alpha_\kappa$, $\kappa \neq i$ and no period about the other cuts $\beta_i : i = 1, \dots g$. This function may be defined as the following also.

It is clear that to each Riemann surface of finite connectivity $2g+n$, there correspond a symmetric Riemann surface of genus $4g+n-1$. This surface defined by taking two replicas $F$ and $\bar{F}$ and identifying corresponding points of boundary $\Gamma_i$. We denote by $\alpha_i, \beta_i, \bar{\alpha}_i, \bar{\beta}_i$ in $F$ and $\bar{F}$ and ${}^{\alpha}w_i^{\cdot}, {}^{\alpha}\bar{w}_i^{\cdot}$ the first Abel's integral which are harmonic and one valued except period 1 about $\bar{\alpha}_i$ and $\alpha_i$ respectively, then ${}^{\alpha}w_i^{\cdot} - {}^{\alpha}\bar{w}_i^{\cdot}$ are zero on the boundary $\Gamma$, and ${}^{\alpha}w_i^{\cdot} - {}^{\alpha}\bar{w}_i^{\cdot}$ is symmetric with respect to $F$ and $\bar{F}$ then $\int_{\Gamma_i} \frac{\partial\, {}^{\alpha}w_i^{\cdot}}{\partial n} \alpha s - \int_{\Gamma_i} \frac{\partial \bar{w}_i^{\cdot}}{\partial n} ds = 0$ therefore the conjugate harmonic function of ${}^{\alpha}w_i^{\cdot} - {}^{\alpha}\bar{w}_i$ has no period about common boundary of $F$ and $\bar{F}$. It is clear that ${}^{\alpha}w_i^{\cdot} - {}^{\alpha}w_i^{\cdot}$ is identical with ${}^{\alpha}w_i$ in $F$.

---

1) Z. Nehari: Extremal problems in the theory of bounded analytic function, American Journal of Math. 1951, Vol. LXXIII, No. 1 pp. 78-106.

$B_R$ denoted the family of analytic function $f(z)$ which are regular and single valued and have a bounded real part, say $|Re\{f(z)\}] < 1$, in $F$.

The Schwarz Lemma.

We consider the generalization of classical Schwarz Lemma, that is the problem to ask the function such that, $f(z) \in B_R$, $|Re\{f'(\zeta)\}| = \max$ (where the simbol of the differentiation means one corresponding to the local parameter defined in the neighbourhood of $\zeta$, which was treated in the case of planer by Z. Nehari and the little changed problem about Riemann surface was done by L. Ahlfors[2]. Z. Nehari proved that maximizing function must satisfies that $Re\{f(\zeta)\} = 0$, therefore the above problem is equivalent to the following problem,

$$f(\zeta) = 0 \qquad Re\ f'(\zeta) = \max.$$

$p(z, \zeta)$ denotes the second kind harmonic function with respect to the point and the local parameter; i.e. $p(z, \zeta) + Re\frac{1}{z-\zeta}$ is regular harmonic in $F$ and $p(z, \zeta)$ vanishes for $z \in \Gamma$, then we have by Green's formula,

$$(1) \qquad u'(\zeta) = -\frac{1}{2\pi}\int_{\Gamma} u(z)\frac{\partial p}{\partial n}ds\,;\ ds = |dz|:\ f(z) = u(z) + iv(z),$$

$\frac{\partial}{\partial n}$ denotes differentiation with respect to the outerward pointing normal. And by the same formula

$$(2) \qquad u(\zeta) = -\frac{1}{2\pi}\int_{\Gamma} u(z)\frac{\partial g(z, \zeta)}{\partial n}ds.$$

And in the same manner, by the Cauchy Riemann-equation, and the boundary properties of $\omega_i(z)$,

$$(3) \qquad \int_{\Gamma} u(z)\frac{\partial \omega_i(z)}{\partial n}ds = \int_{\Gamma}\cdot\omega_i(z)\frac{\partial u}{\partial n}ds = \int_{\Gamma_i}\frac{\partial u}{\partial n}ds = \int_{\Gamma_i} dv.$$

We form an analytic function from $w_i^{\alpha}(z)$ with its conjugate harmonic function ${}^{\alpha}\tilde{w}(z)$

$${}^{\alpha}W_i(z) = {}^{\alpha}w_i(z) + i\,{}^{\alpha}\tilde{w}_i(z)\,;\quad i = 1, \dots, g, \text{ for } \alpha \text{ and } \beta.$$

These are not single valued and have period corresponding to $\alpha_i$ cuts buts have no periods about the boundary.

Let $F^k$ denotes the surface of connectivity $2g+n-1$ into which $F$

2) L. Ahlfors: Extremal problems and Compact subregion of open Riemann surface (Comm. Math. Helv. 1949).

is transformed by a system of canonical cuts $\alpha_i$, $\beta_i$ and by $\Gamma^*$ its boundary, we have in considering the single valudeness of $f(z)$ and ${}^\alpha W_i(z)$ in $^*F$,

$$\int_{\Gamma^*} ({}^\alpha w_i(z)+i{}^\alpha\tilde{w}_i(z)) \quad (du+iv)=0$$

The integration path $\Gamma^*$ is composed of the boundary components $\Gamma_i$, $i=1,\dots,n$, and the closed curve $\alpha_i$, $\beta_i$, $i=1,\dots,g$, the latter are taken twice, in differenti directions. Along $\beta_j$, we have, since ${}^\alpha w_i(z)$ is free of periods about $\beta_j$, ${}^\alpha w_i(z)$ has the same value on both edges of $\beta_j$ and $u(z)$ is one valued,

$$(4) \qquad \int_{\beta_j(-)} ({}^\alpha w_i+i{}^\alpha\tilde{w}_i)(dv+idv)+\int_{\beta_\sigma(+)} ({}^\alpha w_i+i{}^\alpha\tilde{w}_i)(dv+idv)$$
$$=i{}^\alpha\hat{w}_i{}^j\int_{\beta_j} dU-{}^\alpha\hat{w}_i{}^j\int_{\beta_j} dv=-{}^\alpha\hat{w}_i{}^j\int_{\beta_j} dv$$

where ${}^\alpha\hat{w}_i{}^j$ is the period of ${}^\alpha\tilde{w}_i$ about $\alpha_j$. In the same manner

$$(5) \qquad \int_{\alpha_j(-)} {}^\alpha w_i+i{}^\alpha\tilde{w}_i)(dU+idv)+\int_{\alpha_j(+)} ({}^\alpha w_i+i{}^\alpha\tilde{w}_i)(dU+idv)$$
$$=-{}^\alpha\hat{w}_i\int_{\beta_j}\int_{\alpha_j} dv+i\delta_i{}^i\int_{\beta_j} dv$$

where $\delta_j{}^i=1$, if $i=j$, $=0$ if $i\neq i$.

By assumption that $f(z)$ is single valued and ${}^\alpha w_i(z)$ is a constant on $\Gamma_i$ of $F$ and remembering of $\int_{\Gamma_i} d{}^\alpha w_i=0$, $\int_{\Gamma_i} du=0$, we have then

$$(6) \qquad \int_{\Gamma_i} ({}^\alpha w_j+i{}^\alpha\tilde{w}_j)(dv+idv)=i\int_{\Gamma_i} {}^\alpha\tilde{w}_j dv+i\int_{\Gamma_i} {}^\alpha w_j dv-\int_{\Gamma_i} {}^\alpha\tilde{w}_j dv$$
$$=-i\int_{\Gamma_i} v\frac{d{}^\alpha\tilde{w}_i}{dz}dz-i\int_{\Gamma} {}^\alpha w_j dv-\int \tilde{w}_j dv=-i\int v\frac{\partial {}^\alpha w_j}{\partial n}ds\,.$$

If we take the real imaginal of (4), (5) and (6)

$$\int_{\beta_i} dv=-\int_{\Gamma} v\frac{\partial {}^\alpha w_i}{\partial n}ds\,,\qquad \int_{\alpha_i} dv=-\int_{\Gamma} v\frac{\partial {}^\beta w_i}{\partial n}ds\,.$$

Since $u(\zeta)=0$, and by single valuedness of $f(z)$ we hav$\partial$

$$v'(\zeta)=-\frac{1}{2\pi}\int_{\Gamma} v(z)\left[\frac{\partial p}{\partial n}+\alpha\frac{\partial g}{\partial n}+\sum^{g}\lambda_n\left(\frac{\partial {}^\alpha w_v}{\partial n}\right)+\sum^{g}\mu_n\frac{\partial {}^\beta w_v}{\partial n}+\sum^{n-1}\nu_n\frac{\partial\omega}{\partial n}\right]ds$$

where $\alpha$, $\lambda_n$, $\mu_n$, $\nu_n$ are arbitrary real constants.

Since $|u(z)|\leq 1$ in $F$, we have

$$(7)\quad u'(\zeta)\leqq-\frac{1}{2\pi}\int_{\Gamma} v\left|\frac{\partial p}{\partial n}+\alpha\frac{\partial g}{\partial n}+\sum\lambda\frac{\partial\, {}^{\alpha}w}{\partial n}+\sum\mu\frac{\partial\, {}^{\beta}w}{\partial n}+\sum\nu\frac{\partial w}{\partial u}\right] ds\,.$$

Let us denote by $q(z,\zeta)$, $h(z,\zeta)$, ${}^{\alpha}\tilde{w}_i(z)$ and ${}^{\beta}\tilde{w}_i(z)$ the conjugate harmonic functions of $p(z,\zeta)$, $g(z,\zeta)$, ${}^{\alpha}w_i(z)$, ${}^{\beta}w_i(z)$ respectively, the differential of the next form $\alpha X=(p'+ig'+g'+ih'(z)+\sum_i\lambda_i({}^{\alpha}w_i+i\tilde{w})$ $+\sum\mu({}^{\beta}w(+i{}^{\beta}\tilde{w}_i)+\sum(\nu(w+i\tilde{w}))\,dz$ has $2g+n$ zero points in $F$, where each critical point is counted with its multiplicity.

In fact, since this differential is analytic, clearly it does never have limiting points of zero in $F$, let $\tilde{F}$ be the planer surface of connectivity $2g+n$ into which $F$ is transformed by a system of $\alpha,\beta$ cuts which does not pass any one of zero points of the differential. We can map $\tilde{F}$ conformaly onto $\tilde{\tilde{F}}$ on the $z$-plane, then $dx$ is transformed conformaly into $d\tilde{x}$ also. The boundary components of $\tilde{\tilde{F}}$ are composed of ones corresponding to $\Gamma$ and the system of $\alpha,\beta$ cuts, for simplicity we dence them by the same notation with the simbol $\sim$.

Since the real part of $dx$ is constant of $\Gamma$, then

$$\int_{\tilde{\Gamma}_i} d\arg\frac{dx}{dz}=-\int_{\Gamma_i} d\arg dz=2\pi(i=2\dots n),\quad \int_{\tilde{\Gamma}_1} d\arg\frac{dx}{dz}=-\int_{\Gamma_1} d\arg dz=-2\pi\,.$$

Since in $\tilde{F}$ the direction of integration of arg $dx$ along $\tilde{\alpha}_+$ and $\tilde{\alpha}_-$ are the same with respect to the interior of $\tilde{\tilde{F}}$, which corresponds in $F$ inversed direction along i.e. $\alpha_+$ and $\alpha_-$, and since $d\tilde{X}$ and $dX$ are conformally equivalent then we have

$$\int_{\tilde{\alpha}_+} d\arg\frac{d\tilde{x}}{dz}+\int_{\tilde{\alpha}_-} d\arg\frac{d\tilde{x}}{dz}=\int_{\tilde{\alpha}_+} d\arg dz-\int_{\tilde{\alpha}_-} d\arg dz+\int_{\alpha_-} d\arg dx+\int_{\alpha_-} d\arg ax$$
$$=2\int_{\tilde{\alpha}} d\arg dz=4\pi\,.$$

Since $dX$ has double pole then

$$\int_{\tilde{\Gamma}} d\arg ax=2(2g+n)\pi\,.$$

Hence, the number of zero points in $F$, counted with their multiplicity, tourned out to be $2g+n$, the results remains true even if there are zero points on the boundary, provided they are counted with half their multiplicity.

We minimize the right hand side (7) with respect to the parameter $\alpha,\lambda,\mu,\nu$. This problem has at least one solution as proved by Z. Nehari in the planer domain.

Let now $\alpha, \lambda, \mu, \nu$ denote the particular values of these paramet which minimizes the right hand sided of (7). With the abreviation

$$P=\left[\frac{\partial p}{\partial n}+\alpha\frac{\partial g}{\partial n}+\sum\lambda\frac{\partial^{\alpha} w}{\partial n}+\sum\mu\frac{\partial^{\beta} w}{\partial n}+\sum\nu\frac{\partial w}{\partial n}\right].$$

It follows from the minimum property that

$$\int_{\Gamma}\frac{|p|}{P}\frac{\partial g}{\partial n}ds=0\,,\quad\int\frac{|p|}{P}\frac{\partial^{\alpha} w}{\partial n}ds=0\,,\quad\int\frac{|p|}{P}\frac{\partial\omega}{\partial n}ds=0\,.$$

We now introduce the bounded function $U(z)$ which is harmonic and has the boundary value $u(z)=-\frac{|P|}{P}\ P\neq 0$. Although the boundary value at the finite number of points at which $P=0$ are not specified, $U(z)$ is uniquely determined.

In term of this function $U(z)$

$$U(z)=\frac{1}{2\pi}\int_{\Gamma}\frac{|p|}{P}\frac{\partial g}{\partial n}(z,\zeta)\,ds\,,\quad\int U\frac{\partial^{\alpha}\omega}{\partial n}ds=0\,,\quad\int U\frac{\partial\omega}{\partial n}ds=0\,.$$

If $V(z)$ denotes the harmonic conjugate of $U(z)$, if follows that $V(z)$ is free of periods about $\alpha, \beta$ cuts and the boundary components, $V(z)$ is only determind up to an arbitrary constant, which will be so chosen that $V(\zeta)=0$.

$$U_{\zeta}(\zeta)\leqq U_{\zeta}(\zeta)\leqq\frac{1}{2\pi}\int\frac{|p|}{P}\frac{\partial g}{\partial n}(z,\zeta)\,ds\,.$$

whence $u_{\zeta}'(\zeta)\leqq U'(\zeta)$. which proves our assertion.

It is clear $P(z)\cdot U(z)\leqq 0$. This shows that $u(z)+iV(z)$ yields a $(1,m)$; $n\leqq m\leqq 2g+n$, conformal maps of $F$ onto the strip $-1<Re\{w\}<1$.

Because the minimum property of $\int|P|ds$ follows that $P$ has at least one zero point on $\Gamma_i$ and by its one valuedness the nuber of zeros on $\Gamma_i$ must be even, therefore $P$ has at least $2n$ zero points on the boundary components, and since the property of $dX$, it has at most $4G+2n$ zero points on the boundary. Since each sheets of the conformal map of $F$ given by $w=U+iV$ corresponds two change of sign of $U(z)$ on, it follows that the map has $m$ sheets: $n\leqq m\leqq 2g+n$ on the strip we form the function $F(z)$ from $w(z)$ such that

$$F(z)=\tan\left\{\frac{\pi}{4}w(z)\right\}$$

of $\Gamma$, the values of $U(z)$ is either 1 or $-1$, that is, $w(z)$ is the of the form $w(z)=1+it$ where $t$ is a real parameter, it follows from $\frac{F(z)}{1+F^2(z)}$ $=\pm\cosh t$ whence

$$\frac{F(z)}{U(z)(1+F^2(z))} \geqq 0\ ;\ z \in \Gamma$$

From

$$Pds = -i\left[p'(z,\zeta)+\alpha g'(z,\zeta)+\sum \lambda_i{}^{\alpha} w_i'(z)+\sum^{g} \mu_i{}^{\beta} w_i'(z)+\sum^{g} \nu_i\, \omega^i(z)\right] dz$$
$$= R'(z)\, dz$$

$$\frac{iF(z)\, R'(z)\, dz}{1+F^2(z)} \geqq 0$$

$\dfrac{R'(z)}{1+F^2(z)}$ is also the analytic differential, except its double pole at $\zeta$, since the zero of $1+F^2(z)$ obviously coincide with the point at which $U(z)$ changes it sign; these points, in turn, coincide with the zeros of $dR(z)$, and there also with those of the differential. Hence, $\dfrac{F(z)\, dR(z)}{1+F^2(z)}$ is the differential which L. Ahlfors obtained.

These consideration can be applied easily to the problems of Z. Nehari, for instance, on the upper bound for higher derivaties, interpolation problems, etc.

(Received May 9, 1952)