Osaka Mathematical Journal
Vol. 3, No. 1, March, 1951

A Topological Characterization of Pseudo-Harmonic Functions

By Yukinari TÔKI

Introduction. M. Morse and M. Heins[1] studied the relations among the zeros, poles and branch points of the "*pseudo-harmonic*" functions defined as follows:

Let $u(x, y)$ be a function which is harmonic and not identically constant in the neighbourhood N of a point (x_0, y_0) in $z(=x+iy)$-plane. Let the points of N be subjected to an arbitrary homeomorphism T in which N corresponds to another neighbourhood N' of (x_0, y_0) and the point (x, y) on N corresponds to a point (x', y') on N'.

Under T set

$$u(x, y) = U(x', y') .$$

Then the function $U(x', y')$ is called *pseudo-harmonic* on N'.

A function $U(x, y)$ is called *pseudo-harmonic* on a domain D, if $U(x, y)$ is pseudo-harmonic in some neighbourhood of each point of D.

We shall slightly extend the definition of the pseudo-harmonic function as follows:

Let F be a surface, i. e., a 2-dimensional and separable manifold. Let $U(p)$ be a real-valued function in the neighbourhood N of a point p on F, where N corresponds to $x^2+y^2<1$ in the z-plane by a homeomorphism $T(x, y)$.

Set

$$U(p) = U[T(x, y)] \equiv u(x, y).$$

Then $U(p)$ is called *pseudo-harmonic* in N, if $u(x, y)$ is harmonic and not identically constant. A function $U(p)$ is called *pseudo-harmonic* on F, if $U(p)$ is pseudo-harmonic in some neighbourhood of each point of F.

---

1) M. Morse, The topology of pseudo-harmonic functions, Duke Math. Jour. 13 (1947) pp. 21-42. M. Morse and M. Heins, Topological methods in the theory of functions of a single complex variable, Annals of Math. 46 (1945), pp. 600-666, 47 (1946), pp. 233-274.

We study in § 1 the topological characterization of the pseudo-harmonic functions, in § 2 conjugate pseudo-harmonic functions, together with their relations to interior transformations.

It is convenient to introduce here some notations and terminologies which we use in the following.

$S_1 \cdot S_2$, $S_1 + S_2$ denote the meet and join of two point sets S_1 and S_2 respectively, and $S_1 - S_2$ the meet of S_1 and the complymentary set of S_2. $\overline{D}$ denotes the closure of a point set D and βD its boundary.

We understand by a neighbourhood N_p of a point p on F a neighbourhood, whose closure $\overline{N}_p$ is homeomorphic to $|z| \leqq 1$ in the z-plane.

If c is a real number, the set of all points with $U = c$ will be called the *level* c, and denoted by L_c:

$$L_c = \{p : U(p) = c\} .$$

Points of F at which $U > c$ or $U < c$ will said the points *above* c or *below* c respectevely. Further we call the family of levels

$$\{L_c\} \quad c : \text{parameter}$$

equi-locally-conneted at a point $p \in F$, when for any N_p on F there exists another $N'_p \subset N_p$, so that any pair of points of each level L_c in N'_p can be joined by a connected subset of L_c in the interior of N_p. When $\{L_c\}$ is equi-locally-connected at all points of F, $\{L_c\}$ is *equi-locally-connected* on F.

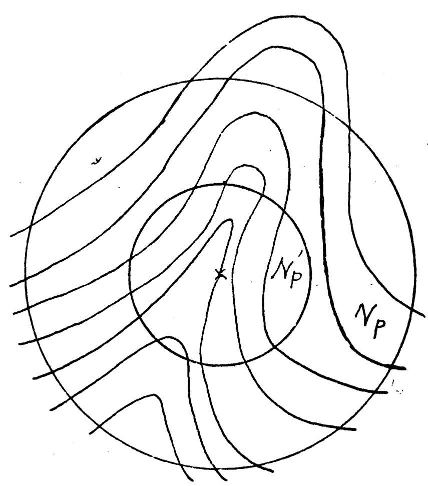

Fig. 1

§ 1. The topological characterization of the pseudo-harmonic functions

From the preceding definition follows directly:

If the family of levels $\{L_c\}$ is equi-locally-connected on F, each level L_c is locally connected.

Let $u(p)$ be a one valued real function, satisfying the following conditions:

(1) $u(p)$ is continuous.

(2) $u(p)$ is an open transformation.

Then we obtain the following lemmas.

Lemma 1. *$u(p)$ never attains its relative extremum on F.*

Lemma 2. *Each neighbourhood of p contains both points above $u(p)$ as well as below $u(p)$.*

This is evident from the condition (2) and Lemma 1.

Lemma 3. *$N_p - L_{u(p)}$ and $F - L_{u(p)}$ are open sets.*

For $L_{u(p)}$ is a closed set.

Each component of $N_p - L_{u(p)}$ or $N_p - L_{u(p)}$ is evidently a domain by Lemma 3.

Lemma 4. *Each domain Ω of $N_p - N_{u(p)}$ and $F - L_{u(p)}$ consists of points above (below) $u(p)$ only.*

If $q_1 \in \Omega$ is above $u(p)$, $q_2 \in \Omega$ below $u(p)$, we can join these two points with a Jordan arc C within Ω. Then there must exist at least one point of $L_{u(p)}$ on C [(1)], which contradicts the definition of Ω. Such Ω is called the *domain above* or *below* $u(p)$.

Lemma 5. *Any component of $F - L_c$ is not compact with respect to F.*

Let Ω be a component of $F - L_c$, then Ω is a domain above or below c [Lemma 4]. If a domain Ω above (below) c is compact with respect to F, there exists at least such a point q on $\overline{\Omega}$ that $u(p)$ attains the maximal (minimal) value there. Since $\beta\Omega \subset L_c$, q must be a point of Ω, which contradicts Lemma 1.

Lemma 6. *Any component of the level c is not entirely confined in any neighbourhood N.*

If $L_c \subset N$ it is possible to enclose L_c with a Jordan curve C lying inside N and $C \cdot L_c = 0$, since each component of $N - L_c$ constitutes a domain above and below c [Lemma 4], if a point on C is above (below) c, all the points of C are also above (below) c. But in any neighbourhood of a point on L_c there necessarily exist points below (above) c [Lemma 2]. Therefore there must exist a (with respect to F) compact domain below (above) c in the interior of C, which is contrary to Lemma 5.

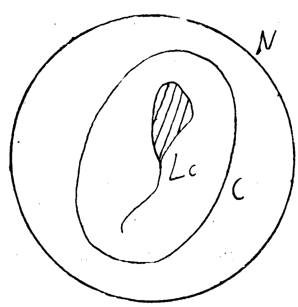

Fig. 2

Now we have the following theorem which plays the most important rôle in this paper.

Theorem 1. *For a one-valued real function $u(p)$ to be pseudo-harmonic on F, it is necessary and sufficient that*

(1) *$u(p)$ is continuous,*

(2) *$u(p)$ is an open transformation,*

(3) *the family of levels $\{L_c\}$ is equi-locally-connected on F with possible exception of a discontinuum E.*

We first derive some properties from the conditions (1), (2) and (3).

i) *Each L_c is locally connected.*

Suppose that L_c is not locally connected at $p \in L_c$. Then since each component of L_c is not entirely contained in any neighhourhood N we can choose a suitable neighbourhood N_p with the following property:

In N_p there are at least a countable number of components $\{L^i\}$ $(i = 1, 2, \ldots)$ of $L_c \cdot N_p$, which do not contain p but possess it as an accumulating point. Since each L^i has point in common with βN_p [Lemma 6], $\{L^i\}$ $(i = 1, 2, \ldots)$ accumulates to a continuum K containing p and having a point in common with βN_p. Consider a point q on K not belonging to E and βN_p, then the family of levels $\{L_c\}$ is not equi-locally-connected at q, while the family of levels $\{L_c\}$ is by condition (3) equi-locally-connected at q, which is a contradiction.

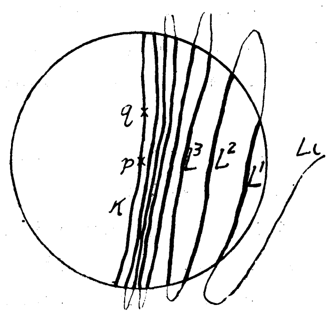

Fig. 3

ii) *Even though $N_p - L_{u(p)}$ consists of an infinity of its components $\{D_n\}$ $(n = 1, 2, \ldots)$, any sequence of points $\{p_n\}$ $(p_n \in D_n;\ n = 1, 3, \ldots)$ has no accumulating point in N_p.*

If $p_0 \in N_p$ is an accumulating point of $\{p_n\}$, we can choose a subsequence of $\{p_{n_\nu}\}$ converging to p_0, which we will denote again by $\{p_n\}$ for the sake of convenience. Let C be a Jordan arc possessing p_1 and p_0 as end points and passing through all p_n $(n = 2, 3, \ldots)$. Since each component of $F - L_{u(p)}$ intersects βN_p [Lemma 5], we can join p_n and a point q_n suitably chosen on βN_p with a Jordan arc C_n in the interior of D_n, so that $C \cdot C_n = p_n$. The sequence of points $\{q_n\}$ has at least one accumulating point

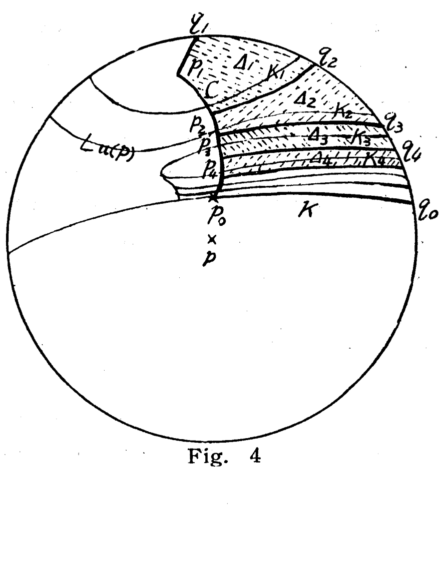

Fig. 4

q, then there is a certain subsequence $\{q_{n_\nu}\}$ of $\{q_n\}$, so that it converges along βN_p in positive or negative sense; we may once more write it by $\{q_n\}$. For fixed n, four kinds of Jordan arcs:

I) subarc $\widehat{p_n p_{n+1}}$ of C,

II) C_{n+1},

III) subarc $\widehat{q_n q_{n+1}}$ of βN_p that excludes the point q_{n+2},

IV) C_n

bound a domain Δ_n, and we have

$$\Delta_i \cdot \Delta_j = O \quad (i, j = 1, 2, \dots ;\ i \neq j).$$

Then there exists in Δ_n a subcontinuum K_n of level c which attains a point $q_n' \in \widehat{q_n q_{n+1}}$ from $p_n' \in \widehat{p_n p_{n+1}}$. $\{K_n\}$ $(n = 1, 2, \dots)$ converges, however, to a subcontinuum K of level c containing p_0 and q. This contradicts i).

iii) *$p \in L_c$ is, in any neighbourhood N_p, a common boundary point of at least one domain above as well as below c, and yet of at most finite number of them.*

If p does not belong to the boundary of any domain above c, N_p must have the common parts with infinite number of domains Ω_n above c $(n = 1, 2, \dots)$ [Lemma 2]. Suitable choice of $p_n \in \Omega_n$ causes $p_n \to p$ for $n \to \infty$, hence $\{p_n\}$ becomes compact. This is impossible. Therefore p is a boundary point of a certain domain above c. It is the same with the domain below c. While, if p is a common boundary point of an infinite number of domains Ω_n $(n = 1, 2, \dots)$ above (below) c, we can choose $p_n \in \Omega_n$ so that $\{p_n\}$ may converge to p, which is also contrary to ii).

Definition: In case $p \in L_c$ is a common boundary point of the sole domain above c and a domain below c, it will be called an *ordinary* point, otherwise a *saddle* point.

iv) *Let Ω denote one of the domains above (below) c. Then every point of $\beta\Omega$ is accessible from the interior of Ω.*

Suppose that $p \in \beta\Omega$ is an inaccessible boundary point of Ω and the decomposition

$$\Omega \cdot U_p = \sum \Omega_n$$

were possible for any N_p. Suitably chosen partial sequence of $\{p_n\}$ $(p_n \in \Omega,\ n = 1, 2, \dots)$ will converge to p. Join all these points in succession with a Jordan arc C ending at p, and we shall be lead to a contradiction in the same way as in ii).

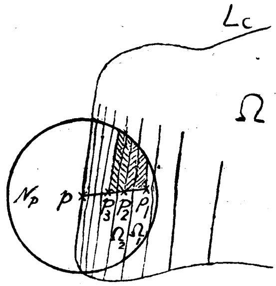

Fig. 5

v) *The set S_c of all saddle points on*

L_c has no accumlating point on L_c.

Let $\{\Omega_i\}$ denote the family above and blow c lying inside an arbitrary neighbourhood N_p of $p\in L_c$ and having p as their boundary point, the number of which must be finite, say n [(iii)]. Let p be a *saddle* point, $n\geqq 3$ results. Let Ω', Ω'', Ω''' be any triple of members belonging to $\{\Omega_i\}$ $(i=1, 2, \dots, n)$. Suppose that they have another boundary point p_1 in common inside N_p. Then it will be possible to join p, p_1 with certain Jordan arcs C', C'', C''' respectively in the interior of Ω', Ω'', Ω''' [iv)]. One of these arcs, say C', is enclosed by the others except for both end points. C' consists, however, only of the points belonging to Ω except the both ends p, p_1, while C'', C''' contain no points of Ω'. Therefore Ω' must be compact with respect to F, which is contrary to Lemma 5. This shows that three domains can posses only one common boundary point. Since number m of domains above or below c on F, which intersect the neighbourhood N_p, is finite [ii)], the number of the saddle points inside N_p does not exceed ${}_mH_3$.

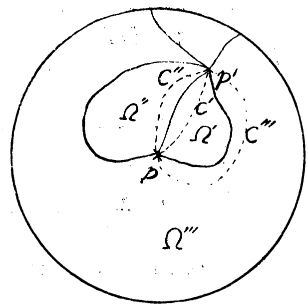

Fig. 6

vi) *Every component of L_c-S_c is homeomorphic to an open interval or closed Jordan curve.*

Let $p\in L_c-S_c$ be an ordinary paint, p becomes the common boundary point of the sole domain Ω^+ above c and the sole domain Ω^- below c. Then we can properly choose N_p, so that $L_c\cdot N_p$ may contain no boundary points of domains other than Ω^+ and Ω^- [v)]. Therefore every point of $L_c\cdot N_p$ is the accessible boundary point of Ω^- and ${}^+\Omega$ [iv)]. Thus we know in vertue of Schönflies' theorem that $\overline{L_c\cdot N_p}$ is a Jordan arc. Owing to Lindelöf's covering theorem L_c-S_c can be covered by at most a countable number of neighbourhoods, i. e. it is a union of a countable number of open Jordan arcs.

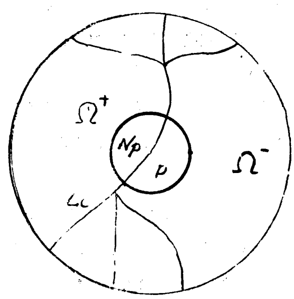

Fig. 7

Definition: If every components of $N_p-L_{u(p)}$ have the point p as their common boundary point, N_p is called a *simple neighbourhood* of the point p.

vii) *There exists a simple neighbourhood N_p for any point p on F, and each component of $N_p-L_{u(p)}$ is a Jordan domain. Moreover any two domains above (below) $u(p)$ do not neighbour one another.*

There exists a neighbourhood N'_p for the point p such that N'_p does not contain any saddle point on the level $u(p)$ with possible exception of p itself [v)]. Each component of $N'_p \cdot L_{u(p)} - p$, which we denote by C_i $(i=1,2, \ldots, n)$, is homeomorphic to an open interval [vi)] and $L_{u(p)}$ is locally connected [i)], so $\overline{C}_i$ is a Jordan arc and the number of $\overline{C}_i$ having p as its end point is finite, which we denote by $\overline{C}_1, \overline{C}_2, \ldots, \overline{C}_n$ in the order of positive sense.

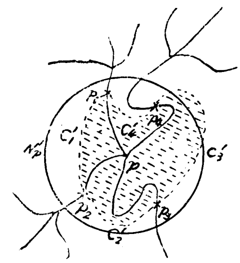

Fig. 8

Let p_i be a point of C_i. Then we can join the points p_i and p_{i+1} $(i=1, 2, \ldots, n;\ p_{n+1}=p_1)$ with a Jordan arc C'_i in the domain above or below $u(p)$. Let the domain enclosed by the Jordan curve $\sum_{i=1}^{n} C'_i$ be the neighbourhood N_p of p. Then N_p is a simple neighbourhood of p.

Next if two domains above (below) c have an arc in common on their boundary, $u(p)$ must take the relative minimum (maximum) on it. This is impossible, i.e., the same kinds of the domains cannot neighbour each other.

Definition. When $N_p - L_{u(p)}$ contains n domains above $u(p)$ holding p in common, $(n-1)$ is called the *order* of the saddle point p.

viii) *The set S of all saddle points on F has no accumulating point.*

Suppose the set S has a point p on F as an accumulating point, to which a certain sequence $\{p_\nu\}$ $(\nu = 1, 2, \ldots)$ of saddle points converges. Let N_p be any one of neighbourhoods of p, p_ν $(\nu \geq n)$ are all contained in its interior so far as n is taken sufficiently large. From each point p_ν issue at least four subarcs of the level $u(p_\nu)$ arriving at βN_p, which we denote by C_{ν_1}, C_{ν_2}, C_{ν_3}, C_{ν_4} respectively, and their end points on βN_p we denote by $p_\nu^1, p_\nu^2, p_\nu^3, p_\nu^4$ respectively.

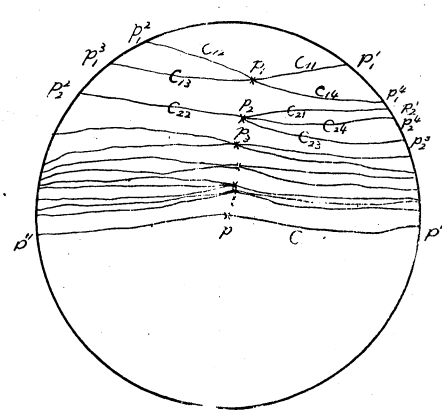

Fig. 8

The set $\{p_\nu^i\}$ $(\nu=1,2,\ldots;\ i=1,2,3,4)$ has at most two accumulating points p' and p'' on βN_p. Then we can choose a subsequence $\{C^i_{\nu_j}\}$ $j=1, 2, \ldots;\ i=1, 2, 3, 4)$ of $C^i_{\nu_j}$, so that at least two arcs among

$C^1_{\nu_j}$, $C^2_{\nu_j}$, $C^3_{\nu_j}$ and $C^4_{\nu_j}$ converge to the arc C of the level $u(p)$ from p to p' or p'' with $j \to \infty$. Let q be an inner point on C not belonging to E. It is evident that the family $u(p_{\nu_j})$ $(j=1, 2, \ldots)$ is not equi-locally-connected at q. This is a contradiction.

ix) *The family* $\{L_c\}$ *of levels is equi-locally-connected on* F *except for the saddle points.*

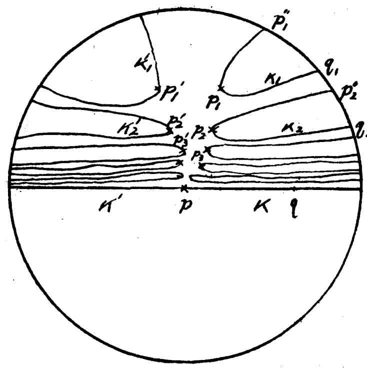

Fig. 10

Let S be the set of all saddle points on F. If $\{L_c\}$ is not equi-locally-connected at $p \in F-S$, there exists a neighbourhood N_p of p which has the following property:

It contains two sequences of points $\{p_i\}$, $\{p'_i\}$ converging to p and satisfying the relation $u(p_i)=u(p'_i)=c_i$, while they are not connected by L_{c_i} in N_p.

Take a simple neighbourhood as N_p, and βN_p intersects each L_{c_i} [Lemma 6], from which L_{c_i} is divided into at least two Jordan arcs. Each of them, that contains p_i, p'_i, shall be denoted by K_i, K'_i respectively. Then $\{K_i\}$ accumulates to the subarc K of the level $u(p)$, which contains p and has two end points on βN_p. For if the sequences of two end points p''_i and q_i of K_i converge to one point on βN_p, there exists a point q on $K-E$, with respect to which $\{K_i\}$ is not equi-locally-connected, but this contradicts the condition (3). Hence K must be a cross-cut of N_p. It is the same with K' derived from $\{K'_i\}$, and yet these two have no common point except for p. For if they have a common point $q' \neq p$, K must coinside with K' [Lemma 5], this contradicts the condition (3). Hence p must be a saddle point, which is a contradiction.

Definition: Let N_p be the neighbourhood of a point p on F. When the neighbourhood A_p of p satisfies the following property:

Let q_1 and q_2 be any two points of the level $u(p)$ in $\Omega \cdot N_p$, where Ω is the domain above or below $u(p)$, then q_1 can be connected with q_2 along $L_{u(p)}$ in N_p.

A_p is then called an *admissible neighbourhood* of N_p.

x) *If* N_p *is a neighbourhood of an arbitrary point* p *on* F, *there exists an admissible neighbourhood of* N_p.

When p is an ordinary point, the family of levels is equi-locally-connected at p [ix)]. Therefore there exists an admissible neighbourhood A_p of N_p.

When p is a saddle point, suppose that there exists no admissible neighbourhood of N_p. Then there exist sequences of points $\{p_i\}$, $\{q_i\}$ in the domain of above (below) $u(p)$, where $u(p_i)=u(q_i)$ $(i=1,2,...)$ and p_i is not connected with q_i along the level $u(p_i)$ in N_p. We see easily in the same way as in ix) that this is a contradiction.

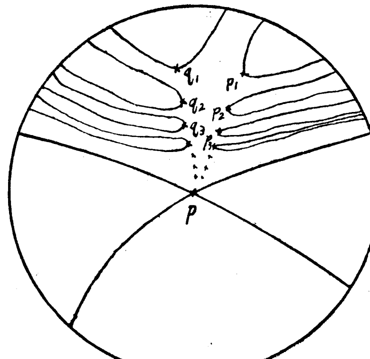

Fig. 11

Lemma 7. *Let F be the Gaussian plane and let q be an arbitrary point of the admissible neighbourhood A_p of N_p different from p. Then there exists a chain*

$$(p, p_1, p_2, ..., p_n=q)\in N_p$$

satisfying the following properties:

$$\text{I)} \quad u(p)<u(p_1)<...<u(p_{n-1})<u(p_n)$$
$$\text{or } u(p)>u(p_1)>...>u(p_{n-1})>u(p_n),$$

II) *for any given positive number* ε

$$|p_i-p_{i-1}|\leqq\varepsilon \quad (i=1,2,...,n).$$

Proof. Let C be a Jordan arc joining p and q (for example $u(p)<u(q)$):

$$C: p=p(t)\ (0\leqq t\leqq 1),\ p(0)=p,\ p(1)=q.$$

Set

$$t'=\sup\ \{t: u(p(t))=u(p)\}$$
$$p'=p(t').$$

Then the following four cases are possible:

a) $|q-p'|\leqq\frac{\varepsilon}{2},\quad p=p'$

b) $|q-p'|\leqq\frac{\varepsilon}{2},\quad p\neq p'$

c) $|q-p'|>\frac{\varepsilon}{2},\quad p\neq p'$

d) $|q-p'|>\frac{\varepsilon}{2},\quad p=p'$

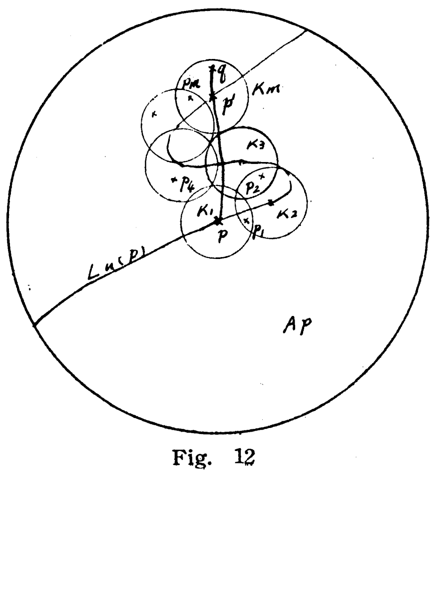

Fig. 12

In the case a), the chain (p, q) satisfies the conditions I, II.

In the case b), p is connected

with p' along the level $u(p)$ in N_p. Let ρ be the distance between βN_p and the subarc $\widehat{pp'}$ of the level $u(p)$. Arc $\widehat{pp'}$ is covered with a finite number of open disks, $K_1, K_2, \dots, K_m$, whose diameters are less than both $\frac{\varepsilon}{4}$ and ρ, and $K_i \cdot K_{i+1} \neq 0$ $(i=1, 2, \dots, m-1)$, $p \in K_1$, $p' \in K_m$, $\widehat{pp'} \cdot K_i \neq 0$.

Set

$$a = \min \left[\max_{p \in K_2} u(p), \max_{p \in K_3} u(p), \dots, \max_{p \in K_m} u(p), u(q)\right].$$

Then we can choose points $p_1, p_2, \dots, p_m$, where $p_i \in K$ and $u(p_i) = u(p) + \frac{i(a-u(p))}{m}$ $(i=1, 2, \dots, m)$. Therefore the chain $(p, p_1, \dots, p_m, q)$ satisfies the conditions I, II.

In the case c), let p'' be such a point on C that $|p''-p'| \leqq \frac{\varepsilon}{2}$.

Set

$$p'' = p(t''),$$

$$b = \min_{t'' \leqq t \leqq 1} u(p(t)),$$

$$t''' = \sup \{t : u(p(t)) = \tfrac{1}{2}(b+u(p'))\}.$$

Then $|p'''-p'| < \frac{\varepsilon}{2}$.

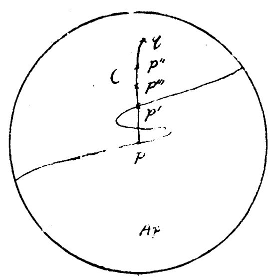

Fig. 13

Therefore we can reduce our case to the case b) for the subarc $\widehat{pp'''}$ of C and to the case d) or a) for the subarc $\widehat{p'''p}$ of C.

In the case d), let p'' be such a point on C that $|p''-p| = \frac{\varepsilon}{2}$. Then the following two cases are possible:

d′) $\quad u(p'') = u(q)$,

d″) $\quad u(p'') < u(q)$.

In the case d′) we can choose the required chain in the same way as in the case b).

In the case d″) repeat the above process about the subarc $\widehat{p''q}$ of C instead of C. After a finite number of times we can get the required chain.

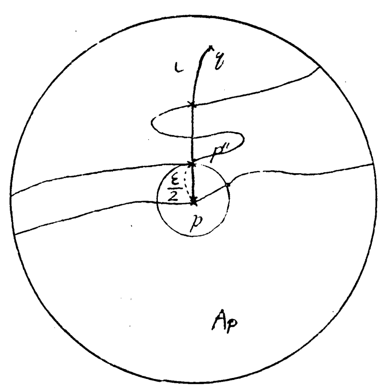

Fig. 14

Lemma 8. *Let q be an arbitrary point of an admissible neighbour-*

hood A_p of N'_p different from p and $N'_p \subset N_p$. Then there exists a Jordan arc from p to q which intersects each level at most once in N_p.

Proof. Let D^r_p be the open disk with radius r and centre p, A^r_p be the maximal open disk which is an admissible neighbourhood of D^r_p. Without loss of generality we suppose that N_p is D^1_p and N'_p is $D^{\frac{1}{2}}_p$. Therefore by Lemma 7 there exists a chain

$$(p,\ p_1,\ \ldots,\ p_n = q) \in D^{\frac{1}{2}}_p$$

satisfying the following property;

I′) $$u(p) < u(p_1) < \ldots < u(p_n)\ .$$

II′) $$|p_i - p_{i-1}| < \varepsilon\left(\frac{1}{4}\right), \text{ where } \varepsilon\left(\frac{1}{4}\right) = \inf_{p \in D^{\frac{1}{2}}_p} (\text{radius of } A^{\frac{1}{4}}_p)$$

Apply Lemma 7 to the pairs of points p_{i-1} and p_i, and we have the chain

$$(p = p',\ p'_1,\ p'_2,\ p,\ \ldots,\ p'_\alpha = p_1,\ p'_{\alpha+1},\ \ldots,\ p'_\beta = p_2,\ p'_{\beta+1}, \ldots,\ p'_\mu = q) \in D^{\frac{1}{2}+\frac{1}{4}}_p$$

satisfying the following properties;

I″) $$u(p') < u(p'_1) < \ldots < u(q)\ .$$

II″) $$|p'_i - p'_{i-1}| < \varepsilon\left(\frac{1}{8}\right), \text{ where } \varepsilon\left(\frac{1}{8}\right) = \inf_{p \in D^{\frac{1}{2}+\frac{1}{4}}_p} (\text{radius of } A^{\frac{1}{8}}_p)\ .$$

If we continue this process indefinitely, we have a countable number of points whose closure C is homeomorphic to the interval $(u(p),\ u(q))$ by the function u(p). Then C is the required Jordan arc.

Lemma 9. *Let Ω be one of the domains above (below) c_0 possessing p_0 as a boundary point. Then it is possible to choose the subdomain D of Ω satisfying the following conditions:*

$\overline{D}$ can be mapped by some homeomorphism onto the rectangle $\boldsymbol{R}$ in the z-plane bounded by $x = \pm 1$, $y = c'$, so that each level L_c contained in $\overline{D}$ corresponds to the segment $y = c$ cut off by $x = \pm 1$.

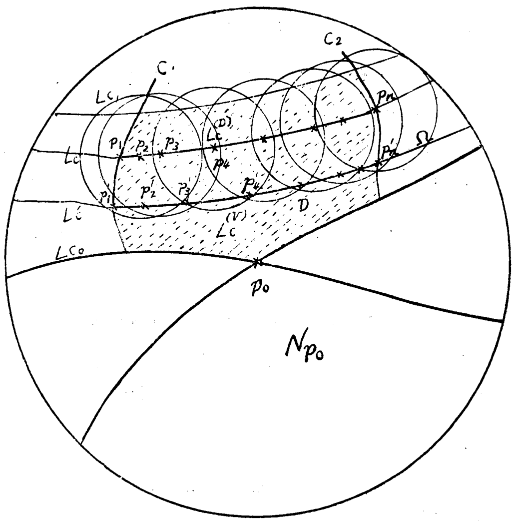

Fig. 15

Proof. Let N_{p_0} be a *simple* neighbourhood of p_0. Without loss of generality we can suppose that N_{p_0} is an open disk in the z-plane. Let p' and q' be points of $\beta\Omega$ in N_{p_0}. Two Jordan arcs C_1, C_2 ($C_1 \cdot C_2 = 0$) can be drawn from p' and q' so that they intersect each level at most once respectively [Lemma 8]. Let L_{p_1} be one of the levels intersecting C_1 and C_2. Then the domain D bounded by L_{c_0}, C_1, C_2 and L_{c_1} will be the required one. Let $L_c^{(D)} = L_c \cdot \overline{D}$.

We shall show that $L_{c'}^{(D)}$ converges to $L_c^{(D)}$ with $c' \to c$ in the sense of Fréchet [2].

Let ε be an arbitrary positive number. $L_c^{(D)}$ is covered by a finite number of $\{A_p^{\frac{\varepsilon}{2}}\}$, where $p \in L_c^{(D)}$.

Set

$$L_c^{(D)} : p = p_c(t) \quad 0 \leqq t \leqq 1.$$

Let $A_{p_i}^{\frac{\varepsilon}{2}}$ be an admissible neighbourhood of $D_{p_i}^{\frac{\varepsilon}{2}}$ ($i = 1, \ldots, n$), where $p_i = p_c(t_i)$, such that $0 < t_1 < \ldots < t_n = 1$.

There exist points $p_i = p_c(t_i)$ and admissible neighbourhoods $A_{p_i}^{\frac{\varepsilon}{2}}$ of $D_{p_i}^{\frac{\varepsilon}{2}}$ satisfying the following conditions :

1. $$0 = t_1 < t_2 < \ldots < t_n = 1 .$$
2. $$p_i \in A_{p_{i+1}}^{\frac{\varepsilon}{2}} \quad (i = 1, 2, \ldots, n-1) .$$

Let C_1' be the subarc possessing p_1 of C_1 in $A_{p_1}^{\frac{\varepsilon}{2}}$, C_2' the subarc possessing p_n of C_2 in $A_{p_n}^{\frac{\varepsilon}{2}}$. Let D_i be $A_{p_i}^{\frac{\varepsilon}{2}} \cdot A_{p_{i+1}}^{\frac{\varepsilon}{2}}$ ($i = 2, \ldots, n-1$).

There exists a positive number δ such that the arc $L_{c'}^{(D)}$ intersects C_1', C_2' and all D_i ($i = 2, \ldots, n-1$) for $|c - c'| < \delta$.

Let $p_2' = p_{c'}(t_1)$, $p_n' = p_{c'}(t_n)$ and $p_i' = p_{c'}(t_i)$ be the point on C_1', C_2' and D_i ($i = 2, \ldots, n-1$) respectively such that $0 = t_1 < t_2 < \ldots < t_n = 1$.

Then there exists a homeomorphism T such that subarcs $\widehat{p_i p_{i+1}}$

---

2) It means that Fréchet distance between $L_{c'}^{(D)}$ and $L_c^{(D)}$ tends to zero with $c' \to c$, where the Fréchet distance is defined as follow: Let T be a homeomorphism between $L_{c'}^{(D)}$ and $L_c^{(D)}$. Then $\inf_T$ [$\max_{p \in L_c^{(D)}}$ (distance between $p' = T(p)$ and p)] is called the Fréchet distance between $L_{c'}^{(D)}$ and $L_c^{(D)}$.

M. Morse, A special parametrization of curves, Bull. Amer. Math. Soc. 42 (1936), 915-922,

$(i=1,2,\ldots,n-1)$ of $L_c^{(D)}$ correspond to subarcs $\widehat{p_i'p_{i+1}'}$ $(i=1, 2, \ldots, n-1)$ of $L_{c'}^{(D)}$ respectively.

Since $\widehat{p_ip_{i+1}}$ and $\widehat{p_i'p_{i+1}'}$ are contained in $C_{p_i}^{\frac{\varepsilon}{2}}$ whose diameter is ε, the Fréchet distance [2] between $L_c^{(D)}$ and $L_{c'}^{(D)}$ is less than ε. Therefore $L_{c'}^{(D)}$ converges to $L_c^{(D)}$ with $c'\to c$ in the sense of Fréchet.

Let μ be the μ-length [3] of subarc $\widehat{p_1 q}$ of L_c and $q(\mu, c)$ be the function corresponding the point q on F and the point $\mu+ic$ on the z-plane. Then $q(\mu, c)$ is continuous.[3]

Setting $\mu^*=2\left(\frac{\mu}{\mu_c}-\frac{1}{2}\right)$, where μ_c denote the μ-length of $L_c^{(D)}$, $q(\mu^*, c)$ maps $\bar{D}$ onto the rectangle $\boldsymbol{R}$.

Thus our conclusion has been verified.

Proof of Theorem 1. Since the necessity of the conditions (1), (2) and (3) is evident, we shall show that they are sufficient. First let p be an ordinary point. There exists a simple neighbourhood N_p as follows:

$$\left.\begin{matrix}\Omega^+\\ \Omega^-\end{matrix}\right\}, \text{ the sole domain of } \left\{\begin{matrix}\text{above}\\ \text{below}\end{matrix}\right\} u(p)=c \text{ in } N_p,$$

is mapped topologically onto the rectangle $\left\{\begin{matrix}\boldsymbol{R}^+\\ \boldsymbol{R}^-\end{matrix}\right\}$ in the z-plane bounded by $x=\pm 1$, $y=c$ and $y=\left\{\begin{matrix}c'\\ c''\end{matrix}\right\}$ $(c''<c<c')$, so that the level $c_0\left\{\begin{matrix}(c\leqq c_0\leqq c')\\ (c''\leqq c_0\leqq c)\end{matrix}\right\}$ may correspond to $y=c_0$ [Lemma 9].

Furthermore, $\boldsymbol{R}=\boldsymbol{R}^++\boldsymbol{R}^-$ becomes a topological image of the whole N_p.

Let $p=T(z)$ denote this homeomorphism, and we have in N_p, i.e. in R

---

3) Let a curve C have a representation $p(t)$, $(0\leqq t\leqq 1)$. Let τ; $0\leqq t_1, \leqq t_2\leqq\cdots\leqq t_n\leqq 1$ be a set of values of t on the interval $(0, 1)$. We introduce the number

$$m_n=\max_{\tau}\left[\min_{1\leqq i\leqq n-1}(\text{dist. } p(t_i)p(t_{i+1}))\right].$$

Set

$$\mu=\frac{m_2}{2}+\frac{m_3}{4}+\frac{m_4}{8}+\cdots\cdots,$$

We call μ the *μ-length* of the curve C.

H. Whitney, Regular families of curves, Annals of Math. 34 (1933), pp. 244-270.

M. Morse, A special parameterization of curves, l. c.

$$u(p)=u(T(z))=U(z)=\Im z\ .$$

Second, let p be a saddle point of order $(n-1)$, N_p be a simple neighbourhood of p.

Then

$$N_p-L_{(u)p}=\sum_{i=1}^{n}(\Omega_i^{+}+\Omega_i^{-})$$

where Ω^+ and Ω^- denote respectively domains above and below c possessing the sole point p as a common boundary point, and situated cyclically in the order Ω_1^+, Ω_1^-, Ω_2^+, Ω_2^-, ..., Ω_n^+, Ω_n^-. The subdomain $\begin{Bmatrix} D_i^+ \\ D_i^- \end{Bmatrix}$ of $\begin{Bmatrix} \Omega_i^+ \\ \Omega_i^- \end{Bmatrix}$ is mapped topologically onto the rectangle $\begin{Bmatrix} \boldsymbol{R}^+ \\ \boldsymbol{R}^- \end{Bmatrix}$ in ζ-plane $(\zeta=\xi+i\eta)$ bounded by $x=\pm 1$, $y=c$ and $y=\begin{Bmatrix} c' \\ c'' \end{Bmatrix}$ $(c''<c<c')$, so that the level c_0 $\begin{Bmatrix} (c\leqq c_0\leqq c') \\ (c''\leqq c_0\leqq c) \end{Bmatrix}$ may correspond to $y=c_0$ [Lemma 9].

$U(x)=\Re z^n+c$ is the harmonic function with a saddle point of order $(n-1)$ at $z=0$. The niveau curve $U=c$ divides any circle $|z|<\rho$ (for sufficiently large ρ) into n sectors above c; σ_1^+, σ_2^+, ..., σ_n^+ and n sectors below c; σ_1^-, σ_2^-, ..., σ_n^- alternately. The subdomain $\begin{Bmatrix} D_i^+ \\ D_i^- \end{Bmatrix}$ of $\begin{Bmatrix} \sigma_i^+ \\ \sigma_i^- \end{Bmatrix}$ topologically onto the rectangle $\begin{Bmatrix} \boldsymbol{R}^+ \\ \boldsymbol{R}^- \end{Bmatrix}$ in the ζ-plane bounded by $x=\pm 1$, $y=c$ and $y=\begin{Bmatrix} c' \\ c'' \end{Bmatrix}$ $(c''<c<c')$, so that the level c_0 $\begin{Bmatrix} (c\leqq c_0\leqq c') \\ (c''\leqq c_0\leqq c) \end{Bmatrix}$ may correspond to $y=c_0$ [Lemma 9].

Hence there exists a topological transformation $p=T(z)$ from the subdomain $|z|<\rho'$ $(\rho'<\rho)$ of $|z|<\rho$ to the subdomain N_p' of N_p, so that the level c_0 with respect to $U(z)$ in $|z|<\rho'$ may correspond to the level c_0 with respect to $u(p)$ in N_p'. Then $u(p)=u(T(z))=U(z)$. Thus the proof is completed.

We see that we can replace the condition (3) in Theorem 1 by the following weaker condition (3)$'$:

(3)$'$ *There is no pair of sequences of continua* $\{C_i\}$ *and* $\{C_i'\}$ *converging to a continuum, where* C_i *and* C_i' *are subcontinua of the same level* c_i *having common point each other.*

Theorem 1$'$. *In order that a real function* $u(p)$ *on* F *is pseudo-harmonic it is necessary and sufficient that* $u(p)$ *satisfies the conditions* (1), (2) *and* (3)$'$.

§ 2. The conjugate pseudo-harmonic functions and its applications.

Let $u(p)$ be a pseudo-harmonic function on F and $v(p)$ be a real valued function on F. When, for a neighbourhood of any point on F, there exists a homeomorphism T by which N corresponds to $x^2+y^2<1$ in the z-plane, and $V(z)\equiv v(T(z))$ is the conjugate harmonic function of $U(z)\equiv u(T(z))$, the function $v(p)$ is called the *conjugate pseudo-harmonic function* of $u(p)$.

Theorem 2. *Let $u(p)$ be a pseudo-harmonic function on F. For a real valued function $v(p)$ to be a conjugate pseudo-harmonic function of $u(p)$ on F it is necessary and sufficient that*

a) *$v(p)$ is continuous,*

b) *$v(p)$ is an open transformation,*

c) *any continuum on each level of $u(p)$ does not correspond to one value by $v(p)$.*

Proof. Since the conditions a), b) and c) are evidently necessary, we shall show that they are sufficient.

Let us denote by L_c^u and L_c^v the levels c of $u(p)$ and $v(p)$ respectively.

i) *Let p be a point on F and N_p be a neighbourhood of p. Each component of L_c^v in the domain above (below) $u(p)$ in N_p intersects every component of $\{L_c^u\}$ at most once.*

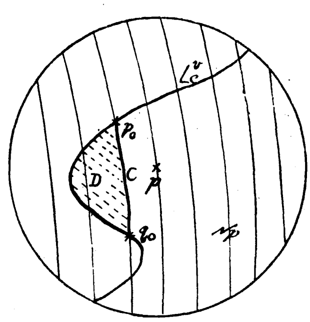

Fig. 16

Suppose that a component of L_c^v intersects a component of $L_{c'}^u$ at two points p and q. There exists at least an open arc C on the subarc $\widehat{pq}$ of $L_{c'}^u$ such that any point on C is not on L_c^v [c)]. Then there exists a domain D bounded by C and L_c^v.

Put $$w(p)=u(p)+iv(p)\ .$$

Then $w(p)$ is a continuous function on $\overline{D}$, therefore $w(p)$ is bounded in D. On the other hand D must be mapped onto the domain bounded by $u=c'$ and $v=c$ by $w(p)$, so that $w(p)$ is not bounded in D, which is impossible.

ii) *Any component C of L_c^v in a suitable neighbourhood N_p of an ordinary point p of $u(p)$ consists of a Jordan arc with its two end points on βN_p.*

We may suppose that $\{L_{c'}^u\}$ are parallel lines in N_p [Lemma 9]. C is distinct from a point [Lemma 6]. Then owing to i) $\overline{C}$ is a Jordan arc. Moreover C separates at least two domain above c and below c [Lemma 2], so that two end points of C is on βN_p.

iii) *Any component C_1^u of $L_{c'}^u$ in a simple neighbourhood N_p of an ordinary point p with respect to $u(p)$ intersects at most one component of L_c^v.*

Suppose that C_1^u intersects two components C_1^v and C_2^v of L_c^v. Since there is no saddle point of $u(p)$ in N_p, there exists such a subarc C_2^u of $L_{c'}^u$ near C_1^u that C_2^u intersects C_1^v and C_2^v.

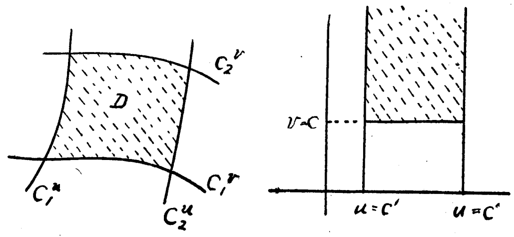

Fig. 17

Put $w(p)=u(p)+iv(p)$.

Let D be the domain bounded by C_1^u, C_2^u, C_1^v and C_2^v. $W(p)$ is continuous on $\bar{D}$. On the other hand D is mapped by $w(p)$ onto the domain bounded by $u=c'$, $u=c''$, and $v=c$, so that $w(p)$ is not bounded in D, which is impossible.

iv) *The family of levels $\{L_c^v\}$ is equi-locally-connected at an ordinary point p of $u(p)$.*

Suppose that the family of levels $\{L_c^v\}$ is not equi-locally-connected at an ordinary poinl p of $u(p)$. If we choose a suitable neighbourhood N_p of p, there exist two sequences of points $\{p_n\}$ and $\{p_n'\}$ $(n=1,2,\ldots)$, such that $v(p_n)=v(p_n')$, $C_n\cdot C_n'=0$ where C_n and C_n' are the subarcs of $L_{u(p_n)}^v$ and $L_{u(p_n')}^v$ containing p_n and p_n' respectively, and $\{C_n\}$ and $\{C_n'\}$ converge to the subarc of the level $v(p)$ containing p. Let q be an inner point of C. Then the Jordan arc of $L_{u(q)}^u$ containing q must intersects C_n and C_n' for sufficiently large number n. This contradicts iii).

v) *$v(p)$ is pseudo-harmonic on F.*

The family of levels $\{L_c^v\}$ is equi-locally-connected on F except for the saddle points of $u(p)$ on F [iv)]. Therefore $v(p)$ is pseudo-harmonic on F [Theorem 1]. Then we can see easily that a compact domain D, bounded by levels $u=c_1$, $u=c_2$, $v=c_1'$ and $v=c_2'$ and containing no saddle point of $u(p)$ and $v(p)$, are mapped topologically onto the rectangle $\boldsymbol{R}$ on the z-plane bounded by $x=c_1$, $x=c_2$, $y=c_1'$ and $y=c_2'$, such that the levels $u=c$ and $v=c'$ correspond to $x=c$ and $y=c'$ respectively. Let p be an arbitrary point on F. As in Theorem 1 there exists such a homeomorphism $p=T(z)$ in a certain N_p that $v(T(z))$ is a conjugate harmonic function of $u(T(z))$ in N_p. Thus the proof is completed.

Definition: Let p_1 and p_2 be two points on F and let N_{p_1} and N_{p_2} be neighbourhoods with $N_{p_1}\cdot N_{p_2}\neq 0$. Let $u_1(p)$ and $u_2(p)$ be pseudo-harmonic in N_{p_1} and in N_{p_2} respectively. When $u_1(p)=u_2(p)$ in N_{p_1}, N_{p_2}, $u_2(p)$ will be called a *direct pseudo-harmonic continuation* of $u_1(p)$.

Let $p_1, p_2, \ldots, p_n$ be the points on F and $N_{p_1}, N_{p_2}, \ldots, N_{p_n}$ be such neighbourhoods that $N_{p_i}\cdot N_{p_{i+1}}\neq 0$ $(i=1, 2, \ldots, n-1)$. Let $u_i(p)$ be pseudo-harmonic in N_{p_i} $(i=1, 2, \ldots, n)$ respectively and let $u_{i+1}(p)$ be a direct pseudo-harmonic continuation of $u_i(p)$.

Set

$$U(z)=\begin{cases} u_1(p) & \text{in } N_{p_1} \\ u_2(p) & \text{in } N_{p_2} \\ \cdot & \cdot \\ \cdot & \cdot \\ u_n(p) & \text{in } N_{p_n} \end{cases}$$

($U(z)$ is pseudo-harmonic in $\sum_{i=1}^{n} N_{p_i}$, but not always one valued.) Then $U(z)$ will be called a *pseudo-harmonic continuation* of $u_1(p)$.

Theorem 3. *Let $u(p)$ be pseudo-harmonic on F. Then there exists always a conjugate pseudo-harmonic function.*

Proof.

I) *There exists a family of curves $\{C\}$ satisfying the following conditions*:

a) *On the ordinary point of $u(p)$ they do not intersect each other, and on the saddle point of $u(p)$ with order $n-1$ just n curves of them intersect each ether.*

b) *Each of them is not compact with respect to F and intersects each level of $u(p)$ at most once.*

c) *They cover every points on F.*

Let a countable number of points $p_1, p_2, \ldots, p_n, \ldots$ be dense on F. There exist a countable number of neighbourhoods $\{N_{q_i}\}$ $(i=1, 2, \ldots)$ such that the levels of $u(p)$ in N_{q_i} correspond to the levels of $\Re z^m$ $(1\leqq m<\infty)$ in $|z|<1$ by homeomorphism T_i [Proof of Theorem 1], and $\sum_{i=1}^{\infty} N_{q_i}=F$, but $\sum_{i=1}^{k} N_{q_i}\neq F$ $(k<\infty)$. Let $p_1\in N_{q_{i_1}}$. We can draw from p_1 a Jordan arc with two end points on $\beta N_{q_{i_1}}$ such that it intersects each level of $u(p)$ at most once. Let one of end points be contained in $N_{q_{i_2}}$. Then we can extend its Jordan arc as far as a point of $\beta N_{q_{i_2}}$

If we continue this process indefinitely, we can get a curve C_1 satisfying the condition b).

Let $C_1, C_2, \ldots, C_n$ be curves satisfying the conditions a) and b). Let $p_1, p_2, \ldots, p_j$ be points on $\sum_{i=1}^{n} C_i$ and p_{i+1} be not on $\sum_{i=1}^{n} C_i$.

We can draw such a Jordan arc that it intersects each level of $u(p)$ at most once and does not intersect C_i $(1 \leqq i \leqq n)$. Therefore we can get the curve C_{n+1} containing p_{j+1} such that it satisfies the condition b) and does not intersect C_i $(1 \leqq i \leqq n)$. Then we can draw a coun table number of curves $\{C_i\}$ $(i=1, 2, \ldots)$ such that they are dense on F and satisfy the conditions a) and b).

There exist consequently a family of curves satisfying the conditions a), b) and c).

II) *There exists a continuous real function $v(p)$ on F such that its levels coincide with the family of curves $\{C\}$ and $v(p)$ is monotone on each level of $u(p)$.*

When we can not draw any closed curve intersecting the family of levels of $u(p)$ at most once, let L^0 be a component of one of the levels of $u(p)$. We can define a continuous and monotone real bounded function $v(p)$ on L^0. Let us extend $v(p)$ to the curves of $\{C\}$ intersecting L^0 such that each value $v(p)$ on C is the same value of $v(p)$ on the point of intersection $L^0 \cdot C$. Let D be the domain on which $v(p)$ was defined. Let $C_1, C_2, \ldots$ be all boundary curves of D and let $L^1, L^2, \ldots$ components of levels of $u(p)$ intersecting $C_1, C_2, \ldots$ respectively. Then be the we can extend a continuous and monotone real bounded function $v(p)$ to the parts L^i contained in the complementary set of D. Let us extend $v(p)$ to the curves of $\{C\}$ intersecting L^i such that each value $v(p)$ on C is the same as $v(p)$ on the point of intersection $L^i \cdot C$.

If we continue this process indefinitely, we can define $v(p)$ on every point of F.

When we can draw a closed curve intersecting the family of levels of $u(p)$ at most once, let F' be the universal covering surface of F. If we define $U(p')$ on F' at p' covering $p \in F$ by $U(p') = u(p)$, we can not draw any closed curves on F' intersecting the family of levels of $U(p')$ at most once.

We can define a continuous monotone real function $V(p')$ on F' such that, if $N_{p_1'}$ and $N_{p_2'}$ are the neighbourhoods on F' covering the neighbourhood of N_p of a point p on F and have no branch point, $V(q_1')-V(q_2')=\text{const.}$, where $q_1'\in N_{p_1'}$, $q_2'\in N_{p_2'}$ and they cover $q\in N_p$. Then we can define the required function $v(p)$, which is many valued on F.

III) *$v(p)$ is the conjugate pseudo-harmonic function of $u(p)$ on F.*

Since $v(p)$ is evidently an open transformation, $v(p)$ satisfies the conditions a), b) and c). Therefore $v(p)$ is the conjugate pseudo-harmonic function of $u(p)$ [Theorem 2.].

Remark: When the function $v(p)$ is a many valued function, let $v_1(p)$ and $v_2(p)$ be two branchs of $v(p)$. We can always choose $v(p)$ such that $v_1(p)-v_2(p)=\text{const.}$, so in this paper the conjugate pseudo-harmonic function of $u(p)$ means such a function $v(p)$.

Theorem 4. *When $u(p)$ is a pseudo-harmonic function on F, we can choose local parameters such that F becomes a Riemann surface and such that $u(p)$ is harmonic on F.*

Proof. Let $v(p)$ be the conjugate pseudo-harmonic function of $u(p)$ on F. Then there exists a homeomorphism T_p between N_p of each point p on F and $|z|<1$ on the z-plane such that $u(T_p(z))$ and $v(T_p(z))$ are conjugate harmonic function in $|z|<1$.

Set

$$W_p(z)=u(T_p(z)) + iv(T_p(z)).$$

Then $W(z)$ is analytic in $|z|<1$.

Let N_{p_1} and N_{p_2} be neighbourhoods of p_1 and p_2 respectively such that $N_{p_1}\cdot N_{p_2}\neq 0$.

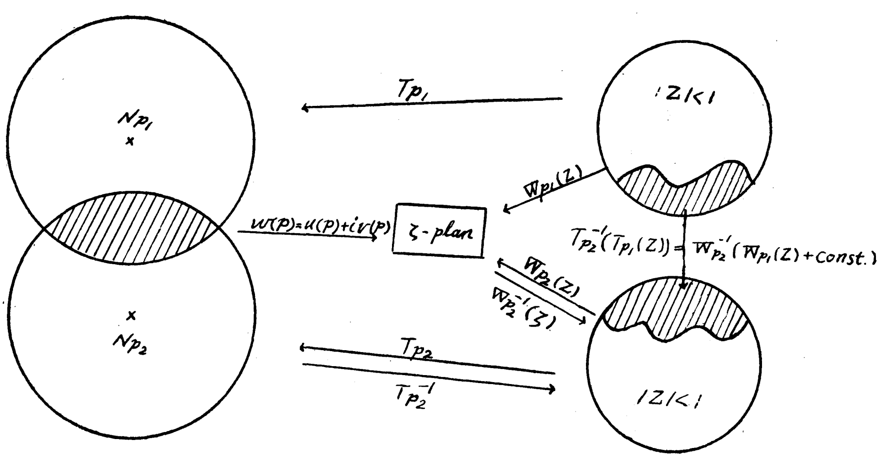

Fig. 18

Then $W_{p_1}(z)$ + const. $= W_{p_2}(z)$ [*Remark*], if $v(p)$ is one valued function the const. $= 0$. Let $T_{p_2}^{-1}$ and $W_{p_2}^{-1}(\zeta)$ be the inverse functions T_{p_2} and $W_{p_2}(z)$ respectively, where $\zeta = u + iv$.

Then $T_{p_2}^{-1}(T_{p_1}(z)) = W_{p_2}^{-1}(W_{p_1}(z) + \text{const.})$ and $W_{p_2}^{-1}(W_{p_1}(z) + \text{const.})$ is analytic. Hence if we choose $\{T_p\}$ as the local parameters on F, F becomes a Riemann surface, and $u(p)$ is harmonic on the Riemann surface F.

Now we shall study the relations between the pseudo-harmonic functions and the interior transformations defined as follows:

The transformation $I(p)$ from the surface F to the surface F' is called an *interior transformation*, when $I(p)$ satisfies the following conditions:

1. *$I(p)$ is continuous on F.*

2. *$I(p)$ is an open transformation.*

3. *$I(p)$ does not transform any continuum on F to one point on F'.*

Theorem 5. *In order that the complex valued function $I(p)$ is an interior transformation it is necessary and sufficient that $\Re I(p)$ and $\Im I(p)$ are the conjugate pseudo-harmonic functions of each other.*

Proof. Since the sufficiency of the condition is evident, we shall show that it is necessary. It is evident the $\Re I(p)$ and $\Im I(p)$ satisfy the conditions (1) and (2) in Theorem 1.

Suppose that $\Re I(p)$ does not satisfy the condition (3)′ in Theorem 1′.

Then there exist a pair of sequences of continua $\{C_i\}$ and $\{C_i'\}$ converging to a continuum C_0 where C_i and C_i' are subcontinua of of the level c_i of $\Re I(p)$ having no common point each other and C_0 is a subcontinuum of the level c_0 of $\Re I(p)$.

Let p_0 and q_0 be points on C_0 in a neighbourhood of $p \in C_0$ such that $\Im I(p) \neq \Im I(q)$. We may suppose for the sake of convenience that

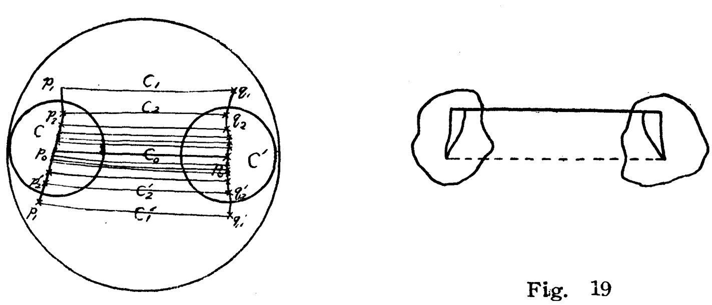

Fig. 19

$\{C_i\}$ and $\{C'_i\}$ are in N_p. Let $\begin{Bmatrix} p_i \\ p'_i \end{Bmatrix}$ and $\begin{Bmatrix} q_i \\ q'_i \end{Bmatrix}$ be points on $\begin{Bmatrix} C_i \\ C'_i \end{Bmatrix}$ respectively such that $\begin{Bmatrix} p_i \\ p'_i \end{Bmatrix} \to p_0$ and $\begin{Bmatrix} q_i \\ q'_i \end{Bmatrix} \to q_0$ with $i \to \infty$.

Let C be a Jordan arc such that C passes through all points p_i and p'_i, and that any subarc $\widehat{p_i\, p'_i}$ of C has a point p_0 on it. Let C' be a Jordan are such that it passes through all points q_i and q'_i and that any subarc $\widehat{p_i\, p'_i}$ of it has a point q_0.

We can choose the neighbourhoods N_{p_0} and N_{q_0} such that

$$|I(p)-I(p_0)| < \frac{1}{3}|I(p_0)-I(q_0)| \quad p \in N_{p_0},$$

$$|I(q)-I(q_0)| < \frac{1}{3}|I(p_0)-I(q_0)| \quad q \in N_{q_0}.$$

For sufficiently large number n the images of arc $\widehat{p_n\, p'_n}$ and arc $\widehat{q_n\, q'_n}$ by $I(p)$ are in the circles

$$|w-I(p_0)| < \frac{1}{3}|I(p_0)-I(q_0)|$$

and

$$|w-I(q_0)| < \frac{1}{3}|I(p_0)-I(q_0)|$$

respectively. Therefore $I(p)$ are unbounded in the compact domain bounded by the subcontinua of C_n and C'_n, arc $\widehat{p_n\, p'_n}$ and arc $\widehat{q_n\, q'_n}$, which contradicts the condition **1**. Since $\Re I(p)$ satisfies the condition (3′), it is a pseudo-harmonic function. By the condition **3** $\Im I(p)$ satisfies the condition (c) in Theorem 2. Therefore $\Im I(p)$ is a conjugate pseudo-harmonic function of $\Re I(p)$.

Then we can easily proof the following Stoïlow's theorems.[4)]

Theorem I. *Let $I(p)$ be an interior transformation from F to F_0, and let q be a point on F_0 such that $q = I(p_0)$. There exist neighbourhoods N_{p_0} and N_q such that N_{p_0} corresponds by $I(p)$ topologically to N_q or the island (Insel) on N_q consisting of a finite number of sheets with only one branch point.*

For we can apply Theorem 5, if we map topologically N_q to $|z| < 1$ in the z-plane.

Theorem II. *Let $I(p)$ be an interior transformation from F to the Gaussian plane. Then there exists a transformation T from F to a Riemann surface R' such that $I(T(p'))$, $p' \in F$, is analytic.*

This is evident from Theorem 5 and Theorem 4.

(Received January 26, 1951)

4) S. Stoïlow, Leçons sur les principes topologiques de la théorie des fonctions analytiques, Paris, Gauthier-Villars, 1938.