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On Linearization of Ordered Groups

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In his recent paper, "Note on a result of L. Fuchs on ordered groups",[1] C. J. Everett has shown: any partial order on an *abelian* group G can be extended into a linear one, if (and trivially only if) every element of G except the unit is of infinite order.

Let us discuss here the non-abelian case in a similar way as Everett. By a *partial order on a group* G, we require that *this order should be preserved under the group operation,* i. e.

$$a > b \textit{ implies } ax > bx \textit{ and } xa > xb \textit{ for all } x \textit{ in } G.$$

Such a partial order on G is completely determined by the set $\mathfrak{P}$ of all elements p 1 (the unit) of G. $\mathfrak{P}$ has namely the following characterizing properties:

1) $\mathfrak{P}$ is a self-conjugate semi-group with 1,

2) $\mathfrak{P}$ contains no element along with its inverse except 1.

Now we are to enlarge this set $\mathfrak{P}$ until for every $x(\neq 1)$ in G either x or x^{-1} belongs to $\mathfrak{P}$, that is to extend the given partial order into a linear one.

First we need some preliminary notations and remarks. Let

$$C_a = \{xax^{-1};\ x \in G\},$$

and further

$$\mathfrak{C}_a = \sum_{n=1}^{\infty} (C_a)^n,$$

where $(C_a)^n$ denotes the set of all elements of the form $a_1 a_2 \dots a_n$, $a_i \in C_a$, and $\sum$ means the set-union.

If G admits a linear order at all, then

(I) $\mathfrak{C}_a$ *and* $\mathfrak{C}_a^{-1}$ *are disjoint for every* a.

Moreover, calling two elements a and b *equivalent,* if there exists a finite chain $\mathfrak{C}_{a_1}, \mathfrak{C}_{a_2}, \dots, \mathfrak{C}_{a_k}$, where $a \in \mathfrak{C}_{a_1}$, $\mathfrak{C}_{a_1} \wedge \mathfrak{C}_{a_2} \neq 0$, $\mathfrak{C}_{a_2} \wedge \mathfrak{C}_{a_3} \neq 0$, $\dots$, $\mathfrak{C}_{a_{k-1}} \wedge \mathfrak{C}_{a_k} \neq 0$, $\mathfrak{C}_{a_k} \ni b$, we easily see that this equivalence satisfies the usual relations of equivalence, and we get the following necessary

1): Amer. J. Math. 72, p. 216 (1950).

condition for linearization of G:

(II) *a and a^{-1} is not equivalent for every a.*

To see the necessity of the above conditions (I) and (II), we have only to notice the fact that by a linear order on G the elements in a single set $\mathfrak{C}_a$ should be made either all greater or all smaller than 1. Evidently (II) is stronger than (I).

Each of these necessary conditions corresponds to that of the abelian case, where we can derive also its sufficiency (see the beginning of this paper). To extend an arbitrarily given partial order in the general case, however, we are in want of the following additional condition, which turns out to be necessary as well:

(III) *For any pair of elements x and y in a set $\mathfrak{C}_a$ for any a, $\mathfrak{C}_x$ and $\mathfrak{C}_y$ have elements in common.*

From the definition of $\mathfrak{C}_a$, it follows that if x is in $\mathfrak{C}_a$ $\mathfrak{C}_x$ is contained in $\mathfrak{C}_a$, therefore to (III) may be given the following expression:

(III)′ *If both $\mathfrak{C}_x$ and $\mathfrak{C}_y$ have elements in common with $\mathfrak{C}_a$, then so do $\mathfrak{C}_x$ and $\mathfrak{C}_y$ themselves.*

For, assuming (III), if in (III)′ $\mathfrak{C}_x$ and $\mathfrak{C}_y$ should be disjoint, then $\mathfrak{C}_a$ would contain two disjoint subsets $\mathfrak{C}_u$ and $\mathfrak{C}_v$ contained in the intersection of $\mathfrak{C}_a$ with $\mathfrak{C}_x$ and $\mathfrak{C}_y$ respectively, which contradicts (III). Conversely if (III)′ holds, then $\mathfrak{C}_x$ and $\mathfrak{C}_y$ in (III) certainly intersect with each other, since they both intersect $\mathfrak{C}_a$.

Furthermore, under the assumption of (III), (II) can be deduced, as is easily seen, from (I).

All this adds up to:

(III) *and* (III)′,

(I) & (III) *and* (II) & (III)

are equivalent respectively.

Now we are in a position to enlarge the set $\mathfrak{P}$. First we can establish the following convenient property of $\mathfrak{P}$ by virtue of the additional condition (III):

(*) *If $\mathfrak{P}$ intersects $\mathfrak{C}_a$, then $\mathfrak{P}$ and $\mathfrak{C}_a^{-1}$ are disjoint.*

Proof: The definition of $\mathfrak{P}$ and of $\mathfrak{C}$ ascertain that for an element p of $\mathfrak{P}$ $\mathfrak{C}_p$ is contained wholly in $\mathfrak{P}$. If therefore p is a common element of $\mathfrak{P}$ and $\mathfrak{C}_a$, $\mathfrak{C}_p$ is contained in both $\mathfrak{P}$ and $\mathfrak{C}_a$, therefore $\mathfrak{C}_p^{-1}$ is also contained in both $\mathfrak{P}^{-1}$ and $\mathfrak{C}_a^{-1}$, since each of these sets is consisted of the inverse elements of $\mathfrak{C}_p$, $\mathfrak{P}$ and $\mathfrak{C}_a$ respectively. Now, if $\mathfrak{C}_a^{-1}$ should contain an element q of $\mathfrak{P}$, then $\mathfrak{C}_q$ would intersect $\mathfrak{C}_p^{-1}$ by virtue of (III). But $\mathfrak{C}_q$ is contained in $\mathfrak{P}$, and $\mathfrak{C}_p^{-1}$ in $\mathfrak{P}^{-1}$, therefore $\mathfrak{P}$ would have some element in common with $\mathfrak{P}^{-1}$, which is certainly different from 1, since any $\mathfrak{C}$-set can not contain 1 because of (I). This contradicts the property 2) of $\mathfrak{P}$. Thus (*) is proved.

Hence, if neither x nor x^{-1} is contained in $\mathfrak{P}$, three mutually exclusive cases may occur:

i) $\mathfrak{P}$ intersects $\mathfrak{C}_x$,

ii) $\mathfrak{P}$ intersects $\mathfrak{C}_x^{-1}$,

and

iii) neither $\mathfrak{C}_x$ nor $\mathfrak{C}_x^{-1}$ intersects $\mathfrak{P}$.

Denoting by $\mathfrak{C}_x'$ the set which is obtained by adjoining 1 to $\mathfrak{C}_x$, we define a new set $\mathfrak{P}^*$ as follows:

In the case i), $\mathfrak{P}^*$ should be $\mathfrak{P}\mathfrak{C}_x'$, that is the set of all elements of the form px_1, where p in $\mathfrak{P}$, x_1 in $\mathfrak{C}_x'$:

In the case ii), similarly as in i) but with x replaced by x^{-1}; in the case iii), in either way.

It is trivial that $\mathfrak{P}^*$ contains $\mathfrak{P}$ properly. We shall show now that $\mathfrak{P}^*$ satisfies the above mentioned characterizing properties 1) and 2). Considering that $\mathfrak{C}_x'$ has also exactly the same properties as $\mathfrak{P}$, it is almost evident that the property 1) hold for $\mathfrak{P}^*$. If $px_1=1$, p in $\mathfrak{P}$, x_1 in $\mathfrak{C}_x'$, then $p=x_x^{-1}$, therefore $\mathfrak{P}$ intersects $\mathfrak{C}'^{-1}_x$. Namely in either of the three cases an element of $\mathfrak{P}^*$ coincides with 1, only if its factors are both 1. Moreover, $px_1\cdot qx_2=1$ implies

$$1=p\cdot x_1qx_1^{-1}\cdot x_1x_2=pq'\cdot x_1x_2$$

with some q' of $\mathfrak{P}$ (for $\mathfrak{P}$ is self-conjugate) hence $pq'=x_1x_2=1$ (for $\mathfrak{P}$ and $\mathfrak{C}_x'$ are semi-groups), and hence $p=q'=x_1=x_2=1$ (by the property 2) of $\mathfrak{P}$ and of $\mathfrak{C}_x'$), which shows that $\mathfrak{P}^*$ has the property 2). It is noticeable here that in the proof to this we did not use the additional condition (III).

Now $\mathfrak{P}^*$ actually induces a partial order on G, which comes out to be a proper extension of the original one. Moreover by making use of (III) such $\mathfrak{P}^*$ again acquires the (*)-property, which enables us to carry out our process if it is needed at all. Meanwhile we are evidently capable of applying Zorn's lemma here, therefore there exists a maximal set $\mathfrak{P}_0$ containing $\mathfrak{P}$ and satisfying the conditions 1) and 2), which must contain either x or x^{-1} for every x of G, since otherwise we could have enlarged it further because of its (*)-property: thus the corresponding order is the desired linear extension. Hence our assumptions are proved to be sufficient.

Finally we show that the condition (III) is actually necessary for extending an arbitrarily given partial order on G. In fact, if (III) does not hold for G, there exist two disjoint subsets $\mathfrak{C}_x$ and $\mathfrak{C}_y$ of a certain set $\mathfrak{C}_a$. Let us consider the set $\mathfrak{Q}=\mathfrak{C}_x'\mathfrak{C}'^{-1}_y$. We shall show that this set Q defines a partial order which cannot be extended into a linear one; thus will be ascertained the necessity of (III). To this we recall the proof to the corresponding proposition about $\mathfrak{P}^*$, which was, as we have noticed, established without (III). It is then obvious that $\mathfrak{Q}$ satisfies the property 1): as to the property 2) the present case corresponds to the previous case iii), viz. here $\mathfrak{C}_x'$ (considered as P) intersects neither $\mathfrak{C}_y$ nor $\mathfrak{C}_y^{-1}$, therefore we can assert similarly that $\mathfrak{Q}$ satisfies the property 2). Hence $\mathfrak{Q}$ determines a partial order, by which $x>1$, $y^{-1}>1$, or what is the same, $y<1$. On the other hand, since x and y are both contained in a single $\mathfrak{C}_a$, these two elements must be in the same ordering compared with 1, if G admits a linear order. Therefore the above constructed partial order cannot be made into a linear one.

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