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On a Local Property of Absolute Neighbourhood Retracts

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1. In this note we shall prove the following theorem and derive from it several corollaries.

THEOREM. *In order that a separable metric space Y is an absolute neighbourhood retract*[1]) *it is necessary and sufficient that Y is compact and has the following property (L):*

(L) *For each point $p \in Y$ and its neighbourhood U there exists a neighbourhood $V \subset U$ of p such that every continuous mapping f of a closed subset A of a given metric space X into V can be extended over X with respect to U.*

This theorem is an extension of the C. Kuratowski's characterization[2]) of absolute neighbourhood n-retracts and gives a local property of absolute neighbourhood retracts.

2. First of all let us prove[3]) the necessity of the condition. Suppose Y is an absolute neighbourhood retract. Y may be considered[4]) as a neighbourhood retract of Hilbert parallelotope Q. Let r be the retraction. Then $r^{-1}(U)$ is an open set of Q containing p. Let ε be a positive number and

$$K_\varepsilon = \underset{x \in Q}{E}[\rho(x, p) < \varepsilon].$$

Take $\varepsilon > 0$ sufficiently small such that $\bar{K}_\varepsilon \subset r^{-1}(U)$ and put $V = K_\varepsilon Y$. Each mapping $f \in V^A$ has an extension $f_1 \in Q^X$[5]). Let π be the projection of Q onto $\bar{K}_\varepsilon$ such that

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1) In the sense of K. Borsuk. See, K. Borsuk: Über eine Klasse von lokal zusammenhängenden Räumen. Fund. Math. 19 (1932), pp. 220-242.

2) C. Kuratowski: Sur les espaces localement connexes et péaniens de dimension n. Fund. Math. 24 (1935), p. 273, Théorème 1.

3) The proof of C. Kuratowski also holds in the case $n=\infty$ without the assumption of compactness of Y. But if Y is compact we can prove it more simply as in the text.

4) K. Borsuk, loc. cit. p. 223, Section 3.

5) W. Hurewicz and H. Wallman: Dimension Theory, p. 82, Cor. 1.

$\pi(x) =$ the intersection of the segment $\overline{px}$ with the boundary of K_ε

$$\text{for } x \notin \overline{K}_\varepsilon,$$
$$= x \qquad \text{for } x \in \overline{K}_\varepsilon.$$

The continuous mapping

$$f^*(x) = r\,\pi\, f_1(x)$$

maps X into U and we have $f^*(x) = f(x)$ for $x \in A$. Hence $f^* \in U^X$ is the required extension of $f \in V^A$.

3. To prove the sufficiency of the condition we need the following Definition and Lemma.

DEFINITION. Let U, V be two open sets of Y such that $V \subset U$. If for every continuous mapping $f \in V^A$, where A is an arbitrary closed subset of X, there exists an open set E containing A such that f has an extension $f^* \in U^E$, then we say that V is an *associated neighbourhood* of U.

LEMMA. *Let V_1 and V_2 be associated neighbourhoods of U_1, U_2 respectively and let W_2 be an open set such that $\overline{W}_2 \subset V_2$. Then $V_1 + W_2$ is an associated neighbourhood of $U_1 + U_2$.*

Proof. Put $A_1 = f^{-1}(V_1)$, $A_2 = f^{-1}(W_2)$ and $F_1 = A - A_2$, $F_2 = A - A_1$. Then F_1 and F_2 are mutually disjoint closed subsets of X. Therefore there exist an closed set F and two open sets G_1, G_2 such that

$$X - F = G_1 + G_2, \qquad G_1 G_2 = 0, \qquad F_1 \subset G_1, \quad F_2 \subset G_2.$$

Let f_1 be the partial mapping $f \mid (A - G_2)$. Since f_1 maps the closed subset $(A - G_2)$ of X into V_1, there exists an open set $E_1 \supset A - G_2$ such that f_1 can be extended to $f_1^* \in U_1^{E_1}$. Take an open set E_0 containing FA sufficiently small such that $\overline{E}_0 \subset E_1$ and $f_1^*(\overline{E}_0 \subset V_2$. The existence of such an open set E_0 is guaranteed since $\overline{W}_2 \subset V_2$ and $f_1^*(FA) = f(FA) \subset V_1 W_2$. Consider the continuous mapping f_2 defined as follows:

$$f_2(x) = f_1^*(x) \qquad \text{for} \quad x \in \overline{E}_0 F,$$
$$= f(x) \qquad \text{for} \quad x \in A - G_1 - F.$$

Then f_2 maps the closed subset $(A - G_1) + \overline{E}_0 F$ into V_2. Therefore there exists an open set $E_2 \subset \overline{E}_0 F + (A - G_1)$ such that f_2 can be extended to $f_2^* \in U_2^{E_2}$.

Define the mapping f^* as follows:

$$f^*(x) = f_1^*(x) \qquad \text{for} \quad x \in G_1 E_1 + F E_0,$$
$$= f_2^*(x) \qquad \text{for} \quad x \in G_2 E_2.$$

It is obvious that f^* is continuous over $G_1 E_1 + FE_0 + G_2 E_2$, and $G_1 E_1 + F E_0 + G_2 E_2$ is an open subset of X containing A. Moreover if $x \in A$ then $f^*(x) = f(x)$. Hence $V_1 + W_2$ is an associated neighbourhood of $U_1 + U_2$.

4. Now we prove the sufficiency of the condition. Let p be an arbitrary point of Y. There exists by supposition an associated neighbourhood V_p of the open set Y. Let W_p be an open set such that $\overline{W}_p \subset V_p$. Since Y is compact, there exist a finite number of points $p_1, p_2, \dots, p_k$ such that

$$Y \subset \sum_{i=1}^{k} W_{p_i}.$$

By virtue of Lemma $V_{p_1} + W_{p_2}$ is an associated neighbourhood of $Y + Y = Y$; in general $V_{p_1} + W_{p_2} + \dots + W_{p_j}$ is an associated neighbourhood of Y. Consequently $V_{p_1} + W_{p_2} + \dots + W_{p_k} = Y$ is an associated neighbourhood of Y.

5. From our Theorem follows directly the following

COROLLARY. *If every point of a compact metric space Y has a neighbourhood homeomorphic with an open set of an absolute neighbourhood retract, then Y is also an absolute neighbourhood retract.*

6. If X is a compact metric space, then its closed subset A is also compact and so is $f(A)$. Therefore $f(A)$ may be covered with a finite number of W's of section 4. Hence we have the following theorem by virtue of Lemma.

THEOREM. *Let X be a compact metric space and let Y be a separable metric space of the property (L). Then every continuous mapping f of a closed subset A of X into Y can be extended over some neighbourhood of A with respect to Y.*

This theorem and the remark of [3]) shows that under the restriction of the space X in the class of compact metric spaces, absolute neighbourhood retracts in the sense of C. Kuratowski[6]) and separable metric space of the property (L) are equivalent.

7. The local contractibleness is characterized as follows.

In order that a separable metric space Y is locally contractible it is necessary and sufficient that for every point $p \in Y$ and for every neighbourhood $U_p \ni p$ there exists an associated neighbourhood $V_p \ni p$ of U_p of the following property:

Let X be an arbitrary metric space and let f_0, f_1 be two continuous

[6]) C. Kuratowski, loc. cit. p. 276, Remarques.

mappings which map X into V_p. *Then the mapping* $\varphi(x, t)$ *defined on the closed subset* $X\times(0)+X\times(1)$ *of* $X\times\langle 0,1\rangle$ *such that*

$$\varphi(x, 0)=f_0(x), \qquad \varphi(x, 1)=f_1(x)$$

can be extended to $\varphi * \in U_p^{X\times\langle 0,1\rangle}$.

By virtue of this characterization and Theorem of Section 1, the following well known theorem is an immediate consequence.

THEOREM. *If Y is an absolute neighbourhood retract then it is locally contractible (Borsuk)* [7].

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[7] K. Borsuk, loc. cit. p. 237, Section 27.