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On some types of convergence of positive definite functions

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1. INTRODUCTION. Let us denote by G an arbitrary but fixed locally compact group and by $m(\cdot)$ a left-invariant Haar measure on G. A complex-valued function $\varphi(g)$ on G is called positive definite if it satisfies

( ) $$\sum_{j,k} \bar{\xi}_k \xi_j \varphi(g_k^{-1} g_j) \geq 0 \text{ [1]},$$

for complex $\xi_k (1 \leq k \leq n)$. We shall denote by $\boldsymbol{P}$ the set of all continuous [2] positive definite functions. Then we can define in $\boldsymbol{P}$, among others, the following two types of topology:

(P) *Pontrjagin's topology*: the neighbourhood system of φ_0 of $\boldsymbol{P}$ consists of all sets of the following form:

$$\{\varphi \mid |\varphi(g) - \varphi_0(g)| < \varepsilon \text{ on } F\},$$

where $\varepsilon > 0$ and F is a compact set in G.

(W) *Weak topology*: Let $L^1 \equiv L^2(G)$ be the Banach space of all complex-valued m-integrable functions on G. Then to every φ of $\boldsymbol{P}$ there corresponds a functional on L^1 if we define

$$(\varphi, x) = \int x(g) \varphi(g) \, dg \text{ [3]},$$

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1) $\bar{z}$ denotes the conjugate complex number of z.

2) We can also define the positive definiteness of a bounded measurable function φ as follows:

(**) $$\int\int \varphi(h^{-1}g) \overline{x(h)} x(g) \, dh \, dg \geq 0 \quad \text{for } x(\cdot) \text{ of } L^1(G).$$

Every measurable function satisfying (*) satisfies (**), too. On the other hand it is proved that every (**)-positive definite function coincides almost everywhere with a continuous one. (Cf. the immediately preceeding note in this Journal by the present author. We can also prove this continuity by using the arguments in the present paper and refering some of those in that paper, without making explicit use of the unitary representation of G, though, in essential, both proofs depend on the same idea. Cf. also [1] in LITERATURE at the end of this paper.)

3) The integration is relative to $m(\cdot)$ and its domain is G.

for x of L^1. Hence, by considering $\boldsymbol{P}$ as a subset of the conjugate space of L^1, the weak topology is defined in it, i.e., the neighbourhood system $\{\boldsymbol{V}\}$ of φ_0 consists of all sets of the following form:

$$V(\varphi_0) = \{\varphi \mid |(\varphi, x_k) - (\varphi_0, x_k)| < \varepsilon, 1 \leqq k \leqq n\}, \tag{1}$$

where $\varepsilon > 0$ and $x_k(\cdot) \varepsilon L^1(G)$.

The purpose of the present paper is to prove that these two topologies are equivalent in any such part of $\boldsymbol{P}$ that is constituted of φ's of constant $\varphi(e)$[4, 5, 6, 7]. From this we can prove some theorems similar to those for characteristic functions on the reals; for example, it is easy to prove that, if a sequence of continuous positive definite functions converges (in the sense of Moore-Smith) at every point to a continuous function, then not only the limit function is positive definite, but also this convergence is uniform on every compact set.

The present author owes the essential simplification of his proof to Mr. Shizuo Kakutani.

2. PRELIMINARIES. (1°) Let V be a compact neighbourhood of e, and $c_V(\cdot)$ its characteristic function. Put $d_V(\cdot) = c_V(\cdot)/m(V)$; and define

$$x_V(g) = \int d_V(h) d_V(hg)\, dh. \tag{2}$$

Then it is easy to verify that

(3) $x_V(g)$ is continuous, $\geqq 0$ on G and $= 0$ outside V^2,

$$\int x_V(g)\, dg = 1 \tag{4}$$

and

$$(\varphi, x_V) \geqq 0 \text{ for any } \varphi \text{ of } \boldsymbol{P}. \text{[8]} \tag{5}$$

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4) e denotes the identity element of G.

5) Recently, it is reported in Mathematical Reviews that D. Raikov proved the similar result in [4], but this paper is yet unavailable to the present author.

6) A proof for a special case, which concerns the topology of the character group of a locally compact commutative group, was stated by D. Raikov in Lemma of his paper [3].

7) This is a solution of Problem 3 in [1].

8) See, for example, [5].

(2°) *M. Krein's inequality* [9]: For positive definite φ, it holds that

(6) $$|\varphi(h)-\varphi(g)|\leq 2\varphi(e)\{\varphi(e)-\Re(\varphi(g^{-1}h))\}$$ [10].

(3°) Let $\varphi \in \boldsymbol{P}$ and $x \in L^1$. For the later application, we shall introduce the following notation:

$$\varphi^x(g)=\int\int \overline{x(s)}\varphi(s^{-1}gt)x(t)\,ds\,dt.$$ [11]

3. PROOF. We shall denote by $\boldsymbol{P}_1$ the subset of $\boldsymbol{P}$ which consists of such φ' s that $\varphi(e)=1$. In order to prove the equivalency of two topologies (P) and (W) (see 1) in $\boldsymbol{P}_1$, it is sufficient to prove that the weak topology (W) is stronger than the Pontrjagin's one (P), since the converse is evident.

First we shall prove some lemmas.

LEMMA 1. *Let $\boldsymbol{P}'$ be a bounded subset of $\boldsymbol{P}$, i.e., $\{\varphi(e)\,|\,\varphi \in \boldsymbol{P}'\}$ be bounded, and let $x \in L^1$. Then, for any positive ε, there exists a neighbourhood V of e such that $g \in V$ implies $|\varphi^x(g)-\varphi^x(e)|<\varepsilon$ for all φ of $\boldsymbol{P}'$.*

PROOF. Put $M=\sup\{\varphi(e)\,|\,\varphi \in \boldsymbol{P}'\}$. Then, for φ of $\boldsymbol{P}'$, we have

$$|\varphi^x(g)-\varphi^x(e)|$$

$$=\left|\int\int \overline{x(s)}\varphi(s^{-1}gt)x(t)\,ds\,dt-\int\int \overline{x(s)}\varphi(s^{-1}t)x(t)\,ds\,dt\right|$$

$$\leq \int\int |x(gs)-x(s)|\,.\,|\varphi(s^{-1}t)x(t)|\,ds\,dt$$

$$\leq M\cdot\|x\|\cdot\int |x(gs)-x(s)|\,ds,$$

which is $<\varepsilon$ for g of a sufficiently small V, q. e. d.

In the following we shall confine our considerations in $\boldsymbol{P}_1$; hence all the appearing φ's will satisfy $\varphi(e)=1$.

LEMMA 2. *Let $\varphi_0 \in \boldsymbol{P}_1$ and $\varepsilon>0$. Then there exists a neighbourhood $\boldsymbol{V}(\varphi_0)$ [12] and an x of L^1 such that $\varphi \in \boldsymbol{V}(\varphi_0)$ implies $|\varphi^x(g)-\varphi(g)|<\varepsilon$ on G.*

PROOF. Let V be so small that $x_V(\cdot)$ [13] satisfies

9) Stated in [2].

10) German R denotes the real part.

11) $\varphi^x(g)$ is also a positive definite function.

12) See (1).

13) See (2).

$$(\varphi_0, x_V) = \int x_V(g)\varphi_0(g)\,dg > \varphi_0(e) - \left(\frac{\varepsilon}{4}\right)^2 = 1 - \left(\frac{\varepsilon}{4}\right)^2;$$

the existence of such V follows from (3) and (4). Let x be one of these x_V's. Define

$$\boldsymbol{V}(\varphi_0) = \left\{\varphi \,\middle|\, |(\varphi, x) - (\varphi_0, x)| < \left(\frac{\varepsilon}{4}\right)^2\right\}.$$

Then $\varphi \varepsilon \boldsymbol{V}(\varphi_0)$ implies

(7) $$(\varphi, x) > 1 - \frac{\varepsilon^2}{8}$$

and

$$\begin{aligned}
&|\varphi^x(g) - \varphi(g)| \\
&= \left| \int\int x(s)\varphi(s^{-1}g\,t)x(t)\,ds\,dt - \int\int x(s)\varphi(g)x(t)\,ds\,dt \right| && \text{(from (4))} \\
&\leqq \int\int x(s)\left\{ |\varphi(s^{-1}gt) - \varphi(s^{-1}g)| + |\varphi(s^{-1}g) - \varphi(g)| \right\} x(t)\,ds\,dt \\
&\leqq 2\int x(s)\sqrt{2 - 2\Re(\varphi(s))}\,ds && \text{(from (6) and (4))} \\
&\leqq 2\left\{ \int x(s)\,ds \,.\, \int 2x(s)\left(1 - \Re(\varphi(s))\right) ds \right\}^{\frac{1}{2}} \\
& && \text{(by Schwarz' inequality; see (3))} \\
&= 2\sqrt{2}\left\{ 1 - \int x(s)\varphi(s)\,ds \right\}^{\frac{1}{2}} && \text{(from (5))} \\
&< 2\sqrt{2}\sqrt{\frac{\varepsilon^2}{8}} = \varepsilon, && \text{(from (7))}
\end{aligned}$$

q. e. d.

THEOREM. *Let $\varphi_0 \varepsilon \boldsymbol{P}_1$, $\varepsilon > 0$ and F be a compact subset of G. Then there exists a $\boldsymbol{V}(\varphi_0)$ such that $\varphi \varepsilon \boldsymbol{V}(\varphi_0)$ implies $|\varphi(g) - \varphi_0(g)| < \varepsilon$ on F.*

PROOF. From LEMMAS 1 and 2 follows that there exists a $\boldsymbol{U}(\varphi_0)$ and a $V(e)$ such that $g \varepsilon V$ and $\varphi \varepsilon \boldsymbol{U}$ imply $|\varphi(g) - 1| < \varepsilon^2/2^7$. Then, for every φ of $\boldsymbol{U}$, it follows by means of (6) that $|\varphi(a) - \varphi(ag)| < \varepsilon/8$ for any a of G, if $g \varepsilon V$; hence

(8) $$|\varphi(g) - \varphi(h)| < \frac{\varepsilon}{4} \quad \text{for } g \text{ and } h \text{ of } aV.$$

Now let F be covered by the union of $a_kV\,(a_k \varepsilon G,\ 1 \leqq k \leqq n)$; put $x_k(g) = c_V(a_k^{-1}g)/m(V)$, and define

(9) $V(\varphi_0)=\left\{\varphi \mid \varphi \in \boldsymbol{U},\ |(\varphi, x_k)-(\varphi_0, x_k)|<\frac{\varepsilon}{2},\ 1 \leqq k \leqq n\right\}$.

Then this is the required. For: Suppose that $|\varphi_0(g_\varepsilon)-\varphi_\varepsilon(g_\varepsilon)| \geqq \varepsilon$ for some φ_ε of V and some g_ε of F; then, from (8), on some $a_l V$ it should beeither $\varphi_0(g)-\varphi_\varepsilon(g)>\varepsilon-\varepsilon/2$ or $\varphi_0(g)-\varphi_\varepsilon(g)<-(\varepsilon-\varepsilon/2)$, and consequently it should be $|(\varphi_0, x_l)-(\varphi_\varepsilon, x_l)|>\varepsilon/2$, which contradicts to (9), q.e.d.

COROLLARY. *$\varphi(g)$ is continuous in (g, φ) in $G \times \boldsymbol{P}_1$, where $\boldsymbol{P}_1$ is considered as topologized by the weak topology.*

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