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COHEN-MACAULAY LOCAL RINGS OF EMBEDDING DIMENSION $e + d - k$

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Abstract

In this paper, we prove the following. Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with multiplicity e and embedding dimension $v = e + d - k$, where $k \geq 3$ and $e - k > 1$. If $\lambda(\mathfrak{m}^3/J\mathfrak{m}^2) = 1$ and $\mathfrak{m}^3 \subseteq J\mathfrak{m}$, where J is a minimal reduction of $\mathfrak{m}$, then $3 \leq s \leq \tau + k - 1$, where s is the degree of the h-polynomial of R and τ is the Cohen-Macaulay type of R.

1. Introduction

Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring of multiplicity e. The Hilbert function of R is by definition the Hilbert function of the associated graded ring of R:

$$G := \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1},$$

i.e.,

$$H_R(n) = dim_{R/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1}.$$

The Hilbert series of R is the power eries

$$P_R(z) = \sum_{n \geq 0} H_R(n) z^n.$$

It is known that there is a polynomial $h(z) \in \mathbb{Z}[z]$ such that $P_R(z) = h(z)/(1 - z)^d$ and $h(1) = e$. This polynomial $h(z) = h_0 + h_1 z + \cdots + h_s z^s$ is called the h-polynomial of R.

Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with embedding dimension $v = e + d - k$, where $k \geq 3$. Let J be a minimal reduction of $\mathfrak{m}$. Let τ be the Cohen-Macaulay type of R, $h = v - d$ and $v_i = \lambda(\mathfrak{m}^{i+1}/J\mathfrak{m}^i)$ for every i; then there are at least two possible Hilbert series of R/J: $P_{R/J}(z) = 1 + hz + z^2 + \cdots + z^k$ and $P_{R/J}(z) = 1 + hz + (k - 1)z^2$. In the first case, R is stretched (cf. definition below) and we have $\mathfrak{m}^k \not\subseteq J\mathfrak{m}$; in the second case, following [3], we say that R is short and we have $\mathfrak{m}^3 \subseteq J\mathfrak{m}$ and $v_1 = k - 1$.

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Let $(R, \mathfrak{m})$ be a d-dimensional local Cohen-Macaulay ring of multiplicity e and embedding dimension v. If $d = 0$, then R is called *stretched* if $e - v$ is the least integer i such that $\mathfrak{m}^{i+1} = 0$. If $d > 0$, then R is *stretched* if there is a minimal reduction J of $\mathfrak{m}$ such that R/J is stretched (cf. [6]), or equivalently, $(\mathfrak{m}^2 + J)/J$ is principal. Regular local rings are not stretched since fields are not stretched. However, for any d-dimensional local Cohen-Macaulay ring $(R, \mathfrak{m})$ having infinite residue field, if $v = e + d - 1$ with $e > 1$ or $v = e + d - 2$ with $e > 2$, then R is stretched. Moreover, if $v = e + d - 3$ and R is Gorenstein, then R is stretched. These stretched rings have been studied in [6], [7] and [8]. In [4], Rossi and Valla extended the notion *stretched*. There they defined, for each $\mathfrak{m}$-primary ideal I, I is *stretched* if there is a minimal reduction J of I such that $I^2 \cap J = IJ$ and $\lambda(I^2/(JI + I^3)) = 1$.

In [6], Sally studied the structure of stretched local Gorenstein rings, and use it to show in [8] that if $(R, \mathfrak{m})$ is a d-dimensional Gorenstein local ring with embedding dimension $v = e+d-3$, then the associated graded ring of R is Cohen-Macaulay. This result has been generalized by Rossi and Valla in [3] as follows.

Theorem 1.1 ([3, Theorem 2.6]). *If $(R, \mathfrak{m})$ is a d-dimensional Cohen-Macaulay local ring of multiplicity $e = h + 3$ and $\lambda(\mathfrak{m}^3/J\mathfrak{m}^2) = 1$, then $s \leq \tau + 2$, where s is the degree of the h-polynomial of R.*

In [4], Rossi and Valla generalized Theorem 1.1 to stretched $\mathfrak{m}$-primary ideals. In this note, we are able to generalize Theorem 1.1 in a different manner in Section 4 as follows. In which, we do not assume R is stretched. In stead, we assume that R is short and $v_2 = 1$.

Theorem 1.2. *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring of multiplicity $e = h + k$, where $k \geq 3$ and $e - k > 1$. If $\lambda(\mathfrak{m}^3/J\mathfrak{m}^2) = 1$ and $\mathfrak{m}^3 \subseteq J\mathfrak{m}$, where J is a minimal reduction of $\mathfrak{m}$, then $3 \leq s \leq \tau + k - 1$, where s is the degree of the h-polynomial of R.*

In the final section, we provide several examples to answer some questions raised by Rossi and Valla in [3].

2. One dimensional local Cohen-Macaulay ring

We state several facts of one dimensional local Cohen-Macaulay rings. These results can be derived easily from [1] and [5].

Lemma 2.1. *Let $(R, \mathfrak{m})$ be a one dimensional local Cohen-Macaulay ring; then $\lambda(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = e - \lambda(\mathfrak{m}^{n+1}/J\mathfrak{m}^n)$, where J is any minimal reduction of $\mathfrak{m}$.*

Lemma 2.2. *Let $(R, \mathfrak{m})$ be a one dimensional Cohen-Macaulay local ring with embedding dimension 2. Then $G(R)$ is Gorenstein.*

Corollary 2.3. *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with embedding dimension $d+1$. Then $G(R)$ is Gorenstein.*

3. Cohen-Macaulay local rings of embedding dimension $e + d - k$

Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with embedding dimension $v = e + d - k$, where $k \geq 3$ and $e - k > 1$. Let τ be the Cohen-Macaulay type of R, $h = v - d$ and $v_i = \lambda(\mathfrak{m}^{i+1}/J\mathfrak{m}^i)$ for every i. Let J be a minimal reduction of $\mathfrak{m}$; then one of the possible Hilbert series of R/J is $1 + hz + (k-1)z^2$. In this case, $\mathfrak{m}^3 \subseteq J\mathfrak{m}$ and $v_1 = k - 1$. If $k = 3$, it is shown in [3, Theorem 2.6] that if $v_2 = 1$ then $s \leq \tau + 2$, where s is the degree of the h-polynomial of R. We are able to generalize this result in this section.

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring of multiplicity $e = h + k$, where $k \geq 3$ and $e - k > 1$. If $\lambda(\mathfrak{m}^3/J\mathfrak{m}^2) = 1$ and $\mathfrak{m}^3 \subseteq J\mathfrak{m}$, where J is a minimal reduction of $\mathfrak{m}$, then $3 \leq s \leq \tau + k - 1$, where s is the degree of the h-polynomial of R.*

REMARK 3.2. (i) Notice that the assumption $v_2 = 1$ ensures that the depth of G is at leat $d - 1$ (cf. [3]). Therefore to show Theorem 3.1, we need only to consider the case when $d = 1$.
(ii) If $d = 1$, then s is the least integer for which $\lambda(\mathfrak{m}^s/\mathfrak{m}^{s+1}) = e$.
(iii) Notice that $\lambda(\mathfrak{m}^2/J\mathfrak{m}) = k - 1$. Moreover, if $\mathfrak{m}^2 = J\mathfrak{m} + (u_1, \ldots, u_{k-1})$, then $\{u_1, \ldots, u_{k-1}\}$ is part of a generating set of the socle of R.

By Remark 3.2, we may assume from now on that $d = 1$ and $v_2 = 1$.

Lemma 3.3. *Let r be the reduction number of $\mathfrak{m}$ with respect to J. If $r \leq 3$, then* Theorem 3.1 *holds.*

Proof. If $r \leq 3$, then $\mathfrak{m}^4 = J\mathfrak{m}^3$, so that $\lambda(\mathfrak{m}^3/\mathfrak{m}^4) = e$, it follows that $s \leq 3 \leq \tau + k - 1$ by the choice of s. □

By Lemma 3.3, we may assume in the sequel that $r \geq 4$.

Lemma 3.4. *The following hold for R:*
(i) *If $\mathfrak{m}^3 = J\mathfrak{m}^2 + (ab)$ for some $b \in mathfracm^2$ and $a \in \mathfrak{m}$, then $\mathfrak{m}^{i+1} = J\mathfrak{m}^i + (a^{i-1}b)$ for every $i \geq 2$.*
(ii) *If $y\mathfrak{m}^2 \not\subseteq J\mathfrak{m}^2$ for some $y \in \mathfrak{m}$, then $y^3 \notin J\mathfrak{m}^2$. In particular, there is an element $y \in \mathfrak{m}$ such that $\mathfrak{m}^{i+1} = J\mathfrak{m}^i + (y^{i+1})$ for every $i \geq 2$.*

Proof. (i) If $\mathfrak{m}^{i+1} = J\mathfrak{m}^i + (a^{i-1}b)$ for some $i \geq 2$, then $\mathfrak{m}^{i+2} = J\mathfrak{m}^{i+1} + a^{i-1}b\mathfrak{m} \subseteq J\mathfrak{m}^{i+1} + a^{i-1}\mathfrak{m}^3 = J\mathfrak{m}^{i+1} + (a^i b) \subseteq \mathfrak{m}^{i+2}$.

(ii) Suppose that $y\mathfrak{m}^2 \nsubseteq J\mathfrak{m}^2$. Then there are $u, v \in \mathfrak{m}$ such that $uvy \notin J\mathfrak{m}^2$ and $\mathfrak{m}^3 = J\mathfrak{m}^2 + (yuv)$. Therefore, $\mathfrak{m}^4 = J\mathfrak{m}^3 + (y^2uv)$. It follows that $y^2u \notin J\mathfrak{m}^2$ and $\mathfrak{m}^3 = J\mathfrak{m}^2 + (y^2u)$. Thus, $\mathfrak{m}^4 = J\mathfrak{m}^3 + (y^3u)$ and then $y^3 \notin J\mathfrak{m}^2$. Now, choose $y \in \mathfrak{m}$ such that $y\mathfrak{m}^2 \nsubseteq J\mathfrak{m}^2$, then $\mathfrak{m}^{i+1} = J\mathfrak{m}^i + (y^{i+1})$ for every $i \geq 2$. □

Lemma 3.5. *Let $J = (x)$ be a minimal reduction of $\mathfrak{m}$. If there is an element $y \in \mathfrak{m}$ such that $\mathfrak{m}^{i+1} = J\mathfrak{m}^i + (y^{i+1})$ for every $i \geq 2$, then $y^l x^t$ is a generator of the module $(J^t\mathfrak{m}^l + \mathfrak{m}^{l+t+1})/(J^{t+1}\mathfrak{m}^{l-1} + \mathfrak{m}^{l+t+1})$ whenever $2 \leq l < r$, where r is the reduction number of $\mathfrak{m}$ with respect to J.*

Proof. If not, $y^l x^t \in J^{t+1}\mathfrak{m}^{l-1} + \mathfrak{m}^{l+t+1}$, so that $y^r x^t \in x^{t+1}\mathfrak{m}^{r-1}$, it follows that $y^r \in J\mathfrak{m}^{r-1}$, a contradiction. Therefore, the conclusion holds. □

Theorem 3.6. *Let $(R, \mathfrak{m})$ be a one dimensional Cohen-Macaulay local ring of multiplicity $e = h + k$, where $k \geq 3$ and $e - k > 1$. Assume that $\lambda(\mathfrak{m}^3/J\mathfrak{m}^2) = \lambda(\mathfrak{m}^4/J\mathfrak{m}^3) = 1$ and $\mathfrak{m}^3 \subseteq J\mathfrak{m}$, where $J = (x)$ is a minimal reduction of $\mathfrak{m}$. Then there is a basis $\{x, y_1, \ldots, y_\tau, z_1, \ldots, z_{e-\tau-k}\}$ of $\mathfrak{m}$, elements $u_{t+1}, \ldots, u_{k-1}$ contained in $\mathfrak{m}$ and elements $\{c_{ij} \mid i = 1, \ldots, k-1,\ j = 1, \ldots, j_i\}$ contained in the ideal $(y_2, \ldots, y_{k-1}, z_1, \ldots, z_{e-\tau-k})$ with $\sum_{i=1}^{k-1} j_i(k-i) = e - \tau - k$ such that $J = (x)$ and the following hold*:

(i) $\mathfrak{m}^{i+1} = J\mathfrak{m}^i + (y_1^{i+1})$ *for every* $i \geq 2$.

(ii) $\mathfrak{m}^2 = J\mathfrak{m} + (y_1^2, y_1y_2, \ldots, y_1y_t, y_{t+1}u_{t+1}, \ldots, y_{k-1}u_{k-1})$, *where* $t = \lambda((y_1\mathfrak{m} + J\mathfrak{m})/J\mathfrak{m})$.

(iii) $\{y_1^2, y_1y_2, \ldots, y_1y_t, y_{t+1}u_{t+1}, \ldots, y_{k-1}u_{k-1}, y_k, \ldots, y_\tau\}$ *is a generating set of the socle of* R.

(iv) $y_1y_i \in J\mathfrak{m}$ *for* $i \geq t+1$ *and* $y_1z_i \in J\mathfrak{m}$ *for every* i.

(v) $y_i\mathfrak{m}^3 \subseteq J\mathfrak{m}^3$ *for every* $i \geq 2$ *and* $z_i\mathfrak{m}^3 \subseteq J\mathfrak{m}^3$ *for every* $i \geq 1$.

(vi) $\{z_1, \ldots, z_{e-\tau-k}\} = \bigcup_{i,j,k}\{z_{ij}^{(l)}\}$, $\lambda((c_{ij}\mathfrak{m} + J\mathfrak{m})/J\mathfrak{m}) = k - i$ *and* $\mathfrak{m}^2 = J\mathfrak{m} + \sum_{l=1}^{k-i} z_{ij}^{(l)} c_{ij}$ *for every* $i = 1, \ldots, k-1$ *and* $j = 1, \ldots, j_i$.

(vii) $c_{ij}z_{i'j'}^{(l)} \in J\mathfrak{m}$ *if* $i < i'$ *or* $i = i'$ *but* $j < j'$.

(viii) $y_1^3 \notin J(z_1, \ldots, z_{e-\tau-k}) + J\mathfrak{m}^2$.

Proof. By Lemma 3.4, there is an element $y_1 \in \mathfrak{m}$ such that (i) hold. Let $t = \lambda((y_1\mathfrak{m} + J\mathfrak{m})/J\mathfrak{m})$; then there are $y_2, \ldots, y_{k-1}, u_{t+1}, \ldots, u_{k-1} \in \mathfrak{m}$ such that $\mathfrak{m}^2 = J\mathfrak{m} + (y_1^2, y_1y_2, \ldots, y_1y_t, y_{t+1}u_{t+1}, \ldots, y_{k-1}u_{k-1})$ and $y_1\mathfrak{m} + J\mathfrak{m} = (y_1^2, y_1y_2, \ldots, y_1y_t) + J\mathfrak{m}$. We may assume that $y_1^2y_i \in J\mathfrak{m}^2$ for $2 \leq i \leq t$ by replacing y_i by $y_i + \lambda y_1$ if necessary, and assume that $y_1y_j \in J\mathfrak{m}$ for $t+1 \leq j \leq k-1$ by replacing y_j by $y_j + \lambda_1y_1 + \cdots + \lambda_ty_t$ if necessary. It follows that $y_i\mathfrak{m}^3 = (y_iy_1^3) + J\mathfrak{m}^3 = J\mathfrak{m}^3$ for every $i \leq k-1$. Since the Cohen-Macaulay type of R is τ and $\{y_1^2, y_1y_2, \ldots, y_1y_t, y_{t+1}u_{t+1}, \ldots, y_{k-1}u_{k-1}\}$ is part of a generating set of the socle of R, we may choose $y_k, \ldots, y_\tau, z_1, \ldots, z_{e-\tau-k} \in \mathfrak{m}$ such that $\{y_k, \ldots, y_\tau, z_1, \ldots, z_{e-\tau-k}\}$ is part of a generating set of $\mathfrak{m}$ and $\{y_1^2, y_1y_2, \ldots, y_1y_t, y_{t+1}u_{t+1}, \ldots, y_{k-1}u_{k-1}, y_k, \ldots, y_\tau\}$ is a generating set of the socle of R. If $z_iy_1 \notin J\mathfrak{m}$ for some i, then we may replace z_i by

$z_i + \alpha_1 y_1 + \cdots + \alpha_t y_t$ if necessary and assume that $z_i y_1 \in J\mathfrak{m}$ for every i. Therefore $z_i \mathfrak{m}^3 \subseteq J\mathfrak{m}^3 + z_i y_1 \mathfrak{m}^2 \subseteq J\mathfrak{m}^3$. Hence, the basis $\{x, y_1, \ldots, y_\tau, z_1, \ldots, z_{e-\tau-k}\}$ of $\mathfrak{m}$ satisfies (i) to (v) so far.

Claim. *For any integer $i = 1, \ldots, k-1$, there is an integer j_i, a basis $\{x, y_1, \ldots, y_\tau, z_1, \ldots, z_{e-\tau-k}\}$ of $\mathfrak{m}$ and elements $\{c_{ij} \mid j = 1, \ldots, j_i\}$ contained in the ideal $(y_2, \ldots, y_{k-1}, z_1, \ldots, z_{e-\tau-k})$ such that not only (i) to (v) but also the following hold:*
(a) $\lambda((c_{ij}\mathfrak{m} + J\mathfrak{m})/J\mathfrak{m}) = k - i$, $\mathfrak{m}^2 = J\mathfrak{m} + \left(z_{ij}^{(1)} c_{ij}, \cdots, z_{ij}^{(k-i)} c_{ij}\right)$.
(b) $c_{ij} z_{ij'}^{(l)} \in J\mathfrak{m}$ *for every l if $j < j'$ and $c_{ij} z \in J\mathfrak{m}$ for every generator of the ideal generated by S_i, where $S_i = \{z_1, \ldots, z_{e-\tau-k}\} - \left\{z_{i'j}^{(l)} \;\middle|\; 1 \le i' \le i, 1 \le j \le j_i, 1 \le l \le k - i\right\}$.*

Note that (vi) and (vii) follows from the Claim.

Proof of the Claim. We proceed by induction on i. Let z be any generator of the ideal $(z_1, \ldots, z_{e-\tau-k})$. Since $y_1 z, y_i z \in J\mathfrak{m}$ for every $i \ge k$, there is an element $c \in (y_2, \ldots, y_{k-1}, z_1, \ldots, z_{e-\tau-k})$ such that $cz \notin J\mathfrak{m}$. If for any generating set $\{z'_1, \ldots, z'_{e-\tau-k}\}$ of the ideal $(z_1, \ldots, z_{e-\tau-k})$ there is no element $c \in (y_2, \ldots, y_{k-1}, z_1, \ldots, z_{e-\tau-k})$ such that $\mathfrak{m}^2 = (cz'_1, \ldots, cz'_{k-1}) + J\mathfrak{m}$, then the Claim holds for $i = 1$. If not, we may assume that $\mathfrak{m}^2 = (c_{11}z_1, \ldots, c_{11}z_{k-1}) + J\mathfrak{m}$ for some $c_{11} \in (y_2, \ldots, y_{k-1}, z_1, \ldots, z_{e-\tau-k})$. Set $z_{11}^{(l)} = z_l$. Let z be any generator of the ideal $(z_k, \ldots, z_{e-\tau-k})$. If $c_{11}z \notin J\mathfrak{m}$, then there are elements α_i such that $c_{11}z - \left(\sum_{i=1}^{k-1} c_{11} z_{11}^{(i)}\right) \in J\mathfrak{m}$, so that we may replace z by $\sum_{i=1}^{k-1} z_{11}^{(i)}$ if necessary and assume that $c_{11}z \in J\mathfrak{m}$. If for any generating set $\{z'_k, \ldots, z'_{e-\tau-k}\}$ of the ideal $(z_k, \ldots, z_{e-\tau-k})$ there is no element $c \in (y_2, \ldots, y_{k-1}, z_1, \ldots, z_{e-\tau-k})$ such that $\mathfrak{m}^2 = (cz'_k, \ldots, cz'_{2k-2}) + J\mathfrak{m}$, then again the Claim holds for $i = 1$. If not, we may use the same trick to find $c_{12}, c_{13}, \ldots$ so that the Claim holds for $i = 1$.

Suppose now we have shown that the Claim holds for any integer $i' \le i$ for some $i \ge 1$. Let $m = \sum_{i'=1}^{i} j_{i'}(k - i')$ and $S_i = \{z_{m+1}, \ldots, z_{e-\tau-k}\}$. If for any generating set $\{z'_{m+1}, \ldots, z'_{e-\tau-k}\}$ of the ideal generated by S_i there is no element $c \in (y_2, \ldots, y_{k-1}, z_1, \ldots, z_{e-\tau-k})$ such that $\mathfrak{m}^2 = (cz'_{m+1}, \ldots, cz'_{m+k-i-1}) + J\mathfrak{m}$, then the Claim holds for $i + 1$. If not, we may assume that for some $c_{i+1,1} \in (y_2, \ldots, y_{k-1}, z_1, \ldots, z_{e-\tau-k})$, $\mathfrak{m}^2 = (c_{i+1,1}z_{m+1}, \ldots, c_{i+1,1}z_{m+k-i-1}) + J\mathfrak{m}$. Set $z_{i+1,1}^{(l)} = z_{m+l}$. As before, we may assume that $c_{i+1,1}z \in J\mathfrak{m}$ for every generator z of the ideal $(z_{m+k-i}, \ldots, z_{e-\tau-k})$. If for any generating set $\{z'_{m+k-i}, \ldots, z'_{e-\tau-k}\}$ of the ideal $(z_{m+k-i}, \ldots, z_{e-\tau-k})$ there is no element $c \in (y_2, \ldots, y_{k-1}, z_1, \ldots, z_{e-\tau-k})$ such that $\mathfrak{m}^2 = (cz'_{m+k-i}, \ldots, cz'_{m+2k-2i-2}) + J\mathfrak{m}$, then again the Claim holds for $i + 1$. If not, we may use the same trick to find $c_{i+1,2}, c_{i+1,3}, \ldots$ so that the Claim hods for $i + 1$. The Claim is now fulfilled. $\square$

To finish the proof, assume that $y_1^3 \in J(z_1, \ldots, z_{e-\tau-k}) + J\mathfrak{m}^2$. Then there are $\delta_i \in R$ not all in $\mathfrak{m}$ such that $y_1^3 - \sum_{i=1}^{e-\tau-k} \delta_i z_i x \in J\mathfrak{m}^2$. Let t be the smallest integer for which δ_t is a unit; then $y_1^3 - \sum_{i=t}^{e-\tau-k} \delta_i z_i x \in J\mathfrak{m}^2$. Let $z = c_{ij}$ if $z_t = z_{ij}^{(l)}$ for some l; then $z \cdot \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i\right)x - zy_1^3 \in J\mathfrak{m}^3$, so that $z \cdot \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i\right) \in \mathfrak{m}^3 \subseteq J\mathfrak{m}$ as $z \in (y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$. However, $z \cdot \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i\right) \notin J\mathfrak{m}$ by the Claim, a contradiction. Therefore (viii) holds. $\square$

Now, we are ready for:

Proof of Theorem 3.1. From the above, we may assume that $d = 1$, $\tau \geq 2$ and $r \geq 4$, where r is the reduction number of some minimal reduction J of $\mathfrak{m}$. By Theorem 3.6, there is a basis $\{x, y_1, \dots, y_\tau, z_1, \dots, z_{e-\tau-k}\}$ of $\mathfrak{m}$, elements $u_{t+1}, \dots, u_{k-1}$ contained in $\mathfrak{m}$ and elements $\{c_{ij} \mid i = 1, \dots, k-1,\ j = 1, \dots, j_i\}$ contained in the ideal $(y_2, \dots, y_{k-1}, z_1, \dots, z_{e-\tau-k})$ with $\sum_{i=1}^{k-1} j_i(k-i) = e - \tau - k$ such that $J = (x)$ and the following hold:

(i) $\mathfrak{m}^{i+1} = J\mathfrak{m}^i + (y_1^{i+1})$ for every $i \geq 2$.

(ii) $\mathfrak{m}^2 = J\mathfrak{m} + (y_1^2, y_1y_2, \dots, y_1y_t, y_{t+1}u_{t+1}, \dots, y_{k-1}u_{k-1})$, where $t = \lambda((y_1\mathfrak{m} + J\mathfrak{m})/J\mathfrak{m})$.

(iii) $\{y_1^2, y_1y_2, \dots, y_1y_t, y_{t+1}u_{t+1}, \dots, y_{k-1}u_{k-1}, y_k, \dots, y_\tau\}$ is a generating set of the socle of R.

(iv) $y_1y_i \in J\mathfrak{m}$ for $i \geq t+1$ and $y_1z_i \in J\mathfrak{m}$ for every i.

(v) $y_i\mathfrak{m}^3 \subseteq J\mathfrak{m}^3$ for every $i \geq 2$ and $z_i\mathfrak{m}^3 \subseteq J\mathfrak{m}^3$ for every $i \geq 1$.

(vi) $\{z_1, \dots, z_{e-\tau-k}\} = \bigcup_{i,j,k}\left\{z_{ij}^{(l)}\right\}$, $\lambda((c_{ij}\mathfrak{m} + J\mathfrak{m})/J\mathfrak{m}) = k - i$ and $\mathfrak{m}^2 = J\mathfrak{m} + \sum_{l=1}^{k-i} z_{ij}^{(l)} c_{ij}$ for every $i = 1, \dots, k-1$ and $j = 1, \dots, j_i$.

(vii) $c_{ij} z_{i'j'}^{(l)} \in J\mathfrak{m}$ if $i < i'$ or $i = i'$ but $j < j'$.

(viii) $y_1^3 \notin J(z_1, \dots, z_{e-\tau-k}) + J\mathfrak{m}^2$.

If $\tau \geq h$, then $s \leq e - 1 = h + k - 1 \leq \tau + k - 1$ by [2] and we are done. Therefore, we may assume that $\tau < h$. To show that $s \leq \tau + k - 1$, it is enough to show that $\lambda(\mathfrak{m}^{\tau+k-1}/\mathfrak{m}^{\tau+k}) = e$ by Remark 3.2 (ii). Moreover, by Lemma 3.5, $\{y_1^{\tau+k-1}, y_1^{\tau+k-2}x, \dots, y_1^2x^{\tau+k-3}\}$ are generators of the module $\mathfrak{m}^{\tau+k-1}/(J^{\tau+k-2}\mathfrak{m} + \mathfrak{m}^{\tau+k})$, therefore to show that $\lambda(\mathfrak{m}^{\tau+k-1}/\mathfrak{m}^{\tau+k}) = e$ it is enough to show that

$$\{y_1x^{\tau+k-2}, x^{\tau+k-1}, z_1x^{\tau+k-2}, \dots, z_{e-\tau-k}x^{\tau+k-2}\}$$

is a linearly independent set in $(x^{\tau+k-2}\mathfrak{m} + \mathfrak{m}^{\tau+k})/\mathfrak{m}^{\tau+k}$.

Suppose not, there are α, β, δ_i in R not all in $\mathfrak{m}$ such that

$$\alpha y_1 x^{\tau+k-2} + \beta x^{\tau+k-1} + \sum_{i=1}^{e-\tau-k} \delta_i z_i x^{\tau+k-2} \in \mathfrak{m}^{\tau+k}.$$

Then

$$\alpha y_1^r x^{\tau+k-2} + \beta y_1^{r-1} x^{\tau+k-1} + \sum_{i=1}^{e-\tau-k} \delta_i z_i x^{\tau+k-2} y_1^{r-1} \in \mathfrak{m}^{\tau+r+k-1},$$

so that $\alpha y_1^r x^{\tau+k-2} \in x^{\tau+k-1}\mathfrak{m}^{r-1}$ as $y_1 z_i \in J\mathfrak{m}$, it follows that $\alpha \in \mathfrak{m}$ by the choice of r. Therefore $\beta x^{\tau+k-1} + \sum_{i=1}^{e-\tau-k} \delta_i z_i x^{\tau+k-2} \in \mathfrak{m}^{\tau+k}$. If $\delta_i \in \mathfrak{m}$ for every i, then $x^{\tau+k-1} \in \mathfrak{m}^{\tau+k}$, which is impossible. So, there is an integer i such that δ_i is a unit. By replacing z_i by $z_i + \beta/\delta_i x$, we may assume that $\beta \in \mathfrak{m}$. Hence $\sum_{i=1}^{e-\tau-k} \delta_i z_i x^{\tau+k-2} \in \mathfrak{m}^{\tau+k}$. Let t be the smallest integer for which δ_t is a unit; then $\sum_{i=t}^{e-\tau-k} \delta_i z_i x^{\tau+k-2} \in \mathfrak{m}^{\tau+k}$.

Let $\alpha \leq \tau+k$ be the integer such that $\sum_{i=t}^{e-\tau-k} \delta_i z_i x^{\tau+k-2} \in J^{\tau+k-\alpha}\mathfrak{m}^{\alpha} - J^{\tau+k+1-\alpha}\mathfrak{m}^{\alpha-1}$. If $\sum_{i=t}^{e-\tau-k} \delta_i z_i x^{\tau+k-2} \in J^{\tau+k-3}\mathfrak{m}^3$, then $\sum_{i=t}^{e-\tau-k} \delta_i z_i x \in \mathfrak{m}^3 = (y_1^3) + J\mathfrak{m}^2$, so that $\sum_{i=t}^{e-\tau-k} \delta_i z_i x \in J\mathfrak{m}^2$ by (viii), it follows that $\sum_{i=t}^{e-\tau-k} \delta_i z_i \in \mathfrak{m}^2$, a contradiction. Therefore, $\alpha \geq 4$. Since $\mathfrak{m}^{\alpha} = (y^{\alpha}) + J\mathfrak{m}^{\alpha-1}$ and $\lambda(\mathfrak{m}^{\alpha}/J\mathfrak{m}^{\alpha-1}) = 1$, there is a *unit* λ_1 such that

$$\sum_{i=t}^{e-\tau-k} \delta_i z_i x^{\alpha-2} - \lambda_1 y_1^{\alpha} \in J\mathfrak{m}^{\alpha-1}. \tag{1}$$

Let $z = c_{ij}$ if $z_t = z_{ij}^{(l)}$; then $z \cdot \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i\right) \notin J\mathfrak{m}$ by (vi) and (vii). Moreover,

$$z\left(\sum_{i=t}^{e-\tau-k} \delta_i z_i x^{\alpha-2}\right) - \lambda_1 y_1^{\alpha} z \in J\mathfrak{m}^{\alpha}.$$

Furthermore, $y_1^3 z \in J\mathfrak{m}^3$ by (v), we have $z\left(\sum_{i=t}^{e-\tau-k} \delta_i z_i x^{\alpha-3}\right) \in \mathfrak{m}^{\alpha}$. Therefore, there is an element λ_2 of R such that

$$z\left(\sum_{i=t}^{e-\tau-k} \delta_i z_i x^{\alpha-3}\right) - \lambda_2 y_1^{\alpha} \in J\mathfrak{m}^{\alpha-1}. \tag{2}$$

From (1) and (2), we see that there is an element λ_3 of R such that

$$(z - \lambda_3 x)\left(\sum_{i=t}^{e-\tau-k} \delta_i z_i\right) x^{\alpha-4} \in \mathfrak{m}^{\alpha-1}.$$

Let $\beta \leq \alpha - 4 \leq \tau + k - 4$ be the non-negative integer such that

$$(z - \lambda_3 x)\left(\sum_{i=t}^{e-\tau-k} \delta_i z_i\right) x^{\beta} \in \mathfrak{m}^{\beta+3} \setminus J\mathfrak{m}^{\beta+2}.$$

Since $z \cdot \left(\sum_{i=t}^{e-\tau-k} \delta_i z_i\right) \notin J\mathfrak{m}$, $(z - \lambda_3 x)\left(\sum_{i=t}^{e-\tau-k} \delta_i z_i\right) \notin J\mathfrak{m}^2$, β exists. Moreover, there is a *unit* λ_4 of R such that

$$(z - \lambda_3 x)\left(\sum_{i=t}^{e-\tau-k} \delta_i z_i\right) x^{\beta} - \lambda_4 y_1^{\beta+3} \in J\mathfrak{m}^{\beta+2}. \tag{3}$$

On the other hand, from (1), we have

$$y_1^{r-\alpha+1}\left(\sum_{i=t}^{e-\tau-k}\delta_i z_i\right)x^{\alpha-2}-\lambda_1 y_1^{r+1}\in J\mathfrak{m}^r,$$

or equivalently,

$$y_1^{r-\alpha+1}\left(\sum_{i=t}^{e-\tau-k}\delta_i z_i\right)x^{\alpha-3}\in \mathfrak{m}^r.$$

Since $\mathfrak{m}^r=(y_1^r)+J\mathfrak{m}^{r-1}$, there is an element λ_5 of R such that

$$y_1^{r-\alpha+1}\left(\sum_{i=t}^{e-\tau-k}\delta_i z_i\right)x^{\alpha-3}-\lambda_5 y_1^{r}\in J\mathfrak{m}^{r-1}. \tag{4}$$

However, from (1), we have

$$y_1^{r-\alpha}\left(\sum_{i=t}^{e-\tau-k}\delta_i z_i\right)x^{\alpha-2}-\lambda_1 y_1^{r}\in J\mathfrak{m}^{r-1} \tag{5}$$

Thus, from (4) and (5), we obtain that

$$y_1^{r-\alpha}(y_1-\lambda_6 x)\left(\sum_{i=t}^{e-\tau-k}\delta_i z_i\right)x^{\alpha-4}\in \mathfrak{m}^{r-1}, \tag{6}$$

for some element λ_6 of R. Now, if we can show that

$$\widetilde{y_1^{r-\beta-3}}\left(\sum_{i=t}^{e-\tau-k}\delta_i z_i\right)x^{\beta}\in \mathfrak{m}^{r-1} \tag{7}$$

for some element $\widetilde{y_1^{r-\beta-3}}\in\mathfrak{m}^{r-\beta-3}\setminus J\mathfrak{m}^{r-\beta-4}$, then from (3) and (7), we see that

$$(z-\lambda_3 x)\widetilde{y_1^{r-\beta-3}}\left(\sum_{i=t}^{e-\tau-k}\delta_i z_i\right)x^{\beta}-\lambda_4 y_1^{\beta+3}\widetilde{y_1^{r-\beta-3}}\in J\mathfrak{m}^{r-1}$$

and $(z-\lambda_3 x)\widetilde{y_1^{r-\beta-3}}\big(\sum_{i=t}^{e-\tau-k}\delta_i z_i\big)x^{\beta}\in z\mathfrak{m}^{r-1}+J\mathfrak{m}^{r-1}=J\mathfrak{m}^{r-1}$ by (v), therefore $\lambda_4 y_1^{\beta+3}\widetilde{y_1^{r-\beta-3}}\in J\mathfrak{m}^{r-1}$, which contradicts to the choice of r. Hence, we conclude that $\{y_1x^{\tau+k-2}, x^{\tau+k-1}, z_1x^{\tau+k-2},\ldots,z_{e-\tau-k}x^{\tau+k-2}\}$ is a linearly independent set in $(x^{\tau+k-2}\mathfrak{m}+\mathfrak{m}^{\tau+k})/\mathfrak{m}^{\tau+k}$.

Finally, by (6), we may prove (7) by reverse induction. Suppose we have shown that for some δ, $\beta < \delta \leq \alpha - 4$,

$$\widetilde{y_1^{r-\delta-3}}\left(\sum_{i=t}^{e-\tau-k} \delta_i z_i\right)x^{\delta} \in \mathfrak{m}^{r-1}$$

for some element $\widetilde{y_1^{r-\delta-3}} \in \mathfrak{m}^{r-\delta-3} \setminus J\mathfrak{m}^{r-\delta-4}$. Then there is an element $\lambda_6 \in R$ such that

$$y_1\widetilde{y_1^{r-\delta-3}}\left(\sum_{i=t}^{e-\tau-k} \delta_i z_i\right)x^{\delta} - \lambda_6 y_1^r \in J\mathfrak{m}^{r-1}. \tag{8}$$

From (5) and (8), we see that

$$\widetilde{y_1^{r-\delta-2}}\left(\sum_{i=t}^{e-\tau-k} \delta_i z_i\right)x^{\delta} \in J\mathfrak{m}^{r-1}$$

for some element $\widetilde{y_1^{r-\delta-2}} \in \mathfrak{m}^{r-\delta-2} \setminus J\mathfrak{m}^{r-\delta-3}$, it follows that

$$\widetilde{y_1^{r-\delta-2}}\left(\sum_{i=t}^{e-\tau-k} \delta_i z_i\right)x^{\delta-1} \in \mathfrak{m}^{r-1}. \qquad \square$$

We end this section by providing the following example.

EXAMPLE 3.7. Let K be a field and $R = K[[x, y, z_1, \ldots, z_{k-1}]]/I$, where I is the ideal of R generated by the set

$$\{z_1^3 - xy, y^2, yz_1, \ldots, yz_{k-1}, z_1z_2, \ldots, z_1z_{k-1}\} \cup \{z_iz_j \mid 2 \leq i \leq j \leq k-1\}.$$

The it is easy to see the following hold:

(i) R is a 1-dimensional Cohen-Macaulay local ring with maximal ideal $\mathfrak{m} = (x, y, z_1, \ldots, z_{k-1})/I$.

(ii) x is a regular element of R and xR is a minimal reduction of $\mathfrak{m}$.

(iii) $v = k+1$, $h = k$ and $e = 2k$.

(iv) $\mathfrak{m}^3 \subseteq x\mathfrak{m}$, $\{z_1^3\}$ is a basis of $\mathfrak{m}^3/x\mathfrak{m}^2$ and $\{z_1^2, z_1z_2, \ldots, z_1z_{k-1}\}$ is a basis of $\lambda(\mathfrak{m}^2/x\mathfrak{m})$.

(v) $H_R(z) = 1 + (k+1)z + (2k-1)z^2 + \sum_{i=3}^{\infty} 2kz^i = (1 + kz + (k-2)z^2 + z^3)/(1-z)$ and $H_{R/xR}(z) = 1 + kz + (k-1)z^2$.

(vi) $s = r = 3$.

(vii) depth $G = 0$.

4. Examples

In [3], Rossi and Valla raised the following questions:

QUESTION 1. Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with embedding dimension $v = e + d - 3$. If $\tau \geq h$, then is depth $G \geq d - 1$?

QUESTION 2. If $(R, \mathfrak{m})$ is a d-dimensional Cohen-Macaulay local stretched domain with multiplicity $e = h + 3$ and $\tau = 2$, then is G Cohen-Macaulay?

We give counterexamples to these questions as follows.

EXAMPLE 4.1. Let K be a field and $R = K[[x, y, z, u, v]]/(u^3 - xz, v^3 - yz, u^4, v^4, uv, z^2, zu, zv)$; then $(R, \mathfrak{m})$ is a 2-dimensional Cohen-Macaulay local ring and x, y is a regular sequence of $\mathfrak{m}$, where $\mathfrak{m} = (x, y, z, u, v)R$. Moreover, $h = 3$, $e = 6$ and $\tau = 3$ as $\{u^2, v^2, z\}$ generates the socle of R. However, $z \in (\mathfrak{m}^3 : (x, y))$ and $z \notin \mathfrak{m}^2$, therefore the depth of G is 0.

EXAMPLE 4.2. Let K be a field and $R = K[[t^5, t^6, t^{14}]]$; then $(R, \mathfrak{m})$ is a one-dimensional Cohen-Maculay local domain, where $\mathfrak{m} = (t^5, t^6, t^{14})R$. Let $x = t^5$, $y = t^6$ and $z = t^{14}$; then $h = 2$, $e = 5 = h + 3$ and $\tau = 2$ as $\{z, y^3\}$ generates the socle of R. Moreover,

$$P_{R/xR}(z) = 1 + 2z + z^2 + z^3$$

and

$$P_R(z) = \frac{1 + 2z + z^2 + z^4}{1 - z}.$$

Hence R is stretched and G is not Cohen-Macaulay. In fact, $zx \in (\mathfrak{m}^4 : x)$ and $zx \notin \mathfrak{m}^3$.

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