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HYBRID MEAN VALUE RESULTS FOR A GENERALIZATION ON A PROBLEM OF D.H. LEHMER AND HYPER-KLOOSTERMAN SUMS

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Abstract

The main purpose of this paper is by using the Fourier expansion for character sums and the mean value theorems of Dirichlet L-functions to give some hybrid mean value results for a generalization on a problem of D.H. Lehmer and hyper-Kloosterman sums.

1. Introduction

Let $q > 2$ and c be two integers with $(c, q) = 1$. For each integer a with $1 \leq a \leq q$ and $(a, q) = 1$, we know that there exists one and only one b with $1 \leq b \leq q$ such that $ab \equiv c \bmod q$. Let

$$L(q, k, c) = \sum_{a=1}^{q}{}' \sum_{\substack{b=1 \\ ab\equiv c \bmod q}}^{q}{}' (a-b)^{2k},$$

where $\sum_a'$ denotes the summation over all a such that $(a, q) = 1$. In reference [1], the second author used the estimates for Kloosterman sums and trigonometric sums to obtain a sharp asymptotic formula for $L(q, k, c)$, and prove the following:

Proposition 1. *Let $q > 2$ and c be two integers with $(c, q) = 1$. Then for any positive integer k, we have the asymptotic formula*

$$L(q, k, c) = \frac{1}{(2k+1)(k+1)}\phi(q)q^{2k} + O(4^k q^{(4k+1)/2} d^2(q) \ln^2 q),$$

where $\phi(q)$ is the Euler function, and $d(q)$ is the divisor function.

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The error terms in Proposition 1 is best possible. In fact for $k = 1$, let

$$L(q, 1, c) = \frac{1}{6}\phi(q)q^2 + \frac{1}{3}q \prod_{p|q}(1 - p) + G(q, c),$$

where $\prod_{p|q}$ denotes the product over all distinct prime divisors of q, the second author [2] used the properties of Dedekind sums and Cochrane sums to give a sharp mean value formula for $G(q, c)$. That is the following:

Proposition 2. *For any integer $q > 2$, we have the asymptotic formula*

$$\sum_{c=1}^{q}{}' G^2(q, c) = \frac{5}{36}q^3\phi^3(q) \prod_{p^\alpha \| q} \frac{(p+1)^3/(p(p^2+1)) - 1/(p^{3\alpha-1})}{1 + 1/p + 1/p^2} + O\left(q^5 \exp\left(\frac{4 \ln q}{\ln \ln q}\right)\right),$$

where $\exp(y) = e^y$, $\prod_{p^\alpha \| q}$ *denotes the product over all prime divisors p of q with $p^\alpha | q$ and $p^{\alpha+1} \nmid q$.*

Let $M(q, c)$ be the number of cases in which a and b are of opposite parity. That is,

$$M(q, c) = \sum_{\substack{a=1 \\ ab \equiv c \bmod q \\ 2 \nmid a+b}}^{q}{}' \sum_{b=1}^{q}{}' 1.$$

For $q = p$ an odd prime and $c = 1$, D.H. Lehmer [3] asked us to find $M(p, 1)$ or at least to say something nontrivial about it. For the sake of simplicity, we call such a number as a D.H. Lehmer Number. In references [4] and [5], the second author proved that

$$M(q, 1) = \frac{1}{2}\phi(q) + O(q^{1/2}d^2(q) \ln^2 q). \tag{1}$$

For any nonnegative integer n, let

$$M(q, 1, n) = \sum_{\substack{a=1 \\ ab \equiv 1 \bmod q \\ 2 \nmid a+b}}^{q}{}' \sum_{b=1}^{q}{}' (a - b)^{2n},$$

the second author [6] also proved the following asymptotic formula:

$$M(q, 1, n) = \frac{1}{(2n+1)(2n+2)}\phi(q)q^{2n} + O(4^n q^{2n+1/2} d^2(q) \ln^2 q).$$

For any fixed positive integer c with $(c, q) = 1$, define

$$F(q, c) = M(q, c) - \frac{1}{2}\phi(q).$$

Then the second author showed in [7] and [8] that for any odd number $q > 2$,

$$\sum_{c=1}^{q}{}' |F(q, c)|^2 = \frac{3}{4}\phi^2(q) \prod_{p^\alpha \| q} \frac{(p+1)^3/(p(p^2+1)) - 1/p^{3\alpha-1}}{1 + 1/p + 1/p^2} + O\left(q \exp\left(\frac{4 \ln q}{\ln \ln q}\right)\right).$$

This proved that the error terms in (1) is also best possible.

In [9], the second author found that there exists some close relation between the error terms $F(q, c)$ and the classical Kloosterman sums:

$$K(m, n; q) = \sum_{b=1}^{q}{}' \mathrm{e}\left(\frac{mb + n\overline{b}}{q}\right),$$

where $\mathrm{e}(y) = \mathrm{e}^{2\pi i y}$, $\overline{b}$ is defined by the equation $b\overline{b} \equiv 1 \mod q$, and obtained the following hybrid mean value formula:

$$\sum_{c=1}^{q}{}' F(q, c) K(\overline{4}c, 1; q) = \frac{4}{\pi^2} q\phi(q) \prod_{p \| q}\left(1 - \frac{1}{p(p-1)}\right) + O(q^{3/2+\epsilon}),$$

where ϵ is any fixed positive number.

In [10], Mordell introduced the hyper-Kloosterman sums as follows:

$$K(h, k, q) = \sum_{\substack{a_1, \cdots, a_k \bmod q \\ (a_1, q) = \cdots = (a_k, q) = 1}} \mathrm{e}\left(\frac{a_1 + \cdots + a_k + h\overline{a}_1 \cdots \overline{a}_k}{q}\right),$$

which is the high-dimensional generalization of the Kloosterman sums. Some applications of the hyper-Kloosterman sums were found in the estimation of Fourier coefficients of Maass forms [11] and the work on Selberg's eigenvalue conjecture [12]. Moreover, there exists some interesting connections between the hyper-Kloosterman sums and the Heibronn sums (see reference [13]).

Now we consider a generalization on this problem of D.H. Lehmer. For any integer $k \geq 1$, let

$$N(q, k, c) = \sum_{a_1=1}^{q}{}' \cdots \sum_{a_k=1}^{q}{}' \sum_{\substack{b=1 \\ a_1 \cdots a_k b \equiv c \bmod q \\ 2 \nmid a_1 + \cdots + a_k + b}}^{q}{}' (a_1 + \cdots + a_k - b)^2$$

and

$$E(q, k, c) = N(q, k, c) - \frac{(3k^2 - 5k + 4)}{24}\phi^k(q) q^2 - \frac{(k+1)}{12}\phi^{k-1}(q) q \prod_{p|q}(1 - p).$$

In this paper, we use the Fourier expansion for character sums and the mean value theorems of Dirichlet L-functions to study the hybrid mean value of $E(q, k, c)$ and the hyper-Kloosterman sums, and give an interesting mean value formula. That is, we shall prove the following:

Theorem. *For any odd number $q \geq 3$ and integer $k \geq 1$, we have the asymptotic formula*

$$\sum_{c=1}^{q}{}' E(q, k, c) K(\overline{2}^{k+1} c, k, q) = \frac{c_k q^{k+2} \phi(q)}{\pi^{k+3}} \prod_{p \| q} \left(1 - \frac{p^k - 1}{p^k (p-1)^2}\right) + O(q^{k+5/2+\epsilon}),$$

where ϵ is any fixed positive number, $\prod_{p\|q}$ denotes the product over all prime divisors p of q with $p|q$ and $p^2 \nmid q$, and

$$c_k = \begin{cases} -6, & \text{if} \quad k = 1, \\ i^{k+3} 2^{2k-2} [\pi^2 (k^2 - k + 2) - 8(k+1)], & \text{otherwise}. \end{cases}$$

2. Several lemmas

To complete the proof of the theorem, we need the following lemmas.

Lemma 1. *Let $q \geq 3$ be an odd number. Then for any positive integer c with $(c, q) = 1$, we have*

$$\begin{aligned} E(q, 1, c) = &\frac{1}{\phi(q)} \sum_{\substack{\chi(-1)=1 \\ \chi \neq \chi_0}} \overline{\chi}(c) \left(\sum_{a=1}^{q} (-1)^a a \chi(a) \right)^2 \\ &+ \frac{1}{\phi(q)} \sum_{\chi(-1)=-1} \overline{\chi}(c) (4\chi(4) - 4\chi(2)) \left(\sum_{a=1}^{q} a \chi(a) \right)^2 \\ &- \frac{2}{\phi(q) q} \sum_{\chi(-1)=-1} \overline{\chi}(c) (1 - 2\chi(2)) \left(\sum_{a=1}^{q} a \chi(a) \right) \left(\sum_{b=1}^{q} (-1)^b b^2 \chi(b) \right) + O(q), \end{aligned}$$

where $\sum_{\substack{\chi(-1)=1 \\ \chi \neq \chi_0}}$ denotes the summation over all non-principal even characters modulo q, and $\sum_{\chi(-1)=-1}$ denotes the summation over all odd characters modulo q.

Proof. From the definition of $N(q, 1, c)$ we can get

$$N(q, 1, c) = \sum_{\substack{a=1 \\ ab \equiv c \bmod q \\ 2 \nmid a+b}}^{q}{}' \sum_{b=1}^{q}{}' (a-b)^2 = \frac{1}{2} \sum_{\substack{a=1 \\ ab \equiv c \bmod q}}^{q}{}' \sum_{b=1}^{q}{}' [1 - (-1)^{a+b}] (a-b)^2$$

$$= \frac{1}{2} \sideset{}{'}\sum_{\substack{a=1 \\ ab\equiv c \bmod q}}^{q} \sideset{}{'}\sum_{b=1}^{q} (a-b)^2 - \frac{1}{2} \sideset{}{'}\sum_{\substack{a=1 \\ ab\equiv c \bmod q}}^{q} \sideset{}{'}\sum_{b=1}^{q} (-1)^{a+b}(a-b)^2.$$

By the orthogonality relation for character sums mod q we easily deduce

$$\begin{aligned} &N(q, 1, c) \\ &= \frac{1}{12}\phi(q)q^2 + \frac{1}{6}q\prod_{p|q}(1-p) - \frac{1}{\phi(q)} \sum_{\chi(-1)=-1} \overline{\chi}(c)\left(\sum_{a=1}^{q} a\chi(a)\right)^2 \\ &\quad - \frac{1}{\phi(q)} \sum_{\chi\neq\chi_0} \overline{\chi}(c)\left[\left(\sum_{a=1}^{q}(-1)^a a^2\chi(a)\right)\left(\sum_{b=1}^{q}(-1)^b\chi(b)\right) - \left(\sum_{a=1}^{q}(-1)^a a\chi(a)\right)^2\right] \\ &\quad + O(q), \end{aligned}$$

where we have used the identities (see reference [14])

$$\sideset{}{'}\sum_{a=1}^{q} a^2 = \frac{1}{3}\phi(q)q^2 + \frac{1}{6}q\prod_{p|q}(1-p), \quad \sideset{}{'}\sum_{a=1}^{q} a = \frac{\phi(q)q}{2} \tag{2}$$

and

$$\sideset{}{'}\sum_{a=1}^{q}(-1)^a = 0, \quad \sideset{}{'}\sum_{a=1}^{q}(-1)^a a = -\frac{1}{2}\prod_{p|q}(1-p), \quad \sideset{}{'}\sum_{a=1}^{q}(-1)^a a^2 = -\frac{1}{2}q\prod_{p|q}(1-p)$$

for any odd number $q \geq 3$.

Note that if $\chi(-1) = 1$ then

$$\sum_{b=1}^{q}(-1)^b\chi(b) = 0, \tag{3}$$

and if $\chi(-1) = -1$, then

$$\sum_{a=1}^{q}(-1)^a a\chi(a) = \frac{q}{2}\sum_{a=1}^{q}(-1)^a\chi(a), \quad \sum_{b=1}^{q}(-1)^b\chi(b) = 2\chi(2)\sum_{b=1}^{(q-1)/2}\chi(b). \tag{4}$$

From [15] we also know that for any odd character χ mod q, we have

$$(1-2\chi(2))\sum_{c=1}^{q} c\chi(c) = \chi(2)q\sum_{c=1}^{(q-1)/2}\chi(c). \tag{5}$$

So from the above formulae we can have

$$E(q,1,c)=\frac{1}{\phi(q)}\sum_{\substack{\chi(-1)=1\\ \chi\neq\chi_0}}\overline{\chi}(c)\left(\sum_{a=1}^{q}(-1)^a a\chi(a)\right)^2$$
$$+\frac{1}{\phi(q)}\sum_{\chi(-1)=-1}\overline{\chi}(c)(4\chi(4)-4\chi(2))\left(\sum_{a=1}^{q}a\chi(a)\right)^2$$
$$-\frac{2}{\phi(q)q}\sum_{\chi(-1)=-1}\overline{\chi}(c)(1-2\chi(2))\left(\sum_{a=1}^{q}a\chi(a)\right)\left(\sum_{b=1}^{q}(-1)^b b^2\chi(b)\right)+O(q).$$

This proves Lemma 1. □

Lemma 2. *Let $q\geq 3$ be an odd number and $k\geq 2$ be an integer. Then for any positive integer c with $(c,q)=1$, we have the identity*

$$E(q,k,c)=\frac{-2^{k-2}}{\phi(q)q^{k-1}}\sum_{\chi(-1)=-1}\overline{\chi}(c)(1-2\chi(2))^k\left(\sum_{a=1}^{q}a\chi(a)\right)^k$$
$$\times\left[\frac{2(k+1)}{q}\left(\sum_{b=1}^{q}(-1)^b b^2\chi(b)\right)+k(k-3)(1-2\chi(2))\left(\sum_{b=1}^{q}b\chi(b)\right)\right].$$

Proof. From the definition of $N(q,k,c)$ we can get

$$N(q,k,c)=\sideset{}{'}\sum_{\substack{a_1=1\\ a_1\cdots a_k b\equiv c \bmod q\\ 2\nmid a_1+\cdots+a_k+b}}^{q}\cdots\sideset{}{'}\sum_{a_k=1}^{q}\sideset{}{'}\sum_{b=1}^{q}(a_1+\cdots+a_k-b)^2$$
$$=\frac{1}{2}\sideset{}{'}\sum_{\substack{a_1=1\\ a_1\cdots a_k b\equiv c \bmod q}}^{q}\cdots\sideset{}{'}\sum_{a_k=1}^{q}\sideset{}{'}\sum_{b=1}^{q}[1-(-1)^{a_1+\cdots+a_k+b}](a_1+\cdots+a_k-b)^2$$
$$=\frac{1}{2}\sideset{}{'}\sum_{\substack{a_1=1\\ a_1\cdots a_k b\equiv c \bmod q}}^{q}\cdots\sideset{}{'}\sum_{a_k=1}^{q}\sideset{}{'}\sum_{b=1}^{q}(a_1+\cdots+a_k-b)^2$$
$$-\frac{1}{2}\sideset{}{'}\sum_{\substack{a_1=1\\ a_1\cdots a_k b\equiv c \bmod q}}^{q}\cdots\sideset{}{'}\sum_{a_k=1}^{q}\sideset{}{'}\sum_{b=1}^{q}(-1)^{a_1+\cdots+a_k+b}(a_1+\cdots+a_k-b)^2$$
$$:=P(q,k,c)+E(q,k,c).$$

Then by (2) and the properties of character sums mod q we have

$$
\begin{aligned}
P(q,k,c) &= \frac{1}{2} \sideset{}{'}\sum_{a_1=1}^{q} \cdots \sideset{}{'}\sum_{a_k=1}^{q} \sideset{}{'}\sum_{\substack{b=1 \\ a_1\cdots a_k b \equiv c \bmod q}}^{q} (a_1+\cdots+a_k-b)^2 \\
&= \frac{1}{2\phi(q)} \sum_{\chi \bmod q} \overline{\chi}(c) \sum_{a_1=1}^{q} \cdots \sum_{a_k=1}^{q} \sum_{b=1}^{q} \chi(a_1)\cdots\chi(a_k)\chi(b)(a_1+\cdots+a_k-b)^2 \\
&= \frac{1}{2\phi(q)} \sideset{}{'}\sum_{a_1=1}^{q} \cdots \sideset{}{'}\sum_{a_k=1}^{q} \sideset{}{'}\sum_{b=1}^{q} (a_1+\cdots+a_k-b)^2 \\
&= \frac{1}{2\phi(q)} \sideset{}{'}\sum_{a_1=1}^{q} \cdots \sideset{}{'}\sum_{a_k=1}^{q} \sideset{}{'}\sum_{b=1}^{q} \left(\sum_{1\le i\le k} a_i^2 + b^2 + 2 \sum_{1\le i<j\le k} a_i a_j - 2 \sum_{1\le i\le k} a_i b \right) \\
&= \frac{(k+1)}{2\phi(q)} \sideset{}{'}\sum_{a_1=1}^{q} \cdots \sideset{}{'}\sum_{a_k=1}^{q} \sideset{}{'}\sum_{b=1}^{q} b^2 \\
&\quad + \frac{1}{\phi(q)} ((k-1)+(k-2)+\cdots+1-k) \sideset{}{'}\sum_{a_1=1}^{q} \cdots \sideset{}{'}\sum_{a_k=1}^{q} \sideset{}{'}\sum_{b=1}^{q} a_1 b \\
&= \frac{(k+1)}{2\phi(q)} \sideset{}{'}\sum_{a_1=1}^{q} \cdots \sideset{}{'}\sum_{a_k=1}^{q} \sideset{}{'}\sum_{b=1}^{q} b^2 + \frac{k(k-3)}{2\phi(q)} \sideset{}{'}\sum_{a_1=1}^{q} \cdots \sideset{}{'}\sum_{a_k=1}^{q} \sideset{}{'}\sum_{b=1}^{q} a_1 b \\
&= \frac{(k+1)\phi^{k-1}(q)}{2} \left[\frac{1}{3}\phi(q)q^2 + \frac{1}{6} q \prod_{p\mid q}(1-p) \right] + \frac{k(k-3)}{8} \phi^k(q) q^2 \\
&= \frac{(3k^2-5k+4)}{24} \phi^k(q) q^2 + \frac{(k+1)}{12} \phi^{k-1}(q) q \prod_{p\mid q}(1-p).
\end{aligned}
$$

On the other hand, from (3), (4) and (5) we also have

$$
\begin{aligned}
E(q,k,c) &= -\frac{1}{2} \sideset{}{'}\sum_{a_1=1}^{q} \cdots \sideset{}{'}\sum_{a_k=1}^{q} \sideset{}{'}\sum_{\substack{b=1 \\ a_1\cdots a_k b \equiv c \bmod q}}^{q} (-1)^{a_1+\cdots+a_k+b} (a_1+\cdots+a_k-b)^2 \\
&= -\frac{1}{2\phi(q)} \sum_{\chi \bmod q} \overline{\chi}(c) \sum_{a_1=1}^{q} \cdots \sum_{a_k=1}^{q} \sum_{b=1}^{q} (-1)^{a_1+\cdots+a_k+b} \\
&\qquad\qquad \times \chi(a_1)\cdots\chi(a_k)\chi(b)(a_1+\cdots+a_k-b)^2
\end{aligned}
$$

$$= \frac{-1}{2\phi(q)} \sum_{\chi(-1)=-1} \overline{\chi}(c) \Bigg[(k+1)\left(\sum_{a=1}^{q}(-1)^a\chi(a)\right)^k \left(\sum_{b=1}^{q}(-1)^b b^2\chi(b)\right) + k(k-3)\left(\sum_{a=1}^{q}(-1)^a a\chi(a)\right)^2 \left(\sum_{b=1}^{q}(-1)^b\chi(b)\right)^{k-1} \Bigg]$$

$$= \frac{-2^{k-2}}{\phi(q)q^{k-1}} \sum_{\chi(-1)=-1} \overline{\chi}(c)(1-2\chi(2))^k \left(\sum_{a=1}^{q} a\chi(a)\right)^k \times \left[\frac{2(k+1)}{q}\left(\sum_{b=1}^{q}(-1)^b b^2\chi(b)\right) + k(k-3)(1-2\chi(2))\left(\sum_{b=1}^{q} b\chi(b)\right)\right].$$

This completes the proof of Lemma 2. □

Lemma 3. *Let χ be a primitive character modulo m, and let $m \geq 3$ be an odd number. Then we have*

$$\sum_{a=1}^{m} a\chi(a) = \frac{mi}{\pi}\tau(\chi)L(1, \overline{\chi}), \quad \textit{if} \quad \chi(-1) = -1;$$

$$\sum_{a=1}^{m}(-1)^a a\chi(a) = \frac{m\tau(\chi)(1-4\chi(2))}{\pi^2}L(2, \overline{\chi}) + O(m), \quad \textit{if} \quad \chi(-1) = 1;$$

$$\sum_{a=1}^{m}(-1)^a a^2\chi(a) = \frac{m^2(1-2\chi(2))i}{\pi}\tau(\chi)L(1, \overline{\chi}) + \frac{m^2(8\chi(2)-1)i}{\pi^3}\tau(\chi)L(3, \overline{\chi}) + O(m^2), \quad \textit{if} \quad \chi(-1) = -1,$$

where $L(1, \chi)$ is the Dirichlet L-function corresponding to χ, $\tau(\chi) = \sum_{a=1}^{m} \chi(a)\mathrm{e}(a/m)$ is the Gauss sum, and $|\tau(\chi)| = \sqrt{m}$.

Proof. From Theorem 12.11 and Theorem 12.20 of [14] we know that if χ is a primitive character modulo m with $\chi(-1) = -1$, then

$$\frac{1}{m}\sum_{b=1}^{m} b\chi(b) = \frac{i}{\pi}\tau(\chi)L(1, \overline{\chi}), \tag{6}$$

so we get the first formula.

Now we prove the third formula, since similarly we can deduce the second one. For any odd primitive character χ modulo m, we have

$$\begin{aligned}\sum_{a=1}^{m}(-1)^a a^2\chi(a) &= \sum_{\substack{a=1\\2\mid a}}^{m} a^2\chi(a) - \sum_{\substack{a=1\\2\nmid a}}^{m} a^2\chi(a) = \sum_{\substack{a=1\\2\mid a}}^{m} a^2\chi(a) - \sum_{\substack{a=1\\2\mid a}}^{m}(m-a)^2\chi(m-a)\\ &= 2\sum_{\substack{a=1\\2\mid a}}^{m} a^2\chi(a) - 2m\sum_{\substack{a=1\\2\mid a}}^{m} a\chi(a) + m^2\sum_{\substack{a=1\\2\mid a}}^{m}\chi(a).\end{aligned} \tag{7}$$

Note that

$$\sum_{a=1}^{m} a\chi(a) = \sum_{\substack{a=1\\2\mid a}}^{m} a\chi(a) + \sum_{\substack{a=1\\2\mid a}}^{m}(m-a)\chi(m-a) = 2\sum_{\substack{a=1\\2\mid a}}^{m} a\chi(a) - m\sum_{\substack{a=1\\2\mid a}}^{m}\chi(a),$$

so from (6) and (7) we can get

$$\begin{aligned}\sum_{a=1}^{m}(-1)^a a^2\chi(a) &= 2\sum_{\substack{a=1\\2\mid a}}^{m} a^2\chi(a) - m\sum_{a=1}^{m} a\chi(a)\\ &= 8\chi(2)\sum_{1\le a\le m/2} a^2\chi(a) - \frac{m^2 i}{\pi}\tau(\chi)L(1,\overline{\chi}).\end{aligned} \tag{8}$$

Notice the Fourier expansion for character sums which was first given by Pólya [16]:

$$\sum_{0<n\le my}\chi(n) = \begin{cases}\dfrac{\tau(\chi)}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\sin(2\pi ny)}{n} + O(1), & \text{if}\quad \chi(-1)=1;\\[2ex] \dfrac{\tau(\chi)L(1,\overline{\chi})}{\pi i} - \dfrac{\tau(\chi)}{\pi i}\displaystyle\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\cos(2\pi ny)}{n} + O(1), & \text{if}\quad \chi(-1)=-1,\end{cases}$$

where χ is a primitive character modulo m, and $y>0$ is a real number. Then by Abel's identity we have

$$\begin{aligned}\sum_{0<n\le m/2} n^2\chi(n) &= \frac{m^2}{4}\sum_{0<n\le m/2}\chi(n) - 2\int_0^{m/2} u\sum_{0<n\le u}\chi(n)\,\mathrm{d}u\\ &= \frac{m^2(\overline{\chi}(2)-2)i}{4\pi}\tau(\chi)L(1,\overline{\chi}) - 2\int_0^{m/2} u\sum_{0<n\le u}\chi(n)\,\mathrm{d}u + O(m^2)\\ &= \frac{m^2(\overline{\chi}(2)-2)i}{4\pi}\tau(\chi)L(1,\overline{\chi}) - 2m^2\int_0^{1/2} s\sum_{0<n\le ms}\chi(n)\,\mathrm{d}s + O(m^2)\end{aligned}$$

$$
\begin{aligned}
&= \frac{m^2(\overline{\chi}(2)-2)i}{4\pi}\tau(\chi)L(1,\overline{\chi}) \\
&\quad - 2m^2\int_0^{1/2} s\left[\frac{\tau(\chi)L(1,\overline{\chi})}{\pi i} - \frac{\tau(\chi)}{\pi i}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\cos(2\pi ns)}{n} + O(1)\right] \mathrm{d}s \\
&\quad + O(m^2) \\
&= \frac{m^2(\overline{\chi}(2)-1)i}{4\pi}\tau(\chi)L(1,\overline{\chi}) + 2m^2\frac{\tau(\chi)}{\pi i}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{n}\int_0^{1/2} s\cos(2\pi ns)\,\mathrm{d}s \\
&\quad + O(m^2) \\
&= \frac{m^2(\overline{\chi}(2)-1)i}{4\pi}\tau(\chi)L(1,\overline{\chi}) + \frac{m^2\tau(\chi)}{2\pi^3 i}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)(\cos(\pi n)-1)}{n^3} + O(m^2) \\
&= \frac{m^2(\overline{\chi}(2)-1)i}{4\pi}\tau(\chi)L(1,\overline{\chi}) - \frac{m^2\tau(\chi)}{\pi^3 i}\sum_{\substack{n=1\\ 2\nmid n}}^{\infty}\frac{\overline{\chi}(n)}{n^3} + O(m^2) \\
&= \frac{m^2(\overline{\chi}(2)-1)i}{4\pi}\tau(\chi)L(1,\overline{\chi}) + \frac{m^2(\overline{\chi}(2)-8)}{8\pi^3 i}\tau(\chi)L(3,\overline{\chi}) + O(m^2).
\end{aligned}
$$

Hence from (8) we have

$$
\sum_{a=1}^{m}(-1)^a a^2\chi(a) = \frac{m^2(1-2\chi(2))i}{\pi i}\tau(\chi)L(1,\overline{\chi}) + \frac{m^2(8\chi(2)-1)i}{\pi^3}\tau(\chi)L(3,\overline{\chi}) + O(m^2).
$$

This proves Lemma 3. □

Lemma 4. *Suppose that χ is an even character modulo q, generated by the primitive character χ_m modulo m, and $q \geq 3$ is an odd number. Let l be the largest divisor of q with $(l, m) = 1$. Then we have*

$$
\sum_{a=1}^{q}(-1)^a a\chi(a) = \frac{m\tau(\chi_m)(1-4\chi_m(2))}{\pi^2}L(2,\overline{\chi}_m)\left(\sum_{d|l} d\mu(d)\chi_m(d)\right) + O(q),
$$

where $\mu(n)$ is the Möbius function.

Proof. Note that m and l both are odd numbers, then from (3) we have

$$
\begin{aligned}
\sum_{a=1}^{q}(-1)^a a\chi(a) &= \sum_{i=0}^{(q/ml)-1}\sum_{j=1}^{ml}(-1)^{iml+j}(iml+j)\chi(iml+j) \\
&= \sum_{i=0}^{(q/ml)-1}(-1)^i\sum_{j=1}^{ml}(-1)^j(iml+j)\chi(iml+j) = \sum_{i=0}^{(q/ml)-1}(-1)^i\sum_{j=1}^{ml}(-1)^j j\chi(j)
\end{aligned}
$$

$$= \sum_{j=1}^{ml} (-1)^j j\chi(j) = \sum_{\substack{a=1 \\ (a,l)=1}}^{ml} (-1)^a a\chi_m(a) = \sum_{a=1}^{ml} (-1)^a a\chi_m(a) \sum_{\substack{d|a \\ d|l}} \mu(d)$$

$$= \sum_{d|l} \mu(d) \sum_{\substack{a=1 \\ d|a}}^{lm} (-1)^a a\chi_m(a) = \sum_{d|l} \mu(d) d\chi_m(d) \sum_{b=1}^{lm/d} (-1)^{db} b\chi_m(b).$$

Since d is odd, we have $(-1)^{db} = (-1)^b$. Therefore

$$\sum_{a=1}^{q} (-1)^a a\chi(a) = \sum_{d|l} \mu(d) d\chi_m(d) \sum_{b=1}^{lm/d} (-1)^b b\chi_m(b)$$

$$= \sum_{d|l} \mu(d) d\chi_m(d) \sum_{i=0}^{(l/d)-1} \sum_{j=1}^{m} (-1)^{im+j} (im+j)\chi_m(im+j)$$

$$= \sum_{d|l} \mu(d) d\chi_m(d) \sum_{i=0}^{(l/d)-1} (-1)^i \sum_{j=1}^{m} (-1)^j (im+j)\chi_m(j)$$

$$= \sum_{d|l} \mu(d) d\chi_m(d) \sum_{j=1}^{m} (-1)^j j\chi_m(j).$$

Note that

$$\left| \sum_{d|l} \mu(d) d\chi_m(d) \right| = \left| \prod_{p|l} (1 - p\chi_m(p)) \right| < \prod_{p|l} p \le l,$$

so from Lemma 3 we get

$$\sum_{a=1}^{q} (-1)^a a\chi(a) = \frac{m\tau(\chi_m)(1 - 4\chi_m(2))}{\pi^2} L(2, \overline{\chi}_m) \left(\sum_{d|l} d\mu(d)\chi_m(d) \right) + O(q).$$

This completes the proof of Lemma 4. □

Lemma 5. *Suppose that χ is an odd character modulo q, generated by the primitive character χ_m modulo m, and $q \ge 3$ is an odd number. Let l be the largest divisor of q with $(l, m) = 1$. Then we have*

$$\sum_{a=1}^{q} a\chi(a) = \frac{qi}{\pi} \tau(\chi_m) L(1, \overline{\chi}_m) \left(\sum_{d|l} \mu(d)\chi_m(d) \right).$$

Furthermore, for $q = lm$, we also have

$$\sum_{a=1}^{q}(-1)^a a^2\chi(a) = \frac{q^2(1-2\chi_m(2))i}{\pi}\tau(\chi_m)L(1,\overline{\chi}_m)\left(\sum_{d|l}\mu(d)\chi_m(d)\right) + \frac{m^2(8\chi_m(2)-1)i}{\pi^3}\tau(\chi_m)L(3,\overline{\chi}_m)\left(\sum_{d|l}d^2\mu(d)\chi_m(d)\right) + O(q^2).$$

Proof. The first formula can be easily deduced from Lemma 6 of [5] and Lemma 3. Now let $q = lm$, we have

$$\sum_{a=1}^{q}(-1)^a a^2\chi(a) = \sum_{a=1}^{lm}(-1)^a a^2\chi(a) = \sum_{\substack{a=1\\(a,l)=1}}^{lm}(-1)^a a^2\chi_m(a) = \sum_{d|l}\mu(d)\sum_{\substack{a=1\\d|a}}^{lm}(-1)^a a^2\chi_m(a)$$
$$= \sum_{d|l}\mu(d)d^2\chi_m(d)\sum_{b=1}^{lm/d}(-1)^{db}b^2\chi_m(b)$$
$$= \sum_{d|l}d^2\mu(d)\chi_m(d)\sum_{b=1}^{lm/d}(-1)^{b}b^2\chi_m(b).$$

Since

$$\sum_{b=1}^{lm/d}(-1)^b b^2\chi_m(b) = \sum_{i=0}^{(l/d)-1}\sum_{j=1}^{m}(-1)^{im+j}(im+j)^2\chi_m(im+j)$$
$$= \sum_{i=0}^{(l/d)-1}(-1)^i\sum_{j=1}^{m}(-1)^j(i^2m^2+2imj+j^2)\chi_m(j)$$
$$= m^2\sum_{i=0}^{(l/d)-1}(-1)^i i^2\sum_{j=1}^{m}(-1)^j\chi_m(j) + 2m\sum_{i=0}^{(l/d)-1}(-1)^i i\sum_{j=1}^{m}(-1)^j j\chi_m(j) + \sum_{i=0}^{(l/d)-1}(-1)^i\sum_{j=1}^{m}(-1)^j j^2\chi_m(j)$$
$$= \frac{m^2 l(l-d)}{2d^2}\sum_{j=1}^{m}(-1)^j\chi_m(j) + \frac{m(l-d)}{d}\sum_{j=1}^{m}(-1)^j j\chi_m(j) + \sum_{j=1}^{m}(-1)^j j^2\chi_m(j),$$

and note that

$$\left|\sum_{d|l} d^2\mu(d)\chi_m(d)\right| = \left|\prod_{p|l}(1-p^2\chi_m(p))\right| < \prod_{p|l} p^2 \le l^2,$$

so from (4), (5), (6) and Lemma 3 we have

$$\begin{aligned}\sum_{a=1}^{q}(-1)^a a^2\chi(a) &= \frac{q^2}{2}\left(\sum_{d|l}\mu(d)\chi_m(d)\right)\left(\sum_{j=1}^{m}(-1)^j\chi_m(j)\right)\\ &\quad - \frac{m^2}{2}\left(\sum_{d|l} d^2\mu(d)\chi_m(d)\right)\left(\sum_{j=1}^{m}(-1)^j\chi_m(j)\right)\\ &\quad + \left(\sum_{d|l} d^2\mu(d)\chi_m(d)\right)\left(\sum_{j=1}^{m}(-1)^j j^2\chi_m(j)\right)\\ &= \frac{q^2(1-2\chi_m(2))i}{\pi}\tau(\chi_m)L(1,\overline{\chi}_m)\left(\sum_{d|l}\mu(d)\chi_m(d)\right)\\ &\quad + \frac{m^2(8\chi_m(2)-1)i}{\pi^3}\tau(\chi_m)L(3,\overline{\chi}_m)\left(\sum_{d|l} d^2\mu(d)\chi_m(d)\right) + O(q^2).\end{aligned}$$

This proves Lemma 5. □

Lemma 6. *Let χ be a character modulo q, generated by the primitive character χ_m modulo m. Then we have the identity*

$$\tau(\chi) = \chi_m\left(\frac{q}{m}\right)\mu\left(\frac{q}{m}\right)\tau(\chi_m).$$

Proof. See Lemma 1.3 of reference [17]. □

Lemma 7. *Let m and r be integers with $m \ge 2$ and $(r, m) = 1$, and let χ be a Dirichlet character modulo m. Then we have the identities*

$$\sum_{\chi \bmod m}{}^{*} \chi(r) = \sum_{d|(m,r-1)} \mu\left(\frac{m}{d}\right)\phi(d)$$

and

$$J(m) = \sum_{d|m}\mu(d)\phi\left(\frac{m}{d}\right),$$

where $\sum^*_{\chi \bmod m}$ denotes the summation over all primitive characters modulo m and $J(m)$ denotes the number of primitive characters modulo m.

Proof. This is Lemma 3 of [18]. □

Lemma 8. *Let $q = uv$, where $(u, v) = 1$, u be a square-full number or $u = 1$, v be a square-free number. Then we have the asymptotic formulae*

$$\Psi_1 := \sum_{d|v} u^{k+1} d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \mu(d_1)\cdots\mu(d_{k+1}) \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} \overline{\chi}(d_1\cdots d_{k+1}) L^{k+1}(1,\overline{\chi})$$
$$= \frac{q^k\phi^2(q)}{2} \prod_{p\|q}\left(1 - \frac{p^k-1}{p^k(p-1)^2}\right) + O(q^{k+1+\epsilon});$$
$$\Psi_2 := \sum_{d|v} u^{k+1} d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \frac{\mu(d_1)\cdots\mu(d_{k+1})}{d_1\cdots d_{k+1}} \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=1}} \overline{\chi}(d_1\cdots d_{k+1}) L^{k+1}(2,\overline{\chi})$$
$$= \frac{q^k\phi^2(q)}{2} \prod_{p\|q}\left(1 - \frac{p^k-1}{p^k(p-1)^2}\right) + O(q^{k+1+\epsilon});$$
$$\Psi_3 := \sum_{d|v} u^{k+1} d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \frac{\mu(d_1)\cdots\mu(d_{k+1})}{d_{k+1}^2} \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} \overline{\chi}(d_1\cdots d_{k+1}) L^{k}(1,\overline{\chi}) L(3,\overline{\chi})$$
$$= \frac{q^k\phi^2(q)}{2} \prod_{p\|q}\left(1 - \frac{p^k-1}{p^k(p-1)^2}\right) + O(q^{k+1+\epsilon}).$$

Proof. We only prove the first formula, since similarly we can get the others. For any non-principal character χ modulo ud, and parameter $N \geq ud$, applying Abel's identity we have

$$L(s,\overline{\chi}) = \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n^s} = \sum_{1\le n\le N} \frac{\overline{\chi}(n)}{n^s} + s\int_N^{\infty} \frac{\sum_{N<n\le y}\overline{\chi}(n)}{y^{s+1}}\,dy$$
$$= \sum_{1\le n\le N} \frac{\overline{\chi}(n)}{n^s} + O\left(\frac{\sqrt{ud}\log(ud)}{N^s}\right).$$

Let $\tau_{k+1}(n)$ be the $(k+1)$-th divisor function (i.e., the number of positive integer solutions of the equation $n_1 n_2\cdots n_{k+1} = n$). Then we have

$$\Psi_1 = \sum_{d|v} u^{k+1} d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \mu(d_1)\cdots\mu(d_{k+1}) \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} \overline{\chi}(d_1\cdots d_{k+1})$$
$$\times \left(\sum_{1\le n\le N} \frac{\overline{\chi}(n)}{n} + O\left(\frac{\sqrt{ud}\log(ud)}{N}\right)\right)^{k+1}$$

$$= \sum_{d|v} u^{k+1}d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \mu(d_1)\cdots\mu(d_{k+1}) \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} \overline{\chi}(d_1\cdots d_{k+1})$$

$$\times \left(\sum_{1\le n\le N} \frac{\overline{\chi}(n)}{n}\right)^{k+1} + O\left(\frac{q^{k+5/2+\epsilon}\log^k N}{N}\right)$$

$$= \sum_{d|v} u^{k+1}d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \mu(d_1)\cdots\mu(d_{k+1}) \sum_{1\le n\le N^{k+1}} \frac{\tau_{k+1}(n)}{n}$$

$$\times \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} \overline{\chi}(d_1\cdots d_{k+1})\overline{\chi}(n) + O\left(\frac{q^{k+5/2+\epsilon}\log^k N}{N}\right)$$

$$= \sum_{d|v} u^{k+1}d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \sum_{1\le n\le N^{k+1}} \frac{\mu(d_1)\cdots\mu(d_{k+1})d_1\cdots d_{k+1}\tau_{k+1}(n)}{d_1\cdots d_{k+1}n}$$

$$\times \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} \overline{\chi}(d_1\cdots d_{k+1})\overline{\chi}(n) + O\left(\frac{q^{k+5/2+\epsilon}\log^k N}{N}\right)$$

$$:= \Omega + O\left(\frac{q^{k+5/2+\epsilon}\log^k N}{N}\right).$$

For $(a, m) = 1$, by Lemma 7 we have

$$\sideset{}{^*}\sum_{\substack{\chi \bmod m \\ \chi(-1)=-1}} \chi(a) = \frac{1}{2}\sideset{}{^*}\sum_{\chi \bmod m}(1-\chi(-1))\chi(a) = \frac{1}{2}\sideset{}{^*}\sum_{\chi \bmod m}\chi(a) - \frac{1}{2}\sideset{}{^*}\sum_{\chi \bmod m}\chi(-a) \tag{9}$$

$$= \frac{1}{2}\sum_{s|(m,a-1)} \mu\left(\frac{m}{s}\right)\phi(s) - \frac{1}{2}\sum_{s|(m,a+1)} \mu\left(\frac{m}{s}\right)\phi(s).$$

Therefore

$$\Omega = \frac{1}{2}\sum_{d|v} u^{k+1}d^{k+1} \sum_{s|ud} \mu\left(\frac{ud}{s}\right)\phi(s)$$

$$\times \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \sum_{\substack{1\le n\le N^{k+1} \\ (n,ud)=1 \\ d_1\cdots d_{k+1}n\equiv 1 \bmod s}} \frac{\mu(d_1)\cdots\mu(d_{k+1})d_1\cdots d_{k+1}\tau_{k+1}(n)}{d_1\cdots d_{k+1}n}$$

$$- \frac{1}{2}\sum_{d|v} u^{k+1}d^{k+1} \sum_{s|ud} \mu\left(\frac{ud}{s}\right)\phi(s)$$

$$\times \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \sum_{\substack{1\le n\le N^{k+1} \\ (n,ud)=1 \\ d_1\cdots d_{k+1}n\equiv -1 \bmod s}} \frac{\mu(d_1)\cdots\mu(d_{k+1})d_1\cdots d_{k+1}\tau_{k+1}(n)}{d_1\cdots d_{k+1}n}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{d|v} u^{k+1}d^{k+1}\sum_{s|ud}\mu\left(\frac{ud}{s}\right)\phi(s) \\
&\quad\times \sum_{d_1|(v/d)}\cdots\sum_{d_{k+1}|(v/d)}\sum_{\substack{1\le n\le N^{k+1}\\ (n,ud)=1\\ d_1\cdots d_{k+1}n=ls+1}}\sum_{0\le l\le((Nv/d)^{k+1}-1)/s}\frac{\mu(d_1)\cdots\mu(d_{k+1})d_1\cdots d_{k+1}\tau_{k+1}(n)}{ls+1} \\
&\quad - \frac{1}{2}\sum_{d|v} u^{k+1}d^{k+1}\sum_{s|ud}\mu\left(\frac{ud}{s}\right)\phi(s) \\
&\qquad\times \sum_{d_1|(v/d)}\cdots\sum_{d_{k+1}|(v/d)}\sum_{\substack{1\le n\le N^{k+1}\\ (n,ud)=1\\ d_1\cdots d_{k+1}n=ls-1}}\sum_{2/s\le l\le((Nv/d)^{k+1}+1)/s}\frac{\mu(d_1)\cdots\mu(d_{k+1})d_1\cdots d_{k+1}\tau_{k+1}(n)}{ls-1} \\
&= \frac{1}{2}\sum_{d|v} u^{k+1}d^{k+1}\sum_{s|ud}\mu\left(\frac{ud}{s}\right)\phi(s) \\
&\quad + O\left(\sum_{d|v} u^{k+1}d^{k+1}\sum_{s|ud}\frac{\phi(s)}{s}\sum_{d_1|(v/d)}\cdots\sum_{d_{k+1}|(v/d)}\sum_{1\le l\le((Nv/d)^{k+1}-1)/s}\frac{d_1\cdots d_{k+1}N^{\epsilon}}{l+1/s}\right) \\
&\quad + O\left(\sum_{d|v} u^{k+1}d^{k+1}\sum_{s|ud}\frac{\phi(s)}{s}\sum_{d_1|(v/d)}\cdots\sum_{d_{k+1}|(v/d)}\sum_{2/s\le l\le((Nv/d)^{k+1}+1)/s}\frac{d_1\cdots d_{k+1}N^{\epsilon}}{l-1/s}\right) \\
&= \frac{1}{2}\sum_{d|v} u^{k+1}d^{k+1}J(ud) + O(q^{k+1+\epsilon}N^{\epsilon}) = \frac{u^k\phi^2(u)}{2}\sum_{d|v} d^{k+1}J(d) + O(q^{k+1+\epsilon}N^{\epsilon}) \\
&= \frac{u^k\phi^2(u)}{2}\prod_{p|v}\left[p^k(p-1)^2\left(1-\frac{p^k-1}{p^k(p-1)^2}\right)\right] + O(q^{k+1+\epsilon}N^{\epsilon}) \\
&= \frac{q^k\phi^2(q)}{2}\prod_{p\|q}\left(1-\frac{p^k-1}{p^k(p-1)^2}\right) + O(q^{k+1+\epsilon}N^{\epsilon}),
\end{aligned}
$$

where we have used the estimate $\tau_{k+1}(n) \ll n^{\epsilon}$, the fact that v is a square-free number, u is a square-full number, and the identity $J(u) = \phi^2(u)/u$, if u is a square-full number.

Now taking $N = q^{3/2}$ in the above, we immediately get

$$
\Psi_1 = \frac{q^k\phi^2(q)}{2}\prod_{p\|q}\left(1-\frac{p^k-1}{p^k(p-1)^2}\right) + O(q^{k+1+\epsilon}).
$$

This completes the proof of Lemma 8. □

Lemma 9. *Let $q = uv$, where $(u, v) = 1$, u be a square-full number or $u = 1$, v be a square-free number. Then we have the following estimates*

$$\Psi_4 := \sum_{d|v} u^{k+1}d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \mu(d_1)\cdots\mu(d_{k+1}) \times \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} \overline{\chi}^j(2)\overline{\chi}(d_1\cdots d_{k+1})L^{k+1}(1,\overline{\chi}) \ll q^{k+1+\epsilon}, \quad j = 1, 2, \ldots, k+1;$$

$$\Psi_5 := \sum_{d|v} u^{k+1}d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \frac{\mu(d_1)\cdots\mu(d_{k+1})}{d_1\cdots d_{k+1}} \times \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=1}} \overline{\chi}^j(2)\overline{\chi}(d_1\cdots d_{k+1})L^{k+1}(2,\overline{\chi}) \ll q^{k+1+\epsilon}, \quad j = 1, 2, \ldots, k+1;$$

$$\Psi_6 := \sum_{d|v} u^{k+1}d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \frac{\mu(d_1)\cdots\mu(d_{k+1})}{d_{k+1}^2} \times \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} \overline{\chi}^j(2)\overline{\chi}(d_1\cdots d_{k+1})L^{k}(1,\overline{\chi})L(3,\overline{\chi}) \ll q^{k+1+\epsilon}, \quad j = 1, 2, \ldots, k+1.$$

Proof. We only prove the first formula, since similarly we can get the others. For parameter $N \geq ud$, using the method of Lemma 8 we have

$$\Psi_4 = \sum_{d|v} u^{k+1}d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \sum_{1\leq n\leq N^{k+1}} \frac{\mu(d_1)\cdots\mu(d_{k+1})2^j d_1\cdots d_{k+1}\tau_{k+1}(n)}{2^j d_1\cdots d_{k+1}n} \times \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} \overline{\chi}(2^j)\overline{\chi}(d_1\cdots d_{k+1})\overline{\chi}(n) + O\left(\frac{q^{k+5/2+\epsilon}\log^k N}{N}\right) := \Upsilon + O\left(\frac{q^{k+5/2+\epsilon}\log^k N}{N}\right).$$

Then from (9) we get

$$\begin{aligned}\Upsilon =& \frac{1}{2}\sum_{d|v} u^{k+1}d^{k+1}\sum_{s|ud}\mu\left(\frac{ud}{s}\right)\phi(s)\\ &\times \sum_{\substack{d_1|(v/d)\\ 2^j d_1\cdots d_{k+1}n\equiv 1 \bmod s}}\cdots\sum_{d_{k+1}|(v/d)}\sum_{\substack{1\le n\le N^{k+1}\\ (n,ud)=1}}\frac{\mu(d_1)\cdots\mu(d_{k+1})2^j d_1\cdots d_{k+1}\tau_{k+1}(n)}{2^j d_1\cdots d_{k+1}n}\\ &-\frac{1}{2}\sum_{d|v} u^{k+1}d^{k+1}\sum_{s|ud}\mu\left(\frac{ud}{s}\right)\phi(s)\\ &\times \sum_{\substack{d_1|(v/d)\\ 2^j d_1\cdots d_{k+1}n\equiv -1 \bmod s}}\cdots\sum_{d_{k+1}|(v/d)}\sum_{\substack{1\le n\le N^{k+1}\\ (n,ud)=1}}\frac{\mu(d_1)\cdots\mu(d_{k+1})2^j d_1\cdots d_{k+1}\tau_{k+1}(n)}{2^j d_1\cdots d_{k+1}n}\\ =& \frac{1}{2}\sum_{d|v} u^{k+1}d^{k+1}\sum_{s|ud}\mu\left(\frac{ud}{s}\right)\phi(s)\\ &\times \sum_{d_1|(v/d)}\cdots\sum_{d_{k+1}|(v/d)}\sum_{\substack{1\le n\le N^{k+1}\\ (n,ud)=1\\ 2^j d_1\cdots d_{k+1}n=ls+1}}\sum_{(2^j-1)/s\le l\le(2^j(Nv/d)^{k+1}-1)/s}\frac{\mu(d_1)\cdots\mu(d_{k+1})2^j d_1\cdots d_{k+1}\tau_{k+1}(n)}{ls+1}\\ &-\frac{1}{2}\sum_{d|v} u^{k+1}d^{k+1}\sum_{s|ud}\mu\left(\frac{ud}{s}\right)\phi(s)\\ &\times \sum_{d_1|(v/d)}\cdots\sum_{d_{k+1}|(v/d)}\sum_{\substack{1\le n\le N^{k+1}\\ (n,ud)=1\\ 2^j d_1\cdots d_{k+1}n=ls-1}}\sum_{(2^j+1)/s\le l\le(2^j(Nv/d)^{k+1}+1)/s}\frac{\mu(d_1)\cdots\mu(d_{k+1})2^j d_1\cdots d_{k+1}\tau_{k+1}(n)}{ls-1}\\ \ll& \sum_{d|v} u^{k+1}d^{k+1}\sum_{s|ud}\frac{\phi(s)}{s}\sum_{d_1|(v/d)}\cdots\sum_{d_{k+1}|(v/d)}\sum_{(2^j-1)/s\le l\le(2^j(Nv/d)^{k+1}-1)/s}\frac{d_1\cdots d_{k+1}N^{\epsilon}}{l+1/s}\\ &+\sum_{d|v} u^{k+1}d^{k+1}\sum_{s|ud}\frac{\phi(s)}{s}\sum_{d_1|(v/d)}\cdots\sum_{d_{k+1}|(v/d)}\sum_{(2^j+1)/s\le l\le(2^j(Nv/d)^{k+1}+1)/s}\frac{d_1\cdots d_{k+1}N^{\epsilon}}{l-1/s}\\ \ll& q^{k+1+\epsilon}N^{\epsilon}.\end{aligned}$$

Now taking $N = q^{3/2}$ in the above, we immediately get

$$\Psi_4 \ll q^{k+1+\epsilon}.$$

This proves Lemma 9. □

3. Proof of the theorem

In this section, we complete the proof of the theorem. Let $q \geq 3$ be an odd number and $k \geq 1$ be an integer, for any character $\chi \bmod q$, we have

(10)
$$\sum_{c=1}^{q} \overline{\chi}(c) K\left(\overline{2}^{k+1} c, k, q\right) = \sum_{a_1=1}^{q}{}' \cdots \sum_{a_k=1}^{q}{}' \sum_{c=1}^{q} \overline{\chi}(c) e\left(\frac{a_1 + \cdots + a_k + \overline{2}^{k+1} c \cdot \overline{a}_1 \cdots \overline{a}_k}{q}\right)$$
$$= \overline{\chi}^{k+1}(2)\tau(\overline{\chi})\left(\sum_{a=1}^{q} \overline{\chi}(a) e\left(\frac{a}{q}\right)\right)^k = \overline{\chi}^{k+1}(2)\tau^{k+1}(\overline{\chi}).$$

We first treat the case $k = 1$ of the theorem. From Lemma 1 we get

$$\sum_{c=1}^{q}{}' E(q, 1, c) K(\overline{4}c, 1, q)$$
$$= \frac{1}{\phi(q)} \sum_{\substack{\chi(-1)=1 \\ \chi \neq \chi_0}} \overline{\chi}(4)\tau^2(\overline{\chi})\left(\sum_{a=1}^{q}(-1)^a a\chi(a)\right)^2 + \frac{1}{\phi(q)} \sum_{\chi(-1)=-1} (4-4\overline{\chi}(2))\tau^2(\overline{\chi})\left(\sum_{a=1}^{q} a\chi(a)\right)^2$$
$$- \frac{2}{\phi(q)q} \sum_{\chi(-1)=-1} (\overline{\chi}(4)-2\overline{\chi}(2))\tau^2(\overline{\chi})\left(\sum_{a=1}^{q} a\chi(a)\right)\left(\sum_{b=1}^{q}(-1)^b b^2\chi(b)\right) + O(q^{5/2+\epsilon}).$$

Let $q = uv$, where $(u, v) = 1$, u be a square-full number or $u = 1$, v be a square-free number. Suppose that χ is a character modulo q, generated by the primitive character χ_m modulo m. Note that $\tau(\overline{\chi}) = \overline{\chi}_m(q/m)\mu(q/m)\tau(\overline{\chi}_m) \neq 0$ if and only if $m = ud$, where $d|v$. Then from Lemmas 4, 5, 6 we have

$$\sum_{c=1}^{q}{}' E(q, 1, c) K(\overline{4}c, 1, q)$$
$$= \frac{1}{\pi^4\phi(q)} \sum_{d|v} u^4 d^4 \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=1}} [16 - 8\overline{\chi}(2) + \overline{\chi}(4)]$$
$$\times \overline{\chi}^2\left(\frac{v}{d}\right)\mu^2\left(\frac{v}{d}\right)\left[\sum_{d_1|(v/d)} \frac{v}{dd_1}\mu\left(\frac{v}{dd_1}\right)\chi\left(\frac{v}{dd_1}\right)\right]^2 L^2(2, \overline{\chi})$$
$$+ \frac{q^2}{\pi^2\phi(q)} \sum_{d|v} u^2 d^2 \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} [4 - 4\overline{\chi}(2) + 2\overline{\chi}(4)]\overline{\chi}^2\left(\frac{v}{d}\right)\mu^2\left(\frac{v}{d}\right)$$
$$\times \left[\sum_{d_1|(v/d)} \mu\left(\frac{v}{dd_1}\right)\chi\left(\frac{v}{dd_1}\right)\right]^2 L^2(1, \overline{\chi})$$

$$+\frac{2}{\pi^4\phi(q)}\sum_{d|v}u^4d^4\sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}}[-16+10\overline{\chi}(2)-\overline{\chi}(4)]\overline{\chi}^2\left(\frac{v}{d}\right)\mu^2\left(\frac{v}{d}\right)$$

$$\times\left[\sum_{d_1|(v/d)}\mu\left(\frac{v}{dd_1}\right)\chi\left(\frac{v}{dd_1}\right)\right]\left[\sum_{d_2|(v/d)}\frac{v^2}{d^2d_2^2}\mu\left(\frac{v}{dd_2}\right)\chi\left(\frac{v}{dd_2}\right)\right]L(1,\overline{\chi})L(3,\overline{\chi})$$

$$+O(q^{7/2+\epsilon}).$$

Note that

$$\overline{\chi}\left(\frac{v}{d}\right)=\overline{\chi}\left(\frac{v}{dd_1}\right)\overline{\chi}(d_1),\quad \mu\left(\frac{v}{d}\right)=\mu\left(\frac{v}{dd_1}\right)\mu(d_1), \tag{11}$$

so by Lemma 8 and Lemma 9 we have

$$\begin{aligned}
&\sideset{}{'}\sum_{c=1}^{q}E(q,1,c)K(\overline{4}c,1,q)\\
&=\frac{q^2}{\pi^4\phi(q)}\sum_{d|v}u^2d^2\sum_{d_1|(v/d)}\sum_{d_2|(v/d)}\frac{\mu(d_1)\mu(d_2)}{d_1d_2}\\
&\quad\times\sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=1}}[16-8\overline{\chi}(2)+\overline{\chi}(4)]\overline{\chi}(d_1d_2)L^2(2,\overline{\chi})\\
&\quad+\frac{q^2}{\pi^2\phi(q)}\sum_{d|v}u^2d^2\sum_{d_1|(v/d)}\sum_{d_2|(v/d)}\mu(d_1)\mu(d_2)\\
&\quad\times\sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}}[4-4\overline{\chi}(2)+2\overline{\chi}(4)]\overline{\chi}(d_1d_2)L^2(1,\overline{\chi})\\
&\quad+\frac{2q^2}{\pi^4\phi(q)}\sum_{d|v}u^2d^2\sum_{d_1|(v/d)}\sum_{d_2|(v/d)}\frac{\mu(d_1)\mu(d_2)}{d_2^2}\\
&\quad\times\sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}}[-16+10\overline{\chi}(2)-\overline{\chi}(4)]\overline{\chi}(d_1d_2)L(1,\overline{\chi})L(3,\overline{\chi})+O(q^{7/2+\epsilon})\\
&=-\frac{6q^3\phi(q)}{\pi^4}\prod_{p\|q}\left(1-\frac{p-1}{p(p-1)^2}\right)+O(q^{7/2+\epsilon}).
\end{aligned}$$

This proves the theorem with $k=1$.

Now let $k \geq 2$ be an integer. Then from formulae (10), (11), and Lemmas 2, 4, 5, 6, 8, 9 we can have

$$
\begin{aligned}
&\sideset{}{'}\sum_{c=1}^{q} E(q,k,c)K\left(\overline{2}^{k+1}c,k,q\right)\\
&= \frac{-2^{k-2}}{\phi(q)q^{k-1}} \sum_{\chi(-1)=-1} \tau^{k+1}(\overline{\chi})\overline{\chi}^{k+1}(2)(1-2\chi(2))^k\\
&\quad\times\left(\sum_{a=1}^{q} a\chi(a)\right)^k\left[\frac{2(k+1)}{q}\left(\sum_{b=1}^{q}(-1)^b b^2\chi(b)\right)+k(k-3)(1-2\chi(2))\left(\sum_{b=1}^{q} b\chi(b)\right)\right]\\
&= \frac{i^{k+1}(-1)^{k+2}2^{k-2}(k^2-k+2)q^2}{\pi^{k+1}\phi(q)} \sum_{d\mid v} u^{k+1}d^{k+1} \sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}} \overline{\chi}^{k+1}\left(\frac{v}{d}\right)\mu^{k+1}\left(\frac{v}{d}\right)\\
&\quad\times[\overline{\chi}(2)-2]^{k+1}\left[\sum_{d_1\mid(v/d)}\mu\left(\frac{v}{dd_1}\right)\chi\left(\frac{v}{dd_1}\right)\right]^{k+1} L^{k+1}(1,\overline{\chi})\\
&\quad+\frac{i^{k+1}(-1)^{k+2}2^{k-1}(k+1)}{\pi^{k+3}\phi(q)} \sum_{d\mid v} u^{k+3}d^{k+3} \sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}} \overline{\chi}^{k+1}\left(\frac{v}{d}\right)\mu^{k+1}\left(\frac{v}{d}\right)[\overline{\chi}(2)-2]^k\\
&\qquad\times[8-\overline{\chi}(2)]\left[\sum_{d_1\mid(v/d)}\mu\left(\frac{v}{dd_1}\right)\chi\left(\frac{v}{dd_1}\right)\right]^k\left[\sum_{d_2\mid(v/d)}\frac{v^2}{d^2d_2^2}\mu\left(\frac{v}{dd_2}\right)\chi\left(\frac{v}{dd_2}\right)\right]\\
&\qquad\times L^k(1,\overline{\chi})L(3,\overline{\chi})+O(q^{k+5/2+\epsilon})\\
&= \frac{i^{k+1}(-1)^{k+2}2^{k-2}(k^2-k+2)q^2}{\pi^{k+1}\phi(q)} \sum_{d\mid v} u^{k+1}d^{k+1} \sum_{d_1\mid(v/d)}\cdots\sum_{d_{k+1}\mid(v/d)} \mu(d_1)\cdots\mu(d_{k+1})\\
&\quad\times \sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}} [\overline{\chi}(2)-2]^{k+1}\overline{\chi}(d_1\cdots d_{k+1})L^{k+1}(1,\overline{\chi})\\
&\quad+\frac{i^{k+1}(-1)^{k+2}2^{k-1}(k+1)q^2}{\pi^{k+3}\phi(q)} \sum_{d\mid v} u^{k+1}d^{k+1} \sum_{d_1\mid(v/d)}\cdots\sum_{d_{k+1}\mid(v/d)} \frac{\mu(d_1)\cdots\mu(d_{k+1})}{d_{k+1}^2}\\
&\qquad\times \sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}} [\overline{\chi}(2)-2]^k[8-\overline{\chi}(2)]\overline{\chi}(d_1\cdots d_{k+1})L^k(1,\overline{\chi})L(3,\overline{\chi})+O(q^{k+5/2+\epsilon})\\
&= \frac{i^{k+1}(-1)2^{2k-2}(k^2-k+2)q^{k+2}\phi(q)}{\pi^{k+1}} \prod_{p\| q}\left(1-\frac{p^k-1}{p^k(p-1)^2}\right)\\
&\quad+\frac{i^{k+1}2^{2k+1}(k+1)q^{k+2}\phi(q)}{\pi^{k+3}} \prod_{p\| q}\left(1-\frac{p^k-1}{p^k(p-1)^2}\right)+O(q^{k+5/2+\epsilon})
\end{aligned}
$$

$$= \frac{i^{k+3}2^{2k-2}q^{k+2}\phi(q)}{\pi^{k+3}}[\pi^2(k^2-k+2)-8(k+1)]\prod_{p\|q}\left(1-\frac{p^k-1}{p^k(p-1)^2}\right)+O(q^{k+5/2+\epsilon}).$$

So from the above we have

$$\sum_{c=1}^{q}{}' E(q,k,c)K\left(\overline{2}^{k+1}c,k,q\right)$$
$$=\frac{c_k q^{k+2}\phi(q)}{\pi^{k+3}}\prod_{p\|q}\left(1-\frac{p^k-1}{p^k(p-1)^2}\right)+O(q^{k+5/2+\epsilon}),\quad \text{for}\quad k\geq 1.$$

This completes the proof of the theorem.

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