

NEGATIVELY CURVED HOMOGENEOUS ALMOST KÄHLER EINSTEIN MANIFOLDS WITH NONPOSITIVE CURVATURE OPERATOR

WAKAKO OBATA

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Abstract

Given a homogeneous almost Kähler manifold (M, J, g) with nonpositive curvature operator, we prove that if g is an Einstein metric having negative sectional curvature, then the almost complex structure J must be integrable. Furthermore, such (M, J, g) eventually has constant negative holomorphic sectional curvature and hence is holomorphically isometric to a complex hyperbolic space.

1. Introduction

Let (M, g) be a Riemannian manifold, and let $\bigwedge^2 T_p M$ denote the exterior algebra over the tangent space $T_p M$ of M at $p \in M$, equipped with the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle X \wedge Y, Z \wedge W \rangle = g(X, Z)g(Y, W) - g(X, W)g(Y, Z), \quad X, Y, Z, W \in T_p M.$$

The curvature tensor R of M gives rise to the curvature operator $\hat{R}: \bigwedge^2 T_p M \rightarrow \bigwedge^2 T_p M$ defined by

$$(1) \quad \langle \hat{R}(X \wedge Y), Z \wedge W \rangle = g(R(X, Y)W, Z)$$

for any $X, Y, Z, W \in T_p M$. It is immediate to see that the curvature operator \hat{R} is self-adjoint with respect to $\langle \cdot, \cdot \rangle$, so that the eigenvalues of \hat{R} are all real. We say that M has nonpositive curvature operator if all eigenvalues of \hat{R} are nonpositive.

In 1991, T. Wolter conjectured that a simply connected homogeneous Einstein manifold M with nonpositive curvature operator is symmetric ([6]). We are concerned with this conjecture when M admits an almost Kähler structure.

More precisely, an almost complex manifold (M, J) equipped with an almost Hermitian metric g with the closed fundamental 2-form $\Phi(X, Y) = g(X, JY)$ is called an almost Kähler manifold, and it is called a Kähler manifold if J is integrable, that is, the Nijenhuis tensor N of J defined by $N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] -$

$[X, Y]$ vanishes identically. We say that an almost Kähler manifold $M = (M, J, g)$ is homogeneous if the group of almost complex isometries of M acts transitively on M .

The aim of this paper is to study the geometry of homogeneous almost Kähler manifolds with nonpositive curvature operator. We note that there exist many examples of Kähler symmetric spaces with nonpositive curvature operator.

In connection with the Goldberg conjecture [1], it is plausible that a simply connected homogeneous almost Kähler Einstein manifold with nonpositive curvature operator is a Kählerian symmetric space. In this paper, assuming the negativity of the sectional curvature, we prove the following

Theorem. *Let (M, J, g) be a homogeneous almost Kähler Einstein manifold with nonpositive curvature operator. If the sectional curvature of M is negative, then J is integrable and (M, J, g) is holomorphically isometric to a complex hyperbolic space $(\mathbb{C}H^n, J_0, g_0)$.*

2. Preliminaries

Let $M = (M, J, g)$ be a homogeneous almost Kähler manifold with nonpositive curvature operator $\hat{R} \leq 0$. Then it is immediate from (1) that the nonpositivity of \hat{R} implies that (M, g) has nonpositive sectional curvature $K \leq 0$ everywhere. Hence, by a result of Heintze [3], we may identify M with a solvable Lie group G with a left invariant almost complex structure J and a left invariant metric $\langle \cdot, \cdot \rangle$. Note that, since M is almost Kähler, the left invariant metric $\langle \cdot, \cdot \rangle$ is a Kähler metric on G with the closed fundamental 2-form $\Phi(X, Y) = \langle X, JY \rangle$.

Assume now that (M, g) has negative sectional curvature $K < 0$ everywhere. Then M is known to be simply connected (see [4]), so that M is identified with a simply connected solvable Lie group G .

Let \mathfrak{g} be the Lie algebra of G consisting of left invariant vector fields on G . The left invariant almost complex structure J and the left invariant metric $\langle \cdot, \cdot \rangle$ on G induce an endomorphism J and an inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} satisfying the following conditions:

- (i) $J^2 = -\text{Id}$,
- (ii) $\langle JX, Y \rangle = -\langle X, JY \rangle$,
- (iii) $\langle [X, Y], JZ \rangle + \langle [Y, Z], JX \rangle + \langle [Z, X], JY \rangle = 0$

for any $X, Y, Z \in \mathfrak{g}$. Moreover, the Levi-Civita connection ∇ is given by

$$(2) \quad \begin{aligned} \nabla_X Y &= \frac{1}{2}[X, Y] + U(X, Y), \\ U(X, Y) &= -\frac{1}{2}((\text{ad } X)^* Y + (\text{ad } Y)^* X) \end{aligned}$$

for all $X, Y \in \mathfrak{g}$, where ad is the adjoint representation of \mathfrak{g} and $*$ denotes transpose with respect to $\langle \cdot, \cdot \rangle$. As a result, the curvature tensor $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z$

is determined by the bracket product, so that we have

$$(3) \quad \begin{aligned} \langle R(X, Y)Y, X \rangle &= \|U(X, Y)\|^2 - \langle U(X, X), U(Y, Y) \rangle - \frac{3}{4}\|[X, Y]\|^2 \\ &\quad - \frac{1}{2}\langle [X, [X, Y]], Y \rangle - \frac{1}{2}\langle [Y, [Y, X]], X \rangle. \end{aligned}$$

Since $(G, \langle \cdot, \cdot \rangle)$ has negative sectional curvature $K < 0$, the derived algebra $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ of \mathfrak{g} gives rise to a subspace of codimension 1 in \mathfrak{g} . Moreover, there is a unit vector A in \mathfrak{g} orthogonal to \mathfrak{n} such that if we denote by D and S the symmetric and the skew-symmetric part of the restriction $\text{ad } A|_{\mathfrak{n}}: \mathfrak{n} \rightarrow \mathfrak{n}$, then D and $D^2 + [D, S]$ are both positive definite (see Heintze [3]). Also, it follows from (2) that

$$(4) \quad \nabla_A A = 0, \quad \nabla_A X = SX, \quad \nabla_X A = -DX$$

for any $X \in \mathfrak{n}$.

Suppose further that $(G, \langle \cdot, \cdot \rangle)$ is an Einstein manifold, that is, the Ricci tensor Ric of G satisfies $\text{Ric}(x, y) = c\langle x, y \rangle$ for some constant c . Then it is proved by Heber [2] that D and S are derivations of \mathfrak{n} , and commute with each other ($DS = SD$). Moreover, by a straightforward computation we see that

$$(5) \quad R(A, X)Y = -\nabla_{DX} Y$$

for any $X, Y \in \mathfrak{n}$.

3. Proof of Theorem

Let (M, J, g) be a homogeneous almost Kähler manifold with nonpositive curvature operator $\hat{R} \leq 0$. Since we assume that it has negative sectional curvature $K < 0$ everywhere, we may identify (M, J, g) with $(G, J, \langle \cdot, \cdot \rangle)$, where G is a simply connected solvable Lie group, J is a left invariant almost complex structure and $\langle \cdot, \cdot \rangle$ is a left invariant Kähler metric on G .

Let \mathfrak{g} be the Lie algebra of G . Note that since \mathfrak{g} is solvable, \mathfrak{n} is nilpotent, so that the center \mathfrak{z} of \mathfrak{n} is nontrivial. Recall that \mathfrak{g} admits an inner product $\langle \cdot, \cdot \rangle$ and an endomorphism J on \mathfrak{g} satisfying Conditions (i), (ii) and (iii) in Section 2. Also, \mathfrak{g} is decomposed into the direct sum $\mathfrak{g} = \mathbb{R}\{A\} \oplus \mathfrak{n}$, where A is a unit vector orthogonal to the derived algebra $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$.

Let \mathfrak{b} be an orthogonal complement of \mathfrak{z} in \mathfrak{n} . It is proved in Heintze [3] that if $(G, J, \langle \cdot, \cdot \rangle)$ is a Kähler manifold with negative curvature, then $(\mathfrak{g}, J, \langle \cdot, \cdot \rangle)$ is isomorphic to the Lie algebra of a solvable Lie group of holomorphic isometries which acts simply transitively on the complex hyperbolic space $(\mathbb{C}H^n, J_0, g_0)$. In particular, $(\mathfrak{g}, J, \langle \cdot, \cdot \rangle)$

satisfies the following condition:

$$\begin{aligned}
 (6) \quad & \mathfrak{g} = \mathbb{R}\{A\} \oplus \mathfrak{b} \oplus \mathfrak{z}, & \mathfrak{z} &= \mathbb{R}\{JA\} \\
 & [A, X] = \frac{1}{2}\lambda X + SX, & [A, JA] &= \lambda JA, \\
 & [X, Y] = \lambda\langle JX, Y \rangle JA, & [X, JA] &= 0
 \end{aligned}$$

for any $X, Y \in \mathfrak{b}$ and some $\lambda \in \mathbb{R}$ (for details, see [3]).

On the other hand, in the case when $(G, J, \langle \cdot, \cdot \rangle)$ is almost Kähler, we may prove the following

Proposition 1. *Let $(G, J, \langle \cdot, \cdot \rangle)$ be a homogeneous almost Kähler manifold with nonpositive curvature operator. Suppose that $\langle \cdot, \cdot \rangle$ is an Einstein metric with negative curvature. Then $(\mathfrak{g}, J, \langle \cdot, \cdot \rangle)$ satisfies Condition (6).*

Proof. From Conditions (i) through (iii) together with the fact that D is positive definite, it follows that $\mathfrak{z} = \mathbb{R}\{JA\}$ (for details, see [5]). Since D and S are both derivations of \mathfrak{n} , we see that \mathfrak{z} and \mathfrak{b} are invariant by D and S , respectively. Hence there exists $\lambda > 0$ such that $\text{ad } A(JA) = \lambda JA$.

Let $\mu_1 < \dots < \mu_s$ be the eigenvalues of $D|_{\mathfrak{b}}$ and \mathfrak{b}_α the eigenspace associated with μ_α , for each $\alpha = 1, \dots, s$. By virtue of Condition (iii) with $X_\alpha \in \mathfrak{b}_\alpha$, JX_α and A , we then obtain

$$(7) \quad \langle JA, [X_\alpha, JX_\alpha] \rangle = \langle JX_\alpha, DJX_\alpha \rangle + \langle X_\alpha, DX_\alpha \rangle > 0.$$

Hence, for any $\alpha \in \{1, \dots, s\}$, there exists $\alpha^* \in \{1, \dots, s\}$ such that $\mu_\alpha + \mu_{\alpha^*} = \lambda$. Indeed, it holds that $\mu_\alpha + \mu_{s+1-\alpha} = \lambda$, since we assume $\mu_1 < \dots < \mu_s$.

Let $X \in \mathfrak{b}_s$ be a unit vector, and define a quadratic function f on \mathbb{R} by

$$f(x) = \langle \hat{R}(xA \wedge X + JX \wedge JA), xA \wedge X + JX \wedge JA \rangle, \quad x \in \mathbb{R}.$$

Using (5), we see that f is given by

$$\begin{aligned}
 f(x) &= x^2 \langle \hat{R}(A \wedge X), A \wedge X \rangle + 2x \langle \hat{R}(A \wedge X), JX \wedge JA \rangle + \langle \hat{R}(JX \wedge JA), JX \wedge JA \rangle \\
 &= x^2 \langle R(A, X)X, A \rangle + 2x \langle R(A, X)JA, JX \rangle + \langle R(JX, JA)JA, JX \rangle \\
 &= -x^2 \langle \nabla_{DX} X, A \rangle - 2x \langle \nabla_{DX} JA, JX \rangle \\
 &\quad + \langle \nabla_{JX} \nabla_{JA} JA - \nabla_{JA} \nabla_{JX} JA - \nabla_{[JX, JA]} JA, JX \rangle \\
 &= -x^2 \mu_s \langle \nabla_X X, A \rangle - 2x \mu_s \langle \nabla_X JA, JX \rangle - \langle \nabla_{JA} JA, \nabla_{JX} JX \rangle + |\nabla_{JX} JA|^2 \\
 &= -x^2 \mu_s^2 + x \mu_s \langle JA, [X, JX] \rangle - \lambda \langle JX, DJX \rangle + |\nabla_{JA} JX|^2,
 \end{aligned}$$

where $|\cdot|$ denotes the norm defined by $\langle \cdot, \cdot \rangle$.

Then, using Condition (iii), for the discriminant \mathcal{D} of f , we obtain

$$\begin{aligned} \mathcal{D} &= \mu_s^2 \langle JA, [X, JX] \rangle^2 - 4(-\mu_s^2)(-\lambda \langle JX, DJX \rangle + |\nabla_{JA} JX|^2) \\ &\geq \mu_s^2 \langle JA, [X, JX] \rangle^2 - 4\lambda \langle JX, DJX \rangle + 4 \langle \nabla_{JA} JX, X \rangle^2 \\ &= \mu_s^2 (2 \langle JA, [X, JX] \rangle^2 - 4\lambda \langle JX, DJX \rangle) \\ &= 2\mu_s^2 \{ (\langle JX, DJX \rangle + \mu_s)^2 - 2\lambda \langle JX, DJX \rangle \} \\ &= 2\mu_s^2 \{ (\langle JX, DJX \rangle - (\lambda - \mu_s))^2 + \lambda(2\mu_s - \lambda) \}. \end{aligned}$$

Since the curvature operator \hat{R} is assumed nonpositive, $f(x)$ must be nonpositive for all $x \in \mathbb{R}$, and hence $\mathcal{D} \leq 0$. Note that, if $s \geq 2$, then $2\mu_s - \lambda > 0$, which implies that $\mathcal{D} > 0$. Therefore, $s = 1$, and $D|_{\mathfrak{b}}$ has only one eigenvalue $1/2\lambda$. Note that $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{z}$, since D is derivation.

Let $\{E_1, JE_1, \dots, E_m, JE_m\}$ be an orthonormal basis of \mathfrak{b} with respect to $\langle \cdot, \cdot \rangle$. Using Condition (iii) for E_i, JE_i and A , we obtain $[E_i, JE_i] = \lambda JA$ for each $i = 1, \dots, m$. Also, it follows that

$$\begin{aligned} \langle R(X, JA)JA, X \rangle &= \langle \nabla_X \nabla_{JA} JA - \nabla_{JA} \nabla_X JA - \nabla_{[X, JA]} JA, X \rangle \\ &= \lambda \langle \nabla_X X, X \rangle + \langle \nabla_{JA} X, \nabla_X JA \rangle = -\frac{1}{2}\lambda^2 |X|^2 + |\nabla_X JA|^2 \\ &= -\frac{1}{2}\lambda^2 |X|^2 + \left| \sum_{j=1}^m (\langle \nabla_X JA, E_j \rangle E_j + \langle \nabla_X JA, JE_j \rangle JE_j) \right|^2 \\ &= -\frac{1}{2}\lambda^2 |X|^2 + \left| \sum_{j=1}^m \left(-\frac{1}{2} \langle JA, [X, E_j] \rangle E_j - \frac{1}{2} \langle JA, [X, JE_j] \rangle JE_j \right) \right|^2 \\ &= -\frac{1}{2}\lambda^2 |X|^2 + \frac{1}{4} \sum_{j=1}^m (|[X, E_j]|^2 + |[X, JE_j]|^2) \end{aligned}$$

for any $X \in \mathfrak{b}$. Then the Ricci curvature $\text{Ric}(JA, JA)$ in the direction JA is given by

$$\begin{aligned} \text{Ric}(JA, JA) &= \langle R(A, JA)JA, A \rangle + \sum_{i=1}^m (\langle R(E_i, JA)JA, E_i \rangle + \langle R(JE_i, JA)JA, JE_i \rangle) \\ &= -\lambda^2 + \sum_{i=1}^m \left(-\frac{1}{2}\lambda^2 + \frac{1}{4} \sum_{j=1}^m (|[E_i, E_j]|^2 + |[E_i, JE_j]|^2) \right. \\ &\quad \left. - \frac{1}{2}\lambda^2 + \frac{1}{4} \sum_{j=1}^m (|[JE_i, E_j]|^2 + |[JE_i, JE_j]|^2) \right) \end{aligned}$$

$$\begin{aligned}
 &= -(m+1)\lambda^2 + \frac{1}{4} \sum_{i,j} (|[E_i, E_j]|^2 + |[E_i, JE_j]|^2 + |[JE_i, E_j]|^2 + |[JE_i, JE_j]|^2) \\
 &= -\frac{m+2}{2}\lambda^2 + \frac{1}{4} \sum_{i \neq j} (|[E_i, E_j]|^2 + |[E_i, JE_j]|^2 + |[JE_i, E_j]|^2 + |[JE_i, JE_j]|^2).
 \end{aligned}$$

On the other hand, we also see that the Ricci curvature $\text{Ric}(A, A)$ in the direction A is given by

$$\begin{aligned}
 \text{Ric}(A, A) &= \sum_{i=1}^m \langle R(A, E_i)E_i, A \rangle + \sum_{i=1}^m \langle R(A, JE_i)JE_i, A \rangle + \langle R(A, JA)JA, A \rangle \\
 &= \sum_{i=1}^m \left(-\frac{\lambda}{2} \langle \nabla_{E_i} E_i, A \rangle - \frac{\lambda}{2} \langle \nabla_{JE_i} JE_i, A \rangle \right) - \lambda \langle \nabla_{JA} JA, A \rangle \\
 &= \sum_{i=1}^m \left(-\frac{1}{4}\lambda^2 - \frac{1}{4}\lambda^2 \right) - \lambda^2 = -\frac{m+2}{2}\lambda^2.
 \end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
 &\text{Ric}(JA, JA) - \text{Ric}(A, A) \\
 (8) \quad &= \frac{1}{4} \sum_{i \neq j} (|[E_i, E_j]|^2 + |[E_i, JE_j]|^2 + |[JE_i, E_j]|^2 + |[JE_i, JE_j]|^2).
 \end{aligned}$$

Since G is Einstein, we have $\text{Ric}(A, A) = \text{Ric}(JA, JA)$. Hence it follows from the above equations that

$$\frac{1}{4} \sum_{i \neq j} (|[E_i, E_j]|^2 + |[E_i, JE_j]|^2 + |[JE_i, E_j]|^2 + |[JE_i, JE_j]|^2) = 0,$$

which implies that $[E_i, E_j] = [JE_i, JE_j] = [E_i, JE_j] = 0$ for $i \neq j$. Finally, we remark that $[X, Y] = \lambda \langle JX, Y \rangle JA$ for any $X, Y \in \mathfrak{b}$. Indeed, it follows from the above observations that

$$\begin{aligned}
 [X, Y] &= \left[\sum_i (\langle X, E_i \rangle E_i + \langle X, JE_i \rangle JE_i), \sum_j (\langle Y, E_j \rangle E_j + \langle Y, JE_j \rangle JE_j) \right] \\
 &= \sum_i (\langle X, E_i \rangle \langle Y, JE_i \rangle [E_i, JE_i] + \langle X, JE_i \rangle \langle Y, E_i \rangle [JE_i, E_i]) \\
 &= \lambda \sum_i (\langle X, E_i \rangle \langle Y, JE_i \rangle - \langle X, JE_i \rangle \langle Y, E_i \rangle) JA \\
 &= \lambda \langle JX, Y \rangle JA.
 \end{aligned}$$

Consequently, $(\mathfrak{g}, \langle \cdot, \cdot \rangle, J)$ satisfies Condition (6). □

Lemma 2. *Let $(G, J, \langle \cdot, \cdot \rangle)$ be a simply connected homogeneous almost Kähler manifold. If $(\mathfrak{g}, J, \langle \cdot, \cdot \rangle)$ satisfies Condition (6), then J is integrable. Moreover, $(G, J, \langle \cdot, \cdot \rangle)$ is holomorphically isometric to a complex hyperbolic space $(\mathbb{C}H^n, J_0, g_0)$.*

Proof. If $(\mathfrak{g}, J, \langle \cdot, \cdot \rangle)$ satisfies Condition (6), it can be verified by a straightforward computation that the Nijenhuis tensor N of J vanishes identically (see [5, Lemma 4]). Hence J is integrable. Moreover, the sectional curvature of $(G, \langle \cdot, \cdot \rangle)$ is given by

$$\begin{aligned} \langle R(Y, X)X, Y \rangle &= -\frac{1}{4}\lambda^2 (\langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2) - \frac{3}{4}\lambda^2 \langle JX, Y \rangle^2 \\ &= -\frac{1}{4}\lambda^2 - \frac{3}{4}\lambda^2 \langle JX, Y \rangle^2, \end{aligned}$$

where $X, Y \in \mathfrak{g}$ are orthonormal vectors. In particular, substituting JX for Y , we obtain

$$\langle R(JX, X)X, JX \rangle = -\frac{1}{4}\lambda^2 - \frac{3}{4}\lambda^2 \langle JX, JX \rangle^2 = -\lambda^2,$$

which shows that the holomorphic sectional curvature is constant curvature $-\lambda^2$. Hence $(G, J, \langle \cdot, \cdot \rangle)$ must be holomorphically isometric to a complex hyperbolic space $(\mathbb{C}H^n, J_0, g_0)$ with constant holomorphic sectional curvature $-\lambda^2$. □

It follows from Proposition 1 together with Lemma 2 that $(G, J, \langle \cdot, \cdot \rangle)$ is holomorphically isometric to a complex hyperbolic space. This completes the proof of Theorem.

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Mathematical Institute
Tohoku University
Sendai 980–8578
Japan