González-Acuña, F. and Ramírez, A.
Osaka J. Math.
44 (2007), 471–482

JØRGENSEN SUBGROUPS OF THE PICARD GROUP

FRANCISCO GONZÁLEZ-ACUÑA and ARTURO RAMÍREZ

(Received December 19, 2005, revised July 27, 2006)

Abstract

Let G be a subgroup of rank two of the Möbius group $PSL(2,\mathbb{C})$. The Jørgensen number $J(G)$ of G is defined by

$$J(G) = \inf\{|\mathrm{tr}^2 A - 4| + |\mathrm{tr}[A, B] - 2| \colon \langle A, B\rangle = G\}.$$

We describ e all subgroups G of the Picard group $PSL(2, \mathbb{Z} + i\mathbb{Z})$ with $J(G) = 1$.

1. Introduction

Let G be a subgroup of rank two of the Möbius group Möb $= PSL(2,\mathbb{C})$. The Jørgensen number $J(G)$ of G is dened by

$$J(G) = \inf\{|\mathrm{tr}^2 A - 4| + |\mathrm{tr}[A, B] - 2| \colon \langle A, B\rangle = G\}.$$

A subgroup G of Möb is elementary if the cardinality of its limit set $\Lambda(G)$ is at most 2 see [8, p.266]. If $G = \langle A, B\rangle$ is a discrete group with A parabolic, then G is elementary iff $\mathrm{tr}[A, B] = 2$ (that is, iff $J(A, B) = 0$).

Jørgensen has proved that if G is a discrete nonelementary rank two subgroup of Möb then $J(G) > 1$.

It has been conjectured [10, p.273] that if G is nonelementary rank two subgroup of Möb which does not contain elliptic elements of infinite order and $J(G) = 1$ then G is discrete.

Groups G with $J(G) = 1$ have been studied in the literature ([3], [4], [13], [10], [12]). Following [10] we call a discrete nonelementary rank two subgroup G of Möb with $J(G) = 1$ a *Jørgensen group*.

An important subgroup of Möb is the Picard group Pic $= PSL(2, \mathbb{Z} + i\mathbb{Z})$. We are interested in the Jørgensen numbers of rank two subgroups of Pic.

Our motivation for the present paper is the article [12] by H. Sato in which he considers the Whitehead link group $\mathcal{W} = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1-i & 1 \end{pmatrix} \right\rangle \subset$ Pic (see [5], [9], [15]) and proves that $J(W) = 2$. Here we will give a brief proof of this result.

2000 Mathematics Subject Classification. Primary 30F40; Secondary 20E06.

We now describe a family of rank two subgroups of Pic. Let

$$\mathrm{Mod} = \mathrm{Mod}^1 = PSL(2,\mathbb{Z}) \simeq \mathbb{Z}_2 * \mathbb{Z}_3,$$
$$\mathrm{Mod}^i = \left\{ \begin{pmatrix} a & -ib \\ ic & d \end{pmatrix} \in PSL(2,\mathbb{C})\colon a,b,c,d \in \mathbb{Z} \right\} \simeq \mathbb{Z}_2 * \mathbb{Z}_3,$$
$$\mathcal{G}_k^{\alpha,\beta} = \left\langle \mathrm{Mod}^\alpha, \begin{pmatrix} \alpha\beta & -k\alpha^2\beta i \\ 0 & (\alpha\beta)^{-1} \end{pmatrix} \right\rangle \quad \text{where} \quad \alpha,\beta \in \{1,i\} \quad \text{and} \quad k \in \mathbb{Z}.$$

For example $\mathcal{G}_0^{1,1} = \mathrm{Mod}$, $\mathcal{G}_0^{i,i} = \mathrm{Mod}^i$ and one can show that $\mathcal{G}_1^{1,1} = \mathcal{G}_1^{i,i} = \mathrm{Pic}$.

The group Pic is generated by Mod and Mod^i, and these two subgroups are conjugate in Möb by a 90° rotation $R = \begin{pmatrix} (1+i)/\sqrt{2} & 0 \\ 0 & (1-i)/\sqrt{2} \end{pmatrix}$.

Denoting by $\mathbb{D}_\infty$ the infinite dihedral group, we will see that (Theorem 11) for $k \geq 2$ we have $\mathcal{G}_k^{\alpha,\beta} \simeq \begin{cases} \mathrm{Mod} *_{\mathbb{Z}} \mathbb{Z}^2 & \text{if } \alpha\beta = \pm 1 \\ \mathrm{Mod} *_{\mathbb{Z}} \mathbb{D}_\infty & \text{if } \alpha\beta = i \end{cases}$, where the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of Mod is amalgamated to a primitive element of infinite order of $\mathbb{Z}^2$ or $\mathbb{D}_\infty$.

The symbol $\overset{\mathrm{Pic}}{\sim}$ denotes conjugation in Pic, the symbol $\sim$ denotes conjugation in Möb.

Our main result is the following:

Theorem 1. *Let $\mathcal{I} = \{(\alpha,\beta,k)\colon \alpha,\beta \in \{1,i\},\ k$ a non negative integer$\}$. Let G be a rank two subgroup of* Pic, *with $J(G) = 1$. Then*:

1) *G is conjugate, in* Pic, *to $\mathcal{G}_k^{\alpha,\beta}$ for some $(\alpha,\beta,k) \in \mathcal{I}$.*
2) *G is isomorphic to exactly one of the groups $\mathcal{G}_0^{1,1}$, $\mathcal{G}_1^{1,1}$, $\mathcal{G}_2^{1,1}$, $\mathcal{G}_1^{1,i}$, $\mathcal{G}_1^{1,i}$ and $\mathcal{G}_2^{1,i}$.*
3) *If $\mathcal{G}_k^{\alpha,\beta} \overset{\mathrm{Pic}}{\sim} \mathcal{G}_{k'}^{\alpha',\beta'}$ where (α,β,k) and (α',β',k') are different elements of $\mathcal{I}$ then $k = k' = 1$, $\alpha = \beta$ and $\alpha' = \beta'$.*
4) *$\mathcal{G}_k^{\alpha,\beta} \sim \mathcal{G}_{k'}^{\alpha',\beta'}$ (with $(\alpha,\beta,k),(\alpha',\beta',k') \in \mathcal{I}$) iff $k = k'$ and $\alpha\beta = \pm\alpha'\beta'$.*

Notice that no Jørgensen subgroup G of Pic is the group of a link in S^3, because $G \supset \mathbb{Z}_2 * \mathbb{Z}_3$.

In Section 1 we give another proof of Sato's theorem.

In Section 2 we give a different description of $\mathcal{G}_k^{\alpha,\beta}$ which shows its rank is two. With this description we extend our family to a family of rank two subgroups $\mathcal{G}_k^{\alpha,\beta}$ with α,β and $k \in \mathbb{C} - \{0\}$ and compare it with a family dened by Sato ([10], [12]). At the end of the section we prove Theorem 1 1).

In Section 3 we prove Theorem 1 4).

In Section 4, using the structure of Pic as an amalgamated product, we prove Theorem 1 3) and 2).

In Section 5 we exhibit a table that gives algebraic information of the groups $\mathcal{G}_k^{\alpha,\beta}$, as their abelianizations, their images under the abelianization map of Pic, and the number of conjugacy classes of elements of order two. These facts are proved in Sections 3 and 4 and are used in the proof of Theorem 1 3) 2).

Some of the results of this paper were presented at the Tôhoku Seminar in Akita, in February 2005, while the first author was visiting Osaka City University. He would like to thank this institution for its kind hospitality.

We would like to thank the referee. His suggestions improved the presentation of the paper.

2. Section 1

Proposition 2. *If G is a rank two subgroup of* Pic *then* $J(G) \in \{0,1,2\} \cup [3,\infty)$.

Proof. Let A and $B \in GL(2,\mathbb{C})$ and $C = AB$. Then

$$\operatorname{tr}[A, B] - 2 = \operatorname{tr}^2 A + \operatorname{tr}^2 B + \operatorname{tr}^2 C - \operatorname{tr} A \operatorname{tr} B \operatorname{tr} C - 4$$

in particular if $\operatorname{tr} A = 2$ one has then $\operatorname{tr}[A, B] - 2 = (\operatorname{tr} C \operatorname{tr} B)^2$ and therefore $J(A, B) = |\operatorname{tr} C - \operatorname{tr} B|^2$. Hence if A is parabolic and $A, B \in$ Pic, we have that $J(A, B)$ is the modulus of the square of an element of $\mathbb{Z} + i\mathbb{Z}$, that is, an integer that is the sum of two squares. If $A \in$ Pic and A is not parabolic then $J(A,B) \geq |\operatorname{tr} A - 2|\,|\operatorname{tr} A + 2| \geq 3$. □

Proposition 3. *Let $\tilde{\phi}\colon PSL(2,\mathbb{Z}+i\mathbb{Z}) \to PSL(2,\mathbb{Z}_2)$ be the homomorphism induced by the ring homomorphism $\phi\colon \mathbb{Z} + i\mathbb{Z} \to \mathbb{Z}_2$. If G is a nonelementary e rank two subgroup of* Pic *and $|\tilde{\phi}(G)| < 6$ then $J(G) \geq 2$.*

Proof. Suppose $\mathcal{G} = \langle A, B\rangle$. Notice that $\tilde{\phi}(G)$ is Abelian since it is a proper subgroup of $PSL(2,\mathbb{Z}_2) \approx S_3$. Hence $\tilde{\phi}(\operatorname{tr}[A, B]) = \operatorname{tr}[\tilde{\phi}(A), \tilde{\phi}(B)] = \operatorname{tr} I = 2 = 0$ and so $\operatorname{tr}[A, B] \in \ker\phi = \langle 1 + i\rangle$. Therefore $|\operatorname{tr}[A, B] - 2| \neq 1$ and also $|\operatorname{tr}[A, B] - 2| \neq 0$ since $\mathcal{G}$ is nonelementary. Hence $J(G) \geq |\operatorname{tr}[A, B] - 2| > 1$ and, by Proposition 2, $J(G) \geq 2$. □

Corollary 4 (H. Sato). *If $\mathcal{W}$ is the Whitehead link group then $J(\mathcal{W}) = 2$.*

Proof. If $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1-i & 1 \end{pmatrix}$, then $\mathcal{W} = \langle A, B\rangle$ and $J(A, B) = 2$. Since $\tilde{\phi}(B) = I$ then $|\tilde{\phi}(\mathcal{W})| = 2$ and therefore, by Proposition 3, $J(\mathcal{W}) = 2$. □

3. Section 2

Proposition 5. *Let $\alpha, \beta \in \{1, i\}$ and $k \in \mathbb{Z}$. Then*:

i) $\mathcal{G}_k^{\alpha,\beta} = \left\langle \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & k\alpha\beta i \end{pmatrix} \right\rangle$.

ii) $\mathcal{G}_k^{\alpha,\alpha} = \mathcal{G}_{-k}^{\alpha,\alpha}$ *and* $\mathcal{G}_k^{\alpha,\beta} \overset{\text{Pic}}{\sim} \mathcal{G}_{-k}^{\alpha,\beta}$.

iii) *The rank of $\mathcal{G}_k^{\alpha,\beta}$ is two and $J\left(\mathcal{G}_k^{\alpha,\beta}\right) = 1$.*

Proof. Write $A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & k\alpha\beta i \end{pmatrix}$.

i) Since $BAB^{-1} = \begin{pmatrix} 1 & 0 \\ -\beta^2\alpha & 1 \end{pmatrix}$ and this matrix together with A generates Mod^α it follows that $\langle A, B\rangle = \langle \mathrm{Mod}^\alpha, B\rangle$. As

$$\begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & -\beta^{-1} \\ \beta & k\alpha\beta i \end{pmatrix} = \begin{pmatrix} \alpha\beta & ik\alpha^2\beta \\ 0 & (\alpha\beta)^{-1} \end{pmatrix}$$

and $\begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix} \in \mathrm{Mod}^\alpha$, i) follows.

ii) If $\alpha = \beta$, then $\begin{pmatrix} \alpha\beta & -k\alpha^2\beta i \\ 0 & (\alpha\beta)^{-1} \end{pmatrix}$ is the inverse of $\begin{pmatrix} \alpha\beta & k\alpha^2\beta i \\ 0 & (\alpha\beta)^{-1} \end{pmatrix}$ and so $\mathcal{G}_k^{\alpha,\alpha} = \mathcal{G}_{-k}^{\alpha,\alpha}$. Else if $\alpha \neq \beta$ conjugating Mod^α and B^{-1} with $\begin{pmatrix} 0 & -\beta^{-1} \\ \beta & 0 \end{pmatrix}$ we obtain Mod^α and $\begin{pmatrix} 0 & -\beta^{-1} \\ \beta & -k\alpha\beta i \end{pmatrix}$; hence $\mathcal{G}_k^{\alpha,\beta} \overset{\mathrm{Pic}}{\sim} \mathcal{G}_{-k}^{\alpha,\beta}$.

iii) As $\mathcal{G}_k^{\alpha,\beta}$ is a discrete group $J(\mathcal{G}_k^{\alpha,\beta}) \geq 1$. Since A is parabolic we have $J(A,B) = |\mathrm{tr}\,AB - \mathrm{tr}\,B|^2 = |\alpha\beta|^2 = 1$ (see the proof of Proposition 2). Hence $J(\mathcal{G}_k^{\alpha,\beta}) = 1$. □

We now compare our groups $\mathcal{G}_k^{\alpha,\beta}$ with groups considered by Sato. Suppose a pair of elements of Möb generates a nonelementary subgroup and the first element is parabolic. Then his pair is conjugate to a pair $(A, B_{\sigma,\mu})$ where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} \mu\sigma & (\mu^2\sigma^2 - 1)/\sigma \\ \sigma & \mu\sigma \end{pmatrix}$ and $\sigma \neq 0$ (see [10], [12]). Define $\mathcal{G}_{\sigma,\mu} = \langle A, B_{\sigma,\mu}\rangle$.

Notice that $\mathcal{G}_{\sigma,\mu} = \mathcal{G}_{-\sigma,\mu} = \mathcal{G}_{\sigma,-\mu} = \mathcal{G}_{\sigma,\mu+1}$ and $\mathcal{G}_{\sigma,\mu}$ is conjugate in Möb to $\mathcal{G}_{\sigma,\mu+1/2}$. This follows from $\langle A, B\rangle = \langle A, -B\rangle = \langle A, -B^{-1}\rangle = \langle A, ABA\rangle$ and $\langle A, B\rangle \sim \langle A, (A^{1/2})^{-1}BA^{1/2}\rangle$ where $B = B_{\sigma,\mu}$ and $A^{1/2} = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$.

For example the Whitehead link group $\mathcal{W}$ is

$$\begin{aligned}\mathcal{W} &= \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1-i & 1 \end{pmatrix} \right\rangle \\ &= \mathcal{G}_{1-i,(1+i)/2} \sim \mathcal{G}_{1-i,i/2} = \mathcal{G}_{1-i,-i/2}\end{aligned}$$

(cf. [12, Theorem 2]).

We now extend our definition of $\mathcal{G}_k^{\alpha,\beta}$. If $\alpha, \beta, k \in \mathbb{C} - \{0\}$ define

$$\mathcal{G}_k^{\alpha,\beta} = \left\langle \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & k\alpha\beta i \end{pmatrix} \right\rangle.$$

Because of the last proposition this definition coincides with the one given in the introduction if $\alpha, \beta \in \{1, i\}$ and $k \in \mathbb{Z}$. Conjugating with $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, where $\lambda^2 = \alpha$, we see that $\mathcal{G}_k^{\alpha,\beta} \sim \mathcal{G}_k^{1,\alpha\beta}$, and conjugating with $\begin{pmatrix} 1 & -ki/2 \\ 0 & 1 \end{pmatrix}$, we get $\mathcal{G}_k^{1,\sigma} \sim \mathcal{G}_{\sigma,ki/2}$.

We have following equalities $\mathcal{G}_k^{\alpha,\beta} = \mathcal{G}_k^{\alpha,-\beta} = \mathcal{G}_{-k}^{-\alpha,\beta} = \mathcal{G}_{k+1}^{\alpha,\beta}$ (the last equality follows from $\langle A, B\rangle = \langle A, BA\rangle$) and, conjugating with $\begin{pmatrix} 1 & -k\alpha i \\ 0 & 1 \end{pmatrix}$, we get $\mathcal{G}_k^{\alpha,\beta} \sim \mathcal{G}_{-k}^{\alpha,\beta}$.

We now describe which of the groups $\mathcal{G}_k^{\alpha,\beta}$ are subgroups of Pic. First, if $\mathcal{G}_k^{\alpha,\beta} \subset$ Pic we must have $\alpha, \beta, k \in \mathbb{Z} + i\mathbb{Z}$ and $|\beta| = 1$. Since $\mathcal{G}_k^{\alpha,\beta} = \mathcal{G}_k^{\alpha,-\beta} = \mathcal{G}_{k+1}^{\alpha,\beta}$ we may assume $k \in \mathbb{Z}$ and $\beta \in \{1, i\}$.

The following theorem describes all the Jørgensen subgroups of Pic, up to conjugation in Pic.

Theorem 6. *If G is a rank two subgroup of* Pic *with $J(G) = 1$ then G is conjugate in* Pic *to $\mathcal{G}_k^{\alpha,\beta}$ where $\alpha, \beta \in \{1, i\}$ and k is a nonnegative integer.*

Proof. Let A and B be generators of G such that

$$J(A, B) = |\mathrm{tr}^2 A - 4| + |\mathrm{tr}[A, B] - 2| = 1.$$

If $\mathrm{tr}\, A \neq \pm 2$ then $|\mathrm{tr}^2 A - 4| \geq 3$ hence $|\mathrm{tr}^2 A - 4| = 0$ and $|\mathrm{tr}[A, B] - 2| = 1$. A is then parabolic with fixed point a/c where a and c are relatively prime Gaussian integers. Let b and d be Gaussian integers such that $ad - bc = 1$. Conjugating A with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ Pic we obtain a parabolic element which fixes ∞. Hence we can assume that $A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, with α a nonzero Gaussian integer.

Write $B = \begin{pmatrix} x & y \\ \beta & z \end{pmatrix} \in$ Pic. Then, as in the proof of Proposition 2,

$$1 = |\mathrm{tr}[A, B] - 2| = |\mathrm{tr}(AB) - \mathrm{tr}\, B|^2 = |\alpha\beta|^2.$$

Hence $|\alpha| = |\beta| = 1$. Conjugating with $\begin{pmatrix} 1 & x\beta^{-1} \\ 0 & 1 \end{pmatrix}$ we see that the pair (A, B) is conjugate in Pic to the pair $\left(A, \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & k\alpha\beta i \end{pmatrix}\right)$ where $k\alpha\beta i = x + z$ and k is Gaussian integer. Than $\mathcal{G} \overset{\mathrm{Pic}}{\sim} \mathcal{G}_k^{\alpha,\beta}$ and since $\mathcal{G}_k^{\alpha,\beta} = \mathcal{G}_k^{\alpha,-\beta} = \mathcal{G}_{-k}^{-\alpha,\beta} = \mathcal{G}_{k+i}^{\alpha,\beta}$ and $\mathcal{G}_k^{\alpha,\beta} \overset{\mathrm{Pic}}{\sim} \mathcal{G}_{-k}^{\alpha,\beta}$ we may assume that $\alpha, \beta \in \{1, i\}$ and k is a nonnegative integer. □

4. Section 3

In this section and the next one we will use free products with amalgamation (see [6]).

Also we will use the 90° rotation $R \in$ Möb. Let $R = \begin{pmatrix} (1+i)/\sqrt{2} & 0 \\ 0 & (1-i)/\sqrt{2} \end{pmatrix} \in$ Möb (multiplication by i); this element does not belong to Pic. Then $R^{-1}\mathrm{Pic}R = \mathrm{Pic}$, $R^{-1}\mathrm{Mod}^{\alpha} R = \mathrm{Mod}^{\alpha'}$, $R^{-1}\mathcal{G}_k^{\alpha,\beta} R = \mathcal{G}_k^{\alpha',\beta'}$ where $\{\alpha, \alpha'\} = \{\beta, \beta'\} = \{1, i\}$; thus $\mathcal{G}_k^{\alpha,\beta} \sim \mathcal{G}_k^{\alpha',\beta'}$. This proves the if part of Theorem 1 4).

A presentation of Pic can be given as follows (see [1], [14]):

$$\mathrm{Pic} = \langle x, y, u, v : x^3 = y^3 = u^2 = v^2 = (uy)^2 = (yx)^2 = (xv)^2 = (vu)^2 = 1\rangle$$

where $u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $x = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & i \\ i & 1 \end{pmatrix}$.

From this one can show $\mathrm{Pic} = V *_{\mathrm{Mod}} Y$ where $V = \langle \mathrm{Mod}, v\rangle$ and $Y = \langle \mathrm{Mod}, y\rangle$. We have also $\mathrm{Mod} = \langle u, x\rangle$ and $\mathrm{Mod}^i = \langle v, y\rangle$ and of course $\mathrm{Pic} = \langle \mathrm{Mod}, \mathrm{Mod}^i\rangle$.

We have the following presentations:

$$\begin{aligned}
V &= \langle x, u, v : x^3 = u^2 = v^2 = (xv)^2 = (vu)^2 = 1\rangle \\
&= \langle u, v\rangle *_{\langle v\rangle} \langle v, x\rangle = \mathbb{Z}_2^2 *_{\mathbb{Z}_2} \mathbb{D}_3, \\
Y &= \langle x, y, u : x^3 = y^2 = u^2 = (uy)^2 = (yx)^2 = 1\rangle \\
&= \langle u, y\rangle *_{\langle y\rangle} \langle y, x\rangle = \mathbb{D}_3 *_{\mathbb{Z}_3} A_4, \\
\mathrm{Mod} &= \langle x, u : x^3 = u^2 = 1\rangle = \mathbb{Z}_2 * \mathbb{Z}_3
\end{aligned}$$

where $\mathbb{D}_3$ is the dihedral group of order six and A_4 is the alternating group in four elements.

We have that $\mathcal{G}_1^{i,i} = \mathrm{Pic}$ because

$$\begin{aligned}
\mathcal{G}_1^{i,i} &= \left\langle \mathrm{Mod}^i, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle \\
&= \left\langle \mathrm{Mod}^i, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle \\
&= \langle \mathrm{Mod}^i, \mathrm{Mod}\rangle = \mathrm{Pic}
\end{aligned}$$

since $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = v\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}v$. This implies $\mathcal{G}_1^{1,1} = R^{-1}\mathcal{G}_1^{i,i}R = \mathrm{Pic}$.

Notice also that $\mathcal{G}_0^{1,1} = \mathrm{Mod} = \langle x, u\rangle$, $\mathcal{G}_0^{i,i} = \mathrm{Mod}^i = \langle y, v\rangle$, $\mathcal{G}_0^{1,i} = V$ and $\mathcal{G}_1^{i,1} \overset{\mathrm{Pic}}{\sim} \mathcal{G}_{-1}^{i,1} = Y$. One can see that $\mathcal{G}_k^{1,1} = \langle u, x, (vy)^k\rangle$, $\mathcal{G}_k^{i,i} = \langle v, y, (xu)^k\rangle$ and, using Proposition 5, $\mathcal{G}_k^{i,1} = \langle v, y, u(xu)^k\rangle$ and $\mathcal{G}_k^{1,i} = \langle u, x, (vy)^k v\rangle$.

The abelianizations of Pic, V, Y and Mod are $\mathbb{Z}_2^2, \mathbb{Z}_2^2, \mathbb{Z}_2, \mathbb{Z}_6$ respectively.

Denote by $\overline{\mathrm{Pic}}$ the abelianization of Pic and by $\mathrm{ab} : \mathrm{Pic} \to \overline{\mathrm{Pic}}$ the abelianizaction map. We will write $\overline{w} = \mathrm{ab}(w)$. We have that $\overline{\mathrm{Pic}} = \mathrm{Pic}/\langle x, y\rangle = \langle \overline{u}, \overline{v}\rangle \simeq \mathbb{Z}_2^2$.

Proposition 7. *If $\alpha = \beta$ and k is odd or if $\alpha \neq \beta$ and k is even then* $\mathrm{ab}\big(\mathcal{G}_k^{\alpha,\beta}\big) = \overline{\mathrm{Pic}}$. *Otherwise*

$$\mathrm{ab}\big(\mathcal{G}_k^{\alpha,\beta}\big) = \begin{cases} \langle \overline{u}\rangle & \text{if } \alpha = 1 \\ \langle \overline{v}\rangle & \text{if } \alpha = i \end{cases}.$$

Proof. We have $\mathrm{ab}\big(\mathcal{G}_k^{1,1}\big) = \langle \overline{u}, \overline{v}^k \rangle$, $\mathrm{ab}\big(\mathcal{G}_k^{1,i}\big) = \langle \overline{u}, \overline{v}^{k+1} \rangle$, $\mathrm{ab}\big(\mathcal{G}_k^{i,1}\big) = \langle \overline{v}, \overline{u}^{k+1} \rangle$ and $\mathrm{ab}\big(\mathcal{G}_k^{i,i}\big) = \langle \overline{v}, \overline{u}^k \rangle$. From these equalities the proposition follows. □

Corollary 8. $\mathcal{G}_k^{1,1} \overset{\mathrm{Pic}}{\approx} \mathcal{G}_k^{i,i}$ *(resp.* $\mathcal{G}_k^{1,i} \overset{\mathrm{Pic}}{\approx} \mathcal{G}_k^{i,1}$*) if k is even (resp. k is odd).*

The following lemma will be used in the classification of the groups $\mathcal{G}_k^{\alpha,\beta}$ in Möb.

Lemma 9. i) *The trace of any element of $\mathcal{G}_k^{1,\beta}$ is of the form $a+kbi$ or $ka+bi$ where $a, b \in \mathbb{Z}$.*
ii) *The trace of any element of $\mathcal{G}_k^{1,1}$ is of the form $a+kbi$ where $a, b \in \mathbb{Z}$.*
iii) *$\pm(1+ki)$ is the trace of an element of $\mathcal{G}_k^{1,1}$ and $\pm(i+k)$ is the trace of an element of $\mathcal{G}_k^{1,i}$.*

Proof. The natural ring homomorphism from $\mathbb{Z}+i\mathbb{Z} \approx \mathbb{Z}[X]/(X^2+1)$ onto $\mathbb{Z}_k + i\mathbb{Z}_k \approx \mathbb{Z}_k[X]/(X^2+1)$ induces a group homomorphism $PSL(2,\mathbb{Z}+i\mathbb{Z}) \xrightarrow{\psi} PSL(2,\mathbb{Z}_k+i\mathbb{Z}_k)$. As $\mathcal{G}_k^{1,\beta} \supset \mathrm{Mod}$ we have, by Proposition 5,

$$\begin{aligned} \mathcal{G}_k^{1,\beta} &= \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & k\beta i \end{pmatrix} \right\rangle \\ &= \left\langle \mathrm{Mod}, \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & k\beta i \end{pmatrix} \right\rangle. \end{aligned}$$

Then $\psi\big(\mathcal{G}_k^{1,\beta}\big) = \left\langle \psi(\mathrm{Mod}), \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & 0 \end{pmatrix} \right\rangle$ which is contained in

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}_k + i\mathbb{Z}_k) \colon a,b,c,d \in \mathbb{Z}_k \text{ or } a,b,c,d \in i\mathbb{Z}_k \right\}$$

so the trace of any element of $\psi\big(\mathcal{G}_k^{1,\beta}\big)$ lies in $\mathbb{Z}_k \cup i\mathbb{Z}_k$. From this i) follows.

ii) is proved similarly.

To prove iii) observe that the trace of $\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\beta^{-1} \\ \beta & k\beta i \end{pmatrix}$ is $\beta^{-1}+k\beta i$. □

The following theorem gives the classification of the groups $\mathcal{G}_k^{\alpha,\beta}$, up to conjugation in Möb.

Theorem 10. *If $\mathcal{G}_k^{1,\beta} \sim \mathcal{G}_{k'}^{1,\beta'}$ where $\beta, \beta' \in \{1, i\}$, $k \geq 0$ and $k' \geq 0$ then $\beta = \beta'$ and $k = k'$.*

Proof. As $\mathcal{G}_1^{1,1}$ (= Pic) and $\mathcal{G}_1^{1,i}$ (= Y) have nonisomorphic abelianizations the case $k = k' = 1$ follows. If $(k, k') \neq (1, 1)$ and $(k, \beta) \neq (k', \beta')$ then, using the lemma, one sees that

$$\left\{\text{traces of elements of } \mathcal{G}_k^{1,\beta}\right\} \neq \left\{\text{traces of elements of } \mathcal{G}_{k'}^{1,\beta'}\right\}$$

and therefore $\mathcal{G}_k^{1,\beta} \overset{\text{Möb}}{\not\approx} \mathcal{G}_{k'}^{1,\beta'}$. □

This completes the proof of Theorem 1 4).

5. Section 4

In this section we will think of Pic as $V *_{\text{Mod}} Y$. Define an integer valued function on Pic as follows:

$$\lambda(w) = \begin{cases} 1 & \text{if } w \overset{\text{Pic}}{\sim} w' \in V \cup Y \\ 2n & \text{if } w \overset{\text{Pic}}{\sim} v_1 y_1 \cdots v_n y_n,\ n \geq 1,\ v_i \in V,\ y_i \in Y\ (i = 1, \ldots, n). \end{cases}$$

The function is well defined (see for example [7, Theorems 4.4 and 4.6] or [6, Chapter IV, Theorems 2.6 and 2.8]). Clearly if $w \overset{\text{Pic}}{\sim} w'$, $\lambda(w) = \lambda(w')$.

Recall that $\mathcal{G}_k^{1,\beta} = \left\langle \text{Mod}, \begin{pmatrix} \beta & -k\beta i \\ 0 & \beta^{-1} \end{pmatrix} \right\rangle$. Write and $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T = \langle t \rangle$ and

$$s = \begin{pmatrix} \beta & -k\beta i \\ 0 & \beta^{-1} \end{pmatrix} = \begin{cases} (vy)^k & \text{if } \beta = 1 \\ u(yv)^{k-1}y & \text{if } \beta = i \end{cases}.$$

Proposition 11. *Consider the groups $\mathcal{G}_k^{1,\beta}$ with $k \geq 2$. Then*:

i) *If $\beta = 1$ (resp. $\beta = i$) then $\langle s, t \rangle \simeq \mathbb{Z}^2$ (resp. $\langle s, t \rangle \simeq \mathbb{D}_\infty$).*

ii) *There is an isomorphism $\mathcal{G}_k^{1,\beta} \simeq \text{Mod} *_T \langle s, t \rangle$.*

iii) *$\lambda(s^{e_1} m_1 s^{e_2} m_2 \cdots s^{e_r} m_r) > 1$ where $e_j \neq 0$ (resp. $e_j = 1$) if $\beta = 1$ (resp. if $\beta = i$), $r \geq 1$, $m_j \in \text{Mod} - T$ $(j = 1, \ldots, r)$.*

Proof. Write $w = s^{e_1} m_1 s^{e_2} m_2 \cdots s^{e_r} m_r$. Let $\beta = 1$ so that $s = \begin{pmatrix} 1 & -ki \\ 0 & 1 \end{pmatrix}$ and $\langle s, t \rangle \simeq \mathbb{Z}^2$. Using the matrix expressions for the elements one can see that, if $m \in \text{Mod} - T$, then $ymv \notin \text{Mod}$, $ymy^{-1} \notin \text{Mod}$, $vmy^{-1} \in \text{Mod}$, $vmv \in \text{Mod}$ and $y^{-1}vmvy \notin \text{Mod}$. Using these facts we see that

$$\begin{aligned} \lambda(w) &= \lambda((vy)^{e_1} m_1 (vy) s^{e_2} m_2 \cdots (vy)^{e_r} m_r) \\ &= 2k \sum_{j=1}^{r} |e_j| - \#\{l : e_l e_{l+1} < 0\} \geq 2kr - r > 1. \end{aligned}$$

Let $\beta = i$ so that $s = \begin{pmatrix} i & k \\ 0 & -i \end{pmatrix}$, $\langle s, t\rangle \simeq \mathbb{D}_\infty$. Then

$$\begin{aligned} w &= sm_1 sm_2 \cdots sm_r \\ &= u(yv)(yv)^{k-2} ym_1 u(yv)(yv)^{k-2} ym_2 u(yv)(yv)^{k-2} ym_3 \cdots u(yv)(yv)^{k-2} ym_r \\ &\overset{\text{Pic}}{\sim} v(yv)^{k-2} y_1 v(yv)^{k-2} y_2 v(yv)^{k-2} y_3 \cdots v(yv)^{k-2} y_r \end{aligned}$$

where $y_j = ym_j uy$. As $m_j \in \text{Mod} - T$, one can verify that $y_j \in Y - \text{Mod}$. Therefore $\lambda(w) = r(2k-2) > 1$.

This proves i) and iii). Assertion ii) follows from iii). □

Corollary 12. *For $k > 2$, $\mathcal{G}_k^{1,1} \simeq \mathcal{G}_k^{i,i} \simeq \mathcal{G}_2^{1,1} \simeq \text{Mod} *_{\mathbb{Z}} \mathbb{Z}_2$ and $\mathcal{G}_k^{1,i} \simeq \mathcal{G}_k^{i,1} \simeq \mathcal{G}_2^{1,i} \simeq \text{Mod} *_{\mathbb{Z}} \mathbb{D}_\infty$.*

Corollary 13. *If $k \geq 2$ and $w \in \mathcal{G}_k^{1,\beta} - \text{Mod}$ then $\lambda(w) > 1$.*

Proof. It follows from the proposition observing that $\lambda(w) = 1$ if $w \in \text{Mod}$, $\lambda((vy)^{mk}) = 2|m|k > 1$ and $\lambda(u(yv)^{k-1}y) = 2k-1 > 1$. □

Corollary 14. *The abelianization of $\mathcal{G}_2^{1,1}$ is $\mathbb{Z} \oplus \mathbb{Z}_6$ and the abelianization of $\mathcal{G}_2^{1,i}$ is $\mathbb{Z}_6$.*

We will use the number of conjugacy classes of elements of order two in $\mathcal{G}_k^{\alpha,\beta}$; we will denote it by $c_2\left(\mathcal{G}_k^{\alpha,\beta}\right)$.

Corollary 15. *We have $c_2\left(\mathcal{G}_0^{1,1}\right) = 1$, $c_2\left(\mathcal{G}_0^{1,i}\right) = 3$, $c_2\left(\mathcal{G}_1^{1,1}\right) = 4$, $c_2\left(\mathcal{G}_1^{1,i}\right) = 2$, $c_2\left(\mathcal{G}_2^{1,1}\right) = 1$ and $c_2\left(\mathcal{G}_2^{1,i}\right) = 2$.*

Proof. Recall that $\mathcal{G}_0^{1,1} = \text{Mod} = \mathbb{Z}_2 * \mathbb{Z}_3$, $\mathcal{G}_0^{1,i} = V = \mathbb{Z}_2^2 *_{\mathbb{Z}_2} \mathbb{D}_3$, $\mathcal{G}_1^{1,1} = \text{Pic} = V *_{\text{Mod}} Y$, $\mathcal{G}_1^{1,i} \approx \mathcal{G}_{-1}^{1,i} = Y = \mathbb{D}_3 *_{\mathbb{Z}_3} A_4$, $\mathcal{G}_2^{1,1} \simeq \text{Mod} *_{\mathbb{Z}} \mathbb{Z}^2$ and $\mathcal{G}_2^{1,i} = \text{Mod} *_{\mathbb{Z}} \mathbb{D}_\infty$. Using the fact that an element of finite order in a free product with amalgamation is conjugate to an element in a factor and using ab the corollary follows. □

The following theorem states that if $(\alpha, \beta, k) \neq (\alpha', \beta', k')$ then $\mathcal{G}_k^{\alpha,\beta} \overset{\text{Pic}}{\not\sim} \mathcal{G}_{k'}^{\alpha',\beta'}$ with one exception (namely $\mathcal{G}_1^{1,1} = \mathcal{G}_1^{i,i} = \text{Pic}$).

Theorem 16. *Let $\alpha, \beta, \alpha', \beta' \in \{1, i\}$, $k \geq 0$ and $k' \geq 0$. Suppose $\mathcal{G}_k^{\alpha,\beta} \overset{\text{Pic}}{\sim} \mathcal{G}_{k'}^{\alpha',\beta'}$ with $(\alpha, \beta, k) \neq (\alpha', \beta', k')$. Then $k = k' = 1$, $\alpha = \beta$ and $\alpha' = \beta'$.*

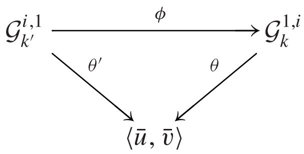
Proof. As $\mathcal{G}_k^{\alpha,\beta} \sim \mathcal{G}_{k'}^{\alpha',\beta'}$ we have, by Theorem 1 4), that $k = k'$ and $\alpha\beta = \pm\alpha'\beta'$ and so we may assume that $\alpha = 1$ and $\alpha' = i$. Hence $k \leq 1$.

Suppose $k \geq 2$. No conjugate, in Pic, of v lies in Mod because ab(Mod) $= \langle\overline{u}\rangle$. Therefore, by Corollary 13, no conjugate, in Pic, of v lies in $\mathcal{G}_k^{1,\beta}$. As $v \in \mathcal{G}_k^{i,\beta'}$, we have $\mathcal{G}_k^{1,\beta} \overset{\text{Pic}}{\not\sim} \mathcal{G}_k^{i,\beta'}$.

Suppose $k = 1$, and $\beta = i$. Then $\beta' = 1$ and $\text{ab}(\mathcal{G}_k^{\alpha,\beta}) = \langle\overline{u}\rangle \neq \langle\overline{v}\rangle = \mathcal{G}_{k'}^{\alpha',\beta'}$ so $\mathcal{G}_k^{\alpha,\beta} \overset{\text{Pic}}{\not\sim} \mathcal{G}_{k'}^{\alpha',\beta'}$.

Suppose $k = 0$, and $\beta = 1$. Then $\beta' = 1$ and $\text{ab}\big(\mathcal{G}_k^{\alpha,\beta}\big) = \langle\overline{u}\rangle \neq \langle\overline{v}\rangle = \mathcal{G}_k^{\alpha',\beta'}$ so $\mathcal{G}_k^{\alpha,\beta} \overset{\text{Pic}}{\not\sim} \mathcal{G}_{k'}^{\alpha',\beta'}$.

Finally suppose $k = 0$, and $\beta = i$. Then $\beta' = 1$. We have $\mathcal{G}_k^{\alpha,\beta} = V = \langle v, u, x\rangle$ and $\mathcal{G}_{k'}^{\alpha',\beta'} = \langle v, y, u\rangle$. There is an inner automorphism ϕ of Pic such that $\phi(\mathcal{G}_k^{\alpha,\beta}) = \mathcal{G}_{k'}^{\alpha',\beta'}$ and we have a commutative diagram



where θ' and θ are the restrictions of ab. Then $\theta'^{-1}(\langle\overline{u}\rangle) \simeq \theta^{-1}(\langle\overline{u}\rangle)$ which is impossible because, since $[V, \text{Mod}] = 2$, $\theta^{-1}(\langle\overline{u}\rangle) = \text{Mod}$ and $\theta'(\langle u\rangle) \supset \langle y, u\rangle \simeq \mathbb{D}_3$ and $\mathbb{D}_3$ is not isomorphic to a subgroup of Mod. $\square$

Theorem 17. *Let* $\alpha, \beta \in \{1, i\}$, $k \geq 0$. *Then* $\mathcal{G}_k^{\alpha,\beta}$ *is isomorphic to one of the groups* Mod, V, Pic, Y, $\text{Mod} *_{\mathbb{Z}} \mathbb{Z}^2$ *and* $\text{Mod} *_{\mathbb{Z}} \mathbb{D}_\infty$. *These six groups are pairwise nonisomorphic*.

Proof. The first assertion is a consequence of $\mathcal{G}_k^{i,\beta} \simeq \mathcal{G}_k^{1,\beta'}$, where $\{\beta, \beta'\} = \{1, i\}$, and Corollary 12. Now V, Pic and $\text{Mod} *_{\mathbb{Z}} \mathbb{D}_\infty$ have abelianization $\mathbb{Z}_2^2$ while Mod, Y and $\text{Mod} *_{\mathbb{Z}} \mathbb{Z}^2$ have pairwise non isomorphic abelianizations different from $\mathbb{Z}_2^2$. Since $c_2(V) = 3$, $c_2(\text{Pic}) = 4$ and $c_2(\text{Mod} *_{\mathbb{Z}} \mathbb{D}_\infty) = 2$, the theorem follows. $\square$

6. Section 5

In what follow ab: Pic $\to \overline{\text{Pic}} = \langle\overline{u}, \overline{v}\rangle$ is the abelianization map, where $u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $\mathbb{D}_3$ is the dihedral group of order six and A_4 is the alternating group in four letters.

In what follows $\mathbb{D}_3$ is the dihedral group of order six, A_4 is the alternating group in for letters ab: Pic $\to \overline{\text{Pic}} = \langle\overline{u}, \overline{v}\rangle$ is the abelianization map, where $u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

The following table has the information

#{conjugacy classes of elements of order two}	a group isomorphic to $\mathcal{G}_k^{\alpha,\beta}$
image under ab of $\mathcal{G}_k^{\alpha,\beta}$	abelianization of $\mathcal{G}_k^{\alpha,\beta}$

for the group $\mathcal{G}_k^{\alpha,\beta}$.

<table>
<tr><th>$\mathcal{G}_k^{\alpha,\beta}$</th><th>$(\alpha, \beta) = (1, 1)$ (resp. (i, i))</th><th>$(\alpha, \beta) = (1, i)$ (resp. $(i, 1)$)</th></tr>
<tr><td>$k = 0$</td><td><table><tr><td>1</td><td>Mod</td></tr><tr><td>$\langle\bar{u}\rangle$ (resp. $\langle\bar{v}\rangle$)</td><td>$\mathbb{Z}_6$</td></tr></table></td><td><table><tr><td>3</td><td>$\mathbb{Z}_2^2 *_{\mathbb{Z}_2} \mathbb{D}_3$</td></tr><tr><td>$\langle\bar{u}, \bar{v}\rangle$</td><td>$\mathbb{Z}_2^2$</td></tr></table></td></tr>
<tr><td>$k = 1$</td><td><table><tr><td>4</td><td>Pic</td></tr><tr><td>$\langle\bar{u}, \bar{v}\rangle$</td><td>$\mathbb{Z}_2^2$</td></tr></table></td><td><table><tr><td>2</td><td>$\mathbb{D}_3 *_{\mathbb{Z}_3} A_4$</td></tr><tr><td>$\langle\bar{u}\rangle$ (resp. $\langle\bar{v}\rangle$)</td><td>$\mathbb{Z}_2$</td></tr></table></td></tr>
<tr><td>$K = 2, 4, \ldots$</td><td><table><tr><td>1</td><td>$\mathrm{Mod} *_{\mathbb{Z}} \mathbb{Z}^2$</td></tr><tr><td>$\langle\bar{u}\rangle$ (resp. $\langle\bar{v}\rangle$)</td><td>$\mathbb{Z} \oplus \mathbb{Z}_6$</td></tr></table></td><td><table><tr><td>2</td><td>$\mathrm{Mod} *_{\mathbb{Z}} \mathbb{D}_\infty$</td></tr><tr><td>$\langle\bar{u}\rangle$ (resp. $\langle\bar{v}\rangle$)</td><td>$\mathbb{Z}_2^2$</td></tr></table></td></tr>
<tr><td>$k = 3, 5, \ldots$</td><td><table><tr><td>1</td><td>$\mathrm{Mod} *_{\mathbb{Z}} \mathbb{Z}^2$</td></tr><tr><td>$\langle\bar{u}, \bar{v}\rangle$</td><td>$\mathbb{Z} \oplus \mathbb{Z}_6$</td></tr></table></td><td><table><tr><td>2</td><td>$\mathrm{Mod} *_{\mathbb{Z}} \mathbb{D}_\infty$</td></tr><tr><td>$\langle\bar{u}, \bar{v}\rangle$</td><td>$\mathbb{Z}_2^2$</td></tr></table></td></tr>
</table>

References

[1] A.M. Brunner, M.L. Frame, Y.W. Lee and N.J. Wielenberg: *Classifying torsion-free subgroups of the Picard Group*, Trans. Amer. Math. Soc. **282** (1984), 205–235.

[2] T. Jørgensen: *On discrete groups of Möbius transformations*, Amer. J. Math. **98** (1976), 739–749.

[3] T. Jørgensen and M. Kiikka: *Some extreme discrete groups*, Ann. Acad. Sci. Fenn. Ser. A I Math. **1** (1975), 245–248.

[4] T. Jørgensen, A. Lascurain and T. Pignataro: *Translation extensions of the classical modular group*, Complex Variables Theory Appl. **19** (1992), 205–209.

[5] S.L. Krushkal′, B.N. Apanasov and N.A. Gusevskiĭ: Kleinian Groups and Uniformization in Examples and Problems, Trans. Math. Monographs **62**, Amer. Math. Soc., Providence, RI, 1986.

[6] R.C. Lyndon and P.E. Schupp: Combinatorial Group Theory, Springer, Berlin, 1977.

[7] W. Magnus, A Karrass and D. Solitar: Combinatorial Group Theory, second revised edition, Dover, New York, 1976.

[8] A. Marden: *Geometrically finite Kleinian groups and their deformation spaces*; in Discrete Groups and Automorphic Functions, (Proc. Conf., Cambridge, 1975), Academic Press, London, 1977, 259–293.

[9] M.H.A. Newman and J.H.C. Whitehead: On the group of a certain linkage, Quart. J. Math. Oxford **8** (1937), 14–21.

[10] H. Sato: *One-parameter families of extreme discrete groups for Jørgensen's inequality*; in In the Tradition of Ahlfors and Bers, (Stony Brook, NY, 1998), Contemp. Math. **256**, Amer. Math. Soc., Providence, RI., 2000, 271–287

[11] H. Sato: *The Picard group, the Whitehead link and Jørgensen group*; in Progress in Analysis, Vol. I, (Berlin, 2001), World Sci. Publ., River Edge, NJ. 2003, 149–158.

[12] H. Sato: *The Jørgensen number of the Whitehead link group*, Boletin de Soc. Mat. Mex. (to appear).
[13] H. Sato and R. Yamada: *Some extreme Kleinian groups for Jrgensen's inequality*, Rep. Fac. Sci. Shizuoka Univ. **27** (1993), 1–8.
[14] R.G. Swan: *Generators and relations for certain special linear groups*, Advances in Math. **6** (1970), 1–77.
[15] N. Wielenberg: *The structure of certain subgroups of the Picard group*, Math. Proc. Cambridge Philos. Soc. **84** (1978), 427–436.

Francisco González-Acuña
Instituto de Matemáticas
UNAM and CIMAT Guanajuato
México

Arturo Ramírez
CIMAT Guanajuto
México