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Osaka J. Math.
44 (2007), 415–439

AN INVERSE PROBLEM FOR THE ONE-DIMENSIONAL WAVE EQUATION IN MULTILAYER MEDIA

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(Received November 25, 2005, revised June 19, 2006)

Abstract

We consider half-line media which consist of many kinds of substances. We assume that the waves through this media are described by the one-dimensional wave equation. We can directly observe the data near the boundary point of the half-line, but we cannot directly observe the data of things away from the boundary point. In this situation, we try to identify these unknown things by creating an artificial explosion and observing on the boundary point the waves generated by the explosion. In the previous works related to this problem, only the speeds of the waves were treated, but we also take into account the impedances of the media in our setting.

1. Introduction

We consider half-line media which consist of many kinds of substances. We can directly observe the data near the boundary point of the half-line, but we cannot directly observe the data of things away from the boundary point. In this situation, we perform the following experiment in order to investigate them: We first create an artificial explosion at a point near the boundary point. Waves generated by this explosion travel in the media. Then we observe the waves at the boundary point, and guess the situation away from the boundary point.

This problem has been studied by Bartoloni-Lodovici-Zirilli [1], for example. However, from the experimental point of view, this result has some problem with respect to the formulation of the situation. Indeed, in [1], they deal with

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial}{\partial x}\left(S(x)\frac{\partial u}{\partial x}(t,x)\right), \quad t > 0, \quad x > 0$$

in order to express behavior of the waves inside the half-line, where $S(x)$ is a piecewise constant function. In this case, the interface or transmission conditions are determined by only the speeds of the waves. However this is not natural since the interface or transmission conditions depend on not only the speeds of the waves but also the

2000 Mathematics Subject Classification. Primary 35R30; Secondary 35L05.

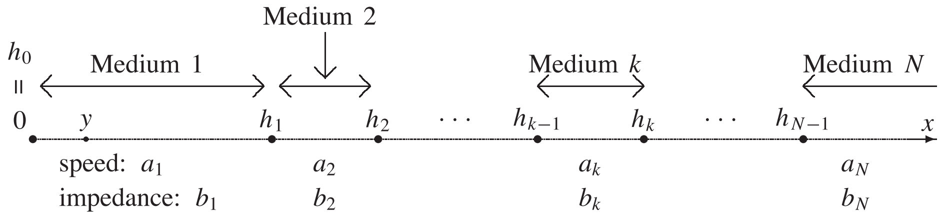


Fig. 1. The situation which we consider.

impedances of the substances. Then we consider this problem in consideration of the impedances, and we try to reconstruct the unknown data concretely.

Now, we introduce the notations and formulate this problem. Put $h_0 := 0$. Let h_k be a positive constant and $h_k > h_{k-1}$ for $k = 1, \ldots, N-1$. We call the interval (h_{k-1}, h_k) Medium k for $k = 1, \ldots, N-1$ and the interval (h_{N-1}, ∞) Medium N. Let a_k and b_k be positive constants for $k = 1, \ldots, N$. The positive number a_k describes the speed of the waves through Medium k, and b_k the impedance of Medium k. Put $D_t := (1/i)(\partial/\partial t)$ and $D_x := (1/i)(\partial/\partial x)$, where i is the imaginary unit. We define $P_k(D_t, D_x) = a_k^2 D_x^2 - D_t^2$ for $k = 1, \ldots, N$. Suppose $0 < y < h_1$.

We consider the following equations:

$$P_1(D_t, D_x)u(t,x) = \delta(t, x-y), \quad 0 < x < h_1, \tag{1}$$

$$P_k(D_t, D_x)u(t,x) = 0, \quad h_{k-1} < x < h_k \quad (2 \le k \le N-1), \tag{2}$$

$$P_N(D_t, D_x)u(t,x) = 0, \quad h_{N-1} < x, \tag{3}$$

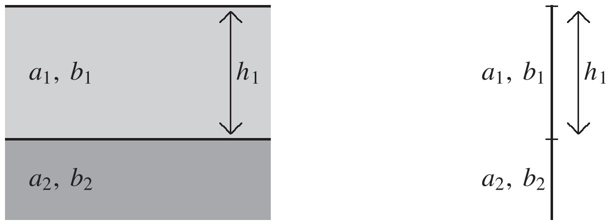
$$D_x u(t,x)|_{x=0+0} = 0, \tag{4}$$

$$u(t,x)|_{x=h_k-0} = u(t,x)|_{x=h_k+0} \quad (1 \le k \le N-1), \tag{5}$$

$$a_k b_k D_x u(t,x)|_{x=h_k-0} = a_{k+1} b_{k+1} D_x u(t,x)|_{x=h_k+0} \quad (1 \le k \le N-1). \tag{6}$$

The equation (4) means the free boundary condition at the point $x = 0$. The equations (5) and (6) for k express the conditions at the point $x = h_k$ which is the joining of Medium k and Medium $k+1$. The equation (5) describes the continuity of the displacement of the waves, and (6) the continuity of the stress. The equations (1)–(6) express the situation that the initial data is the delta function at the point y in Medium 1 at the time $t = 0$ with the boundary condition (4) and the interface or transmission conditions (5) and (6) at the joining point between Medium k and Medium $k+1$.

The following main result says that we can reconstruct the impedances b_{k+1} and the ratios $(h_k - h_{k-1})/a_k$ of the width to the speeds of the waves by the observation data $u(t, 0)$ when the data a_1, b_1 of Medium 1 are known.



The n-dimensional case ($n \geq 2$). The one-dimensional case.

Fig. 2. The two-layer case.

Main result. *Suppose that the constants a_1, b_1, y are known. Assume $b_j \neq b_{j+1}$ for $j = 1, \ldots, N-1$. Assume that the observation data $v(t) := u(t, 0)$ are given on $[0, T)$, where $u(t, x)$ denotes the solution of the equations* (1)–(6). *Then b_{k+1} and $(h_k - h_{k-1})/a_k$ are reconstructed by the following process*:

- *The first step*: *Put $v_1(t) := (1/a_1)H(t - y/a_1) - v(t)$, where H is the Heaviside function*.
- *The $(k+1)$-st step $(k = 1, 2, \ldots)$: If $v_k(t) \equiv 0$ then the process is finished. If $v_k(t) \not\equiv 0$, then put $t_k := \inf\{t \in [0, T) \colon v_k(t) \neq 0\}$, reconstruct the constants $(h_k - h_{k-1})/a_k$ and b_{k+1} by*

$$\frac{h_k - h_{k-1}}{a_k} := \frac{1}{2}\left(t_k + \frac{y}{a_1}\right) - \sum_{j=1}^{k-1} \frac{h_j - h_{j-1}}{a_j},$$

$$b_{k+1} := \frac{2^{2k-2} \prod_{j=1}^{k-1}(b_j b_{j+1}) + v_k(t_k + 0) a_1 \prod_{j=1}^{k-1}(b_j + b_{j+1})^2}{2^{2k-2} \prod_{j=1}^{k-1}(b_j b_{j+1}) - v_k(t_k + 0) a_1 \prod_{j=1}^{k-1}(b_j + b_{j+1})^2} b_k,$$

define $v_{k+1}(t)$ by the known data and the reconstructed data, and go the next step.

We state the concrete way of determining $v_{k+1}(t)$ in Theorem 13. We remark that we can reconstruct the impedances b_{k+1} but we cannot identify the speeds a_k themselves of the waves. This result is not obtained by [1].

On the other hand, our main result is also the expansion of Nagayasu [4] for the one-dimensional case. In [4], the author considers the situation that the half-line consists of two layers, and determine the unknown data by using the observation data on the whole time. However, our main result says that we can reconstruct the unknown data by the observation data on the finite time, and how many data we can reconstruct is determined as to the observation time.

We remark that the one-dimensional case differs from the n-dimensional case ($n \geq 2$) in that the speeds themselves cannot or can be reconstructed. Indeed, we obtain the following result from [4] for example. We consider the two-layer case (see Fig. 2), and assume that a_1 and b_1 are known. Let observation data be given. Then, we can identify a_2, b_2 and h_1 when the physical space dimension is greater than or equal to two. However, we can identify b_2 and h_1/a_1 (namely h_1 itself) but cannot identify a_2 when the physical space dimension is one.

Finally, we explain the plan of this paper. In Section 2, we construct the solution formula of the equations (1)–(6). In Section 3, we state our main result concretely and give its proof. In Appendix, we discuss the case that the impedance of the adjacent media may be equal, that is, $b_j = b_{j+1}$ may hold.

2. The solution formula

In this section, we construct the explicit solution formula in Medium 1 of the equations (1)–(6). In order to make the dependence of the solution on the coefficients clearly, we denote the solution of (1)–(6) by

$$u(t, x) = u_N(t, x; a_1, \dots, a_N; b_1, \dots, b_N; h_1, \dots, h_{N-1}; y).$$

In Section 2.1, we express it in the case of $N = 1$. In Section 2.2, we construct it for $N \geq 2$.

2.1. The solution formula for $N = 1$. The equations which we deal with are as following:

$$P_1(D_t, D_x)u_1(t, x) = \delta(t, x - y), \quad x > 0,$$
$$D_x u_1(t, x)|_{x=0+0} = 0.$$

By Matsumura [2], we find the solution

$$u_1(t, x; a_1; b_1; \ \cdot\ ; y) = \frac{1}{2a_1} H\left(t - \frac{|x - y|}{a_1}\right) + \frac{1}{2a_1} H\left(t - \frac{x + y}{a_1}\right).$$

We remark that its Fourier-Laplace transform along $\rho = \tau - im\log(2 + |\tau|)$ with respect to t is

$$\widehat{u}_1(\rho, x) = \frac{1}{2a_1 i\rho}\{e^{-i\rho|x-y|/a_1} + e^{-i\rho(x+y)/a_1}\}, \quad x > 0.$$

2.2. The solution formula for $N \geq 2$. We construct the solution of (1)–(6) by induction on N. Then we first define $F_k^{(N)}(t, x) = F_k^{(N)}(t, x; a_1, \dots, a_N; b_1, \dots, b_N; h_1, \dots, h_{N-1}; y)$ by

$$F_k^{(N)}(t, x) = u_{N-1}(t, x) - u_N(t, x), \quad h_{k-1} < x < h_k \quad (1 \leq k \leq N - 1),$$

$$F_N^{(N)}(t,x) = u_N(t,x), \quad h_{N-1} < x,$$

where we write

$$u_{N-1}(t,x) = u_{N-1}(t,x;a_1,\dots,a_{N-1};b_1,\dots,b_{N-1};h_1,\dots,h_{N-2};y),$$
$$u_N(t,x) = u_N(t,x;a_1,\dots,a_N;b_1,\dots,b_N;h_1,\dots,h_{N-1};y)$$

for short notation. The distribution $F_k^{(N)}(t,x)$ expresses the behavior of the waves in Medium k which are affected by Medium N. We remark that this $F_k^{(N)}(t,x)$ and $F_1(t,x)$ in Matsumura [3] are different. By the definition of $F_k^{(N)}(t,x)$, the equations (1)–(6) are changed for

$$P_k F_k^{(N)} = 0 \quad \begin{pmatrix} h_{k-1} < x < h_k & (1 \le k \le N-1) \\ h_{N-1} < x & (k = N) \end{pmatrix}, \tag{7}$$

$$D_x F_1^{(N)}\big|_{x=0} = 0, \tag{8}$$

$$\left(F_k^{(N)} - F_{k+1}^{(N)}\right)\big|_{x=h_k} = 0 \quad (1 \le k \le N-2), \tag{9}$$

$$\left(a_k b_k D_x F_k^{(N)} - a_{k+1} b_{k+1} D_x F_{k+1}^{(N)}\right)\big|_{x=h_k} = 0 \quad (1 \le k \le N-2), \tag{10}$$

$$\left(F_{N-1}^{(N)} + F_N^{(N)}\right)\big|_{x=h_{N-1}} = u_{N-1}|_{x=h_{N-1}}, \tag{11}$$

$$\left(a_{N-1} b_{N-1} D_x F_{N-1}^{(N)} + a_N b_N D_x F_N^{(N)}\right)\big|_{x=h_{N-1}} = a_{N-1} b_{N-1} D_x u_{N-1}|_{x=h_{N-1}}, \tag{12}$$

where $P_k = P_k(D_t, D_x)$, $F_k^{(N)} = F_k^{(N)}(t,x)$ and

$$u_{N-1} = u_{N-1}(t,x;a_1,\dots,a_{N-1};b_1,\dots,b_{N-1};h_1,\dots,h_{N-2};y)$$

for short notation. We solve these equations. We apply the Fourier-Laplace transformation along $\rho = \tau - im\log(2+|\tau|)$ with respect to t to these equations as in Matsumura [3], where m is a positive real large enough. Then by (7) we can write

$$\widehat{F}_k^{(N)}(\rho,x) = \Phi_k^{(N)}(\rho) e\left(-\frac{x}{a_k}\right) + \Psi_k^{(N)}(\rho) e\left(\frac{x}{a_k}\right) \quad (1 \le k \le N-1), \tag{13}$$

$$\widehat{F}_N^{(N)}(\rho,x) = \Phi_N^{(N)}(\rho) e\left(-\frac{x}{a_N}\right), \tag{14}$$

where $e(s) := e(s;\rho) := \exp(i\rho s)$. In the same way as $F_k^{(N)}(t,x)$, we write

$$\Phi_k^{(N)}(\rho) = \Phi_k^{(N)}(\rho;a_1,\dots,a_N;b_1,\dots,b_N;h_1,\dots,h_{N-1};y),$$
$$\Psi_k^{(N)}(\rho) = \Psi_k^{(N)}(\rho;a_1,\dots,a_N;b_1,\dots,b_N;h_1,\dots,h_{N-1};y)$$

in order to make the dependence on the coefficients clearly. We define K_M [resp. L_M] $(\rho; a_1, \dots, a_M; b_1, \dots, b_M; h_1, \dots, h_{M-1}, h_M; y)$ by

$$K_M(\rho; a_1, \dots, a_M; b_1, \dots, b_M; h_1, \dots, h_{M-1}, h_M; y)$$
$$:= \widehat{u}_M(\rho, x; a_1, \dots, a_M; b_1, \dots, b_M; h_1, \dots, h_{M-1}; y)|_{x=h_M},$$
$$L_M(\rho; a_1, \dots, a_M; b_1, \dots, b_M; h_1, \dots, h_{M-1}, h_M; y)$$
$$:= -\frac{a_M b_M}{\rho} D_x \widehat{u}_M(\rho, x; a_1, \dots, a_M; b_1, \dots, b_M; h_1, \dots, h_{M-1}; y)|_{x=h_M}$$

for $M = 1, 2, \dots$. Now, we substitute (13) and (14) into the Fourier-Laplace transform of the equations (8)–(12) and simplify them. Then we have

$$\mathcal{Z}_N \begin{bmatrix} \Phi_1^{(N)} \\ \Psi_1^{(N)} \\ \Phi_2^{(N)} \\ \Psi_2^{(N)} \\ \vdots \\ \Phi_{N-1}^{(N)} \\ \Psi_{N-1}^{(N)} \\ \Phi_N^{(N)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ \vdots \\ \\ 0 \\ K_{N-1} \\ L_{N-1} \end{bmatrix}, \tag{15}$$

where we define the (j, l)-components

$$\mathcal{Z}_N(\rho; a_1, \dots, a_N; b_1, \dots, b_N; h_1, \dots, h_{N-1})_{jl}$$

of the $(2N-1) \times (2N-1)$ matrix

$$\mathcal{Z}_N(\rho; a_1, \dots, a_N; b_1, \dots, b_N; h_1, \dots, h_{N-1})$$

by

$$
\mathcal{Z}_N(\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1})_{jl}
:= \begin{cases}
1, & j = 1,\ l = 1, \\
-1, & j = 1,\ l = 2, \\
e\left(-\dfrac{h_k}{a_k}\right), & j = 2k,\ l = 2k-1 \quad (k = 1, \ldots, N-1), \\
e\left(\dfrac{h_k}{a_k}\right), & j = 2k,\ l = 2k \quad (k = 1, \ldots, N-1), \\
b_k e\left(-\dfrac{h_k}{a_k}\right), & j = 2k+1,\ l = 2k-1 \quad (k = 1, \ldots, N-1), \\
-b_k e\left(\dfrac{h_k}{a_k}\right), & j = 2k+1,\ l = 2k \quad (k = 1, \ldots, N-1), \\
-e\left(-\dfrac{h_k}{a_{k+1}}\right), & j = 2k,\ l = 2k+1 \quad (k = 1, \ldots, N-2), \\
-e\left(\dfrac{h_k}{a_{k+1}}\right), & j = 2k,\ l = 2k+2 \quad (k = 1, \ldots, N-2), \\
-b_{k+1} e\left(-\dfrac{h_k}{a_{k+1}}\right), & j = 2k+1,\ l = 2k+1 \quad (k = 1, \ldots, N-2), \\
b_{k+1} e\left(\dfrac{h_k}{a_{k+1}}\right), & j = 2k+1,\ l = 2k+2 \quad (k = 1, \ldots, N-2), \\
e\left(-\dfrac{h_{N-1}}{a_N}\right), & j = 2N-2,\ l = 2N-1, \\
b_N e\left(-\dfrac{h_{N-1}}{a_N}\right), & j = 2N-1,\ l = 2N-1, \\
0, & \text{otherwise}
\end{cases}
$$

and we write

$$
\begin{aligned}
&\mathcal{Z}_N = \mathcal{Z}_N(\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}), \\
&\Phi_k^{(N)}\big[\Psi_k^{(N)}\big] = \Phi_k^{(N)}\big[\Psi_k^{(N)}\big](\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y), \\
&K_{N-1}[L_{N-1}] = K_{N-1}[L_{N-1}](\rho; a_1, \ldots, a_{N-1}; b_1, \ldots, b_{N-1}; h_1, \ldots, h_{N-1}; y)
\end{aligned}
$$

for short notation in the equation (15).

We need to express the explicit formula of $u(t, 0)$ in order to discuss our inverse problem. Then we construct $\Phi_1^{(N)}(\rho)$ and $\Psi_1^{(N)}(\rho)$. Now, for short notation we write

$$
\begin{aligned}
&K_N[L_N] = K_N[L_N](\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}, h_N; y), \\
&\Phi_N^{(N)} = \Phi_N^{(N)}(\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y),
\end{aligned}
$$

$$\mathcal{Z}_N = \mathcal{Z}_N(\rho; a_1, \dots, a_N; b_1, \dots, b_N; h_1, \dots, h_{N-1})$$

in Lemmas 1, 2, 3 and Corollary 4. We first express $\det \mathcal{Z}_N$ explicitly.

Lemma 1. *Let $N \geq 2$. Then*

$$\det \mathcal{Z}_N = (-1)^N e\left(-\frac{h_{N-1}}{a_N}\right) \times Z_N\left(\rho; b_1, \dots, b_N; \frac{h_1}{a_1}, \frac{h_2 - h_1}{a_2}, \dots, \frac{h_{N-1} - h_{N-2}}{a_{N-1}}\right) \tag{16}$$

holds, where we denote

$$Z_N(\rho; b_1, b_2, \dots, b_N; \Theta_1, \Theta_2, \dots, \Theta_{N-1}) := \sum_{\substack{\alpha_k = \pm 1 \\ (1 \leq k \leq N-1)}} \alpha_1 \left\{ \prod_{j=1}^{N-2} (b_j + \alpha_j \alpha_{j+1} b_{j+1}) \right\} (b_{N-1} + \alpha_{N-1} b_N) e\left(\sum_{j=1}^{N-1} \alpha_j \Theta_j \right)$$

for $N \geq 2$, and we define $\prod_{j=1}^{N-2}(b_j + \alpha_j \alpha_{j+1} b_{j+1}) = 1$ when $N = 2$.

Proof. We prove this lemma by induction on N. It is easy to obtain the equation (16) for the case of $N = 2$. Then we assume that the equation (16) for N (≥ 2) holds, and we show the equation (16) for $N + 1$. We first expand $\det \mathcal{Z}_{N+1}$ along the $(2N + 1)$-st column, and expand them along the $(2N)$-th row. Then we have

$$\begin{aligned}
&\det \mathcal{Z}_{N+1}(\rho; a_1, \dots, a_N, a_{N+1}; b_1, \dots, b_N, b_{N+1}; h_1, \dots, h_{N-1}, h_N) \\
&= -e\left(-\frac{h_N}{a_{N+1}}\right) \left\{ b_N e\left(-\frac{h_N}{a_N}\right) e\left(\frac{2h_{N-1}}{a_N}\right) \det \mathcal{Z}_N^- + b_N e\left(\frac{h_N}{a_N}\right) \det \mathcal{Z}_N^+ \right\} \\
&\quad + b_{N+1} e\left(-\frac{h_N}{a_{N+1}}\right) \left\{ e\left(-\frac{h_N}{a_N}\right) e\left(\frac{2h_{N-1}}{a_N}\right) \det \mathcal{Z}_N^- - e\left(\frac{h_N}{a_N}\right) \det \mathcal{Z}_N^+ \right\} \\
&= -e\left(-\frac{h_N}{a_{N+1}}\right) \left\{ (b_N + b_{N+1}) e\left(\frac{h_N}{a_N}\right) \det \mathcal{Z}_N^+ \right. \\
&\qquad\qquad \left. + (b_N - b_{N+1}) e\left(\frac{2h_{N-1} - h_N}{a_N}\right) \det \mathcal{Z}_N^- \right\} \\
&\overset{(*)}{=} -e\left(-\frac{h_N}{a_{N+1}}\right) \left\{ (b_N + b_{N+1}) e\left(\frac{h_N}{a_N}\right) (-1)^N e\left(-\frac{h_{N-1}}{a_N}\right) Z_N^+ \right. \\
&\qquad\qquad \left. + (b_N - b_{N+1}) e\left(\frac{2h_{N-1} - h_N}{a_N}\right) (-1)^N e\left(-\frac{h_{N-1}}{a_N}\right) Z_N^- \right\}
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{N+1} e\left(-\frac{h_N}{a_{N+1}}\right) \\
&\quad \times \sum_{\substack{\alpha_k=\pm 1 \\ (1\le k\le N-1)}} \alpha_1 \left\{\prod_{j=1}^{N-2}(b_j + \alpha_j\alpha_{j+1}b_{j+1})\right\} e\left(\sum_{j=1}^{N-1}\alpha_j \frac{h_j - h_{j-1}}{a_j}\right) \\
&\qquad \times \left\{\sum_{\alpha_N=\pm 1}(b_{N-1} + \alpha_{N-1}\alpha_N b_N)(b_N + \alpha_N b_{N+1}) e\left(\alpha_N \frac{h_N - h_{N-1}}{a_N}\right)\right\} \\
&= (-1)^{N+1} e\left(-\frac{h_N}{a_{N+1}}\right) Z_{N+1}\Bigg(b_1, \dots, b_N, b_{N+1}; \\
&\qquad \frac{h_1}{a_1}, \frac{h_2 - h_1}{a_2}, \dots, \frac{h_{N-1} - h_{N-2}}{a_{N-1}}, \frac{h_N - h_{N-1}}{a_N}\Bigg),
\end{aligned}
$$

where we write

$$
\begin{aligned}
\mathcal{Z}_N^{\pm} &= \mathcal{Z}_N(\rho; a_1, \dots, a_{N-1}, a_N; b_1, \dots, b_{N-1}, \pm b_N; h_1, \dots, h_{N-1}), \\
Z_N^{\pm} &= Z_N\left(\rho; b_1, \dots, b_{N-1}, \pm b_N; \frac{h_1}{a_1}, \frac{h_2 - h_1}{a_2}, \dots, \frac{h_{N-1} - h_{N-2}}{a_{N-1}}\right)
\end{aligned}
$$

for short notation and we use the inductive hypothesis at $(*)$. Hence we obtain the equation (16) for $N+1$. □

Next, we express K_N and L_N explicitly.

Lemma 2. *For* $N \geq 2$

$$
K_N = \Phi_N^{(N)} e\left(-\frac{h_N}{a_N}\right), \quad L_N = b_N K_N
$$

hold.

Proof. Because of $h_{N-1} < h_N$, we have

$$
D_x^j \widehat{u}_N(\rho, x)\Big|_{x=h_N} = D_x^j \widehat{F}_N^{(N)}(\rho, x)\Big|_{x=h_N}
$$

for $j = 0, 1$. From this equation we can obtain this lemma easily. □

Lemma 3. *For* $N \geq 2$ *we have*

$$
\Phi_N^{(N)} = \frac{-(-2)^{N-1}}{2a_1 i\rho} \frac{1}{\det \mathcal{Z}_N} \left(\prod_{j=1}^{N-1} b_j\right) \sum_{\nu=\pm 1} e\left(\nu \frac{y}{a_1}\right). \tag{17}
$$

Proof. We prove the equation (17) by induction on N. First we consider the case of $N = 2$. We remark that we obtain

$$K_1(\rho; a_1; b_1; h_1; y) = \frac{1}{2a_1 i\rho} e\left(-\frac{h_1}{a_1}\right) \sum_{\nu=\pm 1} e\left(\nu \frac{y}{a_1}\right),$$
$$L_1(\rho; a_1; b_1; h_1; y) = b_1 K_1(\rho; a_1; b_1; h_1; y)$$

from the definition of K_1 and L_1, and Section 2.1. By these equations, we have this lemma for $N = 2$. Then we assume that the equation (17) for N (≥ 2) holds, and we show the equation (17) for $N + 1$. We have

$$\begin{aligned}
&\Phi^{(N+1)}_{N+1}(\rho; a_1, \ldots, a_N, a_{N+1}; b_1, \ldots, b_N, b_{N+1}; h_1, \ldots, h_{N-1}, h_N; y) \\
&= \left(\text{the } (2N+1)\text{-st component of } \left(\mathcal{Z}_{N+1}^{-1}\right) \cdot \begin{bmatrix} \mathbf{0} \\ K_N \\ L_N \end{bmatrix}\right) \\
&= \frac{1}{\det \mathcal{Z}_{N+1}} \{(\text{the } (2N, 2N+1)\text{-cofactor of } \mathcal{Z}_{N+1}) K_N \\
&\qquad\qquad + (\text{the } (2N+1, 2N+1)\text{-cofactor of } \mathcal{Z}_{N+1}) L_N\} \\
&\overset{(\#)}{=} \frac{1}{\det \mathcal{Z}_{N+1}} \left\{ -b_N e\left(-\frac{h_N}{a_N}\right) e\left(2\frac{h_{N-1}}{a_N}\right) \det \mathcal{Z}_N^- - b_N e\left(\frac{h_N}{a_N}\right) \det \mathcal{Z}_N \right. \\
&\qquad\qquad \left. + b_N e\left(-\frac{h_N}{a_N}\right) e\left(2\frac{h_{N-1}}{a_N}\right) \det \mathcal{Z}_N^- - b_N e\left(\frac{h_N}{a_N}\right) \det \mathcal{Z}_N \right\} \\
&\quad \times \Phi^{(N)}_N e\left(-\frac{h_N}{a_N}\right) \\
&= -2 \frac{\det \mathcal{Z}_N}{\det \mathcal{Z}_{N+1}} b_N \Phi^{(N)}_N \\
&\overset{(*)}{=} 2 \frac{\det \mathcal{Z}_N}{\det \mathcal{Z}_{N+1}} b_N \frac{(-2)^{N-1}}{2a_1 i\rho} \frac{1}{\det \mathcal{Z}_N} \left(\prod_{j=1}^{N-1} b_j\right) \sum_{\nu=\pm 1} e\left(\nu \frac{y}{a_1}\right) \\
&= \frac{-(-2)^N}{2a_1 i\rho} \frac{1}{\det \mathcal{Z}_{N+1}} \left(\prod_{j=1}^{N} b_j\right) \sum_{\nu=\pm 1} e\left(\nu \frac{y}{a_1}\right),
\end{aligned}$$

where we write

$$\mathcal{Z}_N^- = \mathcal{Z}_N(\rho; a_1, \ldots, a_N; b_1, \ldots, b_{N-1}, -b_N; h_1, \ldots, h_{N-1}),$$
$$\mathcal{Z}_{N+1} = \mathcal{Z}_{N+1}(\rho; a_1, \ldots, a_N, a_{N+1}; b_1, \ldots, b_N, b_{N+1}; h_1, \ldots, h_{N-1}, h_N)$$

for short notation, and we expand the determinant along the $(2N)$-th row and use Lemma 2 at (#), and we use the inductive hypothesis at $(*)$. □

Corollary 4. *For* $N \geq 2$,

$$K_N = \frac{-(-2)^{N-1}}{2a_1 i\rho} \frac{1}{\det \mathcal{Z}_N} \left(\prod_{j=1}^{N-1} b_j \right) \sum_{\nu=\pm 1} e\left(\nu \frac{y}{a_1} - \frac{h_N}{a_N} \right), \quad L_N = b_N K_N$$

hold.

Proof. By Lemmas 2 and 3, we obtain this corollary easily. □

REMARK 5. We define $\mathcal{Z}_1(\rho; a_1; b_1; \cdot) = (-1)$, $Z_1(\rho; b_1; \cdot) = 1$. Then Lemma 1 and Corollary 4 hold also for $N = 1$, where we define $\prod_{j=1}^{N-1} b_j = 1$ for $N = 1$.

Now, we express $\Phi_1^{(N)}$ and $\Psi_1^{(N)}$ explicitly.

Lemma 6. *For* $N \geq 2$,

$$\begin{aligned}
&\Phi_1^{(N)}(\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y) \\
&= \Psi_1^{(N)}(\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y) \\
&= \frac{2^{2N-4}}{2a_1 i\rho} \frac{b_{N-1} - b_N}{\det \mathcal{Z}_N \det \mathcal{Z}_{N-1}} \left\{ \prod_{j=1}^{N-2} (b_j b_{j+1}) \right\} \sum_{\nu=\pm 1} e\left(\nu \frac{y}{a_1} - \frac{h_{N-1}}{a_{N-1}} - \frac{h_{N-1}}{a_N} \right)
\end{aligned}$$

holds, where we write

$$\begin{aligned}
\mathcal{Z}_N &= \mathcal{Z}_N(\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}), \\
\mathcal{Z}_{N-1} &= \mathcal{Z}_{N-1}(\rho; a_1, \ldots, a_{N-1}; b_1, \ldots, b_{N-1}; h_1, \ldots, h_{N-2})
\end{aligned}$$

for short notation, and we define $\prod_{j=1}^{N-2}(b_j b_{j+1}) = 1$ *for* $N = 2$.

Proof. It is easy to obtain $\Phi_1^{(N)}(\rho) = \Psi_1^{(N)}(\rho)$ from the equation (15). Then we find the explicit formula of $\Phi_1^{(N)}(\rho)$. We have

$$\begin{aligned}
&\Phi_1^{(N)}(\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y) \\
&= \left(\text{the first component of } \left(\mathcal{Z}_N^{-1} \right) \cdot \begin{pmatrix} \mathbf{0} \\ K_{N-1} \\ L_{N-1} \end{pmatrix} \right) \\
&= \frac{1}{\det \mathcal{Z}_N} \{ (\text{the } (2N-2, 1)\text{-cofactor of } \mathcal{Z}_N) K_{N-1} \\
&\qquad + (\text{the } (2N-1, 1)\text{-cofactor of } \mathcal{Z}_N) L_{N-1} \}
\end{aligned}$$

$$\stackrel{(\#)}{=} \frac{b_N - b_{N-1}}{\det \mathcal{Z}_N} e\left(-\frac{h_{N-1}}{a_N}\right) K_{N-1}$$

$$\times \prod_{j=1}^{N-2} \det \begin{bmatrix} -e\left(-\frac{h_j}{a_{j+1}}\right) & -e\left(\frac{h_j}{a_{j+1}}\right) \\ -b_{j+1} e\left(-\frac{h_j}{a_{j+1}}\right) & b_{j+1} e\left(\frac{h_j}{a_{j+1}}\right) \end{bmatrix}$$

$$= \frac{b_N - b_{N-1}}{\det \mathcal{Z}_N} e\left(-\frac{h_{N-1}}{a_N}\right) K_{N-1} (-2)^{N-2} \left(\prod_{j=1}^{N-2} b_{j+1}\right)$$

$$\stackrel{(\#)}{=} \frac{b_N - b_{N-1}}{\det \mathcal{Z}_N} e\left(-\frac{h_{N-1}}{a_N}\right) \frac{(-2)^{N-2} i}{2a_1 \rho} \frac{1}{\det \mathcal{Z}_{N-1}} \left(\prod_{j=1}^{N-2} b_j\right)$$

$$\times \sum_{\nu=\pm 1} e\left(\nu \frac{y}{a_1} - \frac{h_{N-1}}{a_{N-1}}\right) (-2)^{N-2} \left(\prod_{j=1}^{N-2} b_{j+1}\right)$$

$$= \frac{2^{2N-4}}{2a_1 i \rho} \frac{b_{N-1} - b_N}{\det \mathcal{Z}_N \det \mathcal{Z}_{N-1}} \left\{\prod_{j=1}^{N-2} (b_j b_{j+1})\right\} \sum_{\nu=\pm 1} e\left(\nu \frac{y}{a_1} - \frac{h_{N-1}}{a_{N-1}} - \frac{h_{N-1}}{a_N}\right),$$

where we use Corollary 4 at (#) and we write

$$\mathcal{Z}_N = \mathcal{Z}_N(\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}),$$

$$\mathcal{Z}_{N-1} = \mathcal{Z}_{N-1}(\rho; a_1, \ldots, a_{N-1}; b_1, \ldots, b_{N-1}; h_1, \ldots, h_{N-2}),$$

$$K_{N-1} = K_{N-1}(\rho; a_1, \ldots, a_{N-1}; b_1, \ldots, b_{N-1}; h_1, \ldots, h_{N-2}, h_{N-1}; y),$$

$$L_{N-1} = L_{N-1}(\rho; a_1, \ldots, a_{N-1}; b_1, \ldots, b_{N-1}; h_1, \ldots, h_{N-2}, h_{N-1}; y)$$

for short notation. □

Proposition 7. *For* $N \geq 2$,

$$F_1^{(N)}(t, x; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y)$$

$$= f^{(N)}\left(t, x; b_1, \ldots, b_N; \frac{h_1}{a_1}, \frac{h_2 - h_1}{a_2}, \ldots, \frac{h_{N-1} - h_{N-2}}{a_{N-1}}; y\right)$$

holds, where we put

$$f^{(N)}(t, x; b_1, \ldots, b_N; \Theta_1, \ldots, \Theta_{N-1}; y)$$

$$:= -\frac{1}{2a_1} \sum_{\substack{0 \leq m_k < \infty \\ (k=1,\ldots,N-1)}} \psi_N(m_1, \ldots, m_{N-1}; b_1, \ldots, b_N)$$

$$\times \sum_{\nu, \tilde{\nu}=\pm 1} H\left(t - \left(\nu \frac{y}{a_1} + \tilde{\nu} \frac{x}{a_1} + 2 \sum_{J=1}^{N-1} (m_J + 1) \Theta_J\right)\right),$$

and define ψ_N *by*

$$\psi_2(m_1; b_1, b_2) = \left(\frac{b_1 - b_2}{b_1 + b_2}\right)^{m_1+1}$$

for $N = 2$ *and as following for* $N \geq 3$:

$$\begin{aligned}
&\psi_N(m_1, \dots, m_{N-1}; b_1, \dots, b_N) \\
&= \sum_{\{(j_\alpha)_{\alpha\in C_N}, (i_\beta)_{\beta\in A_{N-1}}\}\in G_N} 2^{2N-4}(-1)^{\sum_{k=2}^{N-1} m_k + \sum_{\alpha\in C_N}(1-\#\{k:\,\alpha_k=-1\})j_\alpha + \sum_{\beta\in A_{N-1}}(1-\#\{k:\,\beta_k=-1\})i_\beta} \\
&\quad \times \frac{\left(\sum_{k=1}^{N-1} m_k + \sum_{\alpha\in C_N}(1-\#\{k:\alpha_k=-1\})j_\alpha - \sum_{\beta\in A_{N-1}}\#\{k:\beta_k=-1\}i_\beta\right)!}{\left[\prod_{k=1}^{N-2}\left\{\left(m_k - \sum_{\alpha\in C_N^{(k)-}} j_\alpha - \sum_{\beta\in A_{N-1}^{(k)-}} i_\beta\right)!\right\}\right]\left(m_{N-1} - \sum_{\alpha\in C_N^{(N-1)-}} j_\alpha\right)!} \\
&\quad \times \frac{\left(\sum_{\beta\in A_{N-1}} i_\beta\right)!}{\left\{\prod_{\alpha\in C_N}(j_\alpha!)\right\}\left\{\prod_{\beta\in A_{N-1}}(i_\beta!)\right\}} \\
&\quad \times \left\{\prod_{J=1}^{N-3} \frac{b_J b_{J+1}}{(b_J + b_{J+1})^2}\left(\frac{b_J - b_{J+1}}{b_J + b_{J+1}}\right)^{m_J + m_{J+1} - 2\sum_{\alpha\in C_N^{(J,J+1)-}} j_\alpha - 2\sum_{\beta\in A_{N-1}^{(J,J+1)-}} i_\beta}\right\} \\
&\quad \times \frac{b_{N-2} b_{N-1}}{(b_{N-2} + b_{N-1})^2}\left(\frac{b_{N-2} - b_{N-1}}{b_{N-2} + b_{N-1}}\right)^{m_{N-2} + m_{N-1} - 2\sum_{\alpha\in C_N^{(N-2,N-1)-}} j_\alpha} \\
&\quad \times \left(\frac{b_{N-1} - b_N}{b_{N-1} + b_N}\right)^{m_{N-1}+1}.
\end{aligned}$$

Here we define $\prod_{J=1}^{N-3}(*) = 1$ *for* $N = 3$ *and we put*

$$\begin{aligned}
A_N &= \{\alpha = (\alpha_1, \dots, \alpha_{N-1}) \colon \alpha_k = \pm 1,\ \alpha \neq (1, 1, \dots, 1)\}, \\
B_N &= \{\alpha \in A_N \colon\ \#\{k \colon \alpha_k = -1\} = 1\}, \\
C_N &= A_N \setminus B_N, \\
A_N^{(k_1,\dots,k_\upsilon)\pm} &= \{\alpha \in A_N \colon \alpha_{k_1} = \cdots = \alpha_{k_\upsilon} = \pm 1\}, \\
C_N^{(k_1,\dots,k_\upsilon)\pm} &= \{\alpha \in C_N \colon \alpha_{k_1} = \cdots = \alpha_{k_\upsilon} = \pm 1\}, \\
G_N &= G_N(m_1, \dots, m_{N-1}) \\
&= \left\{\begin{array}{l}
\{(j_\alpha)_{\alpha\in C_N},\ (i_\beta)_{\beta\in A_{N-1}}\} \colon \\
j_\alpha \geq 0,\ i_\beta \geq 0, \\
\displaystyle\sum_{\alpha\in C_N^{(k)-}} j_\alpha + \sum_{\beta\in A_{N-1}^{(k)-}} i_\beta \leq m_k \ (1 \leq k \leq N-2), \\
\displaystyle\sum_{\alpha\in C_N^{(N-1)-}} j_\alpha \leq m_{N-1}
\end{array}\right\}.
\end{aligned}$$

REMARK 8. For example, ψ_3 and ψ_4 are as following:

$$\psi_3(m_1, m_2; b_1, b_2, b_3) = \sum_{\substack{j,\iota \geq 0,\\ j+\iota \leq m_1,\\ j \leq m_2}} (-1)^{m_2-j} 2^2 \frac{(m_1+m_2-j-\iota)!}{(m_1-j-\iota)!\,(m_2-j)!\,j!} \times \frac{b_1 b_2}{(b_1+b_2)^2}\left(\frac{b_1-b_2}{b_1+b_2}\right)^{m_1+m_2-2j}\left(\frac{b_2-b_3}{b_2+b_3}\right)^{m_2+1},$$

$$\begin{aligned}\psi_4(m_1, m_2, m_3; b_1, b_2, b_3, b_4) &= \sum_{\substack{j_1,j_2,j_3,j_4,i_1,i_2,i_3 \geq 0\\ j_2+j_3+j_4+i_2+i_3 \leq m_1\\ j_1+j_3+j_4+i_1+i_3 \leq m_2\\ j_1+j_2+j_4 \leq m_3}} (-1)^{m_2+m_3-j_1-j_2-j_3-2j_4-i_3} 2^4\\ &\quad \times \frac{(m_1+m_2+m_3-j_1-j_2-j_3-2j_4-i_1-i_2-2i_3)!}{(m_1-j_2-j_3-j_4-i_2-i_3)!\,(m_2-j_1-j_3-j_4-i_1-i_3)!}\\ &\quad \times \frac{(i_1+i_2+i_3)!}{(m_3-j_1-j_2-j_4)!\,j_1!\,j_2!\,j_3!\,j_4!\,i_1!\,i_2!\,i_3!}\\ &\quad \times \frac{b_1 b_2}{(b_1+b_2)^2}\left(\frac{b_1-b_2}{b_1+b_2}\right)^{m_1+m_2-2(j_3+j_4+i_3)}\\ &\quad \times \frac{b_2 b_3}{(b_2+b_3)^2}\left(\frac{b_2-b_3}{b_2+b_3}\right)^{m_2+m_3-2(j_1+j_4)}\left(\frac{b_3-b_4}{b_3+b_4}\right)^{m_3+1}.\end{aligned}$$

In the case of ψ_3, the indices j and ι correspond to the indices j_α and i_β in Proposition 7, respectively. We remark that $A_2 = \{(-1)\}$ and $C_3 = \{(-1,-1)\}$. In the same way, the indices j_κ ($\kappa = 1,2,3,4$) and i_κ ($\kappa = 1,2,3$) correspond to the indices j_α and i_β in Proposition 7 as following, respectively: $j_1 = j_{(1,-1,-1)}$, $j_2 = j_{(-1,1,-1)}$, $j_3 = j_{(-1,-1,1)}$, $j_4 = j_{(-1,-1,-1)}$, $i_1 = i_{(1,-1)}$, $i_2 = i_{(-1,1)}$, $i_3 = i_{(-1,-1)}$.

Proof of Proposition 7. By the equation (13) for $k = 1$ and Lemmas 6 and 1, we have

$$\begin{aligned}&\widehat{F}_1^{(N)}(\rho, x; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y)\\ &= -\frac{1}{2a_1}\frac{1}{i\rho} 2^{2N-4}(b_{N-1}-b_N)\left\{\prod_{J=1}^{N-2}(b_J b_{J+1})\right\}\\ &\quad \times \frac{1}{Z_N}\frac{1}{Z_{N-1}}\sum_{\nu,\widetilde{\nu}=\pm 1} e\left(\nu\frac{y}{a_1} + \widetilde{\nu}\frac{x}{a_1} - \frac{h_{N-1}-h_{N-2}}{a_{N-1}}\right),\end{aligned} \tag{18}$$

where we write

$$Z_N = Z_N\left(\rho; b_1, \ldots, b_N; \frac{h_1}{a_1}, \frac{h_2-h_1}{a_2}, \ldots, \frac{h_{N-1}-h_{N-2}}{a_{N-1}}\right),$$

$$Z_{N-1} = Z_{N-1}\left(\rho; b_1, \ldots, b_{N-1}; \frac{h_1}{a_1}, \frac{h_2 - h_1}{a_2}, \ldots, \frac{h_{N-2} - h_{N-3}}{a_{N-2}}\right)$$

for short notation. Then we discuss $1/Z_M(\rho; b_1, \ldots, b_M; \Theta_1, \ldots, \Theta_{M-1})$. We first have

$$\begin{aligned}
&Z_M(\rho; b_1, \ldots, b_M; \Theta_1, \ldots, \Theta_{M-1}) \\
&= \left\{\prod_{J=1}^{M-1}(b_J + b_{J+1})\right\} e\left(\sum_{J=1}^{M-1}\Theta_J\right) \\
&\quad\times\left[1 - \left\{-\sum_{\alpha\in A_M}\alpha_1\left(\prod_{J=1}^{M-2}\frac{b_J + \alpha_J\alpha_{J+1}b_{J+1}}{b_J + b_{J+1}}\right)\frac{b_{M-1} + \alpha_{M-1}b_M}{b_{M-1} + b_M} e\left(\sum_{J=1}^{M-1}(\alpha_J - 1)\Theta_J\right)\right\}\right]
\end{aligned}$$

for $M \geq 2$. Here, we remark that the absolute value of

$$\sum_{\alpha\in A_M}\alpha_1\left(\prod_{J=1}^{M-2}\frac{b_J + \alpha_J\alpha_{J+1}b_{J+1}}{b_J + b_{J+1}}\right)\frac{b_{M-1} + \alpha_{M-1}b_M}{b_{M-1} + b_M} e\left(\sum_{J=1}^{M-1}(\alpha_J - 1)\Theta_J\right)$$

can be small enough when the positive number m is large enough. Then we obtain

$$\begin{aligned}
&\frac{1}{Z_M(b_1, \ldots, b_M; \Theta_1, \ldots, \Theta_{M-1})} \\
&= \frac{1}{\prod_{J=1}^{M-1}(b_J + b_{J+1})} e\left(-\sum_{J=1}^{M-1}\Theta_J\right) \\
&\quad\times\sum_{K=0}^{\infty}\left\{\sum_{\alpha\in A_M}(-\alpha_1)\left(\prod_{J=1}^{M-2}\frac{b_J + \alpha_J\alpha_{J+1}b_{J+1}}{b_J + b_{J+1}}\right)\right. \\
&\qquad\qquad\left.\times\frac{b_{M-1} + \alpha_{M-1}b_M}{b_{M-1} + b_M} e\left(\sum_{J=1}^{M-1}(\alpha_J - 1)\Theta_J\right)\right\}^K \\
&= \frac{1}{\prod_{J=1}^{M-1}(b_J + b_{J+1})}\sum_{\substack{0\leq j_\alpha<\infty \\ (\alpha\in A_M)}}\frac{\left(\sum_{\alpha\in A_M} j_\alpha\right)!}{\prod_{\alpha\in A_M}(j_\alpha!)} \\
&\qquad\times(-1)^{\sum_{\alpha\in A_M^{(1)+}} j_\alpha}\left\{\prod_{J=1}^{M-2}\left(\frac{b_J - b_{J+1}}{b_J + b_{J+1}}\right)^{\sum_{\substack{\alpha\in A_M \\ \alpha_J\alpha_{J+1}=-1}} j_\alpha}\right\} \\
&\qquad\times\left(\frac{b_{M-1} - b_M}{b_{M-1} + b_M}\right)^{\sum_{\alpha\in A_M^{(M-1)-}} j_\alpha} e\left(-\sum_{J=1}^{M-1}\left\{2\sum_{\alpha\in A_M^{(J)-}} j_\alpha + 1\right\}\Theta_J\right).
\end{aligned}$$

We substitute this equation into the equation (18). Then we have

$$
\begin{aligned}
&\widehat{F}_1^{(N)}(\rho, x; a_1, \dots, a_N; b_1, \dots, b_N; h_1, \dots, h_{N-1}; y) \\
&= -\frac{1}{2a_1}\frac{1}{i\rho} 2^{2N-4} \left\{ \prod_{J=1}^{N-2} \frac{b_J b_{J+1}}{(b_J + b_{J+1})^2} \right\} \\
&\quad \times \sum_{\substack{0 \le j_\alpha < \infty \\ (\alpha \in A_N)}} \sum_{\substack{0 \le i_\beta < \infty \\ (\beta \in A_{N-1})}} \frac{\left(\sum_{\alpha \in A_N} j_\alpha\right)! \left(\sum_{\beta \in A_{N-1}} i_\beta\right)!}{\left\{\prod_{\alpha \in A_N} (j_\alpha !)\right\}\left\{\prod_{\beta \in A_{N-1}} (i_\beta !)\right\}} \\
&\qquad \times (-1)^{\sum_{\alpha \in A_N^{(1)+}} j_\alpha + \sum_{\beta \in A_{N-1}^{(1)+}} i_\beta} \left\{ \prod_{J=1}^{N-3} \left(\frac{b_J - b_{J-1}}{b_J + b_{J+1}} \right)^{\sum_{\substack{\alpha \in A_N \\ \alpha_J \alpha_{J+1} = -1}} j_\alpha + \sum_{\substack{\beta \in A_{N-1} \\ \beta_J \beta_{J+1} = -1}} i_\beta} \right\} \\
&\qquad \times \left(\frac{b_{N-2} - b_{N-1}}{b_{N-2} + b_{N-1}} \right)^{\sum_{\substack{\alpha \in A_N \\ \alpha_{N-2}\alpha_{N-1} = -1}} j_\alpha + \sum_{\beta \in A_{N-1}^{(N-2)-}} i_\beta} \\
&\qquad \times \left(\frac{b_{N-1} - b_N}{b_{N-1} + b_N} \right)^{\sum_{\alpha \in A_N^{(N-1)-}} j_\alpha + 1} \\
&\qquad \times \sum_{\nu, \widetilde{\nu} = \pm 1} e\left(\nu \frac{y}{a_1} + \widetilde{\nu} \frac{x}{a_1} - 2 \sum_{J=1}^{N-2} \left(\sum_{\alpha \in A_N^{(J)-}} j_\alpha + \sum_{\beta \in A_{N-1}^{(J)-}} i_\beta + 1 \right) \frac{h_J - h_{J-1}}{a_J} \right. \\
&\qquad\qquad \left. - 2 \left(\sum_{\alpha \in A_N^{(N-1)-}} j_\alpha + 1 \right) \frac{h_{N-1} - h_{N-2}}{a_{N-1}} \right)
\end{aligned}
$$

for $N \ge 3$. Now, we apply the inverse Fourier-Laplace transformation with respect to τ to this equation, and we change the indices from j_α $(\alpha \in B_N)$ to m_k by the relations

$$
\begin{aligned}
j_{(1,\dots,1,\overset{(k)}{-1},1,\dots,1)} &= m_k - \sum_{\alpha \in C_N^{(k)-}} j_\alpha - \sum_{\beta \in A_{N-1}^{(k)-}} i_\beta \quad (1 \le k \le N-2), \\
j_{(1,\dots,1-1)} &= m_{N-1} - \sum_{\alpha \in C_N^{(N-1)-}} j_\alpha .
\end{aligned}
$$

Here we remark that

$$
\mathcal{F}_\tau^{-1}\left[\frac{e^{i\rho s}}{i\rho} \right](t) = H(t+s).
$$

Then we obtain this proposition for $N \ge 3$. We can also prove the case of $N = 2$ in the same way. $\square$

3. The proof of the main result

In this section, we prove our main result. We first discuss the behavior of the function $f^{(p)}(t,0)$ near $t=0$ in Lemmas 9 and 10.

Lemma 9. *For $p \geq 2$ and $\Theta_1 > y/a_1$, $\Theta_j > 0$ $(j=2,\dots,p-1)$,*

$$f^{(p)}(t,0;b_1,\dots,b_p;\Theta_1,\dots,\Theta_{p-1};y) = \begin{cases} 0, & t \in \left[0, -\dfrac{y}{a_1} + 2\displaystyle\sum_{j=1}^{p-1}\Theta_j\right), \\ -\dfrac{1}{a_1} q_p(b_1,\dots,b_p), & t \in \left(-\dfrac{y}{a_1} + 2\displaystyle\sum_{j=1}^{p-1}\Theta_j, -\dfrac{y}{a_1} + 2\displaystyle\sum_{j=1}^{p-1}\Theta_j + \varepsilon_p\right) \end{cases}$$

holds, where we define

$$q_p(b_1,\dots,b_p) = 2^{2p-4}\left\{\prod_{j=1}^{p-2}\frac{b_j b_{j+1}}{(b_j+b_{j+1})^2}\right\}\frac{b_{p-1}-b_p}{b_{p-1}+b_p},$$

$$\varepsilon_p = \varepsilon_p(a_1;\Theta_1,\dots,\Theta_{p-1};y) = 2\min\left\{\frac{y}{a_1},\Theta_1,\dots,\Theta_{p-1}\right\}.$$

Proof. From

$$\psi_p(0,\dots,0;b_1,\dots,b_p) = q_p(b_1,\dots,b_p)$$

and

$$\nu\frac{y}{a_1} + 2\sum_{j=1}^{p-1}(m_j+1)\Theta_j \geq -\frac{y}{a_1} + 2\sum_{j=1}^{p-1}\Theta_j + \varepsilon_p \quad (\nu = \pm 1)$$

except for $(m_1,\dots,m_{p-1};\nu) = (0,\dots,0;-1)$ we obtain this lemma. □

Lemma 10. *For $N > k+1 \geq 2$ and $\Theta_1 > y/a_1$, $\Theta_j > 0$ $(j=2,\dots,N-1)$,*

$$\sum_{p=k+1}^{N} f^{(p)}(t,0;b_1,\dots,b_p;\Theta_1,\dots,\Theta_{p-1};y) = \begin{cases} 0, & t \in \left[0, -\dfrac{y}{a_1} + 2\displaystyle\sum_{j=1}^{k}\Theta_j\right), \\ -\dfrac{1}{a_1} q_{k+1}(b_1,\dots,b_{k+1}), & t \in \left(-\dfrac{y}{a_1} + 2\displaystyle\sum_{j=1}^{k}\Theta_j, -\dfrac{y}{a_1} + 2\displaystyle\sum_{j=1}^{k}\Theta_j + \widetilde{\varepsilon}_k\right) \end{cases}$$

holds, where we define

$$\widetilde{\varepsilon}_k = \widetilde{\varepsilon}_k(a_1; \Theta_1, \ldots, \Theta_k, \Theta_{k+1}; y) = 2 \min\left\{ \frac{y}{a_1}, \Theta_1, \ldots, \Theta_k, \Theta_{k+1} \right\}.$$

Proof. By Lemma 9, we obtain this lemma easily. □

Here, we state the proposition which is the key of the proof of our main result.

Proposition 11. *Let* $N \geq k+1 \geq 2$ *and* $\Theta_1 > y/a_1$, $\Theta_j > 0$ $(j = 2, \ldots, N-1)$. *Suppose* $b_k \neq b_{k+1}$. *Let* $T > 0$. *Put*

$$\widetilde{v}(t) := \sum_{p=k+1}^{N} f^{(p)}(t, 0; b_1, \ldots, b_p; \Theta_1, \ldots, \Theta_{p-1}; y).$$

Then the following holds:

- *If* $\widetilde{v}(t) \equiv 0$ *on* $[0, T)$ *then*

$$\Theta_k \geq \frac{1}{2}\left(T + \frac{y}{a_1}\right) - \sum_{j=1}^{k-1} \Theta_j. \tag{19}$$

- *Assume* $\widetilde{v}(t) \not\equiv 0$ *on* $[0, T)$. *Put* $t_k := \inf\{t \in [0, T) : \widetilde{v}(t) \neq 0\}$. *Then there exist a constant* c_k *and a positive constant* $\varepsilon'_k > 0$ *such that*

$$\widetilde{v}(t) \equiv c_k \quad on \quad (t_k, t_k + \varepsilon'_k).$$

Furthermore

$$\Theta_k = \frac{1}{2}\left(t_k + \frac{y}{a_1}\right) - \sum_{j=1}^{k-1} \Theta_j, \tag{20}$$

$$b_{k+1} = \frac{2^{2k-2} \prod_{j=1}^{k-1}(b_j b_{j+1}) + c_k a_1 \prod_{j=1}^{k-1}(b_j + b_{j+1})^2}{2^{2k-2} \prod_{j=1}^{k-1}(b_j b_{j+1}) - c_k a_1 \prod_{j=1}^{k-1}(b_j + b_{j+1})^2} b_k \tag{21}$$

hold.

Proof. By Lemmas 9 and 10, there exists $\varepsilon > 0$ such that

$$\widetilde{v}(t) = \begin{cases} 0, & t \in \left[0, -\dfrac{y}{a_1} + 2\displaystyle\sum_{j=1}^{k} \Theta_j\right), \\ -\dfrac{1}{a_1} q_{k+1}(b_1, \ldots, b_{k+1}), & t \in \left(-\dfrac{y}{a_1} + 2\displaystyle\sum_{j=1}^{k} \Theta_j,\ -\dfrac{y}{a_1} + 2\displaystyle\sum_{j=1}^{k} \Theta_j + \varepsilon\right) \end{cases}$$

holds. We remark that $q_{k+1}(b_1,\ldots,b_{k+1}) \neq 0$ since we assume that $b_k \neq b_{k+1}$. If $\widetilde{v}(t) \equiv 0$ on $[0,T)$ then we have

$$T \leq -\frac{y}{a_1} + 2\sum_{j=1}^{k} \Theta_j,$$

namely the equation (19). Hereafter we assume that $\widetilde{v}(t) \not\equiv 0$ on $[0,T)$. Then the constant t_k in this proposition satisfies

$$t_k = -\frac{y}{a_1} + 2\sum_{j=1}^{k} \Theta_j.$$

We obtain the equation (20) from this equation. On the other hand, we can take the constant ε_k' in this proposition as ε, and the constant c_k in this proposition satisfies

$$c_k = -\frac{1}{a_1} q_{k+1}(b_1,\ldots,b_{k+1}).$$

By this equation, we have the equation (21). □

Next, we remark that there is a possibility that the same observation data can be obtained even if the unknown constants are different.

Lemma 12. *Let* a_j, b_j $(j = 1,\ldots,N)$, h_j $(j = 1,\ldots,N-1)$, $\widetilde{a}_j$, $\widetilde{b}_j$ $(j = 1,\ldots,\widetilde{N})$, $\widetilde{h}_j$ $(j = 1,\ldots,\widetilde{N}-1)$ *be positive constants. Assume that* $h_j > h_{j-1}$ $(j = 1,\ldots,N-1)$ *and* $\widetilde{h}_j > \widetilde{h}_{j-1}$ $(j = 1,\ldots,\widetilde{N}-1)$, *where we put* $h_0 := 0$ *and* $\widetilde{h}_0 := 0$. *Let* $T > 0$. *Assume* $a_1 = \widetilde{a}_1$. *Suppose*

$$\frac{h_j - h_{j-1}}{a_j} = \frac{\widetilde{h}_j - \widetilde{h}_{j-1}}{\widetilde{a}_j} \quad (1 \leq j \leq N_T - 1), \quad b_j = \widetilde{b}_j \quad (1 \leq j \leq N_T),$$

where the natural number N_T *satisfies*

$$T \leq -\frac{y}{a_1} + 2\min\left\{\sum_{j=1}^{N_T} \frac{h_j - h_{j-1}}{a_j}, \sum_{j=1}^{N_T} \frac{\widetilde{h}_j - \widetilde{h}_{j-1}}{\widetilde{a}_j}\right\}.$$

Then for $t \in [0,T)$

$$\begin{aligned}&u_N(t,0;a_1,\ldots,a_N;b_1,\ldots,b_N;h_1,\ldots,h_{N-1};y)\\ &\equiv u_{\widetilde{N}}\big(t,0;\widetilde{a}_1,\ldots,\widetilde{a}_{\widetilde{N}};\widetilde{b}_1,\ldots,\widetilde{b}_{\widetilde{N}};\widetilde{h}_1,\ldots,\widetilde{h}_{\widetilde{N}-1};y\big)\end{aligned}$$

holds.

Proof. We remark that we have

$$\begin{aligned}&u_N(t,0;a_1,\dots,a_N;b_1,\dots,b_N;h_1,\dots,h_{N-1};y)\\&=u_1(t,0;a_1;b_1;\ \cdot\ ;y)\\&\quad-\sum_{p=2}^{N}f^{(p)}\biggl(t,0;b_1,\dots,b_p;\frac{h_1}{a_1},\frac{h_2-h_1}{a_2},\dots,\frac{h_{p-1}-h_{p-2}}{a_{p-1}};y\biggr)\end{aligned}$$

by the definition of $F_1^{(p)}(t,x)$ and Proposition 7. In particular, we have

$$\begin{aligned}&u_N(t,0;a_1,\dots,a_N;b_1,\dots,b_N;h_1,\dots,h_{N-1};y)\\&=u_1(t,0;a_1;b_1;\ \cdot\ ;y)\\&\quad-\sum_{p=2}^{N_T}f^{(p)}\biggl(t,0;b_1,\dots,b_p;\frac{h_1}{a_1},\frac{h_2-h_1}{a_2},\dots,\frac{h_{p-1}-h_{p-2}}{a_{p-1}};y\biggr)\\&\quad-\sum_{p=N_T+1}^{N}f^{(p)}\biggl(t,0;b_1,\dots,b_p;\frac{h_1}{a_1},\frac{h_2-h_1}{a_2},\dots,\frac{h_{p-1}-h_{p-2}}{a_{p-1}};y\biggr),\end{aligned}$$

and the last term vanishes for $t\in[0,T)$ by Lemma 10. Then we obtain this lemma. □

By Lemma 12, we cannot identify a_k and h_k themselves even if the observation data on $[0,\infty)$ are given. We can identify only b_k and $(h_k-h_{k-1})/a_k$. Then we try to reconstruct them.

Now, we state the process in order to reconstruct them.

Theorem 13. *Suppose the constants a_1, b_1, y are known. Assume $b_j\neq b_{j+1}$ for $j=1,\dots,N-1$. Assume that the observation data $v(t):=u_N(t,0)$ are given on $[0,T)$, where $u_N(t,x)$ is the solution of (1)–(6). Then b_{k+1} and $(h_k-h_{k-1})/a_k$ $(k=1,\dots,N_0-1)$ are reconstructed by the following process*:

- *The first step*: *Put $v_1(t):=(1/a_1)H(t-y/a_1)-v(t)$.*
- *The $(k+1)$-st step $(k=1,2,\dots)$: If $v_k(t)\equiv 0$ on $[0,T)$ then the process is finished. If $v_k(t)\not\equiv 0$ on $[0,T)$ then we carry out the following process: Put $t_k:=\inf\{t\in[0,T)\colon v_k(t)\neq 0\}$. Then there exist a constant c_k and a positive constant ε_k' such that*

$$v_k(t)\equiv c_k\quad\text{on}\quad(t_k,t_k+\varepsilon_k').$$

The constants $(h_k-h_{k-1})/a_k$ and b_{k+1} are reconstructed by

$$\frac{h_k-h_{k-1}}{a_k}:=\frac{1}{2}\biggl(t_k+\frac{y}{a_1}\biggr)-\sum_{j=1}^{k-1}\frac{h_j-h_{j-1}}{a_j},$$

$$b_{k+1} := \frac{2^{2k-2}\prod_{j=1}^{k-1}(b_j b_{j+1}) + c_k a_1 \prod_{j=1}^{k-1}(b_j + b_{j+1})^2}{2^{2k-2}\prod_{j=1}^{k-1}(b_j b_{j+1}) - c_k a_1 \prod_{j=1}^{k-1}(b_j + b_{j+1})^2} b_k.$$

We define

$$v_{k+1}(t) := v_k(t) + \frac{1}{a_1} \sum_{\substack{m_l\ (1\le l\le k): \\ \sum_{j=1}^{k}(m_j+1)(h_j-h_{j-1})/a_j \le (1/2)(T+y/a_1)}} \psi_{k+1}(m_1, \dots, m_k; b_1, \dots, b_{k+1})$$
$$\times \sum_{\nu=\pm 1} H\left(t - \left(\nu \frac{y}{a_1} + 2\sum_{j=1}^{k}(m_j+1)\frac{h_j - h_{j-1}}{a_j}\right)\right)$$

and go the next step, where ψ_{k+1} *is defined in* Proposition 7.
Furthermore, when the process is finished at the (N_0+1)*-st step, that is to say,* $v_{N_0}(t) \equiv 0$ *on* $[0, T)$*, we have either* $N = N_0$ *or the following*:

$$N > N_0 \quad \text{and} \quad \frac{h_{N_0} - h_{N_0-1}}{a_{N_0}} \ge \frac{1}{2}\left(T + \frac{y}{a_1}\right) - \sum_{j=1}^{N_0-1}\frac{h_j - h_{j-1}}{a_j}.$$

REMARK 14. For $k = 2, 3, \dots,$ we have

$$\frac{1}{2}\left(t_k + \frac{y}{a_1}\right) - \sum_{j=1}^{k-1}\frac{h_j - h_{j-1}}{a_j} = \frac{1}{2}(t_k - t_{k-1}),$$

that is to say, we can also reconstruct $(h_k - h_{k-1})/a_k$ by

$$\frac{h_k - h_{k-1}}{a_k} = \frac{1}{2}(t_k - t_{k-1}).$$

Proof of Theorem 13. We first remark that

$$\begin{aligned} &u_N(t, 0; a_1, \dots, a_N; b_1, \dots, b_N; h_1, \dots, h_{N-1}; y) \\ &= u_1(t, 0; a_1; b_1; \cdot\,; y) \\ &\quad - \sum_{k=2}^{N} f^{(k)}\left(t, 0; b_1, \dots, b_k; \frac{h_1}{a_1}, \frac{h_2 - h_1}{a_2}, \dots, \frac{h_{k-1} - h_{k-2}}{a_{k-1}}; y\right) \end{aligned}$$

holds as the same way in the proof of Lemma 12. Now, we put $v_1(t) := (1/a_1)H(t - y/a_1) - v(t)$. Then we obtain

$$v_1(t) = \sum_{k=2}^{N} f^{(k)}\left(t, 0; b_1, \dots, b_k; \frac{h_1}{a_1}, \frac{h_2 - h_1}{a_2}, \dots, \frac{h_{k-1} - h_{k-2}}{a_{k-1}}; y\right)$$

since $u_1(t, 0; a_1; b_1; \cdot; y) = (1/a_1)H(t - y/a_1)$ holds. From this equation and Propositions 7 and 11, we obtain this theorem. □

4. Appendix

In this section, we discuss the case that the impedances of the adjacent media may be equal. In this case, the following lemma is a key lemma.

Lemma 15. *Let* $N \geq \kappa + 1$. *If* $b_\kappa = b_{\kappa+1}$ *then*

$$
\begin{aligned}
&F_1^{(N)}(t, x; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y) \\
&= \begin{cases} 0 & (N = \kappa + 1), \\ F_1^{(N-1)}(t, x; a_1, \ldots, a_{\kappa-1}, \overset{(\kappa)}{\widetilde{a}}, a_{\kappa+2}, \ldots, a_N; \\ \qquad b_1, \ldots, b_{\kappa-1}, b_\kappa, b_{\kappa+2}, \ldots, b_N; & (N \geq \kappa + 2) \\ \qquad h_1, \ldots, h_{\kappa-1}, h_{\kappa+1}, h_{\kappa+2}, \ldots, h_{N-1}; y) \end{cases}
\end{aligned}
$$

holds, where the constant $\widetilde{a}$ *satisfies*

$$\frac{h_{\kappa+1} - h_{\kappa-1}}{\widetilde{a}} = \frac{h_\kappa - h_{\kappa-1}}{a_\kappa} + \frac{h_{\kappa+1} - h_\kappa}{a_{\kappa+1}}.$$

Proof. We remark that

$$
\begin{aligned}
&\widehat{F}_1^{(N)}(\rho, x; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y) \\
&= -\frac{1}{2a_1}\frac{1}{i\rho}2^{2N-4}(b_{N-1} - b_N)\left\{\prod_{J=1}^{N-2}(b_J b_{J+1})\right\} \\
&\quad \times \frac{1}{Z_N}\frac{1}{Z_{N-1}}\sum_{\nu,\widetilde{\nu}=\pm 1} e\left(\nu\frac{y}{a_1} + \widetilde{\nu}\frac{x}{a_1} - \frac{h_{N-1} - h_{N-2}}{a_{N-1}}\right)
\end{aligned}
$$

which appears in the proof of Proposition 7, where we write

$$
\begin{aligned}
Z_N &= Z_N\left(\rho; b_1, \ldots, b_N; \frac{h_1}{a_1}, \frac{h_2 - h_1}{a_2}, \ldots, \frac{h_{N-1} - h_{N-2}}{a_{N-1}}\right), \\
Z_{N-1} &= Z_{N-1}\left(\rho; b_1, \ldots, b_{N-1}; \frac{h_1}{a_1}, \frac{h_2 - h_1}{a_2}, \ldots, \frac{h_{N-2} - h_{N-3}}{a_{N-2}}\right)
\end{aligned}
$$

for short notation. Hence

$$\widehat{F}_1^{(\kappa+1)}(\rho, x; a_1, \ldots, a_{\kappa+1}; b_1, \ldots, b_{\kappa+1}; h_1, \ldots, h_\kappa; y) \equiv 0 \tag{22}$$

holds since $b_\kappa - b_{\kappa+1} = 0$. Let $N \geq \kappa + 2$. We remark that we obtain

$$Z_{\kappa+1}(\rho; b_1, \ldots, b_{\kappa+1}; \Theta_1, \ldots, \Theta_\kappa) = 2b_\kappa e(\Theta_\kappa)Z_\kappa(\rho; b_1, \ldots, b_\kappa; \Theta_1, \ldots, \Theta_{\kappa-1})$$

and

$$
\begin{aligned}
&Z_M(\rho; b_1, \ldots, b_M; \Theta_1, \ldots, \Theta_{M-1})\\
&= 2b_\kappa Z_{M-1}(\rho; b_1, \ldots, b_\kappa, b_{\kappa+2}, \ldots, b_M; \Theta_1, \ldots, \Theta_\kappa \overset{(\kappa)}{+} \Theta_{\kappa+1}, \ldots, \Theta_{M-1})
\end{aligned}
$$

for $M \geq \kappa + 2$. Then we have

$$
\begin{aligned}
&\widehat{F}_1^{(N)}(\rho, x; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y)\\
&= \widehat{F}_1^{(N-1)}(\rho, x; a_1, \ldots, a_{\kappa-1}, \overset{(\kappa)}{\widetilde{a}}, a_{\kappa+2}, \ldots, a_N;\\
&\qquad b_1, \ldots, b_{\kappa-1}, b_\kappa, b_{\kappa+2}, \ldots, b_N;\\
&\qquad h_1, \ldots, h_{\kappa-1}, h_{\kappa+1}, h_{\kappa+2}, \ldots, h_{N-1}; y).
\end{aligned}
\tag{23}
$$

Hence we have this lemma by applying the inverse Fourier-Laplace transformations with respect to $\rho = \tau - im \log(2 + |\tau|)$ to the equations (22) and (23). □

Lemma 16. *Let $b_k \neq b_{k+1}$ for $k = 1, \ldots, N-1$. Then*

$$
\begin{aligned}
&F_1^{(M)}(t, x; a_{1,1}, \ldots, a_{1,\lambda_1}, \ldots, a_{\kappa,1}, \ldots, a_{\kappa,\lambda_\kappa}, \ldots,\\
&\qquad a_{N-1,1}, \ldots, a_{N-1,\lambda_{N-1}}, a_{N,1}, \ldots, a_{N,\lambda_N};\\
&\qquad \overbrace{b_1, \ldots, b_1}^{\lambda_1}, \ldots, \overbrace{b_\kappa, \ldots, b_\kappa}^{\lambda_\kappa}, \ldots, \overbrace{b_{N-1}, \ldots, b_{N-1}}^{\lambda_{N-1}}, \overbrace{b_N, \ldots, b_N}^{\lambda_N};\\
&\qquad h_{1,1}, \ldots, h_{1,\lambda_1}, \ldots, h_{\kappa,1}, \ldots, h_{\kappa,\lambda_\kappa}, \ldots,\\
&\qquad h_{N-1,1}, \ldots, h_{N-1,\lambda_{N-1}}, h_{N,1}, \ldots, h_{N,\lambda_N-1}; y)\\
&= \begin{cases} 0, & \lambda_N \geq 2,\\ f^{(N)}\left(t, x; b_1, \ldots, b_N; \displaystyle\sum_{\mu=1}^{\lambda_1} \frac{h_{1,\mu} - h_{1,\mu-1}}{a_{1\mu}}, \ldots, \sum_{\mu=1}^{\lambda_{N-1}} \frac{h_{N-1,\mu} - h_{N-1,\mu-1}}{a_{N-1,\mu}}; y\right), & \\ & \lambda_N = 1, \end{cases}
\end{aligned}
$$

where $M := \sum_{k=1}^{N} \lambda_k$, $h_{1,0} := 0$, and $h_{\kappa,0} := h_{\kappa-1,\lambda_{\kappa-1}}$ for $\kappa = 2, \ldots, N-1$.

Proof. We obtain this lemma from repeating Lemma 15. □

By Lemma 16, we can only find out that the situation is as Fig. 3 when the impedances of the adjacent media may be equal, where we reconstruct b_{k+1} and $(h_k - h_{k-1})/a_k$ as Theorem 13 and the constants $\widetilde{a}_{k,\mu}$ and $\widetilde{h}_{k,\mu}$ satisfy

$$
\sum_{\mu=1}^{\lambda_k} \frac{\widetilde{h}_{k,\mu} - \widetilde{h}_{k,\mu-1}}{\widetilde{a}_{k,\mu}} = \frac{h_k - h_{k-1}}{a_k}.
$$

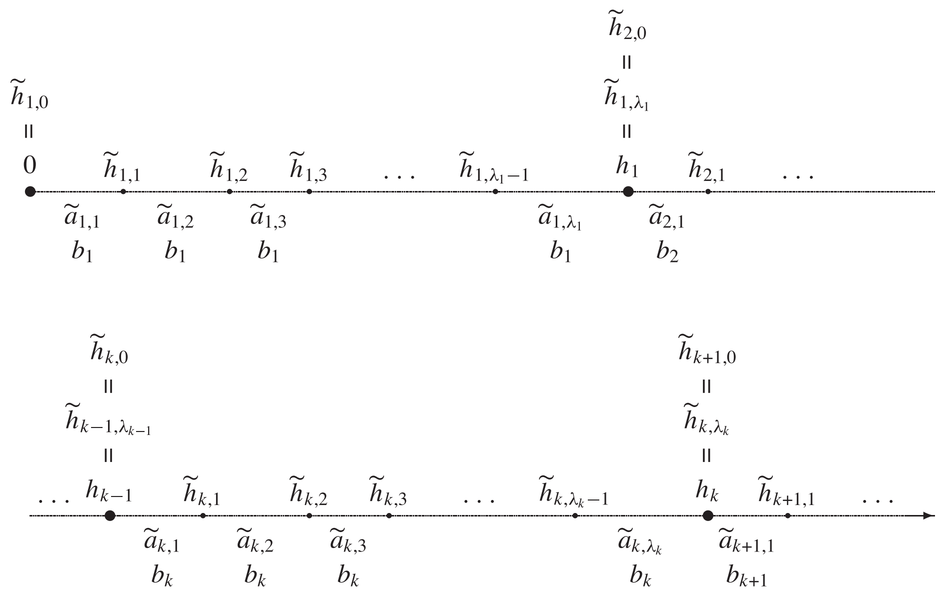


Fig. 3. The situation when the impedances of the adjacent media may be equal.

ACKNOWLEDGEMENTS. This research is partially supported by Grant-in-Aid for JSPS Research Fellowships for Young Scientists. The author would like to express his sincere gratitude to Professor Mitsuru Sugimoto for his unfailing guidance and his useful advice. The author also thanks the referee for useful comments.

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