Watanabe, T.
Osaka J. Math.
44 (2007), 351–362

KNOTTED TRIVALENT GRAPHS AND CONSTRUCTION OF THE LMO INVARIANT FROM TRIANGULATIONS

TADAYUKI WATANABE

(Received January 20, 2006, revised May 30, 2006)

Abstract

We give a Turaev-Viro type construction for the LMO invariant. More precisely, we construct an invariant of closed oriented 3-manifolds from data of their spines or their simplicial decompositions and the values of Kontsevich invariant of the unknotted tetrahedron and the Hopf link by using Bar-Natan and Thurston's operations.

1. Introduction

Reshetikhin and Turaev [13] gave a rigorous definition of quantum invariants of 3-manifolds as linear sums of colored Jones polynomials of surgery presentations. They were extensively generalized by Le, Murakami and Ohtsuki [6]. They constructed a 3-manifold invariant called the LMO invariant with values in certain space of 3-valent graphs by using their operator on the values of Kontsevich invariant of surgery presentations. It is known for integral homology 3-spheres that the LMO invariant is universal among quantum invariants coming from simple Lie groups [9, 1, 5, 3, 4].

On the other hand, Turaev and Viro constructed a 3-manifold invariant (Turaev-Viro invariant) from simplicial decompositions of 3-manifolds [16]. Turaev-Viro invariant is constructed by coloring each edge of simplicial decompositions, associating with each 3-simplex a value called quantum $6j$-symbol determined from the colors of 6 edges of it, multiplying all of them and then summing over all admissible colorings. Although the Turaev-Viro invariant of any closed oriented 3-manifold M is equal to the Reshetikhin-Turaev invariant of $M \# (-M)$ where $-M$ denotes M with its orientation reversed (see e.g. [15, 12]), it is useful when a 3-manifold is presented as a simplicial decomposition.

Knotted trivalent graphs (KTGs) are used in the Turaev-Viro theory. Colored Jones polynomial of links is extended to KTGs by associating with each 3-valent vertex an $U_q(sl_2)$-module of invariants $\mathrm{Hom}_{U_q(sl_2)}(V_k \otimes V_l, V_m)$ (V_k, V_l, V_m: irreducible $U_q(sl_2)$-modules). Similar extension for the Kontsevich invariant was obtained in [8]. Since the quantum $6j$-symbol can be considered to be the value of the colored Jones polynomial of the unknotted tetrahedron , it is natural to expect that the Turaev-Viro theory for

2000 Mathematics Subject Classification. Primary 57M27; Secondary 57M25.

the Kontsevich invariant $\check{Z}$ is similarly obtained by using $\check{Z}(\triangle)$. For the definition of $\check{Z}$, see e.g. [8].

In this paper we give a Turaev-Viro like construction for the LMO invariant. In particular, the main result of this paper is the following:

- We construct an invariant from simplicial decompositions of 3-manifolds and the values $\check{Z}(\triangle)$, $\check{Z}(\boxtimes)$, $\check{Z}(\bigcirc\!\!\!\bigcirc)$ by using some elementary operations.
- The invariant is equal to the LMO invariant of $M\#(-M)$, i.e. even degree part of the LMO invariant of M when M is a rational homology sphere.

Such construction is effective when M is presented as a simplicial decomposition because if we obtained a surgery presentation directly from the simplicial decomposition, then the presentation itself would be very complicated and moreover its sliced q-tangle decomposition to compute its Kontsevich invariant would be surprizingly large if the number of tetrahedra is large. An application of our method that can be considered is deducing formulas of 3-loop part of the LMO invariant for some family of spines, for example a sequence of spines that are generated by some patterns. We will consider this in future work.

We should remark that it is still hard to compute complete values of the LMO invariants since $\check{Z}(\triangle)$ is determined by using the Drinfel'd associator. We can only compute the values up to finite order.

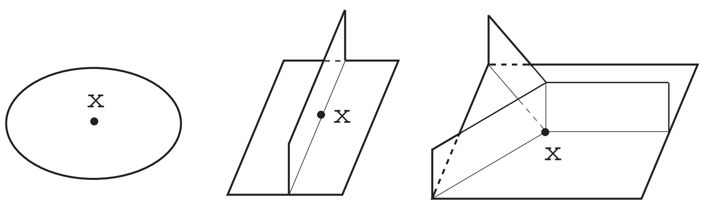
We shall mainly explain the way of construction for special spines. Construction for simplicial decompositions then easily follows as in §5 from the case of special spines.

2. Preliminaries

2.1. Special spines. Now we shall give a construction of an invariant for special spines.

We say that a polyhedron X embedded into a 3-manifold M is a *special spine* if X satisfies the following conditions:

- $M \setminus X$ is homeomorphic to an open 3-ball.
- Each small neighborhood of points in X is homeomorphic to one of the following:



- Each area bounded by edges is homeomorphic to a 2-disc.

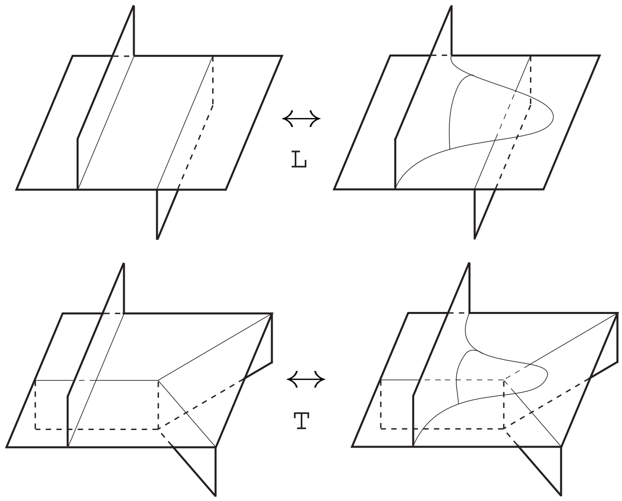
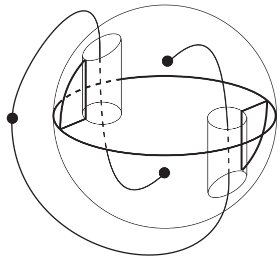


Fig. 1. L-move and T-move

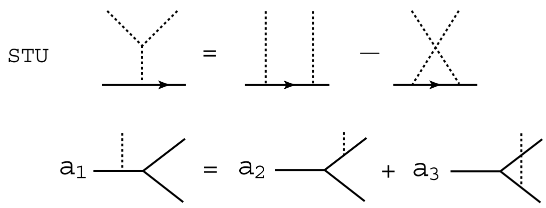
An example of a special spine for S^3 is depicted in the following picture:



The following proposition due to Matveev and Piergallini is a fundamental property of special spines as representations of 3-manifolds.

Proposition 2.1 ([7, 11]). *Two special spines are of the same* 3*-manifold if and only if they are related by moves* L, T *shown in* Fig. 1. *Moreover, we can assume all the intermediate polyhedra are special spines*.

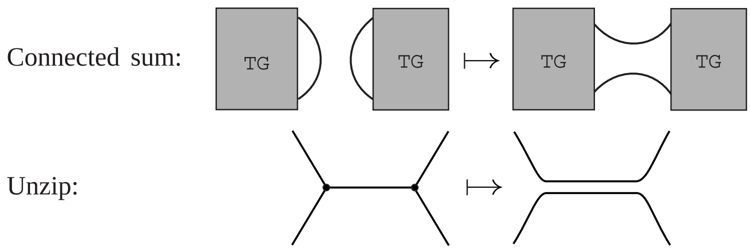
2.2. Trivalent graphs. A *Trivalent graph* (*TG*) is a vertex oriented, edge oriented trivalent graph with fixed vertex orientation. Let Γ be a TG. A *Jacobi diagram based on* Γ is a vertex oriented uni-trivalent graph whose univalent vertices are on Γ. We express the edges of TG by solid lines and those of Jacobi diagrams by dotted lines in pictures. Let $\mathcal{A}(\Gamma)$ be the vector space spanned by all Jacobi diagrams based on Γ modulo the following relations:



where $a_1, a_2, a_3 \in \{1, -1\}$ are determined by the orientation of the edge on which the univalent vertex is landing. If the orientation is right-going then we set the coefficient to be 1 and otherwise to be -1. Each Jacobi diagram has a degree defined by half the number of vertices on its dotted part.

Murakami-Ohtsuki defined an invariant of KTG with values in $\mathcal{A}(\Gamma)$ as an extension of the Kontsevich invariant $\check{Z}\colon \mathcal{KG}(\Gamma) \to \mathcal{A}(\Gamma)$, where $\mathcal{KG}(\Gamma) = \{$framed embeddings$\colon \Gamma \to S^3\}$. For its precise definition, see [8]. In this paper we assume that any KTG is integer framed, i.e. it has a blackboard framing presentation.

2.3. TG operations. *TG operations* are defined by the following picture:



Connected sum is defined by band summing two disjoint connected TGs between edges. Unzip is defined only when the orientation is consistent in the picture. These operations are well defined and have the following good property.

Proposition 2.2 ([14]). *Let G_1, G_2, G_3 be embeddings of connected TGs Γ_1, Γ_2, Γ_3 respectively into S^3. Let X, Y be TGs. If we write $X \#_{ef} Y$ the connected sum between edges e in X and f in Y, and $\mathrm{UZ}_e(X)$ the unzip along the edge e in X, then the following identities hold.*

$$\check{Z}(G_1 \#_{e_1e_2} G_2) = \check{Z}(G_1) \#_{e_1e_2} \check{Z}(G_2),$$
$$\check{Z}(\mathrm{UZ}_{e_3}(G_3)) = \mathrm{UZ}_{e_3}(\check{Z}(G_3)).$$

That is, the following diagrams are commutative.

$$\begin{tikzcd} \mathcal{KG}(\Gamma_1 \cup \Gamma_2) \arrow[r, "\#_{e_1e_2}"] \arrow[d, "\check{Z}"'] & \mathcal{KG}(\Gamma_1 \#_{e_1e_2} \Gamma_2) \arrow[d, "\check{Z}"] \\ \mathcal{A}(\Gamma_1 \cup \Gamma_2) \arrow[r, "\#_{e_1e_2}"] & \mathcal{A}(\Gamma_1 \#_{e_1e_2} \Gamma_2) \end{tikzcd} , \quad \begin{tikzcd} \mathcal{KG}(\Gamma_3) \arrow[r, "\mathrm{UZ}_{e_3}"] \arrow[d, "\check{Z}"'] & \mathcal{KG}(\mathrm{UZ}_{e_3}(\Gamma_3)) \arrow[d, "\mathrm{UZ}_{e_3}"] \\ \mathcal{A}(\Gamma_3) \arrow[r, "\#_{e_1e_2}"] & \mathcal{A}(\mathrm{UZ}_{e_3}(\Gamma_3)) \end{tikzcd} .$$

By using this property, Bar-Natan and Thurston constructed the Kontsevich invariant of links by associating $\check{Z}(\boxtimes)$ with each crossing of a link diagram and doing TG operations. It is different from our construction since we do not use link diagrams. However, the above property is fundamental for ours too.

3. Constructing the invariant

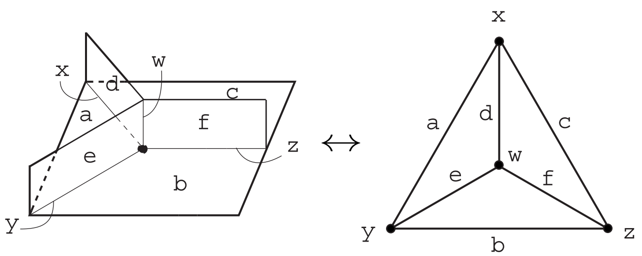
Let M be a closed oriented 3-manifold. We construct the invariant from a special spine X of M by the following procedure. First we shall define a linear map

$$\varphi_n : \mathcal{A}(\triangle)^{\otimes N} \to \mathcal{A}(\emptyset)^{(\deg \leq n)}$$

where N is the number of 4-valent vertices on X and $\mathcal{A}(\emptyset)^{(\deg \leq n)}$ is the space of 3-valent graphs of degree at most n with no underlying TG (the precise definition will be given later).

3.1. Step 1: Associating tetrahedra.

Let X be a special spine of M. First we label each area and each edge of X so that different area (edges) have different labels. By the definition of special spine, the singularity set of X must be a 4-valent graph. We associate a labeled tetrahedron σ_Q with each 4-valent vertex Q as follows:

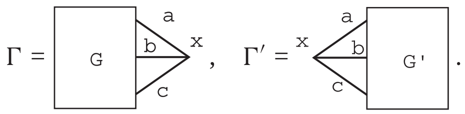


As a result, we get a disjoint union of labeled tetrahedra.

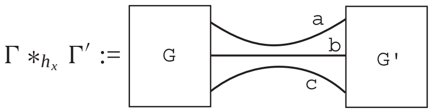
3.2. Step 2: Joining the tetrahedra.

Any point in the interior of an edge x of X joining two 4-valent vertices P, Q, has a neighborhood consisting of exactly 3 sheets meeting at the edge. These sheets determine a bijection h_x between the pair of three edges around vertices each on σ_P and σ_Q corresponding to x.

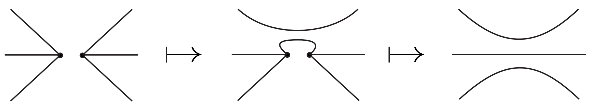
Let Γ, Γ' be connected TGs with the labels as follows:



Then we define their product by



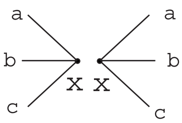
so that the edges are glued together by h_x. This operation is well defined since it is realized by connected sum and unzip as follows:

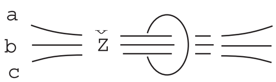


Here we first connect a pair of edges related to each other by h_x. Continuing in this manner, we join the all tetrahedra along a maximal tree $\{x_1, x_2, \ldots, x_{N-1}\}$ in the 1-skeleton of X:

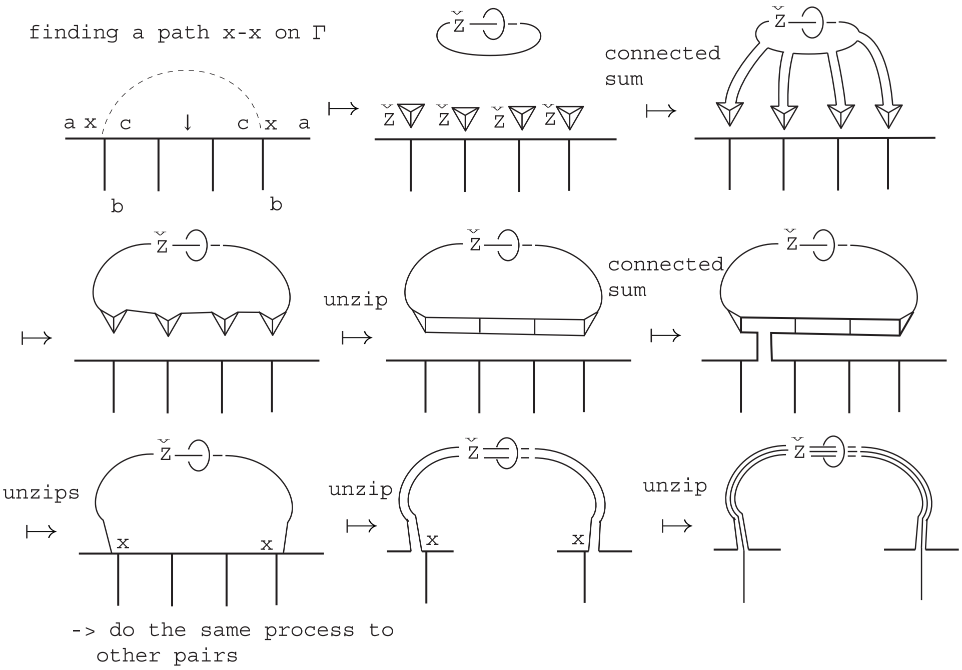
$$Gl_h(\sigma_{Q_1} \otimes \cdots \otimes \sigma_{Q_N}) := \Big(\cdots \Big(\big(\sigma_{Q_1} *_{h_{x_1}} \sigma_{Q_2}\big) *_{h_{x_2}} \sigma_{Q_3}\Big) \cdots \Big) *_{h_{x_{N-1}}} \sigma_{Q_N}$$

to make a connected TG, where the gluing maps $h = (h_{x_1}, \ldots, h_{x_{N-1}})$ are determined by X.

3.3. Step 3: Contraction among the remaining 3-valent vertices. The connected TG obtained in the last step still has 3-valent vertices. Now we shall contract them. If we follow Turaev-Viro's construction, it is natural to replace a pair of remaining vertices 

simply with 

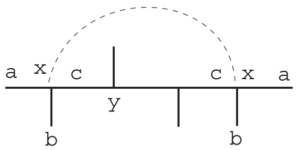
for a contraction because in their formula for the contraction, a value of the invariant of the Hopf link is just multiplied (see [16]). However this is not well defined in this case since the result depends on the positions of dotted legs lying on the three edges incident to x. But this is just a technical problem that we can avoid by defining the contraction as follows:



Dotted parts (1, 3-valent graph parts) are omitted in the picture. In the first picture, we choose a path γ on the TG connecting the pair of vertices that we now want to contract. In the second, disjoint union to it one $\check{Z}(\circledcirc)$ and some $\check{Z}(\triangle)$s as many as the number of vertices on γ. Then we continue unzipping and connect summing to obtain the last picture. Note that the result is still not uniquely determined but is uniquely determined when it is considered modulo the Kirby moves, which we will explain later, along the circle of the Hopf link. Indeed this will suffice for our purpose. We denote by $\mathrm{cntr}_{h'}(\Gamma)$ the result of all contractions among vertices of Γ, where h' is a set of bijections used to contract vertices. Then we obtain an element without 3-valent vertices:

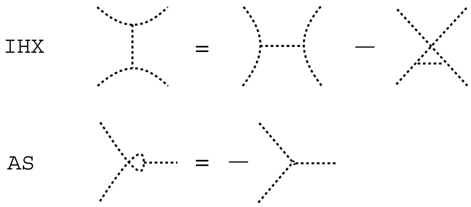
$$\mathrm{cntr}_{h'} \circ Gl_h(\sigma_{Q_1} \otimes \cdots \otimes \sigma_{Q_N}) \in \mathcal{A}(\bigcirc \cdots \bigcirc).$$

Note that in general, vertex orientations of vertices on the path connecting two vertices labeled x may not be as in the picture. For example, it may be as



In such cases, it suffices to disjoint union $\check{Z}(\boxtimes)$ for y insteaed of $\check{Z}(\triangle)$.

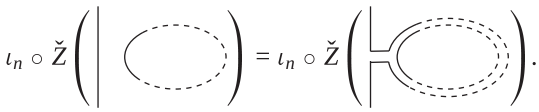
3.4. Step 4: Replacing solid lines with dotted graphs. We then replace solid lines with dotted graphs by the map $\iota_n\colon \mathcal{A}(\bigcirc\cdots\bigcirc)\to\mathcal{A}(\emptyset)^{(\leq n)}$ defined by Le-Murakami-Ohtsuki. Here $\mathcal{A}(\emptyset)^{(\leq n)}$ denotes the vector space spanned by all Jacobi diagrams without univalent vertices with degree at most n quotiented by the IHX, AS relation in the following picture.



The space $\mathcal{A}(\emptyset)^{(\leq n)}$ has an algebraic structure with a product defined by disjoint union.

For the explicit description of ι_n, see [6]. The map ι_n has the following remarkable property.

Proposition 3.1 ([6]). *$\iota_n\circ\check{Z}$ is invariant under the second Kirby move. That is,*



Through the Steps 2 to 4, we have obtained a well defined map

$$\varphi_n\colon \mathcal{A}(\triangle)^{\otimes N}\to\mathcal{A}(\emptyset)^{(\leq n)}$$

defined by

$$\varphi_n(\sigma_{Q_1}\otimes\ldots\otimes\sigma_{Q_N}) := \iota_n\circ\mathrm{cntr}_{h'}\circ Gl_h(\sigma_{Q_1}\otimes\cdots\otimes\sigma_{Q_N})\in\mathcal{A}(\emptyset)^{(\leq n)}.$$

Let g be the number of $\check{Z}(\bigcirc\!\!\bigcirc)$s used in Step 3 and let

$$|X|_n := \iota_n(\check{Z}(\bigcirc\!\!\bigcirc))^{-g}\varphi_n\left(\check{Z}(\triangle)\otimes\cdots\otimes\check{Z}(\triangle)\right)\in\mathcal{A}(\emptyset)^{(\leq n)}.$$

Note that $\iota_n(\check{Z}(\bigcirc\!\!\bigcirc))$ is invertible, which can be verified by direct computations. The details about such computations are found in [10].

Then we have

Theorem 3.2. *$|X|_n$ does not depend on the choice of the special spines of M. So it is an invariant of M.*

The proof of this theorem will be given in the next section.

4. Background and invariance proof

Let Γ_i, $1 \leq i \leq k$ be copies of . Then the construction until Step 3 is summarized as in the following diagram.

$$\begin{tikzcd}
\mathcal{KG}(\Gamma_1 \sqcup \Gamma_2 \sqcup \cdots \sqcup \Gamma_k) \arrow[r, "\#"] \arrow[d, "\check{Z}"'] & \mathcal{KG}(\Gamma_1 \# \Gamma_2 \sqcup \cdots \sqcup \Gamma_k) \arrow[r] \arrow[d, "\check{Z}"] & \cdots \arrow[r] & \mathcal{KG}(\bigcirc \sqcup \cdots \sqcup \bigcirc) \arrow[d, "\check{Z}"] \\
\mathcal{A}(\Gamma_1 \sqcup \Gamma_2 \sqcup \cdots \sqcup \Gamma_k) \arrow[r, "\#"] & \mathcal{A}(\Gamma_1 \# \cdots \# \Gamma_k) \arrow[r] & \cdots \arrow[r] & \mathcal{A}(\bigcirc \sqcup \cdots \sqcup \bigcirc)
\end{tikzcd}$$

More precisely, we go at first from the upper-left corner to the lower-left corner. Then go to the right until getting to the lower right corner. By the commutativity of the diagram, we can obtain the same result by passing the upper-right corner. So the result of Step 3 is equal to the Kontsevich invariant of some framed link. Such obtained framed links have been first studied by Roberts [12][1]. He showed the following strong claim.

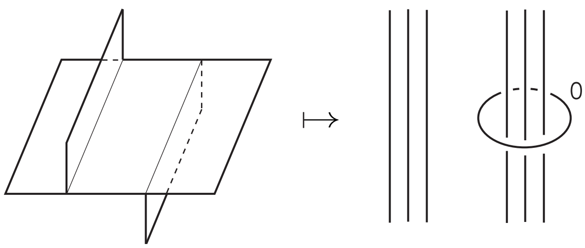
Proposition 4.1 ([12]). *The framed link obtained above is a surgery presentation of $M \# (-M)$.*

This can be proved by using the technique of handle decomposition in 4 dimension. From this proposition, it follows that $|X|_n$ is equal to the n-th part of the LMO invariant of $M \# (-M)$ and hence is an invariant of M. The main point in this paper is that we give a way to compute the LMO invariant for the framed link by using the data of spines and the TG operations and a few ι_n.

Now we shall give an alternative elementary proof of the invariance by using the Matveev-Piergallini theorem (Proposition 2.1) without using Roberts' result.

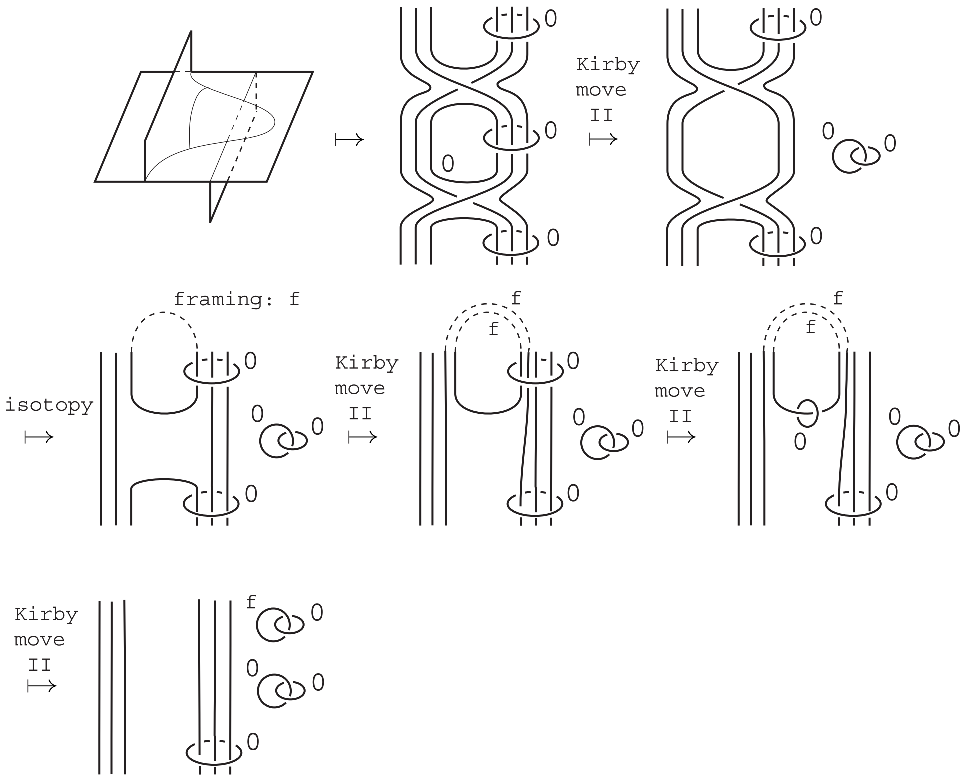
Proof of Theorem 3.2. By the Matveev-Piergallini theorem, it suffices to show that $|X|_n$ is invariant under the L, T-moves in Fig. 1.

We prove only for the L-move since for the T-move is similar. The left hand side of the L-move in Fig. 1 yields the following framed link.



[1]He obtained such framed links, which he calls 'chain mail links', from Heegaard diagrams. They are equivalent to those obtained now.

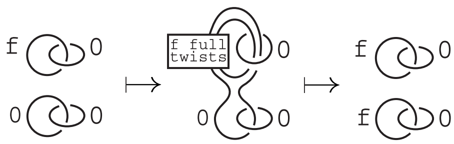
On the other hand, the right hand side of the L-move yields the following framed link.



Lemma 4.2 below allows one to claim that the Hopf links in the picture vanish by the correction term, and hence the both sides of the move are equal. □

Lemma 4.2. $\iota_n\Big(\check{Z}\Big($ f  0 $\Big)\Big) = \iota_n\Big(\check{Z}\Big($ 0  0 $\Big)\Big)$.

Proof. By applying the second Kirby moves, one has the following sequence:



By Proposition 3.1, one has

$\iota_n\Big(\check{Z}\Big($ f  0 $\Big)\Big)\iota_n\Big(\check{Z}\Big($ 0  0 $\Big)\Big) = \iota_n\Big(\check{Z}\Big($ f  0 $\sqcup$ 0  0 $\Big)\Big)$

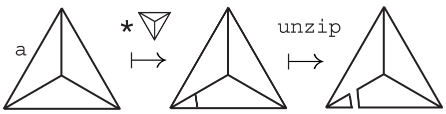
$$= \iota_n\Big(\check{Z}\Big(\mathtt{f}\ \ \mathtt{0} \sqcup \mathtt{f}\ \ \mathtt{0}\Big)\Big) = \iota_n\Big(\check{Z}\Big(\mathtt{f}\ \ \mathtt{0}\Big)\Big)\iota_n\Big(\check{Z}\Big(\mathtt{f}\ \ \mathtt{0}\Big)\Big)$$

Since $\iota_n\Big(\check{Z}\Big(\mathtt{f}\ \ \mathtt{0}\Big)\Big)$ is invertible, the result follows. □

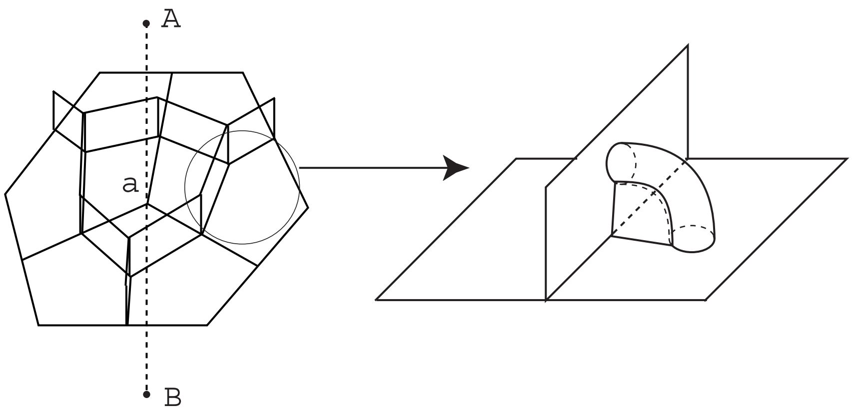
5. Construction from simplicial decomposition

Let X be the 2-skeleton of dual of a simplicial decomposition of M. Note that in this case the same procedure as in §3 does not work since disjoint union of 0-framed unknots may occur (this corresponds to connected sum of $S^2 \times S^1$s). To construct an invariant from X, we make a special spine from X. Since X has at least one closed chamber, we need to join them into a single chamber to make a special spine. This is done as follows.

Let A, B be two vertices of the simplicial decomposition such that they are connected by an edge AB. Let the label of the area dual to AB be a. After we apply Step 1, we do the following modifications to the tetrahedron including an edge labelled by a.



This process corresponds to the following picture. We continue this process until we obtain a special spine. This construction of a spine is due to Casler [2]. Then we apply subsequent steps and get the invariant.



Acknowledgments. I would like to thank my adviser T. Ohtsuki for many helpful advices and comments. I would also like to thank K. Habiro and J.D. Roberts for valuable comments. Especially, Roberts gave much help for improvement of this paper.

References

[1] D. Bar-Natan, S. Garoufalidis, L. Rozansky and D.P. Thurston: *The Århus integral of rational homology* 3*-spheres*. I, Selecta Math. (N.S.) **8** (2002), 315–339.

[2] B.G. Casler: *An imbedding theorem for connected* 3*-manifolds with boundary*, Proc. Amer. Math. Soc. **16** (1965), 559–566.

[3] K. Habiro and T.T.Q. Le: in preparation.

[4] T. Kuriya: *On the LMO conjecture*, preprint, 2003.

[5] T.T.Q. Le: *Quantum invariants of* 3*-manifolds*: *integrality, splittng, and perturbarive expansion*; in Proceedings of the Pacific Institute for the Mathematical Sciences Workshop "Invariants of Three-Manifolds" (Calgary, AB, 1999), Topology Appl. **127** (2003), 125–152.

[6] T.T.Q. Le, J. Murakami and T. Ohtsuki: *On a universal perturbative invariant of* 3*-manifolds*, Topology **37** (1998), 539–574.

[7] S.V. Matveev: *Transformations of special spines, and the Zeeman conjecture*, Math. USSR-Izv. **31** (1988), 423–434.

[8] J. Murakami and T. Ohtsuki: *Topological quantum field theory for the universal quantum invariant*, Comm. Math. Phys. **188** (1997), 501–520.

[9] T. Ohtsuki: *The perturbative* SO(3) *invariant of rational homology* 3*-spheres recovers from the universal perturbative invariant*, Topology **39** (2000), 1103–1135.

[10] T. Ohtsuki: Quantum Invariants—A Study of Knots, 3-Manifolds, and Their Sets, Series on Knots and Everything **29**, World Scientific Publishing Co. Inc., 2002.

[11] R. Piergallini: *Standard moves for the standard polyhedra and spines*, Rend. Circ. Mat. Palermo (2) Suppl. **37** (1988), 391–414.

[12] J. Roberts: *Skein theory and Turaev-Viro invariants*, Topology **34** (1995), 771–787.

[13] N. Reshetikhin and V. Turaev: *Invariants of* 3*-maniolds via link polynomials and quantum groups*, Invent. Math. **103** (1991), 547–597.

[14] D.P. Thurston: *The algebra of knotted trivalent graphs and Turaev's shadow world*; in Invariants of knots and 3-manifolds (Kyoto, 2001), Geom. Topol. Monogr. **4**, Geom. Topol. Publ., Coventry, 2002, 337–362.

[15] V.G. Turaev: Quantum Invariants of Knots and 3-Manifolds, de Gruyter Studies in Mathematics **18**, Walter de Gruyter & Co., Berlin, 1994.

[16] V.G. Turaev and O.Ya. Viro: *State sum invariants of* 3*-manifolds and quantum* $6j$*-symbols*, Topology **31** (1992), 865–902.

[17] E. Witten: *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989), 351–399.

Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606–8502
Japan
e-mail: tadayuki@kurims.kyoto-u.ac.jp