Tomoda, A.
Osaka J. Math.
44 (2007), 231–246

ON THE SPLITTING PRINCIPLE OF BUNDLE GERBE MODULES

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(Received September 20, 2005, revised June 22, 2006)

Abstract

We introduce the notion of an n-trivialization and a compatible curving and construct the splitting of bundle gerbe modules to define the twisted Chern classes and the twisted Chern character for bundle gerbe modules in terms of algebraic topology. Moreover, we prove that the latter coincides with the twisted Chern character due to Bouwknegt-Carey-Mathai-Murray-Stevenson [1] if the bundle gerbe is given an n-trivialization and a compatible curving.

1. Introduction

In order to define the Chern classes for complex vector bundles, there are at least three approaches: the classifying space and the universal bundle; the splitting principle; the Chern-Weil construction.

For a bundle gerbe module W and a curving f, the twisted Chern character $\mathrm{ch}^{\tau}_{\mathrm{DG}}(f, W)$ is introduced by Bouwknegt-Carey-Mathai-Murray-Stevenson [1]. It is an analogue of the Chern-Weil construction and described in terms of differential forms.

In this paper, we shall introduce the twisted Chern classes and the twisted Chern character in terms of algebraic topology by constructing the splitting of a bundle gerbe module into bundle gerbe modules of rank 1. This is an analogue of the splitting principle for complex vector bundles.

Let W be a bundle gerbe module for a bundle gerbe (Y, L) over X. Then we have the projectivization $\widetilde{\mathbb{P}}(W)$ of W. The multiplication of (Y, L) gives rise to a fiber bundle $\widetilde{\mathbb{P}}(W) \to \mathbb{P}(W)$, where $\mathbb{P}(W)$ is a fiber bundle over X with the fiber $\mathbb{CP}^{m-1}$. We should remark that a bundle gerbe $\left(\widetilde{\mathbb{P}}(W), \widetilde{L}\right)$ over $\mathbb{P}(W)$ is induced and that the tautological line bundle $\gamma_W \to \widetilde{\mathbb{P}}(W)$ and the complement $W^{\perp}$ are again bundle gerbe modules for $\left(\widetilde{\mathbb{P}}(W), \widetilde{L}\right)$. So, we obtain the splitting of the bundle gerbe module

$$\overline{p}^* W = \gamma_W \oplus W^{\perp}$$

into bundle gerbe modules for the bundle map $\overline{p}\colon \widetilde{\mathbb{P}}(W) \to Y$. By iterating this con-

2000 Mathematics Subject Classification. Primary 55R65; Secondary 55R20.
The author is partially supported by the 21st century COE program at Keio University.

struction, we have the splitting of a bundle gerbe module into bundle gerbe modules of rank 1.

If an n-trivialization η of (Y, L) is given, it induces a semi-group homomorphism from the isomorphism classes $\mathrm{Mod}(Y, L)$ of bundle gerbe modules to $\mathrm{Vect}(X)$. Via this homomorphism, we define the twisted Euler class $\chi^{\tau}(\eta, \xi)$ for a bundle gerbe module ξ of rank 1. Moreover, the Leray-Hirsch theorem implies the isomorphism of the graded modules

$$H^*(X; \mathbb{Q}) \otimes H^*(\mathbb{CP}^{m-1}; \mathbb{Q}) \cong H^*(\mathbb{P}(W); \mathbb{Q}).$$

Therefore, we can define the twisted Chern class $c^{\tau}(\eta, W)$ for a bundle gerbe module W by using the splitting and the twisted Euler classes in a similar way to the ordinary splitting principle.

The twisted Chern character $\mathrm{ch}^{\tau}_{\mathrm{DG}}(f, W)$ of a bundle gerbe module W for (Y, L) is defined in the situation where a bundle gerbe (Y, L) is given a curving f. On the other hand, the twisted Chern character $\mathrm{ch}^{\tau}_{\mathrm{AT}}(\eta, W)$ in terms of algebraic topology is defined in the situation where (Y, L) is given an n-trivialization η. We introduce the notion of a curving f compatible with an n-trivialization η of (Y, L) in Section 3.

Finally, we prove the following:

Theorem 1. *Let (Y, L) be a bundle gerbe over X endowed with an n-trivialization η and a compatible curving f. For every bundle gerbe module W, we have*

$$\mathrm{ch}^{\tau}_{\mathrm{AT}}(\eta, W) = \mathrm{ch}^{\tau}_{\mathrm{DG}}(f, W).$$

2. Bundle gerbes

We shall devote this section to a brief exposition of the bundle gerbes and bundle gerbe modules and the twisted Chern character. We refer the reader to Murray [2] for bundle gerbes and Bouwknegt-Carey-Mathai-Murray-Stevenson [1] for bundle gerbe modules and the twisted Chern character.

2.1. Bundle gerbes. Let X be a compact oriented smooth manifold and $\pi\colon Y \to X$ a fiber bundle. Then we can consider the fiber product $Y \times_{\pi} Y$ by itself. In general, we denote the k-times iterated fiber product $Y \times_{\pi} \cdots \times_{\pi} Y$ by $Y^{[k]}$. We have a map $\pi_i\colon Y^{[k]} \to Y^{[k-1]}$ which omits the i-th element for $i = 1, \ldots, k$.

DEFINITION 1 (bundle gerbes). Let X and Y be as above and let $L \to Y^{[2]}$ be a Hermitian line bundle. A couple (Y, L) is said to be *a bundle gerbe* over X if L is endowed with a product:

$$L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \xrightarrow{\cong} L_{(y_1,y_3)} \quad \text{for every} \quad (y_1, y_2), (y_2, y_3) \in Y^{[2]}$$

which satisfies the commutative diagram:

$$\begin{array}{ccccc}(L_{(y_1,y_2)} \otimes L_{(y_2,y_3)}) \otimes L_{(y_3,y_4)} & \longrightarrow & L_{(y_1,y_3)} \otimes L_{(y_3,y_4)} & \longrightarrow & L_{(y_1,y_4)} \\ \| & & & & \| \\ L_{(y_1,y_2)} \otimes (L_{(y_2,y_3)} \otimes L_{(y_3,y_4)}) & \longrightarrow & L_{(y_1,y_2)} \otimes L_{(y_2,y_4)} & \longrightarrow & L_{(y_1,y_4)}\end{array}$$

A product of (Y, L) is equivalent to an isomorphism $\pi_3^* L \otimes \pi_1^* L \xrightarrow{\cong} \pi_2^* L$ over $Y^{[3]}$.

A bundle gerbe (Y, L) over X is isomorphic to a bundle gerbe (N, M) over X if there is a bundle isomorphism $g\colon Y \to N$ and $\hat{g}\colon L \to M$ covering the induced map $g^{[2]}\colon Y^{[2]} \to N^{[2]}$ and the product of L corresponds to that of M via $\hat{g}$.

If ξ is a Hermitian line bundle over Y, then we can define a bundle gerbe $(Y, \pi_1^*\xi^* \otimes \pi_2^*\xi)$ whose product is induced by the natural paring.

A bundle gerbe (Y, L) over X is called trivial if (Y, L) is isomorphic to the bundle gerbe $(Y, \pi_1^*\xi^* \otimes \pi_2^*\xi)$ for some ξ. Such a line bundle ξ is called a trivialization of (Y, L). If a Hermitian line bundle $\xi \to Y$ is a trivialization of (Y, L) and ρ is a line bundle over X, then $\xi \otimes \pi^*\rho$ is again a trivialization of (Y, L). Moreover, the set of all trivializations of (Y, L) is $\mathrm{Vect}_1(X)$-torsor, where $\mathrm{Vect}_1(X)$ stands for isomorphism classes of complex line bundles over X.

Let (Y, L) be a bundle gerbe over X. For a smooth map $h\colon Z \to X$ we have the pull-back bundle h^*Y over Z and the bundle map $\overline{h}\colon h^*Y \to Y$ covering h. Moreover, $\overline{h}$ induces a map $\overline{h}^{[2]}\colon h^*Y^{[2]} \to Y^{[2]}$. Here we shall denote $\overline{h}^{[2]}$ by the same symbol $\overline{h}$. If a bundle gerbe (Y, L) is given a trivialization ξ, then h induces a trivialization $\overline{h}^*\xi$ of the pull-back bundle gerbe $\left(h^*Y, \overline{h}^* L\right)$.

Roughly speaking, a bundle gerbe is a higher generalization of a line bundle. A bundle gerbe admits a characteristic class which can be considered as a generalization of the Euler class for a line bundle.

DEFINITION 2 (the Dixmier-Douady class). Take a good cover $\{U_\alpha\}$ of X and local sections $\{s_\alpha : U_\alpha \to P\}$. Then we obtain line bundles $L_{\alpha\beta} = (s_\alpha, s_\beta)^* L$. A choice of sections $z_{\alpha\beta} \in \Gamma(L_{\alpha\beta})$ with $|z_{\alpha\beta}| = 1$ and the induced product give a unique system of maps $\{\varepsilon_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to U(1)\}$ which satisfies the equation

$$z_{\alpha\beta} \cdot z_{\beta\gamma} = \varepsilon_{\alpha\beta\gamma} z_{\alpha\gamma}.$$

Then $\{\varepsilon_{\alpha\beta\gamma}\} \in \check{C}^2\left(X, \underline{U(1)}\right)$ gives a cohomology class $[\{\varepsilon_{\alpha\beta\gamma}\}] \in \check{H}^2\left(X; \underline{U(1)}\right) \cong H^3(X;\mathbb{Z})$, where $\check{C}\left(X, \underline{U(1)}\right)$ is the Čech cochain complex with values in a sheaf of $U(1)$-valued functions and $\check{H}\left(X; \underline{U(1)}\right)$ is the cohomology group of $\check{C}\left(X; \underline{U(1)}\right)$. We denote this by $d(Y, L)$ called the Dixmier-Douady class for the bundle gerbe (Y, L).

REMARK 1. The Dixmier-Douady class $d(Y, L)$ of (Y, L) does not depend on the choice of a good cover of X and local sections $\{s_\alpha\}$ and $\{z_{\alpha\beta}\}$.

Here we shall give an example.

EXAMPLE 1 (lifting bundle gerbes). Consider a central extension:

$$1 \to C \to \widehat{G} \xrightarrow{p} G \to 1$$

with the center $C = U(1)$ or $\mathbb{Z}_n$ and a principal G-bundle Y over X. We define a C-bundle Q over $Y^{[2]}$ by

$$Q = \left\{((y_1, y_2), \widehat{g}) \in Y^{[2]} \times \widehat{G} \;\middle|\; y_1 p(\widehat{g}) = y_2\right\}$$

and obtain the complex line bundle L over $Y^{[2]}$ associated with Q. Then the couple (Y, L) is a bundle gerbe over X and called the lifting bundle gerbe associated with the principal G-bundle Y and the central extension $\widehat{G}$ of G by C.

Let P be a $SO(n)$-bundle over X. Consider a central extension

$$1 \to \mathbb{Z}_2 \to Spin(n) \xrightarrow{p} SO(n) \to 1.$$

Then we obtain the lifting bundle gerbe (P, L) associated with the $SO(n)$-bundle P and the central extension $Spin(n)$ of $SO(n)$ by $\mathbb{Z}_2$. We call (P, L) the spin bundle gerbe of P.

REMARK 2. If $P \to X$ is a $SO(n)$-frame bundle of TX, then the Dixmier-Douady class $d(P, L)$ of the spin bundle gerbe (P, L) of P coincides with the third integral Stiefel-Whitney class $W_3(X)$.

2.2. Bundle gerbe connection. We proceed to the bundle gerbe connection in this subsection. Let (Y, L) be a bundle gerbe over X.

DEFINITION 3 (bundle gerbe connection). Let (Y, L) be a bundle gerbe over X. Then a Hermitian connection ∇ on L is said to be a *bundle gerbe connection* if the endowed product

$$L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \to L_{(y_1,y_3)}$$

preserves the connection ∇ for every $(y_1, y_2, y_3) \in Y^{[3]}$.

REMARK 3. A Hermitian connection ∇ on L induces pull-back connections $\pi_3^*\nabla \otimes 1 + 1 \otimes \pi_1^*\nabla$ on $\pi_3^* L \otimes \pi_1^* L$ and $\pi_2^*\nabla$ on $\pi_2^* L$. The connection ∇ is a bundle gerbe connection if and only if the isomorphism $\pi_3^* L \otimes \pi_1^* L \xrightarrow{\cong} \pi_2^* L$, which is induced by the product of (Y, L), preserves the connections.

Every bundle gerbe has a bundle gerbe connection. For a trivial bundle gerbe we can construct a bundle gerbe connection at ease. Arbitrary bundle gerbe (Y, L) over X is locally trivial. By the use of partition of unity on X, we obtain a bundle gerbe connection on (Y, L).

We have the sequence of fiber products:

$$X \xleftarrow{\pi} Y \leftarrow Y^{[2]} \leftarrow Y^{[3]} \leftarrow \cdots$$

and

$$0 \to \Omega^*(X) \xrightarrow{\delta} \Omega^*(Y) \xrightarrow{\delta} \Omega^*(Y^{[2]}) \to \cdots, \tag{1}$$

where $\delta\colon \Omega^*(Y^{[k]}) \to \Omega^*(Y^{[k+1]})$ is defined by $\delta\omega = \sum_{i=1}^{k+1}(-1)^i \pi_i^* \omega$.

Proposition 1 (Murray [2]). *The sequence* (1) *is exact.*

Consider a bundle gerbe (Y, L) with a bundle gerbe connection ∇. Then the curvature $F(\nabla) \in i\Omega^2(Y)$ satisfies:
(i) $dF(\nabla) = 0$;
(ii) $\delta F(\nabla) = 0$.
(i) is Bianchi's identity. We can prove (ii) by a local discussion. Hence (1) and (ii) imply that there is an imaginary 2-form $f \in i\Omega^2(Y)$ such that $\delta f = F(\nabla)$. Furthermore,

$$\delta\, df = d\, \delta f = dF(\nabla) = 0.$$

Therefore we obtain an imaginary 3-form $\omega \in i\Omega^3(X)$ satisfying $\pi^*\omega = df$. We call f a curving for (Y, L) and ω is called the 3-curvature of (Y, L). The 3-form $\omega/2\pi i$ defines a cohomology class $[\omega/2\pi i] \in H^3(X; \mathbb{R})$ and this is independent of the choices of bundle gerbe connections and curvings.

Theorem 2 (Murray-Stevenson [3]). *For every bundle gerbe* (Y, L) *over* X, *the image of* $d(Y, L)$ *in real cohomology is* $[\omega/2\pi i]$.

2.3. Bundle gerbe modules and the twisted Chern character. We shall introduce the notion of bundle gerbe modules.

DEFINITION 4. Let (Y, L) be a bundle gerbe over X. Then a Hermitian vector bundle W over Y is called a *bundle gerbe module for* (Y, L) if it is endowed with the multiplication of L:

$$L_{(y_1,y_2)} \otimes W_{y_2} \xrightarrow{\cong} W_{y_1} \quad \text{for every} \quad (y_1, y_2) \in Y^{[2]}$$

which satisfies the commutative diagram:

$$\begin{array}{ccccc} (L_{(y_1,y_2)} \otimes L_{(y_2,y_3)}) \otimes W_{y_3} & \longrightarrow & L_{(y_1,y_3)} \otimes W_{y_3} & \longrightarrow & W_{y_1} \\ \| & & & & \| \\ L_{(y_1,y_2)} \otimes (L_{(y_2,y_3)} \otimes W_{y_3}) & \longrightarrow & L_{(y_1,y_2)} \otimes W_{y_2} & \longrightarrow & W_{y_1} \end{array}$$

In other words, a bundle gerbe module $W \to Y$ is a Hermitian vector bundle given an isomorphism $L \otimes \pi_1^* W \cong \pi_2^* W$. It is straightforward that if the rank of a bundle gerbe module W for (Y, L) is equal to 1, W is a trivialization of (Y, L).

We denote by $\mathrm{Mod}(Y, L)$ the isomorphism classes of bundle gerbe modules for (Y, L).

Let (Y, L) be a bundle gerbe over X with $n d(Y, L) = 0$ for some n. Suppose that (Y, L) is endowed with a bundle gerbe connection ∇ and the curving f such that $df = 0$.

Suppose that (Y, L) is a bundle gerbe with a trivialization $\xi \to Y$. Then for every bundle gerbe module W for (Y, L), the multiplication $L \otimes \pi_1^* W \xrightarrow{\cong} \pi_2^* W$ is equivalent to the isomorphism

$$\pi_1^*(\xi^* \otimes W) \cong \pi_2^*(\xi^* \otimes W).$$

This implies that ξ induces a map $/\xi\colon \mathrm{Mod}(Y, L) \to \mathrm{Vect}(X)$ satisfying $\pi^*(W/\xi) = \xi^* \otimes W$ for every bundle gerbe module W.

DEFINITION 5 (bundle gerbe module connection). Let $W \to Y$ be a bundle gerbe module for (Y, L). Then a Hermitian connection ∇^W on W is called a bundle gerbe module connection on W compatible with ∇ if the multiplication

$$\varphi\colon L \otimes \pi_1^* W \xrightarrow{\cong} \pi_2^* W$$

preserves the connections, where $L \otimes \pi_1^* W$ ($\pi_2^* W$ resp.) is endowed with the induced connection $\nabla \otimes 1 + 1 \otimes \pi_1^* \nabla^W$ ($\pi_2^* \nabla^W$ resp.). If there is no confusion, we simply call ∇^W a bundle gerbe module connection on W.

Let ∇^W be a bundle gerbe module connection. Then we have

$$F(\nabla) \otimes 1 + 1 \otimes \pi_1^* F(\nabla^W) = \varphi \circ \pi_2^* F(\nabla^W) \circ \varphi^{-1},$$

therefore since $F(\nabla) = \pi_2^* f - \pi_1^* f$ we obtain

$$\pi_1^*(F(\nabla^W) - f) = \varphi \circ \pi_2^*(F(\nabla^W) - f) \circ \varphi^{-1}.$$

So every invariant polynomial P defines a closed form $P(F(\nabla^W) - f) \in \Omega^{2*}(Y)$ and $\delta P(F(\nabla^W) - f) = 0$. Therefore we obtain $[\eta] \in H^{2*}(X;\mathbb{R})$ such that $\pi^*\eta = P(F(\nabla^W) - f)$. The cohomology class $[\eta]$ does not depend on the bundle gerbe module connection ∇^W. Especially, tr is invariant polynomial and there is a unique cohomology class $[\eta_k] \in H^{2k}(X;\mathbb{R})$ such that

$$\pi^*\eta_k = \mathrm{tr}\left(\left(\frac{-1}{2\pi i}(F(\nabla^W) - f)\right)^k\right).$$

In the same manner as the Chern character of complex vector bundles, we define the twisted Chern character $\mathrm{ch}^\tau_{\mathrm{DG}}(f, W)$ of a bundle gerbe module W by

$$\mathrm{ch}^\tau_{\mathrm{DG}}(f, W) = \mathrm{rank}\, W + \sum_{k=1}^{\infty} \frac{1}{k!}[\eta_k] \in H^{2*}(X;\mathbb{R}).$$

REMARK 4. The Dixmier-Douady class $d(Y, L)$ of a bundle gerbe (Y, L) is independent of the choice of the bundle gerbe connection ∇ and the curving $f \in i\Omega^2(Y)$. On the other hand, the twisted Chern character $\mathrm{ch}^\tau_{\mathrm{DG}}(f, W)$ depends on the choice of the curving $f \in i\Omega^2(Y)$.

If we take another curving f' with $df' = 0$, there is a unique closed 2-form $\phi \in \Omega^2(X)$ satisfying $f' = f + \pi^*(i\phi)$. Then, we have

$$\mathrm{ch}^\tau_{\mathrm{DG}}(f', W) = \mathrm{ch}^\tau_{\mathrm{DG}}(f, W)\exp(-i\phi).$$

Consider a central extension $1 \to \mathbb{Z}_n \to \widehat{G} \to G \to 1$. Then the lifting bundle gerbe (Y, L) associated with a principal G-bundle Y over X has the flat connection ∇ since L is the complex line bundle over $Y^{[2]}$ associated with a $\mathbb{Z}_n$-bundle. The connection ∇ can be regarded as a bundle gerbe connection of (Y, L). So, in this case we can choose $f = 0 \in i\Omega^2(Y)$ as the curving.

3. Splitting principle for bundle gerbe modules

3.1. Construction of splittings. In this section, we shall construct the splitting of bundle gerbe modules and introduce the twisted Chern classes and the twisted Chern character in terms of algebraic topology. For this purpose, the curving of bundle gerbes should be replaced by the *n-trivialization*, which is defined as follows.

DEFINITION 6 (n-trivialization). Let (Y, L) be a bundle gerbe with $n\,d(Y, L) = 0$ for some n. Then $(Y, L^{\otimes n})$ is a trivial bundle gerbe over X. A trivialization $\eta \to Y$ of the trivial bundle gerbe $(Y, L^{\otimes n})$ is called an n-trivialization of (Y, L).

ASSUMPTION 1. In the remainder of this paper, we suppose that a bundle gerbe (Y, L) is given an n-trivialization η.

If W is a bundle gerbe module for (Y, L), then $W^{\otimes n}$ is a bundle gerbe module for $(Y, L^{\otimes n})$ and hence $\eta^* \otimes W^{\otimes n}$ descends to a bundle $W^{\otimes n}/\eta$ over X.

Let (Y, L) be a bundle gerbe and W a bundle gerbe module for (Y, L). The bundle gerbe module W brings us another bundle gerbe $\left(\widetilde{\mathbb{P}}(W), \widetilde{L}\right)$ as follows.

We first define the projectivization $\widetilde{\mathbb{P}}(W)$ of a bundle gerbe module W by

$$\widetilde{\mathbb{P}}(W) = \bigsqcup_{y \in Y} \{y\} \times \mathbb{P}(W_y).$$

For every $(y_1, y_2) \in Y^{[2]}$, the isomorphism $W_{y_1} \cong L_{(y_1,y_2)} \otimes W_{y_2}$ induces the canonical diffeomorphism $\varphi_{(y_1,y_2)} \colon \mathbb{P}(W_{y_2}) \xrightarrow{\cong} \mathbb{P}(W_{y_1})$. Then we have

$$\mathbb{P}(W) = \bigsqcup_{y \in Y} \{y\} \times \mathbb{P}(W_y) \Big/ \sim.$$

Here, $(y_1, [w_1])$ and $(y_2, [w_2])$ are equivalent if and only if

$$\varphi_{(y_1,y_2)}([w_2]) = [w_1]. \tag{2}$$

REMARK 5. $\mathbb{P}(W)$ is a fiber bundle over X with the fiber $\mathbb{CP}^{m-1}$ and the projection is denoted by p, where the rank of W is m. $\widetilde{\mathbb{P}}(W)$ is also a fiber bundle over $\mathbb{P}(W)$ and the projection is denoted by π. Furthermore, we obtain the bundle isomorphism $\widetilde{\mathbb{P}}(W) = p^*Y$. We denote by $\overline{p}$ the bundle map from $\widetilde{\mathbb{P}}(W)$ to Y covering $p \colon \mathbb{P}(W) \to X$.

DEFINITION 7. For the fiber bundle $\widetilde{\mathbb{P}}(W) \to \mathbb{P}(W)$ we have another bundle gerbe $\left(\widetilde{\mathbb{P}}(W), \widetilde{L}\right)$ over $\mathbb{P}(W)$, where $\widetilde{L}$ stands for $\overline{p}^*L$. We call this the bundle gerbe associated with the projectivization of a bundle gerbe module W.

We remark that $\left(\widetilde{\mathbb{P}}(W), \widetilde{L}\right)$ is isomorphic to the pull-back bundle gerbe $p^*(Y, L)$ and that the n-trivialization η of (Y, L) gives rise to an n-trivialization $\overline{p}^*\eta$ of $\left(\widetilde{\mathbb{P}}(W), \widetilde{L}\right)$. Moreover, the bundle gerbe has the tautological bundle gerbe module.

Proposition 2. *The bundle gerbe* $\left(\widetilde{\mathbb{P}}(W), \widetilde{L}\right)$ *has the tautological bundle gerbe module* γ_W.

Proof. First, we have the tautological line bundle γ_W over $\widetilde{\mathbb{P}}(W)$ defined by

$$\gamma_W = \{(y, l, w) \in \overline{p}^*W \mid y \in Y,\ l \in \mathbb{P}(W_y) \quad \text{and} \quad w \in l\}.$$

Then it suffices to show that γ_W is a bundle gerbe module for $\left(\widetilde{\mathbb{P}}(W), \widetilde{L}\right)$.

Choose an arbitrary element $((y_1, l_1), (y_2, l_2)) \in \widetilde{\mathbb{P}}(W)^{[2]}$. Then

$$(y_1, y_2) \in Y^{[2]} \quad \text{and} \quad l_i \in \mathbb{P}(W_{y_i}) \quad \text{for} \quad i = 1, 2.$$

Moreover, we see that the isomorphism

$$L_{(y_1,y_2)} \otimes W_{y_2} \xrightarrow{\cong} W_{y_1}$$

induces $\widetilde{L}_{((y_1,l_1),(y_2,l_2))} \otimes l_2 = L_{(y_1,y_2)} \otimes l_2 \xrightarrow{\cong} l_1$ since (2) implies $\pi(l_1, y_1) = \pi(l_2, y_2) \in \mathbb{P}(W)$. Therefore γ_W is a bundle gerbe module over $\widetilde{\mathbb{P}}(W)$. □

Here, we define a bundle gerbe module $W^\perp$ for $\big(\widetilde{\mathbb{P}}(W), \widetilde{L}\big)$ by

$$W^\perp = \{w \in \overline{p}^* W \mid w \quad \text{is orthogonal to} \quad \gamma_W\}.$$

It is easy to see that $W^\perp$ is a bundle gerbe module. So we have the following:

Proposition 3. *For all bundle gerbe module $W \to Y$ for (Y, L), we have the splitting into bundle gerbe modules for $\big(\widetilde{\mathbb{P}}(W), \widetilde{L}\big)$:*

$$\overline{p}^* W = \gamma_W \oplus W^\perp \to \widetilde{\mathbb{P}}(W).$$

3.2. The twisted Chern classes. We shall consider the cohomology ring of the space $\mathbb{P}(W)$. In the ordinary splitting principle for vector bundles $E \to X$ the projection $p\colon \mathbb{P}(E) \to X$ induces the isomorphism of graded modules:

$$H^*(\mathbb{CP}^{m-1}; \mathbb{Z}) \otimes H^*(X; \mathbb{Z}) \xrightarrow{\cong} H^*(\mathbb{P}(E); \mathbb{Z}),$$

and hence $p^*\colon H^*(X; \mathbb{Z}) \to H^*(\mathbb{P}(E); \mathbb{Z})$ is injective (see Husemoller [4] p.233 for the detail). However, in the case of bundle gerbe modules, these are not true in general. In fact, if (Y, L) is a bundle gerbe over X such that $d(Y, L)$ is a non trivial torsion element and W is a bundle gerbe module for (Y, L), then γ_W is a bundle gerbe module for $\big(\widetilde{\mathbb{P}}(W), \widetilde{L}\big)$ with rank $\gamma_W = 1$, which implies $p^* d(Y, L) = d\big(\widetilde{\mathbb{P}}(W), \widetilde{L}\big) = 0$.

The next proposition is straightforward.

Proposition 4. *For every bundle gerbe module W, we have a line bundle $(\gamma_W^{\otimes n})/\overline{p}^*\eta$ over $\mathbb{P}(W)$. Then, the restriction $((\gamma_W^{\otimes n})/\overline{p}^*\eta)|\mathbb{P}(W)_x$ to a fiber of $\mathbb{P}(W)$ is isomorphic to $\gamma(\mathbb{P}(W)_x)^{\otimes n}$, where $\gamma(\mathbb{P}(W)_x)$ denotes the tautological line bundle over $\mathbb{P}(W)_x$.*

Here we shall define the twisted Euler class $\chi^\tau(\eta, \xi)$ for a bundle gerbe module $\xi \to Y$ with rank 1 by

$$\chi^\tau(\eta, \xi) = \frac{1}{n} \chi((\xi^{\otimes n})/\eta) \in H^2(X; \mathbb{Q}).$$

Proposition 5. *If we change η to $\eta' = \eta \otimes \pi^*\rho$ for a line bundle $\rho \to X$, then we have*

$$\chi^\tau(\eta', \xi) = \chi^\tau(\eta, \xi) - \frac{1}{n}\chi(\rho).$$

Proof. By the definition of the maps $/\eta$ and $/\eta'$, the isomorphism $(\eta')^* \otimes \xi^{\otimes n} = \pi^*\rho^* \otimes (\eta^* \otimes \xi^{\otimes n})$ implies $\xi^{\otimes n}/\eta' = \rho^* \otimes (\xi^{\otimes n}/\eta)$. Therefore, we have

$$\chi^\tau(\eta', \xi) = \chi^\tau(\eta, \xi) - \frac{1}{n}\chi(\rho). \qquad \square$$

This have the naturality. That is, for every C^∞-map $h\colon Z \to X$ and a bundle gerbe module ξ for (Y, L), we have

$$\chi^\tau\left(\overline{h}^*\eta, \overline{h}^*\xi\right) = h^*\chi^\tau(\eta, \xi),$$

where $\overline{h}\colon h^*Y \to Y$ is the bundle map covering $h\colon Z \to X$ and $\overline{h}^*\xi$ is a bundle gerbe module for the pull-back bundle gerbe $\left(h^*Y, \overline{h}^*L\right)$ of (Y, L) by h.

Using this, we can define a homomorphism θ of degree 0:

$$\begin{aligned} \theta\colon H^*(\mathbb{CP}^{m-1}; \mathbb{Q}) &\to H^*(\mathbb{P}(W); \mathbb{Q}) \\ \chi(\gamma(\mathbb{CP}^{m-1}))^k &\mapsto \chi^\tau(\overline{p}^*\eta, \gamma_W)^k \quad \text{for} \quad k = 0, \dots, m-1. \end{aligned}$$

Then we have

$$\begin{aligned} i_x^*\theta(\chi(\gamma(\mathbb{CP}^{m-1}))^k) = i_x^*(\chi^\tau(\overline{p}^*\eta, \gamma_W)^k) &= i_x^*\left(\left(\frac{1}{n}\chi((\gamma_W^{\otimes n})/\overline{p}^*\eta)\right)^k\right) \\ &= \left(\frac{1}{n}\chi(i_x^*((\gamma_W^{\otimes n})/\overline{p}^*\eta))\right)^k = \left(\frac{1}{n}\chi(\gamma(\mathbb{P}(W)_x^{\otimes n}))\right)^k \\ &= \chi(\gamma(\mathbb{P}(W)_x))^k \end{aligned}$$

for $k = 0, \dots, m-1$. This implies that

$$i_x^* \circ \theta\colon H^*(\mathbb{CP}^{m-1}; \mathbb{Q}) \to H^*(\mathbb{P}(W)_x; \mathbb{Q})$$

is an isomorphism for every $x \in X$. Therefore, by using the Leray-Hirsch theorem (see [4] p.231) we have the following:

Theorem 3. *The homomorphism of graded modules*

$$\Phi^*\colon H^*(X; \mathbb{Q}) \otimes H^*(\mathbb{CP}^{m-1}; \mathbb{Q}) \to H^*(\mathbb{P}(W); \mathbb{Q}),$$

which is given by

$$\Phi^*(\beta \otimes \alpha) = p^*\beta \cup \theta(\alpha) \quad (\beta \in H^*(X;\mathbb{Q}),\ \alpha \in H^*(\mathbb{C}\mathbb{P}^{m-1};\mathbb{Q})),$$

is an isomorphism, and therefore p^* is injective.

Hence, there exists a unique m-tuple $(\beta_1, \ldots, \beta_m) \in \prod_{k=1}^m H^{2k}(X;\mathbb{Q})$ satisfying

$$-\chi^\tau(\overline{p}^*\eta, \gamma_W)^m = \sum_{k=1}^m (-1)^k p^*\beta_k \cup \chi^\tau(\overline{p}^*\eta, \gamma_W)^{m-k}.$$

DEFINITION 8 (the twisted Chern class). We define *the k-th twisted Chern class* $c_k^\tau(\eta, W)$ of the bundle gerbe module W for (Y, L) by

$$c_0^\tau(\eta, W) = 1, \quad c_k^\tau(\eta, W) = \beta_k \quad (1 \leqslant k \leqslant m), \quad \text{and} \quad c_k^\tau(\eta, W) = 0 \quad (k > m).$$

The total twisted Chern class c^τ has the naturality.

Proposition 6. *For every bundle gerbe module W for (Y, L) and every smooth map $h\colon Z \to X$ we have*

$$c^\tau\left(\overline{h}^*\eta, \overline{h}^*W\right) = h^*c^\tau(\eta, W),$$

where $\overline{h}$ denotes the bundle map covering h.

Proof. Consider the next commutative diagram:

$$\begin{tikzcd}
& Y \arrow[dd] & & \widetilde{\mathbb{P}}(W) \arrow[ll, "\overline{p}"'] \arrow[dd] \\
h^*Y \arrow[ur, "\overline{h}"] \arrow[dd] & & \widetilde{\mathbb{P}}\left(\overline{h}^*W\right) \arrow[ll, "\overline{q}"'] \arrow[ur, "\overline{h}_1"] \arrow[dd] & \\
& X & & \mathbb{P}(W) \arrow[ll, "p"'] \\
Z \arrow[ur, "h"] & & \mathbb{P}\left(\overline{h}^*W\right) \arrow[ll, "q"'] \arrow[ur, "h_1"] &
\end{tikzcd}$$

We have to note that $h^*\mathbb{P}(W) = \mathbb{P}\left(\overline{h}^*W\right)$ and that $(p \circ h_1)^*(Y, L) = (h \circ q)^*(Y, L)$.

By the definition of the twisted Chern classes c^τ, we have

$$-\chi^\tau(\overline{p}^*\eta,\gamma_W)^m=\sum_{k=1}^m(-1)^k p^*c_k^\tau(\eta,W)\cup\chi^\tau(\overline{p}^*\eta,\gamma_W)^{m-k}$$

and hence the naturality of the twisted Euler classes implies

$$\begin{aligned}-\chi^\tau\left(\overline{h}_1^*\overline{p}^*\eta,\overline{h}_1^*\gamma_W\right)^m&=-h_1^*\chi^\tau(\overline{p}^*\eta,\gamma_W)^m\\&=\sum_{k=1}^m(-1)^k h_1^*(p^*c_k^\tau(\eta,W)\cup\chi^\tau(\overline{p}^*\eta,\gamma_W)^{m-k})\\&=\sum_{k=1}^m(-1)^k h_1^*p^*c_k^\tau(\eta,W)\cup\chi^\tau\left(\overline{h}_1^*\overline{p}^*\eta,\overline{h}_1^*\gamma_W\right)^{m-k}.\end{aligned}$$

It is easy to prove $\overline{h}_1^*\gamma_W=\gamma_{\overline{h}^*W}$. Therefore, we have

$$-\chi^\tau\left(\overline{q}^*\overline{h}^*\eta,\gamma_{\overline{h}^*W}\right)^m=(-1)^k\sum_{k=1}^m q^*(h^*c_k^\tau(\eta,W))\cup\chi^\tau\left(\overline{q}^*\overline{h}^*\eta,\gamma_{\overline{h}^*W}\right)^{m-k}.$$

This equality implies

$$c_k^\tau\left(\overline{h}^*\eta,\overline{h}^*W\right)=h^*c_k^\tau(\eta,W)\quad\text{for every}\quad k.$$ □

Proposition 7. *Let W be a bundle gerbe module for (Y,L) with* $\operatorname{rank} W=m$. *Then we obtain $\widehat{X}$ and $h\colon\widehat{X}\to X$ such that $h^*\colon H^*(X;\mathbb{Q})\to H^*\left(\widehat{X};\mathbb{Q}\right)$ is injective and the bundle gerbe module $\overline{h}^*W$ for the bundle gerbe $\left(h^*Y,\overline{h}^*L\right)$ over $\widehat{X}$ splits into m bundle gerbe modules γ_i with* $\operatorname{rank}\gamma_i=1$, *where $\overline{h}$ denotes the bundle map $\overline{h}\colon h^*Y\to Y$ covering h.*

Proof. First, we construct the bundle gerbe $\left(\widetilde{\mathbb{P}}(W),\widetilde{L}\right)$ associated with the projectivization of a bundle gerbe module W for (Y,L). Then the bundle gerbe module W splits into bundle gerbe modules $\gamma_W\oplus W^\perp$. Next, we take the bundle gerbe associated with the projectivization of $W^\perp$. By iterating this operation, we obtain $\widehat{X}$ and a map $h\colon\widehat{X}\to X$ as we require. □

Proposition 8. *Let ξ_i be a bundle gerbe module for (Y,L) with* $\operatorname{rank}\xi_i=1$ *for* $i=1,\dots,m$. *Then $W=\xi_1\oplus\cdots\oplus\xi_m$ is a bundle gerbe module for (Y,L) and we have*

$$c^\tau(\eta,W)=\prod_{i=1}^m c^\tau(\eta,\xi_i).$$

Proof. Consider the tautological bundle gerbe module γ_W of W. It is easy to see that γ_W^* is a bundle gerbe module for $\big(\widetilde{\mathbb{P}}(W), \widetilde{L}^*\big)$. The n-trivialization η of L induces the n-trivialization η^* of L^* and $\overline{p}^*\eta^*$ of $\widetilde{L}^*$. Then $\overline{p}^*\xi_i \otimes \gamma_W^*$ is a bundle gerbe module for $\big(\widetilde{\mathbb{P}}(W), \underline{\mathbb{C}}\big)$ since $\big(\widetilde{\mathbb{P}}(W), \widetilde{L} \otimes \widetilde{L}^*\big)$ is canonically isomorphic to $\big(\widetilde{\mathbb{P}}(W), \underline{\mathbb{C}}\big)$. A trivialization $\underline{\mathbb{C}}$ of $\big(\widetilde{\mathbb{P}}(W), \underline{\mathbb{C}}\big)$ defines $/\underline{\mathbb{C}}$: $\mathrm{Mod}\big(\widetilde{\mathbb{P}}(W), \underline{\mathbb{C}}\big) \to \mathrm{Vect}(X)$. Here we shall denote the descended bundle $W/\underline{\mathbb{C}}$ by W_0 for every $W \in \mathrm{Mod}\big(\widetilde{\mathbb{P}}(W), \underline{\mathbb{C}}\big)$. Then, we have the isomorphisms:

$$\begin{aligned}
\underline{\mathbb{C}} \oplus (W^\perp \otimes \gamma_W^*)_0 &= (\gamma_W \otimes \gamma_W^*)_0 \oplus (W^\perp \otimes \gamma_W^*)_0 \\
&= ((\gamma_W \oplus W^\perp) \otimes \gamma_W^*)_0 \\
&= (\overline{p}^* W \otimes \gamma_W^*)_0 \\
&= (\overline{p}^*\xi_1 \otimes \gamma_W^*)_0 \oplus \cdots \oplus (\overline{p}^*\xi_m \otimes \gamma_W^*)_0.
\end{aligned}$$

Therefore, we obtain

$$0 = \chi(\underline{\mathbb{C}} \oplus (W^\perp \otimes \gamma_W^*)_0) = \prod_{i=1}^{m} \chi((\overline{p}^*\xi_i \otimes \gamma_W^*)_0). \tag{3}$$

The next isomorphism is straightforward:

$$\begin{aligned}
((\overline{p}^*\xi_i \otimes \gamma_W^*)_0)^{\otimes n} &= ((\overline{p}^*\xi_i \otimes \gamma_W^*)^{\otimes n})_0 = (\overline{p}^*\xi_i \otimes \gamma_W^*)^{\otimes n}/(\overline{p}^*\eta \otimes \overline{p}^*\eta^*) \\
&= ((\overline{p}^*\xi_i)^{\otimes n} \otimes (\gamma_W^*)^{\otimes n})/(\overline{p}^*\eta \otimes \overline{p}^*\eta^*) \\
&= ((\overline{p}^*\xi_i)^{\otimes n}/\overline{p}^*\eta) \otimes ((\gamma_W^*)^{\otimes n}/\overline{p}^*\eta^*) \\
&= ((\overline{p}^*\xi_i)^{\otimes n}/\overline{p}^*\eta) \otimes (\gamma_W^{\otimes n}/\overline{p}^*\eta)^*.
\end{aligned}$$

Hence we have the equality

$$\chi((\overline{p}^*\xi_i \otimes \gamma_W^*)_0) = \chi^\tau(\overline{p}^*\eta, \overline{p}^*\xi_i) - \chi^\tau(\overline{p}^*\eta, \gamma_W). \tag{4}$$

Combining (3) and (4), we obtain

$$\begin{aligned}
0 = (-1)^m &\chi^\tau(\overline{p}^*\eta, \gamma_W)^m \\
&+ (-1)^{m-1}(\chi^\tau(\overline{p}^*\eta, \overline{p}^*\xi_1) + \cdots + \chi^\tau(\overline{p}^*\eta, \overline{p}^*\xi_m))\chi^\tau(\overline{p}^*\eta, \gamma_W)^{m-1} \\
&+ \cdots + \chi^\tau(\overline{p}^*\eta, \overline{p}^*\xi_1) \cdots \chi^\tau(\overline{p}^*\eta, \overline{p}^*\xi_m).
\end{aligned}$$

So by the definition of the twisted Chern classes and $1 + \chi^\tau(\overline{p}^*\eta, \overline{p}^*\xi_i) = c^\tau(\overline{p}^*\eta, \overline{p}^*\xi_i) = p^* c^\tau(\eta, \xi_i)$ we have

$$c^\tau(\eta, W) = \prod_{i=1}^{m} c^\tau(\eta, \xi_i). \qquad \square$$

Proposition 9. *Let V and W be bundle gerbe modules for (Y, L). Then we have $c^\tau(\eta, V \oplus W) = c^\tau(\eta, V) \cup c^\tau(\eta, W)$.*

Proof. Proposition 7 implies that there is a space $\widehat{X}$ and the map $\pi_V \colon \widehat{X} \to X$ such that $\pi_V^* \colon H^*(X; \mathbb{Q}) \to H^*\big(\widehat{X}; \mathbb{Q}\big)$ is injective and $\overline{\pi}_V^* V = \gamma_1 \oplus \cdots \oplus \gamma_l$, where γ_i denotes a bundle gerbe module for the bundle gerbe $(\pi_V^* Y, \overline{\pi}_V^* L)$ over $\widehat{X}$ with $\operatorname{rank} \gamma_i = 1$. So, $\overline{\pi}_V^*(V \oplus W) = \gamma_1 \oplus \cdots \oplus \gamma_l \oplus \overline{\pi}_V^* W$. By applying Proposition 7 again to $(\pi_V^* Y, \overline{\pi}_V^* L)$ and $\overline{\pi}_V^* W$, we obtain the splitting:

$$\overline{\pi}_W^* \overline{\pi}_V^*(V \oplus W) = \overline{\pi}_W^* \gamma_1 \oplus \cdots \oplus \overline{\pi}_W^* \gamma_l \oplus \xi_1 \oplus \xi_m,$$

where ξ_i is a bundle gerbe module with $\operatorname{rank} \xi_i = 1$ for $(\pi_W^* \pi_V^* Y, \overline{\pi}_W^* \overline{\pi}_V^* L)$ over X'. Therefore, by Proposition 8 we obtain

$$\begin{aligned}
&\pi_W^* \pi_V^* c^\tau(\eta, V \oplus W) \\
&= c^\tau(\overline{\pi}_W^* \overline{\pi}_V^* \eta, \overline{\pi}_W^* \overline{\pi}_V^* (V \oplus W)) \\
&= c^\tau(\overline{\pi}_W^* \overline{\pi}_V^* \eta, \overline{\pi}_W^* \gamma_1 \oplus \cdots \oplus \overline{\pi}_W^* \gamma_l \oplus \xi_1 \oplus \cdots \oplus \xi_m) \\
&= c^\tau(\overline{\pi}_W^* \overline{\pi}_V^* \eta, \overline{\pi}_W^* \gamma_1) \cup \cdots \cup c^\tau(\overline{\pi}_W^* \overline{\pi}_V^* \eta, \overline{\pi}_W^* \gamma_l) \\
&\quad \cup c^\tau(\overline{\pi}_W^* \overline{\pi}_V^* \eta, \xi_1) \cup \cdots \cup c^\tau(\overline{\pi}_W^* \overline{\pi}_V^* \eta, \xi_m) \\
&= c^\tau(\overline{\pi}_W^* \overline{\pi}_V^* \eta, \overline{\pi}_W^* \gamma_W \oplus \cdots \oplus \overline{\pi}_W^* \gamma_l) \cup c^\tau(\overline{\pi}_W^* \overline{\pi}_V^* \eta, \xi_1 \oplus \cdots \oplus \xi_m) \\
&= c^\tau(\overline{\pi}_W^* \overline{\pi}_V^* \eta, \overline{\pi}_W^* \overline{\pi}_V^* V) \cup c^\tau(\overline{\pi}_W^* \overline{\pi}_V^* \eta, \overline{\pi}_W^* \overline{\pi}_V^* W) \\
&= \pi_W^* \pi_V^* (c^\tau(\eta, V) \cup c^\tau(\eta, W)).
\end{aligned}$$

□

Let σ_k be the k-th elementary symmetric polynomial and s_k the k-th Newton polynomial in m variables $t_1, \ldots, t_m$. It is well-known that they satisfy the equality

$$t_1^k + \cdots + t_m^k = s_k(\sigma_1, \ldots, \sigma_k).$$

DEFINITION 9 (the twisted Chern character in algebraic topology). Let W be a bundle gerbe module for (Y, L). Then we define the twisted Chern character $\mathrm{ch}_{\mathrm{AT}}^\tau(\eta, W)$ of W by

$$\mathrm{ch}_{\mathrm{AT}}^\tau(\eta, W) = \operatorname{rank} W + \sum_{k>0} \frac{1}{k!} s_k(c_1^\tau(\eta, W), \ldots, c_k^\tau(\eta, W)).$$

So far, we have constructed the twisted Chern classes and the twisted Chern character for bundle gerbe modules in terms of Algebraic Topology. Then the twisted Chern character of bundle gerbe modules could be defined in the situation where bundle gerbe (Y, L) is given a bundle gerbe connection ∇ and the curving f. On the other hand, the twisted Chern character in terms of algebraic topology is defined in the situation where the bundle gerbe is given an n-trivialization.

A curving f for (Y, L) is called compatible with the n-trivialization η if there is a bundle gerbe module connection ∇^{η} on η satisfying the equation $f = F(\nabla^{\eta})/n$.

We need choose a curving f compatible with the n-trivialization in order to make the twisted Chern character $\mathrm{ch}^{\tau}_{\mathrm{DG}}$ satisfy the property

$$\mathrm{ch}^{\tau}_{\mathrm{DG}}(f, W)^n = \mathrm{ch}(W^{\otimes n}/\eta) \quad \text{for every bundle gerbe module} \quad W.$$

We can choose a curving compatible with the n-trivialization. Let ∇ be a bundle gerbe connection for (Y,L) and f a curving for (Y,L) with $df = 0$. The n-trivialization η of (Y,L) can be regarded as a bundle gerbe module for $(Y,L^{\otimes n})$. Then we can take a bundle gerbe module connection ∇^{η} of η compatible with the bundle gerbe connection $\nabla^{\otimes n}$ for $(Y,L^{\otimes n})$. Hence we have $\delta(nf) = F(\nabla^{\otimes n}) = \delta(F(\nabla^{\eta}))$. Since $\delta(nf - F(\nabla^{\eta})) = 0$ and $d(nf - F(\nabla^{\eta})) = 0$, there is a unique closed 2-form ϕ on X satisfying $nf - F(\nabla^{\eta}) = \pi^*(i\phi)$. Then $f' = f - \pi^*(i\phi)/n$ is a compatible curving with η.

Proof of Theorem 1. By Proposition 7, we have $h\colon \widehat{X} \to X$ such that $h^*\colon H^*(X;\mathbb{Q}) \to H^*(\widehat{X};\mathbb{Q})$ is injective and that $\overline{h}^* W$ splits into bundle gerbe modules $\xi_1 \oplus \cdots \oplus \xi_m$ for $(h^*Y, \overline{h}^* L)$. A given n-trivialization η induces an n-trivialization $\overline{h}^*\eta$ of $(h^*Y, \overline{h}^* L)$. The bundle gerbe module connection ∇ and the curving f of (Y, L) induces the bundle gerbe connection $\overline{h}^*\nabla$ and the curving $\overline{h}^* f$ of $(h^*Y, \overline{h}^* L)$.

It suffices to prove $\mathrm{ch}^{\tau}_{\mathrm{AT}}(\overline{h}^*\eta, \xi_i) = \mathrm{ch}^{\tau}_{\mathrm{DG}}(\overline{h}^* f, \xi_i)$ for every i. Take a bundle gerbe module connection ∇^{ξ_i} on ξ_i. In fact, we have an equality

$$\begin{aligned}
\pi^* \mathrm{ch}^{\tau}_{\mathrm{DG}}(\overline{h}^* f, \xi_i) &= \exp\left(\frac{-1}{2\pi i}\left(F(\nabla^{\xi_i}) - \overline{h}^* f\right)\right) \\
&= \exp\left(\frac{1}{n}\left(\frac{-1}{2\pi i}\left(nF(\nabla^{\xi_i}) - \overline{h}^*(nf)\right)\right)\right) \\
&= \exp\left(\frac{1}{n}\left(\frac{-1}{2\pi i}\left(F\left(\nabla^{\xi_i^{\otimes n}\otimes(\overline{h}^*\eta^*)}\right)\right)\right)\right) \\
&= \exp\left(\pi^*\left(\frac{-1}{2n\pi i} F\left(\nabla^{(\xi_i^{\otimes n})/\overline{h}^*\eta}\right)\right)\right) \\
&= \pi^* \exp\left(\frac{-1}{2n\pi i} F\left(\nabla^{(\xi_i^{\otimes n})/\overline{h}^*\eta}\right)\right).
\end{aligned}$$

The induced homomorphism $\pi^*\colon \Omega^*(\widehat{X}) \to \Omega^*(h^*Y)$ is injective. Therefore, we obtain $\mathrm{ch}^{\tau}_{\mathrm{DG}}(\xi_i) = \exp\left((-1/2n\pi i)F\left(\nabla^{(\xi_i^{\otimes n})/\overline{h}^*\eta}\right)\right)$. This coincides with the image of $\exp\left(\chi^{\tau}\left(\overline{h}^*\eta, \xi_i\right)\right) = \mathrm{ch}^{\tau}_{\mathrm{AT}}(\overline{h}^*\eta, \xi_i)$ in real cohomology. Therefore, we have $\mathrm{ch}^{\tau}_{\mathrm{DG}}(\overline{h}^* f, \xi_i) = \mathrm{ch}^{\tau}_{\mathrm{AT}}(\overline{h}^*\eta, \xi_i)$. $\square$

ACKNOWLEDGMENT. I would like to thank Hitoshi Moriyoshi for continuous encouragement and helpful suggestions.

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