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NON-LOCAL ELLIPTIC PROBLEM IN HIGHER DIMENSION

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Abstract

Non-local elliptic problem, $-\Delta v = \lambda\bigl(e^v / \bigl(\int_\Omega e^v\,dx\bigr)^p\bigr)$ with Dirichlet boundary condition is considered on n-dimensional bounded domain Ω with $n \geq 3$ for $p > 0$. If Ω is the unit ball, $3 \leq n \leq 9$ and $2/n \leq p \leq 1$, we have infinitely many bendings in λ of the solution set in $\lambda - v$ plane. Finally if Ω is an annulus domain and $p \geq 1$, we show that a solution exists for all $\lambda > 0$.

1. Introduction

In this paper we consider the following elliptic equation with non-local term:

$$\begin{cases} -\Delta v = \lambda \dfrac{e^v}{\bigl(\int_\Omega e^v\,dx\bigr)^p} & x \in \Omega, \\ v = 0 & x \in \partial\Omega, \end{cases} \tag{1}$$

where λ, p are positive constants and Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. Actually, usual Gel'fand problem, in the theories of thermonic emission ([5]), isothermal gas sphere ([4]), and gas combustion ([1]), is formulated as the nonlinear eigenvalue problem

$$\begin{cases} -\Delta v = \sigma e^v & x \in \Omega, \\ v = 0 & x \in \partial\Omega, \end{cases} \tag{2}$$

with a constant $\sigma > 0$. Problems (1) and (2) are equivalent through the relation

$$\sigma = \frac{\lambda}{\bigl(\int_\Omega e^v\,dx\bigr)^p},$$

and hence some features of the solution set

$$\mathcal{C} = \{(\lambda, v) \mid v = v(x) \text{ is a classical solution of (1) for } \lambda > 0\}$$

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resemble those of the solution set for (2), denoted by $\mathcal{S}$. (1) is the non-local stationary problem of

$$\begin{cases} v_t = \Delta v + \lambda \dfrac{e^v}{\left(\int_\Omega e^v\, dx\right)^p} & x \in \Omega,\ t > 0, \\ v = 0 & x \in \partial\Omega,\ t > 0, \\ v|_{t=0} = v_0(x) & x \in \Omega. \end{cases}$$

Such problems are studied in ([2]). They arise in the study of phenomena associated with the occurrence of shear bands in metals being deformed under high strain rates ([3]) and Ohmic heating ([10], [11]). We note that if $p = 1$, the motivation to study (1) is the Keller-Segel system ([9]) which describes the chemotactic aggregation of cellular slime molds given by

$$\begin{cases} \varepsilon u_t = \nabla \cdot (\nabla u - u \nabla v) & x \in \Omega,\ t \in (0, T), \\ \tau v_t = \Delta v + u & x \in \Omega,\ t \in (0, T), \\ \dfrac{\partial u}{\partial \nu} - u \dfrac{\partial v}{\partial \nu} = v = 0 & x \in \partial\Omega,\ t \in (0, T), \\ u|_{t=0} = u_0(x) \geq 0 & x \in \Omega, \\ v|_{t=0} = v_0(x) & x \in \Omega, \end{cases} \tag{3}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$, τ, ε are positive constants, and ν is the outer unit normal vector, respectively. In the stationary state of (3), it is reduced to (1) ([20]). In fact, since $u = u(x, t) \geq 0$ and

$$\frac{d}{dt} \int_\Omega u\, dx = \frac{1}{\varepsilon} \int_\Omega \nabla \cdot (\nabla u - u \nabla v)\, dx = 0,$$

the total mass is conserved, that is, $\|u(\,\cdot\,, t)\|_1 = \|u_0\|_1$. Here and henceforth, $\|\cdot\|_p$ denotes the standard L^p norm. The system of equations (3) has a Lyapunov function

$$J(u, v) = \int_\Omega \left(u(\log u - 1) - uv + \frac{1}{2} |\nabla v|^2 \right) dx$$

and it holds that

$$\varepsilon \frac{d}{dt} J(u, v) + \tau\varepsilon \|v_t\|_2^2 + \int_\Omega u |\nabla(\log u - v)|^2\, dx = 0.$$

It implies that in the stationary state

$$\log u - v = \log \sigma$$

holds for some constant $\sigma > 0$. In other words

$$u = \sigma e^v$$

holds. Putting $\lambda = \|u_0\|_1$ we have

$$\sigma = \frac{\lambda}{\int_\Omega e^v \, dx}$$

by mass conservation. Thus the second equation in (3) implies that

$$0 = \Delta v + \lambda \frac{e^v}{\int_\Omega e^v \, dx}$$

with Dirichlet boundary condition.

We have known the result on $\mathcal{C}$ when $n = 1, 2$ and Ω is the unit ball $B = \{x \in \mathbf{R}^n \mid |x| < 1\}$. For $n = 1$ with $\Omega = B$, if $p \geq 1$, then (1) has a unique solution for all $\lambda > 0$. On the contrary if $0 < p < 1$, there exists $\overline{\lambda} > 0$ such that (1) has two solutions for $\lambda < \overline{\lambda}$, (1) has one solution for $\lambda = \overline{\lambda}$ and (1) has no solution for $\lambda > \overline{\lambda}$. For $n = 2$ with $\Omega = B$, if $p < 1$, then (1) has a unique solution for all $\lambda > 0$. If $p = 1$, then (1) has a unique solution for all $0 < \lambda < 8\pi$ but no solution for $\lambda \geq 8\pi$. On the contrary if $0 < p < 1$, there exists $\overline{\lambda} > 0$ such that (1) has two solutions for $\lambda < \overline{\lambda}$, (1) has one solution for $\lambda = \overline{\lambda}$ and (1) has no solution for $\lambda > \overline{\lambda}$. These facts are proven in [2].

We consider the $p = 1$ case in [13] and expand their results to the genaral $p > 0$ in this paper.

We also mention the structure of $\mathcal{S}$. For $n \geq 3$ with $\Omega = B$, $\mathcal{S}$ is a one-dimensional open manifold with the end points in $(\sigma, v) = (0, 0)$ and $(\sigma, v) = (2(n-2), 2\log(1/|x|))$, respectively, the latter being a weak solution of (2). Moreover for $3 \leq n \leq 9$, $\mathcal{S}$ bends infinitely many times with respect to σ around $\sigma = 2(n-2)$. Morse indices increases by one whenever it bends. On the other hand if $n \geq 10$, no bending occurs. They are shown in [14], [15].

We define the section of $\mathcal{C}$ cut by $\lambda > 0$ as follows:

$$\mathcal{C}^\lambda = \left\{ v \in C^2(\Omega) \cap C\left(\overline{\Omega}\right) \;\middle|\; v = v(x) \text{ solves (1)} \right\}.$$

The first theorem is concerned with the star-shaped domain, so that $x \cdot \nu > 0$ holds for each $x \in \partial\Omega$.

Theorem 1. *If Ω is star-shaped with respect to the origin with $n \geq 3$ and $p \leq 1$, then there is $\overline{\lambda} \in (0, +\infty)$ such that (1) has no solution for $\lambda > \overline{\lambda}$. Moreover, $\mathcal{C}_0$ is unbounded in $\lambda - v$ plane, where $\mathcal{C}_0$ stands for the connected component of $\mathcal{C}$ satisfying $(0, 0) \in \overline{\mathcal{C}_0}$.*

The second theorem is concerned with the ball case.

Theorem 2. *If Ω is the unit ball $B = \{x \in \mathbf{R}^n \mid |x| < 1\}$ with $n \geq 3$, then $\mathcal{C}$ is a one-dimensional open manifold and can be parametrized as*

$$\mathcal{C} = \{(\lambda(s), v(\,\cdot\,, s)) \mid 0 < s < +\infty\}$$

with the end points in $(\lambda, v) = (0, 0)$ and the weak solution

$$(\lambda, v) = \left(2\omega_n^p(n-2)^{1-p}, 2\log\left(\frac{1}{|x|}\right)\right),$$

respectively, where ω_n denotes the area of the unit sphere in $\mathbf{R}^n$. Moreover for $3 \leq n \leq 9$ and $2/n \leq p \leq 1$, $\mathcal{C}$ bends infinitely many times with respect to λ around $\lambda = 2\omega_n^p(n-2)^{1-p}$. On the other hand if $n \geq 10$ and $p \leq 1$, no bending occurs.

The third theorem is on the spectral property of the linearized operator. To state the result, we define Morse index as follows. For given $(\lambda, v) \in \mathcal{C}$, the linearized eigenvalue problem is given by

$$\begin{cases} \Delta\phi + \lambda \dfrac{e^v}{\left(\int_\Omega e^v\,dx\right)^p}\phi - p\lambda \dfrac{\int_\Omega e^v \phi\,dx}{\left(\int_\Omega e^v\,dx\right)^{p+1}} e^v = -\mu\phi & x \in \Omega, \\ \phi = 0 & x \in \partial\Omega. \end{cases} \tag{4}$$

Then, the Morse index $i = i(\lambda, v)$ and the radial Morse index $i_R = i_R(\lambda, v)$ denote the number of negative eigenvalues and that of radially symmetric eigenfunctions, respectively.

Theorem 3. *If Ω is the unit ball with $3 \leq n \leq 9$, $2/n \leq p \leq 1$ and $n \geq 10$, $p \leq 1$, respectively, then $i = i_R$ holds and $i = i(\lambda, v)$ increases by one at each bending point.*

The last theorem is on the annulus domain $A_a = \{x \in \mathbf{R}^n \mid a < |x| < 1\}$ with $a \in (0, 1)$. We deal with only radial solutions. Then we define the solution set by

$$\mathcal{C}_a = \{(\lambda, v) \mid v = v(|x|) \text{ is a classical solution of (1) for } \lambda > 0\}.$$

Theorem 4. *If Ω is the annulus domain A_a with $n \geq 3$ and $p \geq 1$, then $\mathcal{C}_a$ is a one-dimensional open manifold and can be parametrized as*

$$\mathcal{C}_a = \{(\lambda(s), v(\,\cdot\,, s)) \mid 0 < s < +\infty\}$$

with the end points in $(\lambda, v) = (0, 0)$ and (λ, v) satisfying

$$\lim_{s\uparrow+\infty} \lambda(s) = +\infty \quad \text{and} \quad \lim_{s\uparrow+\infty} \sup_{a<x<1} |v(x, s)| = +\infty.$$

This paper is composed of four sections. In §2, we treat a star-shaped domain and prove Theorem 1. Next in §3, we study the ball case and prove Theorems 2 and 3. Finally, §4 is on the annulus domain case and we prove Theorem 4.

2. Star-shaped domain

In this section, we assume that Ω is a star-shaped bounded domain with respect to the origin in $\mathbf{R}^n$ with $n \geq 3$ with the smooth boundary $\partial\Omega$ and that ν is the outer unit normal vector.

Proof of Theorem 1. The first part of Theorem has already proven in [2], but we provide the proof for completeness. In fact we apply the Pohozaev identity ([16]) to (1).

(5)

$$\begin{aligned}\frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial v}{\partial\nu}\right)^2 ds &= \frac{n\lambda}{\left(\int_\Omega e^v\,dx\right)^p}\int_\Omega (e^v-1)\,dx + \frac{2-n}{2}\frac{\lambda}{\left(\int_\Omega e^v\,dx\right)^p}\int_\Omega e^v v\,dx\\ &\leq \frac{n\lambda}{\left(\int_\Omega e^v\,dx\right)^{p-1}},\end{aligned}$$

where ds is the area element of ∂B with standard metric.

On the other hand it follows from (1) that

$$\frac{\lambda}{\left(\int_\Omega e^v\,dx\right)^{p-1}} = \int_\Omega(-\Delta v)\,dx = \int_{\partial\Omega}\left(-\frac{\partial v}{\partial\nu}\right)ds,$$

and therefore we have

$$\frac{\lambda^2}{\left(\int_\Omega e^v\,dx\right)^{2(p-1)}} \leq \int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial v}{\partial\nu}\right)^2 ds\int_{\partial\Omega}\frac{1}{(x\cdot\nu)}\,ds.$$

Combining this inequality and (5), we have

$$\frac{\lambda^2}{\left(\int_\Omega e^v\,dx\right)^{2(p-1)}} \leq \frac{2n\lambda}{\left(\int_\Omega e^v\,dx\right)^{p-1}}\int_{\partial\Omega}\frac{1}{(x\cdot\nu)}\,ds.$$

Hence since $p \leq 1$ and $v > 0$ in Ω, we have

$$\lambda \leq \frac{2n}{\left(\int_\Omega e^v\,dx\right)^{1-p}}\int_{\partial\Omega}\frac{1}{(x\cdot\nu)}\,ds \leq \frac{2n}{|\Omega|^{1-p}}\int_{\partial\Omega}\frac{1}{(x\cdot\nu)}\,ds,$$

where $|\Omega|$ is the measure of Ω. This gives $\overline{\lambda}$ in the statement.

We denote by $\mathcal{C}_1$ the branch of solutions of (1) starting from $(\lambda, v) = (0,0)$. Supposing that $\mathcal{C}_1$ is bounded, we prove unboundedness of the component $\mathcal{C}_1$ by contradiction by making use of the standard degree argument similar to [18]. We provide the proof for completeness and proceed it in the same mathod as in [19]. Putting

$$F(\lambda, v) = \Delta v + \lambda \frac{e^v}{\left(\int_\Omega e^v \, dx\right)^p},$$

we apply the implicit function theorem. Then there exists a solution (λ, v) for $0 < \lambda \ll 1$. $\mathcal{C}_1$ is compact from the assumption and the existence of the upper bound $\overline{\lambda} < +\infty$. Because

$$\mathcal{C} \cap \{\lambda = 0\} = \{(0, 0)\},$$

we can take open set $\mathcal{U}$ containing $\mathcal{C}_1$ with the properties

$$\partial \mathcal{U}_\lambda \cap \mathcal{C} = \emptyset \quad \text{and} \quad \mathcal{U}_\lambda = \emptyset \quad \text{for } \lambda \gg 1,$$

and $\mathcal{U}_\lambda \cap \mathcal{C}$ is composed of the solution of (1) with $\mu_1(\lambda, v) > 0$ for $0 < \lambda \ll 1$, where

$$\mathcal{U}_\lambda = \left\{ v \in C\left(\overline{\Omega}\right) \,\middle|\, (\lambda, v) \in \mathcal{U} \right\}.$$

In Banach space $C\left(\overline{\Omega}\right)$, Leray-Schauder degree $d(\Psi_\lambda, 0, \mathcal{U}_\lambda)$ is taken for any $\lambda > 0$, where $\Psi_\lambda = I_{C(\overline{\Omega})} - \Phi_\lambda$ with

$$\Phi_\lambda(v) = (-\Delta)^{-1} \lambda \frac{e^v}{\left(\int_\Omega e^v \, dx\right)^p}.$$

From the homotopy invariance ([18]), $d(\Psi_\lambda, 0, \mathcal{U}_\lambda)$ is independent of $\lambda > 0$. However by existence and nonexistence of the solution of (1), we have

$$\begin{cases} d(\Psi_\lambda, 0, \mathcal{U}_\lambda) = 0 & \text{for } \lambda \gg 1, \\ d(\Psi_\lambda, 0, \mathcal{U}_\lambda) = 1 & \text{for } 0 < \lambda \ll 1, \end{cases}$$

which is a contradiction. $\square$

3. Ball case

In this section, we assume that $\Omega = B$, where $B = \{x \in \mathbf{R}^n \mid |x| < 1\}$.

Proof of Theorem 2. According to [6], any solution of (1) is radially symmetric. Hence we have

$$\begin{cases} (r^{n-1} v')' + \lambda r^{n-1} \dfrac{e^v}{\left(\int_\Omega e^v \, dx\right)^p} = 0 & \text{for } r > 0, \\ v(1) = 0, \quad v'(0) = 0, \end{cases}$$

where

$$v = v(r) \quad \text{for} \quad r = |x|.$$

We begin with the parametrization of the solution set $\mathcal{C}$, following [7], [13], [14] and [15]. In fact, any solution is obtained as a solution of the initial value problem

$$\begin{cases} (r^{n-1}v')' + \sigma r^{n-1}e^v = 0 \quad \text{for } r > 0, \\ v(0) = A, \quad v'(0) = 0, \end{cases} \tag{6}$$

with a certain positive constant A. Through the Emden transformation

$$v(r) = w(t) - 2t + A, \quad r = \left\{ \frac{2(n-2)}{\sigma e^A} \right\}^{1/2} e^t, \tag{7}$$

(6) is reduced to the autonomous ordinary differential equation

$$\begin{cases} \ddot{w} + (n-2)\dot{w} + 2(n-2)(e^w - 1) = 0 \\ \lim_{t\to-\infty}(w(t) - 2t) = \lim_{t\to-\infty} e^{-t}(\dot{w}(t) - 2) = 0. \end{cases} \tag{8}$$

Then there exists a unique global solution $w = w(t)$ of (8) by [14]. The orbit $\mathcal{O} = \{(w(t), z(t)) = (w(t), \dot{w}(t)) \mid t \in \mathbf{R}\}$ starts at $t = -\infty$ along and below the line $z = 2$ with $w = -\infty$, and approaches the origin $(0, 0)$ as $t \to +\infty$. If $3 \le n \le 9$, it proceeds clockwise in $\{(w, z) \mid z < 2\}$, crosses infinitely many times z- and w-axes alternately, while it keeps to stay in $\{(w, z) \mid w < 0, \ 0 < z < 2\}$ in the case of $n \ge 10$. Through the Emden transformation (8), the boundary condition in (1) is converted to

$$w(\tau) - 2\tau + A = 0$$

with

$$\left\{ \frac{2(n-2)}{\sigma e^A} \right\}^{1/2} e^\tau = 1.$$

Therefore for any $\tau \in \mathbf{R}$, (σ_τ, v_τ) defined by

$$v_\tau(r) = w(t) - 2t - \{w(\tau) - 2\tau\} = w(\log r + \tau) - w(\tau) - 2\log r \tag{9}$$

with $r = e^{t-\tau}$ and

$$A_\tau = 2\tau - w(\tau), \quad \sigma_\tau = 2(n-2)e^{2\tau - A} = 2(n-2)e^{w(\tau)} \tag{10}$$

satisfies (1). Conversely every solution of (1) can be expressed in the form of (9) and (10). According to [7], [13], [14] and [19], the total set $\mathcal{S}$ of the solution (σ, v)

of (2) is homeomorphic to $\mathcal{O}$ through the relation (7) with the constants A, σ determined by (10). This means that $\mathcal{C}$ is homeomorphic to $\mathcal{O}$. Each point of $\mathcal{O}$ is given as $(w(\tau), z(\tau))$ and hence $\mathcal{C}$ is parametrized by $\tau \in \mathbf{R}$. Then we have

$$\lambda(\tau) = \omega_n^p 2^{1-p}(n-2)^{1-p}(2 - z(\tau))^p e^{(1-p)w(\tau)}. \tag{11}$$

In fact, putting $K = \int_B e^v dx$ we have

$$\lambda^{1/p-1} = K^{1-p}\sigma^{1/p-1}$$

because of $K^p = \lambda/\sigma$. By (10), we have

$$\lambda^{1/p} = \lambda K^{1-p}\sigma^{1/p-1} = \lambda K^{1-p}(2(n-2)e^{w(\tau)})^{1/p-1}.$$

Integrating (1) over B, we have

$$-\lambda K^{1-p} = \omega_n v'(1) = \omega_n(z(\tau) - 2)$$

by (7). Finally combining two equations, we have the desired one. Hence the behaviour of $\tau = +\infty$ follows at once. On the other hand $\lim_{\tau\to-\infty} v_\tau = 0$ and $\lim_{\tau\to-\infty} \sigma_\tau = 0$ imply that $\lim_{\tau\to-\infty} \lambda_\tau = 0$. The rest follows.

In the use of (11), we have

$$\dot{\lambda}(\tau) = \omega_n^p 2^{1-p}(n-2)^{1-p}(2 - z(\tau))^{p-1} e^{(1-p)w(\tau)}\{(1-p)z(\tau)(2 - z(\tau)) - p\dot{z}(\tau)\}.$$

If $n \geq 10$ and $p \leq 1$, $\dot{\lambda}(\tau) > 0$ for all $\tau \in \mathbf{R}$ which proves the statement. Next we concentrate on the case of $3 \leq n \leq 9$ and $2/n \leq p \leq 1$. To do so, we put $g(\tau) = (1-p)z(\tau)(2 - z(\tau)) - p\dot{z}(\tau)$. Then we have

$$\dot{g}(\tau) = 2(1-p)(2 - z(\tau))\dot{z}(\tau) + (pn - 2)\dot{z}(\tau) + 2p(n-2)e^{w(\tau)}z(\tau). \tag{12}$$

Let $\mathcal{O}_k$ $(k \geq 2)$ denote the successive points w-axis and $z = -2(e^w - 1)$ crossed by the orbit $\mathcal{O}$ in $w - z$ plane in order. Moreover we set $\mathcal{O}_1 = (-\infty, 2)$. Then $\dot{\lambda}(\tau) > 0$ and $\dot{\lambda}(\tau) < 0$ on the arc $\mathcal{O}_{4k-3}\mathcal{O}_{4k-2}$ and $\mathcal{O}_{4k-1}\mathcal{O}_{4k}$, respectively for $k \geq 1$. On the other hand $\dot{g}(\tau) < 0$ and $\dot{g}(\tau) > 0$ on the arc $\mathcal{O}_{4k-2}\mathcal{O}_{4k-1}$ and $\mathcal{O}_{4k}\mathcal{O}_{4k+1}$, respectively for $k \geq 1$. Hence there exists a unique point $\tilde{\mathcal{O}} = (w(\tau), z(\tau))$ on the every arc $\mathcal{O}_{4k-2}\mathcal{O}_{4k-1}$ and $\mathcal{O}_{4k}\mathcal{O}_{4k+1}$ respectively for $k \geq 1$ such that $\dot{\lambda}(\tau) = 0$. The proof is complete. $\square$

We proceed the proof of Theorem 3 in the same argument and computation as in [15].

Proof of Theorem 3. As far as we consider negative eigenvalues in (4), the corresponing eigenfunctions are radially symmmetric ([12], [13]). Hence $i(\lambda, v) = i_R(\lambda, v)$ for

$(\lambda, v) \in \mathcal{C}$. We denote by μ^l_τ the l-th eigenvalue of (4) in $(\lambda(\tau), v(\tau)) \in \mathcal{C}$ coresponding to radially symmetric eigenfunctions. Any of them is simple. If $(\lambda(\tau), v(\tau)) \in \mathcal{C}$ is the turning point of $\mathcal{C}$, then there exists $l \geq 1$ such that $\mu^l_\tau = 0$ by the implicit function theorem. On the contrary, $\mu^l_\tau = 0$ for some $l \geq 1$ at $(\lambda(\tau), v(\tau)) \in \mathcal{C}$ implies that it is a turning point by the bifurcation theorem from the critical point of odd multiplicity ([17], [18]). Since $\mu^1_{-\tau} > 0$ for sufficiently large $\tau > 0$, we have $i(\lambda, v) = 0$ for $(\lambda(\tau), v(\tau)) \in \mathcal{C}$ for $n \geq 10$ and $p \leq 1$.

For $3 \leq n \leq 9$ and $2/n \leq p \leq 1$, let $T_k = (\lambda(\tau_k), v(\tau_k))$ for $\tau_1 < \tau_2 < \cdots$ denote the turning point of $\mathcal{C}$. Then we have $\mu^l_{\tau_k} = 0$ for some $l \geq 1$ and we have only to show that $\dot{\mu}^l_{\tau=\tau_k} < 0$ for all $k \geq 1$.

Differentiating (1) with respect to τ, we have

$$\begin{cases} \Delta \dot{v} + \dot{\lambda} \dfrac{e^v}{\left(\int_B e^v\, dx\right)^p} + \lambda \dfrac{e^v \dot{v}}{\left(\int_B e^v\, dx\right)^p} - \lambda p \dfrac{\int_B e^v \dot{v}\, dx}{\left(\int_B e^v\, dx\right)^{p+1}} e^v = 0 & x \in B, \\ \dot{v} = 0 & x \in \partial B, \end{cases} \tag{13}$$

and hence

$$\begin{cases} \Delta \dot{v}_k + \lambda_k \dfrac{e^{v_k} \dot{v}_k}{\left(\int_B e^{v_k}\, dx\right)^p} - \lambda_k p \dfrac{\int_B e^{v_k} \dot{v}_k\, dx}{\left(\int_B e^{v_k}\, dx\right)^{p+1}} e^{v_k} = 0 & x \in B, \\ \dot{v}_k = 0 & x \in \partial B, \end{cases}$$

for $v_k = v(\,\cdot\,, \tau_k)$. Then we have

$$\dot{v}(r, \tau) = \dot{w}(\log r + \tau) - \dot{w}(\tau) \not\equiv 0,$$

and therefore, $\dot{v}_k$ is an eigenfunction of (4) corresponding to $\mu = \mu^l_{\tau_k} = 0$. Then, the standard perturbation theory ([8]) guarantees the existence of $\phi = \phi(\,\cdot\,, \tau)$ and $\mu = \mu(\tau)$ satisfying (4), $\phi(\,\cdot\,, \tau_k) = \dot{v}_k$, and $\mu(\tau_k) = \mu^k_{\tau_k} = 0$. Differentiating (4) and (13) with respect to τ, subtracting each other with $\tau = \tau_k$, multiplying by $\dot{v}$ and integrating it over B, we have

$$\ddot{\lambda} \frac{\int_B e^v \dot{v}\, dx}{\left(\int_B e^v\, dx\right)^p} = \dot{\mu} \int_B \dot{v}^2\, dx, \tag{14}$$

where $\dot{\mu} = \dot{\mu}(\tau_k)$, $\ddot{\lambda} = \ddot{\lambda}(\tau_k)$, $v = v(\,\cdot\,, \tau_k)$ and $\dot{v} = \dot{v}(\,\cdot\,, \tau_k)$. As is stated in the proof of Theorem 2,

$$\dot{\lambda}(\tau) = \omega_n^p 2^{1-p} (2 - z(\tau))^{p-1} e^{(1-p)w(\tau)} g(\tau) = \omega_n^p \sigma(\tau)^{1-p} (2 - z(\tau))^{p-1} g(\tau), \tag{15}$$

where $g(\tau) = (1-p) z(\tau)(2 - z(\tau)) - p\dot{z}(\tau)$. Differentiating (15), we have

$$\begin{aligned} \ddot{\lambda}(\tau) &= \omega_n^p \sigma(\tau)^{1-p} (2 - z(\tau))^{p-2} \\ &\quad \times \{(1-p)(-z(\tau)^2 + 2z(\tau) + \dot{z}(\tau)) g(\tau) + (2 - z(\tau)) \dot{g}(\tau)\}. \end{aligned}$$

Since $\dot{\lambda}(\tau) = 0$ at $\tau = \tau_k$, it holds that $g(\tau_k) = 0$, namely,

$$(1 - p)z(\tau_k)(2 - z(\tau_k)) - p\dot{z}(\tau_k) = 0$$

from (15). We have $\lambda = \omega_n^p(2 - z)^p\sigma^{1-p}$ from (10) and (11),

$$\dot{\lambda} = \omega_n^p(2 - z)^{p-1}\sigma^{1-p}\{(2 - z)\sigma^{-1}\dot{\sigma}(1 - p) - pz\}$$

by differentiating, and

$$\sigma\dot{z} = \frac{(2 - z)(1 - p)}{p}\dot{\sigma}$$

at $\tau = \tau_k$. Hence we have

$$\begin{aligned} \ddot{\lambda}(\tau) &= \omega_n^p\sigma(\tau)^{-p}(2 - z(\tau))^{p-1} \\ &\quad \times \left\{\frac{2(p - 1)^2(2 - z(\tau))^2}{p} + \frac{(pn - 2)(1 - p)(2 - z(\tau))}{p} + p\sigma(\tau)\right\}\dot{\sigma}(\tau) \end{aligned} \tag{16}$$

at $\tau = \tau_k$. Since $\sigma = \lambda\big/\left(\int_B e^v\,dx\right)^p$, it holds that

$$\dot{\sigma} = \frac{\dot{\lambda}}{\left(\int_B e^v\,dx\right)^p} - \lambda p\frac{\int_B e^v\dot{v}\,dx}{\left(\int_B e^v\,dx\right)^{p+1}}. \tag{17}$$

Hence we have

$$\begin{aligned} \dot{\mu}\int_B \dot{v}^2\,dx &= -\lambda p\frac{\left(\int_B e^v\dot{v}\,dx\right)^2}{\left(\int_B e^v\,dx\right)^{2p+1}}\omega_n^p\sigma(\tau)^{-p}(2 - z(\tau))^{p-1} \\ &\quad \times \left\{\frac{2(p - 1)^2(2 - z(\tau))^2}{p} + \frac{(pn - 2)(1 - p)(2 - z(\tau))}{p} + p\sigma(\tau)\right\} \end{aligned}$$

at $\tau = \tau_k$ from (14) and (16). Let $\int_B e^v\dot{v}\,dx = 0$. Then (17) means that $\dot{\lambda} = 0$ and $\dot{\sigma} = 0$ vanish at $\tau = \tau_k$ simultaneously. However it is impossible from (10) and (11). Finally we have $\dot{\mu} < 0$ at $\tau = \tau_k$. $\square$

4. Annulus domain

In this section, we assume that $\Omega = A_a = \{x \in \mathbf{R}^n \mid a < |x| < 1\}$ with $a \in (0, 1)$ and consider radially symmetric solutions of (1).

We cite known results in [14] for the case of $3 \le n \le 9$. Radial solutions of (2) satisfy

$$\begin{cases} (r^{n-1}v')' + \sigma r^{n-1}e^v = 0 & \text{for } a < r < 1, \\ v(a) = v(0) = 0. \end{cases} \tag{18}$$

We can continue the solution of (18) up to $r = +0$ satisfying

$$\lim_{r\downarrow 0}\left(v'(r) - \frac{L}{r^{n-1}}\right) = 0$$

and

$$\lim_{r\downarrow 0}\left(v(r) + \frac{L}{(n-2)r^{n-2}}\right) = M$$

for some $L, M > 0$. Through the modified Emden transformation

$$v(r) = w(t) - 2t + M, \quad r = Be^t$$

with

$$B = \left\{\frac{2(n-2)}{\sigma e^M}\right\}^{1/2},$$

(18) is reduced to the autonomous ordinary differential equation

$$\begin{cases} \ddot{w} + (n-2)\dot{w} + 2(n-2)(e^w - 1) = 0, \\ \lim\limits_{t\to-\infty}\left(w(t) - 2t + \alpha e^{-(n-2)t}\right) = 0, \\ \lim\limits_{t\to-\infty} e^{-t}\left(\dot{w}(t) - 2 - \alpha(n-2)e^{-(n-2)t}\right) = 0, \end{cases} \tag{19}$$

where $\alpha = LB^{-(n-2)}/(n-2)$.

Then there exists a unique global solution $w = w_\alpha(t)$ of (19) for every $\alpha > 0$. The orbit $\mathcal{O}_\alpha = \{(w(t), z(t)) = (w_\alpha(t), \dot{w}_\alpha(t)) \mid t \in \mathbf{R}\}$ starts at $t = -\infty$ above the line $z = 2$ with $w = -\infty$, and approachs the origin $(0,0)$ as $t \to +\infty$. Then the family of orbits $\{\mathcal{O}_\alpha\}_{\alpha\geq 0}$ forms a foliation, that is, $\mathcal{O}_\alpha \cap \mathcal{O}_\beta = \emptyset$ if $\alpha \neq \beta$. Every orbit $\mathcal{O}_\alpha$ with $\alpha > 0$ crosses the line $z = 2$ just once. Moreover if $\alpha > \beta > 0$, O_α lies in the left side of $\mathcal{O}_\beta$ as t increases. For every point $(\eta_0, \zeta_0) \in \Sigma \equiv \{(w,z) \mid w > 0,\ z > 2\}$, there is a unique nonnegative α such that $(\eta_0, \zeta_0) \in \mathcal{O}_\alpha$.

Let $w_\alpha = w_\alpha(t)$ be the solution of (19). Then every point (w, z) on $\mathcal{O}_\alpha$ determines the time t so that the boundary condition

$$v(a) = v(1) = 0$$

is converted into

$$w(t^-) - 2t^- + M = 0 \quad \text{and} \quad w(t^+) - 2t^+ + M = 0 \tag{20}$$

for

$$t^- = \log\frac{a}{B} \quad \text{and} \quad t^+ = \log\frac{1}{B}. \tag{21}$$

Henceforward we omit the subscripts α and so on unless there is any confusion. Further $w^{\pm}$ and $z^{\pm}$ denote $w_\alpha(t^{\pm})$ and $z_\alpha(t^{\pm})$, respectively. From (20) and (21), it holds that

$$w^+ - w^- = -2\log a \quad \text{and} \quad t^+ - t^- = -\log a. \tag{22}$$

Conversely, if there exists a pair of points $P^{\pm}(w^{\pm}, z^{\pm})$ on O_α satisfying (22), we have a radial solution $v = v(r)$ for (18) with some positive constant σ. In fact, we define B, M, L, σ and $v(r)$ as

$$\begin{aligned}
&B = ae^{-t^-} = e^{-t^+},\\
&M = 2t^- - w^- = 2t^+ - w^+,\\
&L = \alpha(n-2)a^{n-2}e^{-(n-2)t^-} = \alpha(n-2)e^{-(n-2)t^+},\\
&\sigma = 2(n-2)e^{w^+} = 2(n-2)a^{-2}e^{w^-},\\
&v(r) = w(t) - 2t + M.
\end{aligned}$$

Therefore the structure of the solution of (18) is reduced to that of pairs of $P^{\pm}$ on $\mathcal{O}_\alpha$. We call $\{P_\alpha^{\pm}\}$ the boundary pair on $\mathcal{O}_\alpha$ associated with the annulus A_a in $\mathbf{R}^n$.

For every $a \in (0,1)$, there exists a unique pair of points $P_\alpha^{\pm} = P^{\pm}(w_\alpha^{\pm}, z_\alpha^{\pm})$ satisfying (22) on each orbit $\mathcal{O}_\alpha$ ($\alpha > 0$). The points P_α^+ and P_α^- lie below and above the line $z = 2$ respectively, and further, these points $P_\alpha^{\pm}$ depend on α continuously. Conversely for each point $(w_\alpha(t), z_\alpha(t))$ on $\mathcal{O}_\alpha$ with $z_\alpha(t) < 2$ ($z_\alpha(t) > 2$), there exists a unique $a^* = a^*(t) \in (0,1)$ ($a_* = a_*(t) \in (0,1)$) such that

$$w_\alpha(t) - w_\alpha(t + \log a^*) = -2\log a^*.$$
$$(w_\alpha(t - \log a_*) - w_\alpha(t) = -2\log a_*.)$$

Hence we have only to study $K_a = \{P_\alpha^-(w_\alpha^-, z_\alpha^-) \mid \alpha > 0\}$ to get the structure of solution of (18). The set K_a forms a continuous curve in $\mathbf{R}^2$, which is homeomorphic to $\mathbf{R}$. Now we have two lemmas.

Lemma 1 ([14], Lemmas 4.8 and 4.9). *For any fixed $a \in (0,1)$, we have*

$$\begin{cases}
\lim_{\alpha\to 0} w_\alpha^+ = \lim_{\alpha\to 0} w_\alpha^- = -\infty,\\
\lim_{\alpha\to 0} z_\alpha^+ = \lim_{\alpha\to 0} z_\alpha^- = 2,\\
\lim_{\alpha\to 0} \|v_\alpha\|_{C(\overline{\Omega})} = 0,\\
\lim_{\alpha\to 0} \sigma_\alpha = 0,
\end{cases}$$

where $P_\alpha^{\pm}(w_\alpha^{\pm}, z_\alpha^{\pm})$ are the pair of boundary points on O_α associated with A_a.

Lemma 2 ([14], Remark 6.1). *For any fixed $a \in (0,1)$, we have*

$$\begin{cases} \lim_{\alpha\to+\infty} w_\alpha^+ = \lim_{\alpha\to+\infty} w_\alpha^- = -\infty, \\ \lim_{\alpha\to+\infty} \|v_\alpha\|_{C(\overline{A_a})} = +\infty, \\ \lim_{\alpha\to+\infty} \sigma_\alpha = 0, \end{cases}$$

where $P_\alpha^\pm(w_\alpha^\pm, z_\alpha^\pm)$ are the pair of boundary points on O_α associated with A_a.

Proof of Theorem 4. We concentrate on the case of $3 \leq n \leq 9$. The behaviour of λ_α and v_α as $\alpha \to +0$ follows from Lemma 1, that is,

$$\lim_{\alpha\to 0} \lambda_\alpha = \lim_{\alpha\to 0} \sigma_\alpha \left(\int_{A_a} e^{v_\alpha}\, dx\right)^p = 0 \quad \text{and} \quad \lim_{\alpha\to 0} \|v_\alpha\|_{C(\overline{A_a})} = 0.$$

Integrating (1) over A_a, we have

$$\lambda = \omega_n^p (2(n-2)a^{-2}e^{w^-})^{1-p}(a^{n-2}(z^- - 2) - (z^+ - 2)) \tag{23}$$

in the same way as we deduce (11). Now it holds that

$$\lim_{\alpha\to+\infty} z_\alpha^- = +\infty.$$

In fact, we assume that $z_\alpha^- < M$ for any $\alpha \in \mathbf{R}$, where M is a positive constant owing to $z_\alpha^- > 2$. Since K_a is homeomorphic to $\mathbf{R}$, there is a constant $K > 0$ such that $w_\alpha^- < K$ for any $\alpha \in \mathbf{R}$ by Lemma 2. By setting $Q = \{(w,z) \mid w < K,\ 2 < z < M\}$, it holds that $\mathcal{O}_\beta \cap \overline{Q} = \emptyset$ for some $\beta \in \mathbf{R}$. Actually, for $(w,z) = (K,M)$ there is $\alpha > 0$ such that $(K,M) \in \mathcal{O}_\alpha$ because of $(K,M) \in \Sigma \subset \bigcup_{\alpha\geq 0} \mathcal{O}_\alpha$. Since z- and w-coordinate are decreasing and increasing with respect to t respectively in $\{(w,z) \mid w > g(z),\ z > 0\}$, $\mathcal{O}_\alpha \cap \overline{Q} = (K,M)$. Hence if we put $\beta = \alpha + \delta$ for any $\delta > 0$, we have $\mathcal{O}_\beta \cap \overline{Q} = \emptyset$ because $\mathcal{O}_\beta$ lies in the left side of $\mathcal{O}_\alpha$ as t increases. The points $\{(w_\alpha^-, z_\alpha^-) \mid \alpha > 0\}$ on $\mathcal{O}_\beta$ don't satisfy $w_\alpha^- < K$, $z_\alpha^- < M$ simultaneously, which is a contradiction. Therefore we have $\lim_{\alpha\to+\infty} z_\alpha^- = +\infty$. Finally from (23), $z_\alpha^+ < 2$ and $p \geq 1$, we have $\lim_{\alpha\to+\infty} \lambda_\alpha = +\infty$. In the case of $n \geq 10$, we chage Σ and K by $\{(w,z) \mid w < 0,\ z > 2\}$ and a negative constant, respectively. □

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