

ADDITIVE GROUP ACTIONS WITH FINITELY GENERATED INVARIANTS

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Abstract

Every locally trivial action of the additive group of complex numbers on a factorial affine variety has finitely generated ring of invariants. A criterion is given for such an action on complex four space to be conjugate to a translation. Restrictions on the nature of the singularities of the variety defined by the ring of invariants of triangular actions are noted.

1. Introduction

Let G_a denote the additive group of complex numbers, and X a complex affine variety. By an action of G_a on X we will mean an algebraic action. It is well known that every such action can be realized as the exponential of some locally nilpotent derivation D of the coordinate ring $\mathbb{C}[X]$ and that every locally nilpotent derivation gives rise to an action. The ring C_0 of G_a invariants in $\mathbb{C}[X]$ is equal to the ring of constants of the generating derivation. While it is known that C_0 need not be finitely generated, even for actions on \mathbb{C}^5 , the question of finite generation is interesting for special kinds of actions. Indeed, while we have finite generation for all actions on normal varieties of dimension ≤ 3 , the known actions on \mathbb{C}^n with nonfinitely generated invariants all are (quasi)homogeneous and therefore have fixed points. It is unknown for $n > 3$ whether every fixed point free action on \mathbb{C}^n has finitely generated invariants, but the ring of G_a invariants is finitely generated for all actions on \mathbb{C}^4 whose generating derivation is triangulable (triangulable actions) [2].

An action is said to be equivariantly trivial if there is a variety Y for which X is G_a equivariantly isomorphic to $Y \times G_a$, the action on $Y \times G_a$ being given by $g \cdot (y, h) = (y, g + h)$. Equivariant triviality of an action on X is equivalent with the existence of a regular function $s \in \mathbb{C}[X]$ for which $Ds = 1$. Such a function is called a slice and, if one exists, $\mathbb{C}[X] = C_0[s]$. In this case Y is a geometric quotient and $C_0 \cong \mathbb{C}[Y]$. The action is locally trivial if there are affine varieties Y_i and a cover of X by G_a stable affine open subsets X_i on which the action is equivariantly trivial.

There are locally trivial actions on normal affine varieties with nonfinitely generated invariants [3]. Throughout this paper, the term “factorial affine variety” means an

affine variety with trivial divisor class group, i.e. one whose coordinate ring is a unique factorization domain. The authors asserted in [4, Prop. 2.9 and Cor. 2.10] the existence of locally trivial actions on factorial affine varieties. The argument was incorrect and it is shown in Section 2 that in fact the ring of invariants for a locally trivial action on a factorial affine variety is finitely generated.

It was shown in [9] that a locally trivial triangular action on \mathbb{C}^4 is equivariantly trivial with quotient isomorphic to \mathbb{C}^3 . In Section 4 a topological criterion for equivariant triviality of a locally trivial action is given.

Given an action $\sigma: G_a \times X \rightarrow X$, let $\bar{\sigma}: G_a \times X \rightarrow X \times X$ denote the graph morphism and $\hat{\sigma}: \mathbb{C}[X] \rightarrow \mathbb{C}[X, t]$ (resp. $\tilde{\sigma}: \mathbb{C}[X \times X] \rightarrow \mathbb{C}[X, t]$) denote the induced maps on coordinate rings. The action is said to be proper if $\bar{\sigma}$ is a proper morphism (i.e. if $\mathbb{C}[X, t]$ is integral over the image of $\tilde{\sigma}$). A proper action on $X = \mathbb{C}^n$ is known to be locally trivial if $\mathbb{C}[X]$ is a flat ring extension of C_0 or if C_0 defines a smooth variety [6, Theorem 2.8]. While a finitely generated C_0 need not define a smooth variety in general, for triangular actions on \mathbb{C}^4 isolated singularities of this variety can be only of a restricted type, namely canonical singularities. An example is given of a fixed point free but nonproper triangular action on \mathbb{C}^4 with nonisolated singularities. The authors know of no example of a proper triangular action on \mathbb{C}^4 with nonregular C_0 , i.e. all known proper actions are equivariantly trivial.

2. Finite generation for locally trivial actions

From [8] we know that the quotient of a locally trivial action on an affine factorial variety X exists as a quasiaffine variety $Y^0 \subset \text{Spec } R^0$, where R^0 is the subring of C_0 constructed as follows: Let $\delta(a_1), \dots, \delta(a_n) \in C_0$ generate the unit ideal in $\mathbb{C}[X]$, and set $R_i = \mathbb{C}[X, 1/\delta(a_i)]^{G_a}$. Note that $\mathbb{C}[X, 1/\delta(a_i)] = R_i[a_i/\delta(a_i)]$ so that R_i is a finitely generated \mathbb{C} algebra, say $R_i = \mathbb{C}[b_{i1}, \dots, b_{im}, 1/\delta(a_i)]$, with $b_{ij} \in C_0$. It is easy to check that $R_i = C_0[1/\delta(a_i)]$. The ring

$$R^0 = \mathbb{C}[b_{ij}, \delta(a_i) \mid 1 \leq i \leq n, 1 \leq j \leq m]$$

is the required subring of C_0 and, with $Y_i \equiv \text{Spec } R_i$, we have $Y^0 = \bigcup_i Y_i$. Note that R_i is a unique factorization domain, so that Y_i and therefore Y^0 are normal.

Denote by R the integral closure of R^0 , which is a finitely generated R^0 module. Factorial closedness of C_0 in $\mathbb{C}[X]$ implies that R is a subring of C_0 , in fact C_0 is the factorial closure of R and therefore the morphism $X \rightarrow \text{Spec } R^0$ factors through $\text{Spec } R$. We can therefore replace R^0 by R and Y^0 by the image Y of X in $\text{Spec } R$. Thus $X \rightarrow Y$ is a geometric quotient and $X \rightarrow \text{Spec } R$ an open morphism. Moreover, C_0 is the ring of global sections of the structure sheaf of Y , and isomorphic to $T_l(R)$, the ideal transform of R with respect to the radical ideal defining the (Zariski closed) complement of Y in $\text{Spec } R$.

REMARK 1. If R is regular then $\text{Spec } R - V(J)$ is affine variety for any height one ideal J e.g. [12]. When R is regular it follows that C_0 is affine and regular, and Y is smooth as well. Since X is the total space of a principal G_a bundle over the smooth quasiaffine Y , X is smooth.

More generally, we have the

Theorem 2.1. *Let G_a act locally trivially on the factorial affine variety X . Then the ring of G_a invariants in $\mathbb{C}[X]$ is finitely generated.*

Proof. With R, Y, I as above, note that if the height of I is at least 2, i.e. prime ideals minimal over I all have height ≥ 2 so that

$$\text{codim}_{\text{Spec } R}(\text{Spec } R - Y) > 1$$

then $C_0 = R$. Assume then that $ht(I) = 1$ and write $I = J \cap K$ where J is the intersection of the height one prime ideals minimal over I and K the intersection of the minimal prime ideals of I of height greater than one. Then $C_0 = T_I(R) = T_J(R)$ since R is integrally closed [13, p.41 Corollary]. We claim that for each maximal ideal m of R , C_{0m} is flat over R_m and therefore that C_0 is flat over R . The assertion then follows from [15, Corollary 3.5].

Denote by Z the subscheme of $\text{Spec } R$ defined by J and by W its complement. In the terminology of [1],

$$Naf(Z) \equiv \{x \in \text{Spec } R : W \cap \text{Spec } R_x \text{ is not affine}\}$$

is a closed subset of Z , since R is noetherian, and empty if and only if W is affine. We claim that $Naf(Z) = \emptyset$. Let m be the maximal ideal of R defining a closed point z of Z , and set $S = R - m \subset C_0$. Since Z is a component of the complement of the image of $f: X \rightarrow \text{Spec } R$, $m\mathbb{C}[X] = \mathbb{C}[X]$. We claim that $qf(R) \subset S^{-1}\mathbb{C}[X]$. Indeed, if $0 \neq r \in R$ is a nonunit in $S^{-1}\mathbb{C}[X]$, let M be a maximal ideal of $S^{-1}\mathbb{C}[X]$ containing $rS^{-1}\mathbb{C}[X]$. Then $\mathbb{C} \cong \mathbb{C}[X]/M \cap \mathbb{C}[X]$ is a field which is finitely generated as a ring over $R/M \cap R$. Thus $R/M \cap R$ is a field, and $M \cap R = m$, contradicting the assumption that $z \notin \text{im}(f)$.

Since $qf(R) = qf(C_0)$, we obtain $S^{-1}\mathbb{C}[X] = qf(C_0)[s]$ where s is transcendental over $qf(C_0)$ and $\delta(s) \in C_0 - \{0\}$. Since $S \subset C_0$,

$$\begin{aligned} (S^{-1}\mathbb{C}[X])^{G_a} &= qf(C_0) \\ &= S^{-1}C_0 \\ &= S^{-1}T_J(R) \\ &= T_{JR_m}(R_m) \\ &= JR_mT_{JR_m}(R_m). \end{aligned}$$

The last equality shows that $W \cap \text{Spec } R_m$ is affine for every maximal ideal of R , from which it follows that $\text{Naf}(Z) = \emptyset$, i.e. that W is affine.

From affineness of W we conclude that $T_J(R)$ is flat over R and therefore finitely generated over R and over \mathbb{C} [15, Corollary 3]. \square

3. A slice criterion

Following Miyanishi [11], a morphism $f: Z \rightarrow W$ of complex algebraic schemes is said to be *geometrically irreducible in codimension one* (GICO) if for any irreducible subvariety T of Z of codimension one the field extension $\mathbb{C}(T)/\mathbb{C}(\overline{f(T)})$ is regular. Here $\overline{f(T)}$ denotes the closure of $f(T)$ in W . An action of G_a on a complex affine variety X is said to be GICO if $\mathbb{C}[X]^{G_a} \cong C_0$ is affine and the induced morphism $\text{Spec } \mathbb{C}[X]^{G_a} \rightarrow \text{Spec } C_0$ is GICO. It was shown in [6] that proper actions on $X = \mathbb{C}^n$ with finitely generated invariants are GICO and that the GICO condition is equivalent with the intersection of the kernel and image of the generating derivation not lying in any height one ideal of $\mathbb{C}[X]$ (equivalently of C_0). A GICO action on \mathbb{C}^n with C_0 finitely generated and regular is locally trivial [5].

An example of a locally trivial G_a action on \mathbb{C}^5 with finitely generated regular invariants but no slice was given in [18]. The next result suggests that such an example might not occur for actions on \mathbb{C}^4 .

Theorem 3.1. *Consider a GICO (e.g. proper) G_a action on $X = \mathbb{C}^4$ and assume that C_0 is finitely generated and regular. Set $W \cong \text{Spec } C_0$, $\pi: X \rightarrow W$ the morphism induced by the ring inclusion $C_0 \subset \mathbb{C}[X]$, and $V = W - \text{im}(\pi)$. If V is smooth and distinct irreducible components of V are disjoint then the action is equivariantly trivial.*

Proof. It is shown in [6] that a proper action is GICO and a GICO action with regular invariants is locally trivial. Thus $U \cong \text{im}(\pi)$ is open in W and $\pi: X \rightarrow U$ is a geometric quotient. If the action doesn't admit a slice, then [11, Theorem 2] yields that V is of pure codimension 2 in W (the flatness of π , missing from the hypothesis of that theorem, holds in the present context [4]). In fact, with $V = \bigcup_{i=1}^k V_i$ the decomposition into irreducible components, we show that the V_i are all isomorphic to the line \mathbb{C}^1 . In the following, we use singular homology with integral coefficients.

From the Lefschetz theorem on the homology of complex affine varieties we obtain $H_j(V_i) = 0 = H_j(V)$ for $j \geq 2$, and $H_j(W) = 0$ for $j \geq 5$. The Thom isomorphism yields $H_j(W, U) \cong H_{j-4}(V)$ so that $H_j(W, U) = 0$ for $j < 4$ and $j > 5$. Also, the local triviality realizes X as a principal G_a bundle over U , and therefore $H_j(U) = 0$ for $j > 0$.

The long exact sequence for the pair (W, U) shows that $H_0(W) \cong H_0(U) \cong \mathbb{Z}$, $H_j(W) = 0$ for $j = 1, 2, 3$ and $H_5(W, U) = 0$. Thus $H_1(V) = 0$, from which we deduce that all $V_i \cong \mathbb{C}^1$.

For each i , let $T_i \subset W$ be a tubular neighborhood of V_i and set $T = \bigcup_{i=1}^k T_i$. From the Mayer-Vietoris sequence for (U, T) it follows that $H_j(U \cap T) = 0$ for all $j > 0$. But each $U \cap T_i$ is homotopic to $\mathbb{R}^4 - \{\text{pt.}\}$, which in turn is homotopy equivalent to S^3 . Since $H_3(S^3) \neq 0$ we obtain a contradiction unless $V = \emptyset$. \square

It should be noted that an example of a proper but not locally trivial action on a smooth factorial fourfold (not isomorphic to \mathbb{C}^4) with finitely generated but nonregular invariants was given in [8].

4. Singularities

Recall the following result from [9]:

Theorem 4.1. *Let X be a smooth factorial quas affine variety. Suppose that G_a acts algebraically on X and that C_0 is finitely generated over \mathbb{C} . If $\dim X \leq 5$ then C_0 is Gorenstein.*

This applies in particular to triangular G_a actions on \mathbb{C}^4 for which the ring of invariants is known to be finitely generated. Since the ring of invariants is identical with the kernel of the generating derivation, the following lemma is easily verified:

Lemma 2. *Let G_a act on \mathbb{C}^n via a nonzero triangular derivation*

$$\delta = \sum_{i=2}^n p_i(x_1, \dots, x_{i-1}) \frac{\partial}{\partial x_i}.$$

Then:

- (1) δ commutes with $\partial/\partial x_n$.
- (2) $\partial/\partial x_n$ restricts to a locally nilpotent derivation on the kernel of δ .
- (3) The associated G_a action on $\ker(\delta)$ is trivial if and only if $\delta = \partial/\partial x_n$.

Flenner and Zaidenberg [10, Corollary 1.13] have shown that an isolated Cohen-Macaulay singularity of a complex affine variety admitting a nontrivial G_a action is a rational singularity. As a consequence we obtain the

Corollary 4.2. *Let G_a act on \mathbb{C}^4 via a triangular derivation δ and let Y denote the affine variety defined by C_0 . If $y \in Y$ is an isolated singularity then y is a rational singularity.*

REMARK 3. (1) The proper but not locally trivial action on \mathbb{C}^5 in [5] has finitely generated ring of invariants defining a variety singular in codimension exactly 3.

(2) Fixed point free triangular actions on \mathbb{C}^4 for which Y has isolated compound du Val singularities have been given in [18] and [7]. A typical example is the following:

EXAMPLE 1. Let G_a act on \mathbb{C}^4 via the triangular derivation

$$\delta = [(x_2^2 - 2x_1x_3) - 1] \frac{\partial}{\partial x_4} + x_2 \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_2}.$$

Setting $c_1 = x_1$, $c_2 = x_2^2 - 2x_1x_3 - 1$, $c_3 = x_1x_4 - x_2c_2$,

$$C_0 = \mathbb{C}[c_1, c_2, c_3, c_4] \quad \text{with the relation} \quad c_1c_4 - c_3^2 - c_2^2(c_2 + 1) = 0.$$

Observe that the origin is the unique singular point. After a change of variables, the completion of the local ring of the singular point is easily seen to be isomorphic to $\mathbb{C}[[c_1, c_2, c_3, c_4]]/(c_1^2 + c_4^2 + c_3^2 + c_2^2)$.

REMARK 4. We know of no proper triangular action on \mathbb{C}^4 for which the variety defined by the ring of invariants is singular (and therefore know of no proper action on \mathbb{C}^4 which is not conjugate to a translation).

With the aid of *Singular* [16] Parag Mehta discovered the following example of a fixed point free nonproper action with maximally singular invariants (i.e. the singular locus has codimension 2, minimal for a factorial variety):

EXAMPLE 2. Let G_a act on \mathbb{C}^4 via the triangular derivation

$$\delta = [x_3(x_2^2 - 2x_1x_3) + 1] \frac{\partial}{\partial x_4} + x_2 \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_2}.$$

Set

$$\begin{aligned} c_1 &= x_1 \\ c_2 &= x_2^2 - 2x_1x_3 \\ c_3 &= x_2^5 - 5x_1x_2^3x_3 + 6x_1^2x_2x_3^2 + 3x_1^2x_4 - 3x_1x_2 \\ c_4 &= x_2^6x_3^2 - \frac{20}{3}x_1x_2^4x_3^3 + \frac{44}{3}x_1^2x_2^2x_3^4 - \frac{32}{3}x_1^3x_3^5 \\ &\quad - 2x_2^5x_4 + 10x_1x_2^3x_3x_4 - 12x_1^2x_2x_3^2x_4 + 2x_2^4x_3 \\ &\quad - 12x_1x_2^2x_3^2 + 16x_1^2x_3^3 - 3x_1^2x_4^2 + 6x_1x_2x_4 - 3x_2^2. \end{aligned}$$

Then $C_0 = \mathbb{C}[c_1, c_2, c_3, c_4]$ and the relation satisfied by the generators is

$$c_3^2 + 6c_1c_2^3 - c_2^5 + 3c_1^2c_4 = 0.$$

With Y denoting as usual the affine variety with coordinate ring C_0 , the singular locus of Y is given by $c_1 = c_2 = c_3 = 0$. The singularities are again seen to be compound du Val by applying the quasihomogeneity criterion for du Val surface singularities [14, p.275] to the surface defined by:

$$c_3^2 + 6c_1c_2^3 - c_2^5 = 0.$$

It is of interest to explore the kinds of isolated singularities that can arise as isolated singularities of Y . A three dimensional rational Gorenstein singularity y is known to be canonical. It follows that the singularity is either compound du Val or the general hyperplane section through y is an elliptic surface singularity [14]. On the other hand, the following proposition indicates that the class of singularities that can arise is even more restricted. The argument is a slight modification of one given in [17].

Proposition 4.3. *Let G_a act via a triangular derivation δ on \mathbb{C}^4 . If the variety Y defined by the ring C_0 of G_a invariants has only isolated singularities, then they are not quotient singularities.*

Proof. Let $\pi: \mathbb{C}^4 \rightarrow Y$ denote the morphism induced by the ring inclusion. With Z_1 and Z_2 denoting the zero loci of $C_0 \cap \text{im}(\delta)$ in \mathbb{C}^4 and Y respectively, $\pi|_{\mathbb{C}^4 - Z_1}: \mathbb{C}^4 - Z_1 \rightarrow Y - Z_2$ is a geometric quotient. In particular π fibers are generically connected. Note that for any point y , $\pi^{-1}(y)$ has codimension at least 2. Indeed, as argued in [17] a codimension 1 component would be the zero locus of an invariant polynomial and therefore not the fiber of a single point.

Suppose that an isolated singularity y of Y is a quotient singularity. Let V be an analytic neighborhood of y analytically isomorphic to \mathbb{C}^3/G for some nontrivial finite subgroup of $GL_3(\mathbb{C})$. Note that the fundamental group $\pi_1(V - \{y\}) \cong G$. Let B be an open ball in $\pi^{-1}(V)$ and consider the morphism $\pi_B \equiv \pi|_{B - \pi^{-1}(y)}: B - \pi^{-1}(y) \rightarrow V - \{y\}$. Since $\text{codim}_B \pi^{-1}(y) \geq 2$, $B - \pi^{-1}(y)$ is simply connected and therefore π_B factors through the simply connected universal covering space U of $V - \{y\}$. This contradicts the generic connectedness of π fibers. \square

We close with the following

Conjecture 1. *A proper G_a action on \mathbb{C}^4 has regular ring of invariants. Thus a proper action is locally trivial, and a proper triangular action is equivariantly trivial.*

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