Shin, D.
Osaka J. Math.
44 (2007), 1–10

RATIONAL CURVES ON GENERAL HYPERSURFACES OF DEGREE 7 IN $\mathbb{P}^5$

DONGSOO SHIN

(Received September 20, 2005, revised February 6, 2006)

Abstract

We prove that there is no smooth irreducible reduced rational curve of degree e, $2 \leq e \leq 11$, on general hypersurfaces of degree 7 in $\mathbb{P}^5$.

1. Introduction

Throughout this paper we work over an algebraically closed field k of characteristic 0.

Let X_d be a general hypersurface in $\mathbb{P}^n$ of degree d. H. Clemens proved in [2] that if $d \geq 2n-1$ and $n \geq 3$ then there is no rational curve in X_d. In [9, 10], C. Voisin sharpened Clemens' lower bound for d by proving that if $d \geq 2n-2$ and $n \geq 4$ then X_d contains no rational curve.

On the other hand, if $d = 2n-3$ and $n \geq 3$, it has been classically known that there always exists a line on X_{2n-3} ([7, Theorem V.4.3.]). Note that for $n = 3$ and $d = 2n-2 = 4$, every surface of degree 4 in $\mathbb{P}^3$ contains a rational curve (although a general such surface contains no *smooth* rational curve). Therefore Voisin's lower bound for d and n are sharp in the sense that there is no rational curve on a general hypersurface $X_d \subset \mathbb{P}^n$.

The number of lines on X_{2n-3} is finite ([7, Theorem V.4.3.]). In [9, 10], C. Voisin extended this classical fact in case $n \geq 5$: If $n \geq 5$ then X_{2n-3} contains at most finite number of rational curves of each degree $e \geq 1$. Note that the analogue of this result for $n = 4$ would solve Clemens' conjecture on the finiteness of rational curves of each degree $e \geq 1$ on general quintic threefolds in $\mathbb{P}^4$.

Recently G. Pacienza extended Voisin's result in [8] by proving that there is, in fact, *no* rational curve of degree $e \geq 2$ on X_{2n-3} if $n \geq 6$. Therefore the only rational curves on X_{2n-3} are lines if $n \geq 6$.

It is natural to raise a question about the case $n = 5$ in Pacienza's result: Is there a rational curve of degree greater than one on general hypersurfaces of degree 7 in $\mathbb{P}^5$?

In this paper we prove

2000 Mathematics Subject Classification. Primary 14N05; Secondary 14N25.
Supported by the Brain Korea 21 Project and KOSEF R01-2002-000-00051-0.

Theorem 1.1. *There is no smooth irreducible reduced rational curve of degree* e*,* $2 \le e \le 11$*, on general hypersurfaces of degree* 7 *in* $\mathbb{P}^5$.

To do so, we count the dimension of the incidence scheme $\{(C, X) \mid C \subset X\}$, where C is a smooth irreducible reduced rational curve of degree e and X is a hypersurface of degree 7 in $\mathbb{P}^5$. We use similar techniques in [6], where the authors treat rational curves of degree at most 9 on general quintic threefolds.

We introduce some notation. For a projective variety Y, let $\mathrm{Hilb}^{et+1}(Y)$ be the Hilbert scheme parametrizing subschemes with the Hilbert polynomial $et + 1$. We define a subscheme $R_e(Y)$ of $\mathrm{Hilb}^{et+1}(Y)$ to be the open subscheme parametrizing smooth irreducible reduced rational curves of degree e.

Let $\mathbb{F} = \mathbb{P}H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(7))$ be the parameter space of hypersurfaces of degree 7 in $\mathbb{P}^5$, i.e., $\mathbb{F} \cong \mathbb{P}^N$, $N = \binom{5+7}{7} - 1$. We define the incidence scheme

$$I_e := \{(C, X) \in R_e(\mathbb{P}^5) \times \mathbb{F} \mid C \subset X\}$$

and let

$$p_R : I_e \to R_e(\mathbb{P}^5) \quad \text{and} \quad p_{\mathbb{F}} : I_e \to \mathbb{F}$$

be the projections. Note that $R_e(X) \cong p_{\mathbb{F}}^{-1}(X)$ for $X \in \mathbb{F}$.

We define $R_{e,i}(\mathbb{P}^5)$ to be the locally closed subset of $R_e(\mathbb{P}^5)$ parametrizing curves C with $h^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) = i$ where $\mathcal{I}_{C,\mathbb{P}^5}$ is the ideal sheaf of C in $\mathbb{P}^5$. Set

$$I_{e,i} := p_R^{-1}(R_{e,i}(\mathbb{P}^5)).$$

Finally let $\mathbb{G}(k, n)$ be the Grassmannian parametrizing k-linear space in $\mathbb{P}^n$.

2. Proof of Theorem 1.1

Throughout this section, X is a general hypersurface in $\mathbb{P}^5$ of degree 7 and C is a smooth irreducible reduced rational curve of degree $e \ge 1$.

Theorem 1.1 is a consequence of the following result.

Proposition 2.1. *For* $e \le 11$*,* I_e *is irreducible of dimension* $1 - e + N$.

Before proving Proposition 2.1, we prove Theorem 1.1 by using the above result.

Proof of Theorem 1.1. By Proposition 2.1, if $2 \le e \le 11$, then $\dim I_e < \dim \mathbb{F} = N$. So $p_{\mathbb{F}}$ is not surjective. Therefore

$$R_e(X) \cong p_{\mathbb{F}}^{-1}(X) = \varnothing$$

for general X. $\square$

To prove Proposition 2.1, we need the following lemma.

Lemma 2.2. *$R_e(\mathbb{P}^n)$ is smooth, irreducible, and of dimension $(n+1)e+n-3$.*

Proof. Fix $C \in R_e(\mathbb{P}^n)$. The restricted Euler sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(1)^{\oplus n+1} \to \mathcal{T}_{\mathbb{P}^n}|_C \to 0$$

yields $H^1(C, \mathcal{T}_{\mathbb{P}^n}|_C) = 0$. The sequence of tangent and normal sheaves

$$0 \to \mathcal{T}_C \to \mathcal{T}_{\mathbb{P}^n}|_C \to \mathcal{N}_{C,\mathbb{P}^n} \to 0$$

yields $H^1(C, \mathcal{N}_{C,\mathbb{P}^n}) = 0$. Hence, by the functorial property of the Hilbert scheme, $R_e(\mathbb{P}^n)$ is smooth at C of dimension $h^0(C, \mathcal{N}_{C,\mathbb{P}^n})$, and

$$\begin{aligned} h^0(C, \mathcal{N}_{C,\mathbb{P}^n}) &= \chi(\mathcal{T}_{\mathbb{P}^n}|_C) - \chi(\mathcal{T}_C) \\ &= \chi(\mathcal{O}_C(1)^{\oplus n+1}) - \chi(\mathcal{O}_C) - \chi(\mathcal{T}_C) \\ &= (n+1)(e+1) - 1 - (2+1) \\ &= (n+1)e + n - 3. \end{aligned}$$

Note that morphisms of degree e from $\mathbb{P}^1$ to $\mathbb{P}^n$ are parametrized by a Zariski open set of the projective space $\mathbb{P}((S^e k^2)^{n+1})$, where $S^e k^2$ is the symmetric product. We denote this quasi-projective variety $\mathrm{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$. Let $\mathrm{RatMor}_e(\mathbb{P}^1, \mathbb{P}^n)$ be the subset of $\mathrm{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$ consisting of all morphisms whose image is a smooth irreducible reduced rational curve. Then $\mathrm{RatMor}_e(\mathbb{P}^1, \mathbb{P}^n)$ is an open subset of $\mathrm{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$. Since $\mathrm{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$ is irreducible, so is $\mathrm{RatMor}_e(\mathbb{P}^1, \mathbb{P}^n)$. There is a surjective morphism from $\mathrm{RatMor}_e(\mathbb{P}^1, \mathbb{P}^n)$ to $R_e(\mathbb{P}^n)$. Therefore $R_e(\mathbb{P}^n)$ is irreducible. □

Proof of Proposition 2.1. Assume $C \in R_{e,i}(\mathbb{P}^5)$. Let

$$r : H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(7)) \to H^0(C, \mathcal{O}_C(7))$$

be the restriction map. Then $p_R^{-1}(C)$ is the projectivation of the kernel of r. From the standard exact sequence

$$\begin{aligned} 0 &\to H^0(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) \to H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(7)) \\ &\to H^0(C, \mathcal{O}_C(7)) \to H^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) \to 0, \end{aligned}$$

we get

$$\dim p_R^{-1}(C) = h^0(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) - 1 = (N + 1 - (7e+1) + i) - 1.$$

Therefore

$$\begin{aligned} \dim I_{e,i} &= \dim R_{e,i}(\mathbb{P}^5) + \dim p_R^{-1}(C) \\ &= \dim R_{e,i} + (N+1-(7e+1)+i) - 1. \end{aligned} \tag{1}$$

Assume that $e \leq 9$. By the regularity theorem in [4], C is 8-regular, i.e., $H^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) = 0$. So $R_{e,0}(\mathbb{P}^5) = R_e(\mathbb{P}^5)$, which has dimension $6e+2$, and the fibers $p_R^{-1}(C)$ are irreducible of same dimension $(N+1-(7e+1)) - 1$. Therefore I_e is irreducible of dimension $1-e+N$. The proof is done in case $e \leq 9$.

Assume that $e = 10$ or 11. The following Lemma 2.3 implies that $R_{e,0}(\mathbb{P}^5)$ is open and nonempty, and hence $R_{e,0}(\mathbb{P}^5)$ is irreducible. So $I_{e,0}$ is irreducible of dimension $1-e+N$ since fibers $p_R^{-1}(C)$ for $C \in R_{e,0}(\mathbb{P}^5)$ are irreducible of same dimension $(N+1-(7e+1)) - 1$.

Also from the following Lemma 2.3 and equation (1)

$$\dim I_{e,i} < 1 - e + N \quad \text{for} \quad i > 0.$$

It is also clear, from the way I_e is defined, that all its components have dimension at least $1-e+N$ because the corresponding incidence in $\mathrm{RatMor}_e(\mathbb{P}^1, \mathbb{P}^n) \times \mathbb{P}^N$ is cut out by $7e+1$ equations, so both this incidence, and I_e, have codimension at most $7e+1$ (locally). Therefore the closure of $I_{e,0}$ is I_e, and hence I_e is irreducible of dimension $1-e+N$. Thus Proposition 2.1 is proved if given Lemma 2.3. □

Lemma 2.3. *For $e = 10, 11$, if $i > 0$ and if $R_{e,i}(\mathbb{P}^5)$ is nonempty, then*

$$\operatorname{codim}(R_{e,i}(\mathbb{P}^5), R_e(\mathbb{P}^5)) > i.$$

Before proving Lemma 2.3, we begin with some general observations.

REMARK 2.4. Suppose $C \in R_e(\mathbb{P}^5)$.
(1) If $e \geq 3$, then C cannot lie in a 2-plane because its arithmetic genus is 0. Moreover, if $e \geq 4$, then C cannot lie in a 2-dimensional quadric cone by [5, V, Ex. 2.9].
(2) If C lies in a k-linear subspace H in $\mathbb{P}^5$ with the ideal sheaf $\mathcal{I}_{C,H}$, then

$$h^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) = h^1(H, \mathcal{I}_{C,H}(7)).$$

We briefly prove this formula. Consider the following exact sequence of twisted ideals

$$0 \to \mathcal{I}_{\mathcal{H},\mathbb{P}^s}(7) \to \mathcal{I}_{C,\mathbb{P}^s}(7) \to \mathcal{I}_{C,\mathcal{H}}(7) \to 0,$$

where $k+1 \leq s \leq 5$ and $\mathcal{H}$ is a hyperplane $\mathbb{P}^{s-1}$ in $\mathbb{P}^s$. Note that $\mathcal{I}_{\mathcal{H},\mathbb{P}^s}(7) = \mathcal{O}_{\mathbb{P}^s}(6)$; hence we have $h^1(\mathbb{P}^{s-1}, \mathcal{I}_{C,\mathbb{P}^{s-1}}(7)) = h^1(\mathbb{P}^s, \mathcal{I}_{C,\mathbb{P}^s}(7))$ because $\mathcal{O}_{\mathbb{P}^s}(6)$ has no H^1 or H^2. Using this formula $5-k$ times proves the desired formula.

We recall the following useful facts which will be used when proving Lemma 2.3.

Lemma 2.5 ([4]). *Let C be a nondegenerate $(e+1-r)$-irregular curve in $\mathbb{P}^r$ $(r \geq 3)$ of degree e. If $e > r+1$, then C is rational, smooth with a $(e+2-r)$-secant line, and one of the following holds;*
(1) *$r = 3$, C is contained in a smooth quadric, and $h^1(\mathbb{P}^r, \mathcal{I}_{C,\mathbb{P}^r}(e-r)) = e-3$, or*
(2) *$r = 3$, C is* not *contained in a smooth quadric, and $h^1(\mathbb{P}^r, \mathcal{I}_{C,\mathbb{P}^r}(e-r)) = 1$, or*
(3) *$r \geq 4$ and $h^1(\mathbb{P}^r, \mathcal{I}_{C,\mathbb{P}^r}(e-r)) = 1$.*

Lemma 2.6 ([3]). *Let C be an irreducible smooth curve in $\mathbb{P}^3$. Suppose C is nondegenerate, of degree e, and of genus g. If $e \geq 6$ and $(e, g) \notin \{(7,0), (7,1), (8,0)\}$, then $h^1(\mathbb{P}^3, \mathcal{I}_{C,\mathbb{P}^3}(e-4)) > 0$ if and only if C has a $(e-2)$-secant line.*

Lemma 2.7 ([6]). *Let $e \geq 4$ and $r \geq 3$. Fix s with $e > s \geq 3$. In $R_e(\mathbb{P}^r)$ the subset of curves with a s-secant line has codimension at least $(r-1)(s-2)-s$.*

Proof of Lemma 2.3. Assume $e = 10$. If C is not contained in any hyperplane, then C is 7-regular and hence 8-regular, i.e., $H^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) = 0$. Therefore $R_{10}(\mathbb{P}^5) - R_{10,0}(\mathbb{P}^5)$ is contained in the closed set $\mathcal{G}$ of curves contained in hyperplanes in $\mathbb{P}^5$. Then

$$\begin{aligned}\operatorname{codim}(\mathcal{G}, R_{10}(\mathbb{P}^5)) &\geq \dim R_{10}(\mathbb{P}^5) - (\dim R_{10}(\mathbb{P}^4) + \dim \mathbb{G}(4,5)) \\ &= (6 \times 10 + 2) - (5 \times 10 + 1 + 5) = 6.\end{aligned}$$

In particular,

$$\operatorname{codim}(R_{10,1}(\mathbb{P}^5), R_{10}(\mathbb{P}^5)) > 1,$$

as asserted.

Suppose $h^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) \geq 2$. Then C must lie in a hyperplane G, since, if not, C is 7-regular. If C is nondegenerate in G, then C is 8-regular, i.e., $h^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) = 0$, which contradicts $h^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) \geq 2$. Therefore C is contained in a 3-linear space H in $\mathbb{P}^5$ and, by Remark 2.4 (2),

$$h^1(H, \mathcal{I}_{C,H}(7)) \geq 2.$$

Then, by Lemma 2.5 (1),

$$h^1(H, \mathcal{I}_{C,H}(7)) = h^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) = 7.$$

Therefore if $R_{10,i}(\mathbb{P}^5)$ is nonempty then i is 0, 1, or 7. So it remains to prove that

$$\operatorname{codim}(R_{10,7}(\mathbb{P}^5), R_{10}(\mathbb{P}^5)) > 7.$$

Since $h^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) = 7$, from Lemma 2.5 (1), C lies in a smooth quadric surface Q contained in H. Since C is smooth, rational, and of degree 10, C is contained in the linear system $|9L + M|$ on Q where L and M are two generators of $\mathrm{Pic}(Q)$. Then Q varies in $\mathbb{P}H^0(H, \mathcal{O}_H(2))$ and H varies in $\mathbb{G}(3, 5)$. Therefore,

$$\begin{aligned}\operatorname{codim}(R_{10,7}(\mathbb{P}^5), R_{10}(\mathbb{P}^5)) &\geq \dim R_{10,7}(\mathbb{P}^5) \\ &\quad - (\dim |9L + M| + \dim \mathbb{P}H^0(H, \mathcal{O}_H(2)) + \dim \mathbb{G}(3, 5)) \\ &= 62 - (19 + 9 + 8) = 26 > 7.\end{aligned}$$

Thus Lemma 2.3 holds for $e = 10$.

Assume that $e = 11$. If C is nondegenerate in $\mathbb{P}^5$, then C is 8-regular, i.e., $h^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) = 0$. Hence $R_{11}(\mathbb{P}^5) - R_{11,0}(\mathbb{P}^5)$ is contained in the closed set $\mathcal{G}$ of curves in hyperplanes in $\mathbb{P}^5$.

$$\begin{aligned}\operatorname{codim}(\mathcal{G}, R_{11}(\mathbb{P}^5)) &\geq \dim R_{11}(\mathbb{P}^5) - (\dim R_{11}(\mathbb{P}^4) + \dim \mathbb{G}(4, 5)) \\ &= (6 \times 11 + 2) - (5 \times 11 + 1 + 5) = 7.\end{aligned}$$

In particular,

$$\operatorname{codim}(R_{11,i}(\mathbb{P}^5), R_{11}(\mathbb{P}^5)) \geq 7 > i \quad \text{for} \quad i = 1, \ldots, 6$$

as asserted.

Assume $h^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) \geq 7$. C lies in a hyperplane G, since, if not, C is 8-regular. By Remark 2.4 (2),

$$h^1(G, \mathcal{I}_{C,G}(7)) = h^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) \geq 7.$$

Suppose that C is nondegenerate in G. Then C is 8-*irregular* in G since $h^1(G, \mathcal{I}_{C,G}(7)) \geq 7$.

However, from Lemma 2.5 (3), we know that

$$h^1(G, \mathcal{I}_{C,G}(7)) = 1,$$

which contradicts our assumption $h^1(G, \mathcal{I}_{C,G}(7)) \geq 7$. Thus C is contained in a 3-linear space H in $\mathbb{P}^5$. There are three possible cases;

(1) C lies in H, and C has *no* 9-secant line,
(2) C lies in some smooth quadric surface Q with ideal $\mathcal{I}_{C,Q}$.
(3) C lies in H, but C lies in *no* smooth quadric surface, and C has a 9-secant line.

In case (1), by Lemma 2.6 and Remark 2.4,

$$h^1(H, \mathcal{I}_{C,H}(7)) = h^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) = 0,$$

which contradicts our assumption $h^1(G, \mathcal{I}_{C,G}(7)) \geq 7$.

In case (2), since C is rational, smooth, and of degree 11, C is contained in the linear system $|10L + M|$ on Q where L and M are two generators of $\mathrm{Pic}(Q)$. Thus

$$\mathcal{I}_{C,Q}(7) = \mathcal{O}_Q(-3, 6).$$

Then the Künneth formula yields

$$h^1(Q, \mathcal{I}_{C,Q}(7)) = 0 \times 0 + 2 \times 7 = 14.$$

Note that

$$h^1(Q, \mathcal{I}_{C,Q}(7)) = h^1(H, \mathcal{I}_{C,H}(7)) = h^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) = 14.$$

Indeed, the second equality is Remark 2.4 (2), and the first can be proved similarly. Therefore

$$h^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) = 14$$

and it remains to prove that

$$\mathrm{codim}(R_{11,14}(\mathbb{P}^5), R_{11}(\mathbb{P}^5)) > 14.$$

Let $\mathcal{G}$ be the subset of $R_{11}(\mathbb{P}^5)$ consisting of all curves C included in the case (2). These C are contained in the linear system $|10L + M|$ on Q and Q varies in $\mathbb{P}H^0(H, \mathcal{O}_H(2))$ and H varies in $\mathbb{G}(3, 5)$. Therefore

$$\begin{aligned}\mathrm{codim}(R_{11,14}(\mathbb{P}^5), R_{11}(\mathbb{P}^5)) &\geq \dim R_{11}(\mathbb{P}^5)\\ &\quad - (\dim|10L + M| + \dim \mathbb{P}H^0(H, \mathcal{O}_H(2)) + \dim \mathbb{G}(3, 5))\\ &= 68 - (21 + 9 + 8) = 30 > 14.\end{aligned}$$

In particular,

$$\mathrm{codim}(R_{11,14}(\mathbb{P}^5), R_{14}(\mathbb{P}^5)) > 14$$

as asserted.

In case (3), let $\mathcal{S}$ be the subset of $R_{11}(H)$ consisting of all C satisfying the conditions in case (3). Then, by Lemma 2.7, $\mathcal{S}$ is of codimension at least 5 in $R_{11}(H)$.

Let $\mathcal{G}$ be the subset of $R_{11}(\mathbb{P}^5)$ consisting of all C satisfying the conditions in case (3) for a 3-linear space H. Note that H varies in $\mathbb{G}(3, 5)$. Therefore

$$\begin{aligned}\mathrm{codim}(\mathcal{G}, R_{11}(\mathbb{P}^5)) &\geq \dim R_{11}(\mathbb{P}^5) - (\dim R_{11}(H) + \dim \mathbb{G}(3, 5))\\ &\quad + \mathrm{codim}(\mathcal{S}, R_{11}(H))\\ &= 68 - (44 + 8) + 5 = 21.\end{aligned}$$

Therefore it suffices to prove that

$$h^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) < 21,$$

or equivalently, by Remark 2.4 (2), that

$$h^1(H, \mathcal{I}_{C,H}(7)) = h^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) < 21.$$

Choose a 2-plane U in H that meets C in 11 distinct points, no three of which are collinear. Such an U exists by [1, Lemma, p.109].

Let $k \geq 5$. These 11 points impose independent conditions on the system of curves of degree k in U by [1, Lemma, p.115]. Therefore, in the long exact sequence

$$\begin{aligned} H^0(U, \mathcal{O}_U(k)) &\to H^0(C \cap U, \mathcal{O}_{C\cap U}(k)) \\ &\to H^1(U, \mathcal{I}_{C\cap U,U}(k)) \to H^1(U, \mathcal{O}_U(k)), \end{aligned}$$

the first map is surjective. However, the last term vanishes. Therefore

$$H^1(U, I_{C\cap U,U}(k)) = 0.$$

Consequently, the exact sequence of sheaves

$$0 \to \mathcal{I}_{C,H}(k-1) \to \mathcal{I}_{C,H}(k) \to \mathcal{I}_{C\cap U,U}(k) \to 0$$

yields

$$h^1(H, \mathcal{I}_{C,H}(4)) \geq h^1(H, \mathcal{I}_{C,H}(5)) \geq h^1(H, \mathcal{I}_{C,H}(6)) \geq \cdots . \tag{2}$$

Consider the standard exact sequence of sheaves

$$0 \to \mathcal{I}_{C,H}(k) \to \mathcal{O}_H(k) \to \mathcal{O}_C(k) \to 0.$$

Since $H^1(H, \mathcal{O}_H(k)) = 0$ for $k \geq 0$, taking cohomology yields

$$h^0(H, \mathcal{I}_{C,H}(k)) = \binom{k+3}{3} - (11k+1) + h^1(H, \mathcal{I}_{C,H}(k)). \tag{3}$$

Proceeding by contradiction, assume $h^1(H, \mathcal{I}_{C,H}(7)) \geq 21$. We will prove that

$$h^0(H, \mathcal{I}_{C,H}(8)) \geq 78.$$

Then, by the equation (3),

$$h^1(H, \mathcal{I}_{C,H}(8)) \geq 2.$$

However, by Lemma 2.5, $h^1(H, \mathcal{I}_{C,H}(8))$ must be one. Therefore there is a contradiction.

By the equation (2) and equation (3), we get

$$h^0(H, \mathcal{I}_{C,H}(4)) \geq 11, \quad h^0(H, \mathcal{I}_{C,H}(7)) \geq 63.$$

Note that $h^0(H, \mathcal{I}_{C,H}(2)) = 0$ since C cannot lie neither on a smooth quadric surface by the assumption of case (3) nor on a quadric cone by Remark 2.4. Therefore every element in $H^0(H, \mathcal{I}_{C,H}(3))$ is irreducible.

Suppose $h^0(H, \mathcal{I}_{C,H}(3)) \geq 2$. Take two independent irreducible cubics F_3 and F_3' in $H^0(H, \mathcal{I}_{C,H}(3))$. Then $\deg(F_3 \cap F_3') = 9$, but $C \subset F_3 \cap F_3'$ and $\deg(C) = 11$, which is impossible. Therefore $h^0(H, \mathcal{I}_{C,H}(3)) \leq 1$.

Suppose there exists a nonzero cubic F_3 in $H^0(H, \mathcal{I}_{C,H}(3))$. Let

$$\alpha \colon H^0(H, \mathcal{O}_H(1)) \to H^0(H, \mathcal{I}_{C,H}(4))$$

be the linear map defined by multiplying with F_3. The image of α is a subspace of $H^0(H, \mathcal{I}_{C,H}(4))$ of dimension 4. Note that

$$h^0(H, \mathcal{I}_{C,H}(1)) = h^0(H, \mathcal{I}_{C,H}(2)) = 0.$$

Therefore there exist irreducible quartics in $H^0(H, \mathcal{I}_{C,H}(4))$.

Suppose $h^0(H, \mathcal{I}_{C,H}(3)) = 0$. Since

$$h^0(H, \mathcal{I}_{C,H}(1)) = h^0(H, \mathcal{I}_{C,H}(2)) = h^0(H, \mathcal{I}_{C,H}(3)) = 0,$$

every element in $H^0(H, \mathcal{I}_{C,H}(4))$ is irreducible.

Therefore, since $h^0(H, \mathcal{I}_{C,H}(3)) \leq 1$, there always exists an irreducible quartic F_4 in $H^0(H, \mathcal{I}_{C,H}(4))$.

Let

$$\alpha \colon H^0(H, \mathcal{O}_H(3)) \to H^0(H, \mathcal{I}_{C,H}(7))$$

be the linear map defined by multiplying with F_4. The image of α is a subspace of $H^0(H, \mathcal{I}_{C,H}(7))$ of dimension 20. Let W be a subspace of $H^0(H, \mathcal{I}_{C,H}(7))$ satisfying

$$H^0(H, \mathcal{I}_{C,H}(7)) = \mathrm{image}(\alpha) \oplus W.$$

Note that $\dim W = h^0(H, \mathcal{I}_{C,H}(7)) - \dim \mathrm{image}(\alpha) \geq 63 - 20 = 43$.

Take a nonzero $L \in H^0(H, \mathcal{O}_H(1))$. Define

$$X := \{F_4 F \colon F \in H^0(H, \mathcal{O}_H(4))\},$$
$$Y := \{FL \colon F \in W\}.$$

X and Y are subspaces of $H^0(H,\mathcal{I}_{C,H}(8))$ of dimension 35 and 43, respectively. Moreover, by the irreducibility of F_4 and by the choice of W, we have $X \cap Y = 0$. Therefore

$$h^0(H,\mathcal{I}_{C,H}(8)) \geq \dim X + \dim Y = 78,$$

as asserted. $\square$

ACKNOWLEDGEMENT. This work is a part of Ph. D. Thesis of the author submitted to Seoul National University. I am extremely grateful to Changho Keem for uncounted discussions and steady encouragement. I thank Gianluca Pacienza, who suggested this problem at a summer school held at Byen-San, Korea, in 2003.

References

[1] E. Arbarello, M. Cornalba, P. Griffiths and J. Harris: Geometry of Algebraic Curves, I, Springer-Verlag, New York, 1985.

[2] H. Clemens: *Curves on generic hypersurfaces*, Ann. Sci. École Norm. Sup. (4) **19** (1986), 629–636.

[3] J. d'Almeida: *Courbes de l'espace projectif* : *séries linéaires incomplètes et multisécantes*, J. Reine Angew. Math. **370** (1986), 30–51.

[4] L. Gruson, R. Lazarsfeld and C. Peskine: *On a theorem of Castelnuovo, and the equations defining space curves*, Invent. Math. **72** (1983), 491–506.

[5] R. Hartshorne: Algebraic Geometry, Springer-Verlag, New York, 1977.

[6] T. Johnsen and S.L. Kleiman: *Rational curves of degree at most* 9 *on a general quintic threefold*, Comm. Algebra **24** (1996), 2721–2753.

[7] J. Kollár: Rational Curves on Algebraic Varieties, Springer-Verlag, Berlin, 1996.

[8] G. Pacienza: *Rational curves on general projective hypersurfaces*, J. Algebraic Geom. **12** (2003), 245–267.

[9] C. Voisin: *On a conjecture of Clemens on rational curves on hypersurfaces*, J. Differential Geom. **44** (1996), 200–213.

[10] C. Voisin: *A correction*: *On a conjecture of Clemens on rational curves on hypersurfaces*, J. Differential Geom. **49** (1998), 601–611.

Department of Mathematical Sciences
Seoul National University
Seoul 151-747
Korea
e-mail: dsshin@math.snu.ac.kr