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SOME QUOTIENT ALGEBRAS ARISING FROM THE QUANTUM TOROIDAL ALGEBRA $U_q(sl_{n+1}(\mathcal{C}_\gamma))$ ($n \geq 2$)

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Abstract

Some quotient algebras arising from the quantum toroidal algebra $U_q(sl_{n+1}(\mathcal{C}_\gamma))$ ($n \geq 2$) are considered. They are related to integrable highest weight representations of the algebra and are shown to be isomorphic to tensor products of two algebras of symmetric Laurent polynomials and Macdonald's difference operators.

1. Introduction

The quantum toroidal algebras were introduced in [1] and [2] as q deformations of the universal enveloping algebras of toroidal Lie algebras [3]. Since then, the algebras and representations of them have been studied in [2], [4]–[11]. In [12], we studied some quotient algebras arising from the quantum toroidal algebra of type sl_2 and found a connection with Macdonald's difference operators [13]. In this paper we extend this result to the case of the quantum toroidal algebra of type sl_{n+1} ($n \geq 2$).

Let $\mathcal{C}_\gamma$ be the algebra over $\mathbf{C}(\gamma)$ of Laurent polynomials in noncommutative variables x, y satisfying $xy = \gamma^2 yx$. The $\mathbf{C}(\gamma)$ Lie algebra $\mathcal{L} = sl_{n+1}(\mathcal{C}_\gamma)$ is defined to be the derived subalgebra of $gl_{n+1}(\mathcal{C}_\gamma)$. Lie algebras of this kind and central extensions of them were considered in the study of extended affine Lie algebras in [14] and representations of these algebras were investigated in [15]–[21]. The quantum toroidal algebra which we study is a q deformation of the universal enveloping algebra of this Lie algebra $\mathcal{L}$. We shall briefly explain what quotient algebras we consider.

As was shown in [14],

$$\mathcal{L} = \left\{ u \in gl_{n+1}(\mathcal{C}_\gamma) \;\middle|\; \operatorname{tr}(u) \in \bigoplus_{(k,l)\neq(0,0)} \mathbf{C}(\gamma) x^k y^l \right\}.$$

Therefore if we let $\mathcal{N}^+$ (resp. $\mathcal{N}^-$) be the subalgebra of strictly upper (resp. strictly lower) triangular matrices and $\mathcal{H}$ the subalgebra of diagonal matrices, then $\mathcal{L} = \mathcal{N}^- \oplus \mathcal{H} \oplus \mathcal{N}^+$. Let Q_n be the root lattice of sl_{n+1} and the E_{ij} matrix units. Since $h_i := E_{ii} - E_{i+1,i+1} \in \mathcal{H}$ for $1 \leq i \leq n$, $\mathcal{L}$ and $U(\mathcal{L})$ are Q_n graded as in the case of the Lie algebra sl_{n+1}. We denote the homogeneous subspace of degree α of $U(\mathcal{L})$ by $U(\mathcal{L})_\alpha$.

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Set

$$I = \sum_{\substack{m_1,\dots,m_n\in\mathbf{Z} \\ m_j>0 \text{ for some } j}} U(\mathcal{L})_{-\sum_{j=1}^n m_j\alpha_j} U(\mathcal{L})_{\sum_{j=1}^n m_j\alpha_j}$$

and $\mathcal{B} = U(\mathcal{L})_0/I$. Then $I = \sum_{\alpha>0} U(\mathcal{N}^-)_{-\alpha}U(\mathcal{H})U(\mathcal{N}^+)_\alpha$ and $\mathcal{B} \simeq U(\mathcal{H})$ as algebras. Let V be an $\mathcal{L}$ module generated by a nonzero vector v such that $\mathcal{N}^+v = 0$ and $h_i v = N_i v$ for some $N_i \in \mathbf{C}(\gamma)$ $(1 \le i \le n)$. Then V admits a weight space decomposition $V = \bigoplus_{\alpha\ge 0} V_{\lambda-\alpha}$ where $\lambda = \sum_{1\le i\le n} N_i\omega_i$ with the ω_i being the fundamental weights of sl_{n+1}. Suppose further that the elements $E_{ij}x^k y^l$ $(i \ne j)$ act locally nilpotently on V. Then the N_i are nonngegative integers and $V_{\lambda-m\alpha_i} = 0$ $(m > N_i)$ for $1 \le i \le n$. Let $I_{N_1,\dots,N_n}$ be the ideal of $U(\mathcal{L})_0$ generated by I, $h_i - N_i$ and $\sum_{m>N_i} U(\mathcal{L})_{m\alpha_i}U(\mathcal{L})_{-m\alpha_i}$ $(1 \le i \le n)$ and set $\mathcal{B}_{N_1,\dots,N_n} = U(\mathcal{L})_0/I_{N_1,\dots,N_n}$. Then V_λ becomes a $\mathcal{B}_{N_1,\dots,N_n}$ module since $U(\mathcal{L})_0$ preserves V_λ and $I_{N_1,\dots,N_n}$ annihilates it.

The quotient algebras which we consider in this paper are q analogues of $\mathcal{B}$ and the $\mathcal{B}_{N_1,\dots,N_n}$. Our main results are that $\mathcal{B}_{N_1,\dots,N_n} = 0$ unless $N_2 = \cdots = N_{n-1} = 0$ and that $\mathcal{B}_{N_1,0,\dots,0,N_n}$ is isomorphic to the tensor product of two algebras of symmetric Laurent polynomials and Macdonald's difference operators. We hope that this result will help us to study integrable highest weight representations of $U_q(\mathcal{L})$.

This paper is organized as follows. After summarizing some notations which we use in this papaer in Section 2, the definitions of the quantum toroidal algebra and their automorphisms are given in Section 3. In Section 4 some results [12] on quotient algebras from $U_q(sl_2(\mathcal{C}_\gamma))$ are reviewed. In Section 5 we study quotient algebra arising from $U_q(sl_{n+1}(\mathcal{C}_\gamma))$ with $n \ge 2$. In Sections 6, 7 and 8, the proofs of some technical details are given.

2. Notations

In this section we summarize several notations which we use in this paper.

2.1. Miscellaneous notations. Let q and γ be formal vairables and set $F = \mathbf{C}(q,\gamma)$. For an integer m and a nonnegative integer l, we set $[m] = (q^m - q^{-m})/(q - q^{-1})$ and $[l]! = [1][2]\cdots[l]$. For a positive integer m we let $\left(a_{ij}^{(m)}\right)_{0\le i,j\le m}$ denote the Cartan matrix of type $A_m^{(1)}$.

For $r \in F$ and elements $a_1,\dots,a_m$ of any F algebra, we define $[a_1,\dots,a_m]_r$ inductively by $[a_1,a_2]_r = a_1a_2 - ra_2a_1$ and

$$[a_1,\dots,a_l]_r = [[a_1,\dots,a_{l-1}]_r, a_l]_r \quad (3 \le l \le m).$$

Note that this satisfies $[a_1,\dots,a_m]_r = [a_1,[a_2,\dots,a_m]_r]_r$ if $[a_i,a_j] = 0$ for $|i-j| > 1$.

For an algebra A and a family of elements $(a_j)_{j\in J}$ of A we let $\langle a_j \mid j \in J\rangle$ denote the ideal of A generated by the elements a_j $(j \in J)$. For any $a \in A$ we shall denote the image of a in a quotient algebra of A simply by a unless otherwise mentioned.

2.2. The algebra $C_{\mathbf{p},N}$. For a nonnegative integer N and $\mathbf{p} = (p_1, \dots, p_N) \in (F^\times)^N$, define the F algebra $C_{\mathbf{p},N}$ to be the vector space $F(y_1,\dots,y_N) \otimes F\left[x_1^{\pm 1},\dots,x_N^{\pm 1}\right]$ with multiplication rule

$$\left(f \otimes \prod_i x_i^{l_i}\right)\left(g \otimes \prod_i x_i^{m_i}\right) = fg' \otimes \prod_i x_i^{l_i+m_i}$$

where $g'(y_1, \dots, y_m) = g\left(p_1^{l_1} y_1, \dots, p_N^{l_N} y_N\right)$. In the case $N = 0$, $C_{\mathbf{p},N}$ should be understood as F. For simplicity we shall write $f \prod_i x_i^{l_i}$ for $f \otimes \prod_i x_i^{l_i} \in C_{\mathbf{p},N}$.

For $\mathbf{p} = (\overbrace{p, \dots, p}^{N})$ we shall write $C_{p,N}$ for $C_{\mathbf{p},N}$. Define elements e_r and D_r $(r \in \mathbf{Z})$ of $C_{p,N}$ by $e_0 = D_0 = 1$, $e_r = D_r = 0$ for $|r| > N$ and

$$e_r = \sum_{\substack{I \subset \{1,2,\dots,N\} \\ |I|=r}} \prod_{i \in I} y_i, \quad D_r = \sum_{\substack{I \subset \{1,2,\dots,N\} \\ |I|=r}} \prod_{\substack{i \in I \\ j \notin I}} \frac{q y_i - q^{-1} y_j}{y_i - y_j} \prod_{i \in I} x_i,$$
$$e_{-r} = e_{N-r}(e_N)^{-1}, \quad D_{-r} = D_{N-r}(D_N)^{-1}$$

for $0 < r \leq N$. We denote the subalgebra of $C_{p,N}$ generated by the elements e_r and D_r $(r \in \mathbf{Z})$ by $\widetilde{C}_{p,N}$.

2.3. The vector space $V_{\mathbf{p},N,m}$. For a nonnegative integer N, $\mathbf{p} = (p_1, \dots, p_N) \in (F^\times)^N$ and a positive integer m, set $V_{\mathbf{p},N,m} = C_{\mathbf{p},N} \otimes (F^m)^{\otimes N}$. We shall write fg for $f \otimes g \in C_{\mathbf{p},N} \otimes \mathrm{End}((F^m)^{\otimes N})$ and regard this as an element of $\mathrm{End}(V_{\mathbf{p},N,m})$ by letting f act on $C_{\mathbf{p},N}$ by left multiplication. For $\mathbf{p} = (p, \dots, p)$ we shall denote $V_{\mathbf{p},N,m}$ simply by $V_{p,N,m}$.

We denote the canonical basis of F^m by $v_1, \dots, v_m$. We set $E_{ij}^{(k)} = 1^{\otimes k-1} \otimes E_{ij} \otimes 1^{N-k} \in \mathrm{End}\left((F^m)^{\otimes N}\right)$ for $1 \leq k \leq N$.

3. Definition of algebras and automorphisms

3.1. The quantum toroidal algebra $U_q\left(\widehat{sl_{n+1}(\mathcal{C}_\xi)}\right)$.

3.1.1. The algebra $U_q\left(\widehat{sl_{n+1}(\mathcal{C}_\xi)}\right)$. For a positive integer n we shall define the quantum toroidal algebra of type sl_{n+1} as follows.

In the case $n \geq 2$, for any $\xi \in F^\times$ we define the F algebra $U_q\left(\widehat{sl_{n+1}(\mathcal{C}_\xi)}\right)$ [1], [2] by generators

$$x_{i,m}^{\pm}, h_{i,r}, k_i^{\pm 1}, C^{\pm 1} \quad (0 \leq i \leq n, \ m \in \mathbf{Z}, \ r \in \mathbf{Z} \setminus \{0\})$$

and relations

(3.1) $C^{\pm 1}$ central, $k_i^{\pm 1} k_i^{\mp 1} = C^{\pm 1} C^{\mp 1} = 1$,

(3.2) $[k_i, k_j] = [k_i, h_{j,r}] = 0$,

(3.3) $k_i x^{\pm}_{j,m} k_i^{-1} = q^{\pm a^{(n)}_{ij}} x^{\pm}_{j,m}$,

(3.4) $\left[x^{+}_{i,m}, x^{-}_{j,l}\right] = \dfrac{\delta_{ij}}{q - q^{-1}} \left(C^{-l} \Phi^{(+)}_{i,m+l} - C^{-m} \Phi^{(-)}_{i,m+l}\right)$,

(3.5) $\left[x^{\pm}_{i,m+1}, x^{\pm}_{i,l}\right]_{q^{\pm 2}} + \left[x^{\pm}_{i,l+1}, x^{\pm}_{i,m}\right]_{q^{\pm 2}} = 0$,

(3.6) $\kappa^{r}_{ij} [h_{i,r}, h_{j,s}] = \delta_{r+s,0} \dfrac{\left[r a^{(n)}_{ij}\right]}{r} \dfrac{C^r - C^{-r}}{q - q^{-1}}$,

(3.7) $\kappa^{r}_{ij} \left[h_{i,r}, x^{\pm}_{j,m}\right] = \pm \dfrac{\left[r a^{(n)}_{ij}\right]}{r} C^{(r \mp |r|)/2} x^{\pm}_{j,r+m}$,

(3.8) If $a^{(n)}_{ij} = 0$, $\left[x^{\pm}_{i,k}, x^{\pm}_{j,l}\right] = 0$,

If $a^{(n)}_{ij} = -1$,

(3.9) $\kappa_{ij} \left[x^{\pm}_{i,l+1}, x^{\pm}_{j,m}\right]_{q^{\mp 1}} + \left[x^{\pm}_{j,m+1}, x^{\pm}_{i,l}\right]_{q^{\mp 1}} = 0$,

(3.10) $x^{\pm}_{i,m_1} x^{\pm}_{i,m_2} x^{\pm}_{j,l} - [2] x^{\pm}_{i,m_1} x^{\pm}_{j,l} x^{\pm}_{i,m_2} + x^{\pm}_{j,l} x^{\pm}_{i,m_1} x^{\pm}_{i,m_2} + (m_1 \leftrightarrow m_2) = 0$

where $\kappa_{ij} = 1$ for $(i, j) \neq (n, 0), (0, n)$, $\kappa_{n0} = \kappa_{0n}^{-1} = \xi^2 / q^{n+1}$, $\Phi^{(\pm)}_{i,\pm r} = 0$ $(r < 0)$ and $\Phi^{(\pm)}_{i,\pm r}$ $(r \geq 0)$ is expressed in terms of $k_i^{\pm 1}$ and the $h_{i,s}$ by

$$\sum_{r \geq 0} \Phi^{(\pm)}_{i,\pm r} z^r = k_i^{\pm 1} \exp\left(\pm\left(q - q^{-1}\right) \sum_{r > 0} h_{i,\pm r} z^r\right).$$

In the case $n = 1$, for any $\xi \in F^{\times}$ the F algebra $U_q\left(\widehat{sl_2(\mathcal{C}_\xi)}\right)$ is defined [6], [12] by generators

$$x^{\pm}_{i,m}, h_{i,r}, k_i^{\pm 1}, C^{\pm 1} \quad (i = 0, 1,\ m \in \mathbf{Z},\ r \in \mathbf{Z} \setminus \{0\})$$

and relations (3.1)–(3.5) and

(3.11) $[h_{i,r}, h_{i,s}] = \delta_{r+s,0} \dfrac{[2r]}{r} \dfrac{C^r - C^{-r}}{q - q^{-1}}$,

(3.12) $[h_{i,r}, h_{1-i,s}] = -\delta_{r+s,0} \dfrac{[r](\xi^r + \xi^{-r})}{r} \dfrac{C^r - C^{-r}}{q - q^{-1}}$,

(3.13) $\left[h_{i,r}, x^{\pm}_{i,m}\right] = \pm \dfrac{[2r]}{r} C^{(r \mp |r|)/2} x^{\pm}_{i,r+m}$,

(3.14) $\left[h_{i,r}, x^{\pm}_{1-i,m}\right] = \mp \dfrac{[r](\xi^r + \xi^{-r})}{r} C^{(r \mp |r|)/2} x^{\pm}_{1-i,r+m}$,

$$\mathrm{Sym}_{m_1,m_2,m_3}\left[x^{\pm}_{i,m_1},\left[x^{\pm}_{i,m_2},\left[x^{\pm}_{i,m_3},x^{\pm}_{1-i,n}\right]_{q^{-2}}\right]\right]_{q^2}=0 \tag{3.15}$$

where $\mathrm{Sym}_{m_1,m_2,m_3}$ means symmetrization in m_1, m_2 and m_3.

Hereafter for $n \geq 1$ and $\xi \in F^\times$ we shall denote $U_q\big(\widehat{sl_{n+1}(\mathcal{C}_\xi)}\big)$ by $\widehat{\mathcal{U}}(n,\xi)$. For $n \geq 1$ let $Q_n = \mathbf{Z}\alpha_1 \oplus \cdots \oplus \mathbf{Z}\alpha_n$ and set $\alpha_0 = -(\alpha_1 + \cdots + \alpha_n) \in Q_n$. We give $\widehat{\mathcal{U}}(n,\xi)$ a structure of $Q_n \oplus \mathbf{Z} \oplus \mathbf{Z}$ graded algebras by assigning

$$(\pm\alpha_i, \pm\delta_{i,0}, m) \text{ to } x^{\pm}_{i,m}, \quad (0,0,r) \text{ to } h_{i,r} \quad \text{and} \quad (0,0,0) \text{ to } k_i^{\pm 1} \text{ and } C^{\pm 1}$$

and denote the homogeneous subspace of degree (α, l, m) by $\widehat{\mathcal{U}}(n,\xi)_{(\alpha,l,m)}$. For $\alpha \in Q_n$ set $\widehat{\mathcal{U}}(n,\xi)_\alpha = \sum_{l,m\in\mathbf{Z}} \widehat{\mathcal{U}}(n,\xi)_{(\alpha,l,m)}$. By declaring $\widehat{\mathcal{U}}(n,\xi)_\alpha$ to be the homogenous subspace of degree α, $\widehat{\mathcal{U}}(n,\xi)$ is endowed with a structure of Q_n graded algebras.

3.1.2. Some automorphisms of $U_q\big(\widehat{sl_{n+1}(\mathcal{C}_\xi)}\big)$. For a positive integer n and $0 \leq j \leq n$ let $\mathcal{X}_j$ be the automorphism of $\widehat{\mathcal{U}}(n,\xi)$ determined by

$$\mathcal{X}_j : x^{\pm}_{i,m} \mapsto (-1)^{j\delta_{i,j}} x^{\pm}_{i,m\mp\delta_{i,j}}, \quad h_{i,r} \mapsto h_{i,r}, \quad k_i \mapsto C^{-\delta_{i,j}} k_i, \quad C \mapsto C$$

and set $\mathcal{Y}_j = \mathcal{X}_j \mathcal{X}_{j-1}^{-1}$ $(1 \leq j \leq n)$.

In the case $n \geq 2$, setting $\kappa = \xi^2/q^{n+1}$, we further define an automorphism ζ of $\widehat{\mathcal{U}}(n,\xi)$ by

$$\zeta : x^{\pm}_{i,m} \mapsto \big((-1)^{n+1}\kappa\big)^{\pm\delta_{i,0}} x^{\pm}_{i,m}, \quad h_{i,r} \mapsto h_{i,r}, \quad k_i \mapsto k_i, \quad C \mapsto C$$

and set $\mathcal{X}_{n+1} = \zeta\mathcal{X}_0$ and $\mathcal{Y}_{n+1} = \mathcal{X}_{n+1}\mathcal{X}_n^{-1}$. In this case, we also need the automorphisms $\mathcal{S}$ and ς_a $(a \in F^\times)$ of $\widehat{\mathcal{U}}(n,\xi)$ determined by

$$\mathcal{S} : x^{\pm}_{i,m} \mapsto (-1)^m \kappa^{-m\delta_{i,n}} x^{\pm}_{\overline{i+1},m}, \quad h_{i,r} \mapsto (-1)^r \kappa^{-r\delta_{i,n}} h_{\overline{i+1},r}, \quad k_i \mapsto k_{\overline{i+1}}, \quad C \mapsto C,$$
$$\varsigma_a : x^{\pm}_{i,m} \mapsto a^m x^{\pm}_{i,m}, \quad h_{i,r} \mapsto a^r h_{i,r}, \quad k_i \mapsto k_i, \quad C \mapsto C.$$

Here $\bar{i}$ denotes the integer between 0 and n which is equal to i mod $n+1$.

In the case $n = 1$, we set $\mathcal{Y} = \mathcal{Y}_1$ and define automorphisms $\mathcal{S}$ and ι_{a,b_0,b_1} $(a, b_0, b_1 \in F^\times)$ of $\widehat{\mathcal{U}}(1,\xi)$ by

$$\mathcal{S} : x^{\pm}_{i,m} \mapsto (-1)^m x^{\pm}_{1-i,m}, \quad h_{i,r} \mapsto (-1)^r h_{1-i,r}, \quad k_i \mapsto k_{1-i}, \quad C \mapsto C,$$
$$\iota_{a,b_0,b_1} : x^{\pm}_{i,m} \mapsto a^m b_i^{\pm 1} x^{\pm}_{i,m}, \quad h_{i,r} \mapsto a^r h_{i,r}, \quad k_i \mapsto k_i, \quad C \mapsto C.$$

3.2. The quantum affine algebra $U_q\big(\widehat{sl_{n+1}}\big)$. To study quotient algebras arising from the quantum toroidal algebra, we need other automorphisms of $\widehat{\mathcal{U}}(n,\xi)$ in addition to those defined in the previous subsection. To define them we need the quantum affine algebras $U_q\big(\widehat{sl_{n+1}}\big)$ with $n \geq 1$ and their (anti)automorphisms.

3.2.1. The algebra $U_q(\widehat{sl_{n+1}})$. For a positive integer n, $U_q(\widehat{sl_{n+1}})$ [22] is defined to be the F algebra generated by $x^{\pm}_{i,m}$, $h_{i,r}$, $k_i^{\pm 1}$ and $C^{\pm 1}$ ($1 \le i \le n$, $m \in \mathbf{Z}$, $r \in \mathbf{Z} \setminus \{0\}$) with relations (3.1)–(3.10) where the subscripts i, j are between 1 and n. This algebra is endowed with a structure of Q_n graded algebras by assigning

$$\pm\alpha_i \text{ to } x^{\pm}_{i,m} \quad \text{and} \quad 0 \text{ to } h_{i,r}, k_i^{\pm 1} \text{ and } C^{\pm 1}.$$

We denote the homogeneous subspace of degree α by $U_q(\widehat{sl_{n+1}})_\alpha$.

As was proved in [23], this algebra admits the presentation [24], [25] in terms of generators e_i, f_i and $t_i^{\pm 1}$ ($0 \le i \le n$) and relations

$$t_i^{\pm 1} t_i^{\mp 1} = 1, \quad t_i t_j = t_j t_i, \tag{3.16}$$

$$t_i e_j t_i^{-1} = q^{a^{(n)}_{ij}} e_j, \tag{3.17}$$

$$t_i f_j t_i^{-1} = q^{-a^{(n)}_{ij}} f_j, \tag{3.18}$$

$$[e_i, f_j] = \frac{\delta_{ij}}{q - q^{-1}} (t_i - t_i^{-1}), \tag{3.19}$$

$$\sum_{s=0}^{1-a^{(n)}_{ij}} (-1)^s e_i^{(s)} e_j e_i^{(1-a^{(n)}_{ij}-s)} = 0 \quad (i \ne j), \tag{3.20}$$

$$\sum_{s=0}^{1-a^{(n)}_{ij}} (-1)^s f_i^{(s)} f_j f_i^{(1-a^{(n)}_{ij}-s)} = 0 \quad (i \ne j) \tag{3.21}$$

where $x^{(l)} = x^l/[l]!$ for $x = e_i, f_i$. We choose the following correspondence of the generators [23]:

$$e_i = x^+_{i,0}, \quad f_i = x^-_{i,0}, \quad t_i = k_i \quad (1 \le i \le n), \quad t_0 \cdots t_n = C, \tag{3.22}$$

$$e_0 = C(k_1 \cdots k_n)^{-1} [x^-_{1,1}, x^-_{2,0}, \ldots, x^-_{n,0}]_q, \tag{3.23}$$

$$f_0 = [x^+_{n,0}, \ldots, x^+_{2,0}, x^+_{1,-1}]_{q^{-1}} k_1 \cdots k_n C^{-1}. \tag{3.24}$$

Here the last two equalities should be understood as $e_0 = C k_1^{-1} x^-_{1,1}$ and $f_0 = x^+_{1,-1} k_1 C^{-1}$ in the case $n = 1$. Note that

$$x^+_{1,-1} = [f_2, \ldots, f_n, f_0]_q t_1^{-1} C, \quad x^-_{1,1} = C^{-1} t_1 [e_0, e_n, \ldots, e_2]_{q^{-1}} \tag{3.25}$$

under this correspondence. Note also that $e_i \in U_q(\widehat{sl_{n+1}})_{\alpha_i}$ and $f_i \in U_q(\widehat{sl_{n+1}})_{-\alpha_i}$ for $0 \le i \le n$ with $\alpha_0 = -(\alpha_1 + \cdots + \alpha_n)$.

3.2.2. Some (anti)automorphisms of $U_q(\widehat{sl_{n+1}})$. For $0 \le i \le n$ let T_i [26] be the automorphism of $U_q(\widehat{sl_{n+1}})$ determined by

$$T_i(e_i) = -f_i t_i, \quad T_i(f_i) = -t_i^{-1} e_i, \quad T_i(t_j) = t_j t_i^{-a^{(n)}_{ij}},$$

$$T_i(e_j) = \sum_{s=0}^{-a_{ij}^{(n)}} (-1)^s q^{-s} e_i^{\left(-a_{ij}^{(n)}-s\right)} e_j e_i^{(s)} \quad \text{if} \quad i \neq j,$$
$$T_i(f_j) = \sum_{s=0}^{-a_{ij}^{(n)}} (-1)^s q^{s} f_i^{(s)} f_j f_i^{\left(-a_{ij}^{(n)}-s\right)} \quad \text{if} \quad i \neq j.$$

Let further σ and η be the antiautomorphisms of $U_q\big(\widehat{sl_{n+1}}\big)$ determined by

$$\sigma : e_i \mapsto e_i, \quad f_i \mapsto f_i, \quad t_i \mapsto t_i^{-1},$$
$$\eta : x_{i,m}^{\pm} \mapsto x_{i,-m}^{\pm}, \quad h_{i,r} \mapsto -C^r h_{i,-r}, \quad k_i \mapsto k_i^{-1}, \quad C \mapsto C.$$

3.3. The automorphisms ψ and $\mathcal{T}_i$ of $U_q\big(\widehat{sl_{n+1}(\mathcal{C}_\xi)}\big)$. Let ϱ_h and ϱ_v be the homomorphisms from $U_q\big(\widehat{sl_{n+1}}\big)$ to $\widehat{\mathcal{U}}(n,\xi)$ determined by

$$\varrho_h : e_i \mapsto x_{i,0}^{+}, \quad f_i \mapsto x_{i,0}^{-}, \quad t_i \mapsto k_i \quad (0 \leq i \leq n)$$

and

$$\varrho_v : x_{i,m}^{\pm} \mapsto x_{i,m}^{\pm}, \quad h_{i,r} \mapsto h_{i,r}, \quad k_i \mapsto k_i \quad (1 \leq i \leq n), \quad C \mapsto C,$$

respectively. Note that these are homomorphisms of Q_n graded algebras.

Now we can define automorphisms ψ and $\mathcal{T}_i$ $(0 \leq i \leq n)$ of $\widehat{\mathcal{U}}(n,\xi)$.

Proposition 1. (1) *For $n \geq 1$ there exists an automorphism ψ of $\widehat{\mathcal{U}}(n,\xi)$ determined by*

$$\psi \circ \varrho_v = \varrho_h, \quad \psi \circ \varrho_h = \varrho_v \circ \eta \circ \sigma.$$

(2) *ψ maps as follows*:

$$x_{i,0}^{\pm} \mapsto x_{i,0}^{\pm}, \quad k_i \mapsto k_i \quad (1 \leq i \leq n),$$
$$x_{1,-1}^{+} \mapsto \big[x_{2,0}^{-}, \ldots, x_{n,0}^{-}, x_{0,0}^{-}\big]_q k_1^{-1}(k_0 \cdots k_n),$$
$$x_{1,1}^{-} \mapsto (k_0 \cdots k_n)^{-1} k_1 \big[x_{0,0}^{+}, x_{n,0}^{+}, \ldots, x_{2,0}^{+}\big]_{q^{-1}},$$
$$x_{0,0}^{+} \mapsto \big[x_{n,0}^{-}, \ldots, x_{2,0}^{-}, x_{1,-1}^{-}\big]_q k_0^{-1}(k_0 \cdots k_n) C,$$
$$x_{0,0}^{-} \mapsto C^{-1}(k_0 \cdots k_n)^{-1} k_0 \big[x_{1,1}^{+}, x_{2,0}^{+}, \ldots, x_{n,0}^{+}\big]_{q^{-1}},$$
$$k_0 \cdots k_n \mapsto C^{-1}, \quad C \mapsto k_0 \cdots k_n.$$

(3) *ψ satisfies $\psi\big(\widehat{\mathcal{U}}(n,\xi)_{(\alpha,l,m)}\big) \subset \widehat{\mathcal{U}}(n,\xi)_{(\alpha,m,-l)}$ for $(\alpha,l,m) \in Q_n \oplus \mathbf{Z} \oplus \mathbf{Z}$. In particular ψ presereves each $\widehat{\mathcal{U}}(n,\xi)_\alpha$.*

Proof. Part (1) was proved in [8] and [10]. Part (2) follows from (1) and (3.22)–(3.25). Since $\widehat{\mathcal{U}}(n,\xi)$ is generated by the elements $x^{\pm}_{i,0}$, $x^{\pm}_{1,\mp 1}$, $k_i^{\pm 1}$ and $C^{\pm 1}$ $(0 \le i \le n)$, part (2) proves (3). □

The following two propositions were proved in [8] and [10].

Proposition 2. *For $n \ge 1$ there exist automorphisms $\mathcal{T}_i$ $(0 \le i \le n)$ of $\widehat{\mathcal{U}}(n,\xi)$ determined by*

$$\mathcal{T}_i \circ \varrho_v = \varrho_v \circ T_i \quad (1 \le i \le n), \quad \mathcal{T}_j \circ \varrho_h = \varrho_h \circ T_j, \quad \mathcal{S} \circ \mathcal{T}_j = \mathcal{T}_{\overline{j+1}} \circ \mathcal{S} \quad (0 \le j \le n).$$

Proposition 3. *Let $n \ge 2$. Set $\widetilde{\mathcal{Y}}_j = \mathcal{T}_{j-1}^{-1} \cdots \mathcal{T}_1^{-1} \mathcal{S} \mathcal{T}_n \cdots \mathcal{T}_j$ for $1 \le j \le n+1$ and $\widetilde{\mathcal{S}} = \mathcal{T}_1^{-1} \cdots \mathcal{T}_n^{-1} \mathcal{Y}_{n+1}^{-1}$. Then the automorphisms $\mathcal{Y}_j$, $\mathcal{S}$, $\mathcal{T}_i$ and ψ of $\widehat{\mathcal{U}}(n,\xi)$ satisfy the following equalities*:

(1) $\psi \circ \mathcal{Y}_j = \widetilde{\mathcal{Y}}_j \circ \psi$ $(1 \le j \le n+1)$.
(2) $\psi \circ \mathcal{S} = \widetilde{\mathcal{S}} \circ \psi$.
(3) $\psi \circ \mathcal{T}_i = \mathcal{T}_i \circ \psi$ $(1 \le i \le n)$.

4. Quotient algebras from $U_q(sl_2(\mathcal{C}_\xi))$

In this section we shall summarize several results [12] on $\widehat{\mathcal{U}}(1,\xi)$ and quotient algebras from it. These results play an essential role in the study of quotient algebras arising from $\widehat{\mathcal{U}}(n,\xi)$ $(n \ge 2)$ in the next section.

In this section, fixing $\xi \in F^\times$, we denote $\widehat{\mathcal{U}}(1,\xi)$ by $\widehat{U}$ and set $U = \widehat{U}/\langle C - 1, k_1 k_0 - 1\rangle$. U inherits a structure of Q_1 graded algebras from $\widehat{U}$. We let U_α signify the homogeneous subspace of degree α. The automorphisms of $\widehat{U}$ in Sections 3.1.2 and 3.3 induce automorphisms of U, which we denote by the same letters. Hereafter, in particular, we let Φ signify the isomorphism ψ of U.

4.1. Notations. First we prepare some notations. Letting $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$, set

$$a_{\alpha,r} = \frac{\xi^{\varepsilon_\alpha r} h_{1,r} + q^{-r} h_{0,r}}{\xi^{\varepsilon_\alpha r} - \xi^{-\varepsilon_\alpha r}} \quad \text{and} \quad \mathbf{a}_{\alpha,r} = \Phi(a_{\alpha,r})$$

for $r \in \mathbf{Z} \setminus \{0\}$ and $\alpha = 1, 2$, so that

$$a_{1,r} + a_{2,r} = h_{1,r} \quad \text{and} \quad \xi^{-r} a_{1,r} + \xi^r a_{2,r} = -q^{-r} h_{0,r}.$$

Define $\Lambda_{\alpha,r} \in U_0$ $(\alpha = 1, 2,\ r \in \mathbf{Z})$ by the generating series

$$\sum_{r \ge 0} \Lambda_{\alpha,\pm r} z^r = \exp\left(-\sum_{r>0} \frac{a_{\alpha,\pm r}}{[r]} z^r\right)$$

and set $\mathbf{\Lambda}_{\alpha,r} = \Phi(\Lambda_{\alpha,r})$.

For a pair of nonnegative integers $\mathbf{r} = (r, s)$ we set

$$P_{\mathbf{r}} = \Lambda_{1,r}\Lambda_{1,-r}\Lambda_{2,s}\Lambda_{2,-s} \quad \text{and} \quad \mathbf{P}_{\mathbf{r}} = \Phi(P_{\mathbf{r}}).$$

For a nonnegative integer N let Z_N be the set of pairs of nonnegative integers $\mathbf{r} = (r_1, r_2)$ such that $r_1 + r_2 = N$.

4.2. The quotient algebras $\mathcal{A}$, $\mathcal{A}_N$ and $\mathcal{A}_{\mathbf{r}}$.

4.2.1. Set $I = \sum_{m>0} U_{-m\alpha_1}U_{m\alpha_1}$. Then I is an ideal of U_0. Set $\mathcal{A} = U_0/I$. Since the automorphisms $\mathcal{Y}$ and Φ of U preserve U_α for any $\alpha \in Q_n$, they induce automorphisms of $\mathcal{A}$, which we denote by the same symbols. For $\alpha = 1, 2$ let $\mathcal{A}^\alpha$ be the subalgebra of $\mathcal{A}$ generated by the elements $a_{\alpha,r}$ and $\mathbf{a}_{\alpha,r}$ $(r \in \mathbf{Z} \setminus \{0\})$.

Proposition 4. *In $\mathcal{A}$ the following hold.*
(1) $\mathcal{Y}(a_{\alpha,r}) = a_{\alpha,r}$, $\mathcal{Y}(\mathbf{a}_{\alpha,r}) = (-q\xi^{\varepsilon_\alpha})^r \mathbf{a}_{\alpha,r}$.
(2) $\Phi(\mathbf{a}_{\alpha,r}) = q^{-2r} a_{\alpha,-r}$.

Proposition 5. (1) *The algebra $\mathcal{A}^\alpha$ is generated by the elements $a_{\alpha,1}$, $a_{\alpha,-1}$, $\mathbf{a}_{\alpha,1}$ and $\mathbf{a}_{\alpha,-1}$.*
(2) *The algebra $\mathcal{A}$ is generated by $\mathcal{A}^1$, $\mathcal{A}^2$, k_1 and k_1^{-1}.*

4.2.2. For a nonnegative integer N let I_N be the ideal of U_0 generated by I, $\sum_{m>N} U_{m\alpha_1}U_{-m\alpha_1}$ and $k_1 - q^N$, and set $\mathcal{A}_N = U_0/I_N$. Note that $\mathcal{A}_N$ can be regarded as a qutotient algebra of $\mathcal{A}$.

Lemma 1. *The following hold in $\mathcal{A}_N$.*
(1) (i) $P_{\mathbf{r}} \in Z(\mathcal{A}_N)$ $(\mathbf{r} \in Z_N)$.
(ii) $\sum_{\mathbf{r}\in Z_N} P_{\mathbf{r}} = 1$.
(iii) $P_{\mathbf{r}}P_{\mathbf{s}} = \delta_{\mathbf{r},\mathbf{s}}P_{\mathbf{r}}$ $(\mathbf{r}, \mathbf{s} \in Z_N)$.
(2) *The $\mathbf{P}_{\mathbf{r}}$ satisfy the above containment and equalities with the $P_{\mathbf{r}}$ replaced by the $\mathbf{P}_r$.*

4.2.3. Let $\mathbf{r}, \mathbf{r}' \in Z_N$ for some nonnegative integer N and set

$$\mathcal{A}_{\mathbf{r},\mathbf{r}'} = \mathcal{A}_N/\langle 1 - P_{\mathbf{r}}, 1 - \mathbf{P}_{\mathbf{r}'}\rangle \quad \text{and} \quad \mathcal{A}_{\mathbf{r}} = \mathcal{A}_{\mathbf{r},\mathbf{r}}.$$

Lemma 2. $\mathcal{A}_{\mathbf{r},\mathbf{r}'} = 0$ *if* $\mathbf{r} \neq \mathbf{r}'$.

Proposition 6. *The following hold in $\mathcal{A}_{\mathbf{r}}$ with $\mathbf{r} = (r_1, r_2)$.*
(1) $\Lambda_{\alpha,\pm l} = 0$, $\mathbf{\Lambda}_{\alpha,\pm l} = 0$ $(l > r_\alpha,\ \alpha = 1, 2)$.
(2) $\Lambda_{\alpha,r_\alpha}\Lambda_{\alpha,-l} = \Lambda_{\alpha,r_\alpha-l}$, $\mathbf{\Lambda}_{\alpha,r_\alpha}\mathbf{\Lambda}_{\alpha,-l} = \mathbf{\Lambda}_{\alpha,r_\alpha-l}$ $(0 \leq l \leq r_\alpha,\ \alpha = 1, 2)$.

Proposition 7. (1) *For* $\mathbf{r} = (N, 0)$ *there exists an isomorphism* $\widetilde{C}_{q^2\xi^2,N} \to \mathcal{A}_{\mathbf{r}}$ *determined by* $e_l \mapsto \Lambda_{1,l}$ *and* $D_l \mapsto \mathbf{\Lambda}_{1,l}$.
(2) *For* $\mathbf{r} = (0, N)$ *there exists an isomorphism* $\widetilde{C}_{q^2\xi^{-2},N} \to \mathcal{A}_{\mathbf{r}}$ *determined by* $e_l \mapsto \Lambda_{2,l}$ *and* $D_l \mapsto \mathbf{\Lambda}_{2,l}$.

4.3. Representations of U.

Proposition 8. *Set* $p_\alpha^{1/2} = q\xi^{\varepsilon_\alpha}$ *for* $\alpha = 1, 2$. *Let* $H_1^{(l)} = E_{11}^{(l)} - E_{22}^{(l)}$ *and* $H_0^{(l)} = -H_1^{(l)}$.
(1) *For* $\alpha = 1, 2$ *and a nonnegative integer* N *there exists a homomorphism* $\tau_{\alpha,\xi,N} : U \to \operatorname{End} V_{p_\alpha,N,2}$ *determined by*

$$
\begin{aligned}
x_{1,m}^+ &\mapsto \sum_{i=1}^N \prod_{i<j} \frac{q^{H_1^{(j)}} y_i - q^{-H_1^{(j)}} y_j}{y_i - y_j} y_i^m E_{12}^{(i)}, \\
x_{1,m}^- &\mapsto \sum_{i=1}^N \prod_{j<i} \frac{q^{H_1^{(j)}} y_i - q^{-H_1^{(j)}} y_j}{y_i - y_j} y_i^m E_{21}^{(i)}, \\
x_{0,m}^+ &\mapsto \sum_{i=1}^N \prod_{i<j} \frac{q^{H_0^{(j)}} p_\alpha^{1/2} y_i - \left(q p_\alpha^{-1/2}\right)^{-H_0^{(j)}} y_j}{p_\alpha^{1/2} y_i - p_\alpha^{H_0^{(j)}/2} y_j} p_\alpha^{m/2} y_i^m x_i E_{21}^{(i)}, \\
x_{0,m}^- &\mapsto \sum_{i=1}^N \prod_{j<i} \frac{q^{H_0^{(j)}} p_\alpha^{-1/2} y_i - \left(q p_\alpha^{-1/2}\right)^{-H_0^{(j)}} y_j}{p_\alpha^{-1/2} y_i - p_\alpha^{H_0^{(j)}/2} y_j} p_\alpha^{-m/2} y_i^m x_i^{-1} E_{12}^{(i)}, \\
h_{1,r} &\mapsto \frac{[r]}{r} \sum_{i=1}^N y_i^r \left(q^{-r} E_{11}^{(i)} - q^r E_{22}^{(i)}\right), \\
h_{0,r} &\mapsto \frac{[r]}{r} \sum_{i=1}^N y_i^r \left(q^{-r} p_\alpha^{r/2} E_{22}^{(i)} - q^r p_\alpha^{-r/2} E_{11}^{(i)}\right), \\
k_1 &\mapsto q^{\sum_{i=1}^N H_1^{(i)}}.
\end{aligned}
$$

(2) *For* $\alpha = 1, 2$ *and a nonnegative integer* N, *there exists an automorphism* $G_{p_\alpha,N}$ *of the vector space* $C_{p_\alpha,N}$ *such that* $\tau'_{\alpha,\xi,N} : U \to \operatorname{End} V_{p_\alpha,N,2}$ $(u \mapsto (G_{p_\alpha,N} \otimes 1)^{-1} \circ \tau_{\alpha,\xi,N}(u) \circ (G_{p_\alpha,N} \otimes 1))$ *satisfies*

$$
\begin{aligned}
&\tau'_{\alpha,\xi,N}(\Lambda_{\alpha,l})\left(f \otimes v_1^{\otimes N}\right) = e_l f \otimes v_1^{\otimes N}, \quad \tau'_{\alpha,\xi,N}(\mathbf{\Lambda}_{\alpha,l})\left(f \otimes v_1^{\otimes N}\right) = D_l f \otimes v_1^{\otimes N}, \\
&\tau'_{\alpha,\xi,N}(\Lambda_{3-\alpha,l})\left(f \otimes v_1^{\otimes N}\right) = f \otimes v_1^{\otimes N}, \quad \tau'_{\alpha,\xi,N}(\mathbf{\Lambda}_{3-\alpha,l})\left(f \otimes v_1^{\otimes N}\right) = f \otimes v_1^{\otimes N}
\end{aligned}
$$

for any $f \in C_{p_\alpha,N}$.

Proof. Part (1) is a special case of [12, Proposition 4.5]. For $a, b \in F^\times$ let $m_{a,b}$ signify the algebra automorphism of $C_{p_\alpha,N}$ determined by $x_i \mapsto a x_i$ and $y_i \mapsto b y_i$.

Part (2) follows from the results in the proof of [12, Proposition 4.6] and the use of the automorphism $m_{a,b}$. □

5. Quotient algebras from $U_q(sl_{n+1}(\mathcal{C}_\gamma))$ $(n \geq 2)$

Now we can start to consider $U_q(sl_{n+1}(\mathcal{C}_\gamma))$, the quantum toroidal algebra of type sl_{n+1} with $\xi = \gamma$, for $n \geq 2$ and study quotient algebras arising from it. Hereafter we fix $n \geq 2$ and set $\widehat{\mathcal{U}} = \widehat{\mathcal{U}}(n,\gamma)$ and $\mathcal{U} = \widehat{\mathcal{U}}/\langle C-1, k_0k_1\cdots k_n - 1\rangle$. We further set $\Psi = \psi \circ \varsigma_{(-q)^{n-1}}$. The algebra $\mathcal{U}$ inherits a structure of Q_n graded algebras from $\widehat{\mathcal{U}}$. We let $\mathcal{U}_\alpha$ signify the homogeneous subspace of degree α. We denote the algebras U, $\mathcal{A}$, $\mathcal{A}_N$ and $\mathcal{A}_{\mathbf{r}}$ in Section 4 by $U(\xi)$, $\mathcal{A}(\xi)$, $\mathcal{A}_N(\xi)$ and $\mathcal{A}_{\mathbf{r}}(\xi)$, respectively, to specify the dependence on the parameter ξ.

5.1. The homomorphism φ_i. In order to study quotient algebras arising from $\mathcal{U}$, the homomorphisms in the following proposition are useful since we already have some results on the quotient algebras arising from U, which we have reviewed in Section 4.

For $1 \leq i \leq n$ set $\mathcal{U}[i] = \sum_{m\in\mathbf{Z}} \mathcal{U}_{m\alpha_i}$ and

$$\mathcal{I}[i] = \sum_{\substack{m_1,\ldots,m_n,m_i'\in\mathbf{Z} \\ m_j>0 \text{ for some } j\in\{1,\ldots,n\}\setminus\{i\}}} \mathcal{U}_{m_i'\alpha_i - \sum_{1\leq j\neq i\leq n} m_j\alpha_j}\mathcal{U}_{m_i\alpha_i + \sum_{1\leq j\neq i\leq n} m_j\alpha_j}.$$

Then $\mathcal{U}[i]$ is a subalgebra of $\mathcal{U}$ and $\mathcal{I}[i]$ is an ideal of $\mathcal{U}[i]$. Since $\mathcal{X}_j$ and Ψ are automorphisms of the Q_n graded algebra $\mathcal{U}$, they preserve both $\mathcal{U}[i]$ and $\mathcal{I}[i]$. Therefore they induce automorphisms of the quotient algebra $\mathcal{U}[i]/\mathcal{I}[i]$, which we denote by the same letters.

Proposition 9. (1) *For $1 \leq i \leq n$ there exists a homomorphism $\varphi_i\colon U(\gamma/q^i) \to \mathcal{U}[i]/\mathcal{I}[i]$ determined by*

$$\begin{aligned}
&x_{1,l}^{\pm} \mapsto x_{i,l}^{\pm}, \quad h_{1,r} \mapsto h_{i,r}, \quad k_1 \mapsto k_i,\\
&x_{0,l}^{+} \mapsto (-1)^{i-1}(q/\gamma)^l \left(x_{i+1,0}^{+}\cdots x_{n,0}^{+}\right)\left(x_{i-1,0}^{+}\cdots x_{1,0}^{+}\right)x_{0,l}^{+},\\
&x_{0,l}^{-} \mapsto (-1)^{i-1}(q/\gamma)^l x_{0,l}^{-}\left(x_{1,0}^{-}\cdots x_{i-1,0}^{-}\right)\left(x_{n,0}^{-}\cdots x_{i+1,0}^{-}\right),\\
&h_{0,r} \mapsto (q/\gamma)^r \left(\sum_{l=0}^{i-1} q^{lr}h_{l,r} + \gamma^{2r}\sum_{l=i+1}^{n} q^{-lr}h_{l,r}\right).
\end{aligned}$$

(2) *The homomorphism φ_i satisfies the following equalities*:

(i) $\Psi \circ \varphi_i = \varphi_i \circ \Phi$.

(ii) $$\mathcal{X}_j \circ \varphi_i = \begin{cases} \varphi_i \circ \mathcal{X}_0 \circ \iota_{1,(-1)^j q^{1+j}\gamma^{-1},1} & \text{if } 0 \leq j < i,\\ \varphi_i \circ \mathcal{X}_1 \circ \iota_{1,1,(-1)^{i-1}} & \text{if } j = i,\\ \varphi_i \circ \mathcal{X}_0 \circ \iota_{1,(-1)^j q^{1-j}\gamma,1} & \text{if } i < j \leq n+1.\end{cases}$$

The proof of this proposition will be given in Section 8.

5.2. The quotitent algebra $\mathcal{B}$. Now let us consider the quotient algebra $\mathcal{B} = \mathcal{U}_0/\mathcal{I}$ where

$$\mathcal{I} = \sum_{\substack{m_1,\dots,m_n \in \mathbf{Z} \\ m_j>0 \text{ for some } j}} \mathcal{U}_{-\sum_{j=1}^n m_j\alpha_j}\mathcal{U}_{\sum_{j=1}^n m_j\alpha_j}.$$

The automorphisms $\mathcal{X}_j$ and Ψ of $\mathcal{U}$ induce automorphisms of $\mathcal{B}$ as before, which we denote by the same symbols.

Since $\mathcal{U}_0 \cap \mathcal{I}[i] \subset \mathcal{I}$, there exists a homomorphism $\pi_i : (\mathcal{U}[i]/\mathcal{I}[i])_0 = \mathcal{U}_0/\mathcal{U}_0 \cap \mathcal{I}[i] \to \mathcal{U}_0/\mathcal{I} = \mathcal{B}$ ($\overline{u} \mapsto \overline{u}$). It is easy to see that for $1 \le i \le n$ the composite map $\pi_i \circ \varphi_i|_{U(\gamma/q^i)_0}$ induces a homomorphism $\mathcal{A}(\gamma/q^i) \to \mathcal{B}$, which we denote by $\widetilde{\varphi}_i$.

Define $b_{l,r} \in \mathcal{U}_0$ ($1 \le l \le n+1$, $r \in \mathbf{Z} \setminus \{0\}$) by

$$b_{l,r} = h_{0,r} + q^r h_{1,r} + \cdots + q^{(l-1)r} h_{l-1,r} + \gamma^{2r}\left(q^{-lr} h_{l,r} + \cdots + q^{-nr} h_{n,r}\right) \tag{5.1}$$

and set $\mathbf{b}_{l,r} = \Psi(b_{l,r})$. Note that $b_{l,r} - b_{l+1,r} = \left(\gamma^{2r}q^{-lr} - q^{lr}\right)h_{l,r}$. Since the elements $b_{l,r}$ commute with each other in $\mathcal{U}$, so do the $\mathbf{b}_{l,r}$.

Proposition 10. (1) *The homomorphism* $\widetilde{\varphi}_i : \mathcal{A}(\gamma/q^i) \to \mathcal{B}$ *is determined by* $k_1 \mapsto k_i$ *and*

$$a_{\alpha,r} \mapsto \varepsilon_\alpha b_{i+\alpha-1,r}\big/\left(\gamma^{2r}q^{-ir} - q^{ir}\right), \quad \mathbf{a}_{\alpha,r} \mapsto \varepsilon_\alpha \mathbf{b}_{i+\alpha-1,r}\big/\left(\gamma^{2r}q^{-ir} - q^{ir}\right).$$

(2) *The homomorphism* $\widetilde{\varphi}_i$ *and the automorphisms of* $\mathcal{A}(\gamma/q^i)$ *and* $\mathcal{B}$ *satisfy the equalities in part* (2) *of* Proposition 9 *with* φ_i *replaced by* $\widetilde{\varphi}_i$.

Proof. By Proposition 5 the homomorphism $\widetilde{\varphi}_i$ is determined by specifying the images of k_1, the $a_{\alpha,r}$ and the $\mathbf{a}_{\alpha,r}$. The expressions for the $\widetilde{\varphi}_i(\mathbf{a}_{\alpha,r})$ follow from part (2)-(i) of Proposition 9. □

Combining the above proposition with Propositions 4 and 5, we obtain the following two propositions, which will be proven in the next subsection.

Proposition 11. *In* $\mathcal{B}$ *the following hold for* $1 \le i, l \le n+1$ *and* $r \in \mathbf{Z} \setminus \{0\}$.

(1) $\Psi(b_{l,r}) = \mathbf{b}_{l,r}$, $\Psi(\mathbf{b}_{l,r}) = -(\gamma/q)^{2r} b_{l,-r}$.

(2) $\mathcal{Y}_i(b_{l,r}) = b_{l,r}$,

$$\mathcal{Y}_i(\mathbf{b}_{l,r}) = (-1)^r \begin{cases} q^{-r}\mathbf{b}_{l,r} & \textit{if} \quad 1 \le l \le i-1, \\ \left(q^{1-2i}\gamma^2\right)^r \mathbf{b}_{i,r} & \textit{if} \quad l = i, \\ q^r \mathbf{b}_{l,r} & \textit{if} \quad i+1 \le l \le n+1. \end{cases}$$

(3) $\widetilde{\mathcal{Y}}_i(\mathbf{b}_{l,r}) = \mathbf{b}_{l,r}$,

$$\widetilde{\mathcal{Y}}_i(b_{l,r}) = (-1)^r \begin{cases} q^r b_{l,r} & \textit{if} \quad 1 \leq l \leq i-1, \\ \left(q^{1-2i}\gamma^2\right)^{-r} b_{i,r} & \textit{if} \quad l = i, \\ q^{-r} b_{l,r} & \textit{if} \quad i+1 \leq l \leq n+1. \end{cases}$$

For $1 \leq l \leq n+1$ let $\mathcal{B}^l$ be the subalgebra of $\mathcal{B}$ generated by the elements $b_{l,r}$ and $\mathbf{b}_{l,r}$ $(r \in \mathbf{Z} \setminus \{0\})$.

Proposition 12. (1) *$\mathcal{B}^l$ is generated by the elements $b_{l,1}$, $b_{l,-1}$, $\mathbf{b}_{l,1}$ and $\mathbf{b}_{l,-1}$.*
(2) *$[\mathcal{B}^l, \mathcal{B}^m] = 0$ if $1 \leq l \neq m \leq n+1$.*
(3) *$\mathcal{B}$ is generated by $\mathcal{B}^l$, k_i and k_i^{-1} $(1 \leq l \leq n+1,\ 1 \leq i \leq n)$.*

5.3. Proof of Propositions 11 and 12. First we prepare the following lemma.

Lemma 3. (1) $\iota_{a,b_0,b_1}(\mathbf{a}_{\alpha,r}) = (b_0 b_1)^r \mathbf{a}_{\alpha,r}$.
(2) $\varsigma_a(\mathbf{b}_{l,r}) = \mathbf{b}_{l,r}$.

Proof. The element $a_{\alpha,r}$ is in $\mathcal{U}(1,\xi)_{(0,0,r)}$ and $\iota_{a,b_0,b_1}(u) = a^l (b_0 b_1)^r b_1^m u$ for $u \in \mathcal{U}(1,\xi)_{(m\alpha_1,r,l)}$. Therefore part (3) of Proposition 1 proves (1). The proof of (2) is similar. □

Proof of Proposition 11. (1) Fixing $i \in \{1, 2, \ldots, n\}$ and $\alpha \in \{1, 2\}$, set $c = \varepsilon_\alpha / ((\gamma/q^i)^r - (\gamma/q^i)^{-r})$. Then by Propositions 10 and 4

$$\begin{aligned} c\gamma^{-r}\Psi(\mathbf{b}_{i+\alpha-1,r}) &= \left(\Psi \circ \widetilde{\varphi}_i\right)(\mathbf{a}_{\alpha,r}) \\ &= \left(\widetilde{\varphi}_i \circ \Phi\right)(\mathbf{a}_{\alpha,r}) \\ &= q^{-2r}\widetilde{\varphi}_i(a_{\alpha,-r}) \\ &= -q^{-2r} c\gamma^r b_{i+\alpha-1,-r}. \end{aligned}$$

This proves the second equality.

(2) The first equality follows from the definitions of $\mathcal{Y}_i$ and $b_{l,r}$. Part (2) of Proposition 10 implies

$$\mathcal{Y}_j \circ \widetilde{\varphi}_i = \begin{cases} \widetilde{\varphi}_i \circ \iota_{1,-q,1} & \text{if} \quad 1 \leq j < i, \\ \widetilde{\varphi}_i \circ \mathcal{Y} \circ \iota_{1,(-1)^{i-1}q^{-i}\gamma,(-1)^{i-1}} & \text{if} \quad j = i, \\ \widetilde{\varphi}_i \circ \mathcal{Y}^{-1} \circ \iota_{1,(-1)^{i-1}q^{-i}\gamma,(-1)^{i-1}} & \text{if} \quad j = i+1, \\ \widetilde{\varphi}_i \circ \iota_{1,-q^{-1},1} & \text{if} \quad i+1 < j \leq n+1. \end{cases}$$

Apply the above equality to $\mathbf{a}_{1,r}$ and $\mathbf{a}_{2,r}$. Then we obtain the second equality by Proposition 10, Lemma 3 (1) and Proposition 4 (1).

(3) By the definition of Ψ and Lemma 3 (2), $\psi(b_{l,r}) = (-q)^{-(n-1)r}\mathbf{b}_{l,r}$ and $\psi(\mathbf{b}_{l,r}) = \Psi(\mathbf{b}_{l,r})$. So by applying ψ to the equalities in (2), we obtain the claim thanks to Proposition 3 (1) and part (1). □

To prove Proposition 12, we need the following lemma, which will be proven in Section 7.

Lemma 4. *For $l \neq m$ and $r, s = \pm 1$, $[b_{l,r}, \mathbf{b}_{m,s}] = 0$ in $\mathcal{B}$.*

Proof of Proposition 12. Part (1) follows from Propositions 5 and 10 since

$$\mathcal{B}^{i+\alpha-1} = \widetilde{\varphi}_i\left(\mathcal{A}^{\alpha}\left(\gamma/q^i\right)\right) \tag{5.2}$$

for $1 \le i \le n$ and $\alpha = 1, 2$. Part (2) follows from part (1) and Lemma 4. Part (3) can be proven by specialization argument as in [12]. We mainly use the argument in the proof of part (2) of [12, Lemma 7.5] and do not need a counterpart of [12, Lemma 7.3]. □

5.4. The quotient algebra $\mathcal{B}_{N_1,\dots,N_n}$. For $(N_1, \dots, N_n) \in \mathbf{Z}^n_{\ge 0}$ let $\mathcal{I}_{N_1,\dots,N_n}$ be the ideal of $\mathcal{U}_0$ generated by $\mathcal{I}$, $\sum_{m>N_i}\mathcal{U}_{m\alpha_i}\mathcal{U}_{-m\alpha_i}$ and $k_i - q^{N_i}$ $(1 \le i \le N)$. Set $\mathcal{B}_{N_1,\dots,N_n} = \mathcal{U}_0/\mathcal{I}_{N_1,\dots,N_n}$. This quotient algebra is the main object of our study. We can and do regard this algebra as a quotient algebra of $\mathcal{B}$.

5.4.1. Main result. Set $p_+ = \gamma^2$ and $p_- = q^{2(n+1)}/\gamma^2$. For $l = 1, n+1$ and $r \in \mathbf{Z}$ define $\Gamma_{l,r}, \mathbf{\Gamma}_{l,r} \in \mathcal{B}$ by

$$\Gamma_{1,r} = \widetilde{\varphi}_1(\Lambda_{1,r}), \quad \mathbf{\Gamma}_{1,r} = \widetilde{\varphi}_1(\mathbf{\Lambda}_{1,r}), \quad \Gamma_{n+1,r} = \widetilde{\varphi}_n(\Lambda_{2,r}), \quad \mathbf{\Gamma}_{n+1,r} = \widetilde{\varphi}_n(\mathbf{\Lambda}_{2,r}). \tag{5.3}$$

Note that $\mathbf{\Gamma}_{l,r} = \Psi(\Gamma_{l,r})$ by Proposition 10 (2).

Theorem 1. (1) *$\mathcal{B}_{N_1,\dots,N_n} = 0$ unless $N_2 = \cdots = N_{n-1} = 0$.*
(2) *$\mathcal{B}_{N_1,0,\dots,0,N_n}$ is generated by $\Gamma_{l,r}$ and $\mathbf{\Gamma}_{l,r}$ $(l = 1, n+1,\ r \in \mathbf{Z})$ and the following relations hold in this algebra for $2 \le j \le n$ and $l = 1, n+1$:*
(i) $b_{j,r} = \mathbf{b}_{j,r} = 0$ $(r \in \mathbf{Z} \setminus \{0\})$.
(ii) $\Gamma_{l,r} = 0$, $\mathbf{\Gamma}_{l,r} = 0$ $(|r| > m_l)$.
(iii) $\Gamma_{l,m_l}\Gamma_{l,-r} = \Gamma_{l,m_l-r}$, $\mathbf{\Gamma}_{l,m_l}\mathbf{\Gamma}_{l,-r} = \mathbf{\Gamma}_{l,m_l-r}$ $(0 \le r \le m_l)$.
Here $m_1 = N_1$ and $m_{n+1} = N_n$.
(3) *There exists an isomorphism $f_{N_1,N_n} : \widetilde{C}_{p_+,N_1} \otimes \widetilde{C}_{p_-,N_n} \to \mathcal{B}_{N_1,0,\dots,0,N_n}$ determined by*

$$e_r \otimes 1 \mapsto \Gamma_{1,r}, \quad D_r \otimes 1 \mapsto \mathbf{\Gamma}_{1,r}, \quad 1 \otimes e_r \mapsto \Gamma_{n+1,r}, \quad 1 \otimes D_r \mapsto \mathbf{\Gamma}_{n+1,r}.$$

The proof of this theorem will be given in the next subsubsection and Section 6.

5.4.2. Proof of Theorem 1 except for the injectivity of f_{N_1,N_n}. First we prepare two lemmas. For $1 \leq i \leq n$ and a pair of nonnegative integers $\mathbf{r}$, set $P_{\mathbf{r}}^{(i)} = \widetilde{\varphi}_i(P_{\mathbf{r}})$ and $\mathbf{P}_{\mathbf{r}}^{(i)} = \widetilde{\varphi}_i(\mathbf{P}_{\mathbf{r}})$.

Lemma 5. *The following hold in $\mathcal{B}_{N_1,\dots,N_n}$ for $1 \leq i \leq n$.*

(1) (i) $P_{\mathbf{r}}^{(i)} \in Z(\mathcal{B}_{N_1,\dots,N_n})$ $(\mathbf{r} \in Z_{N_i})$.

(ii) $\sum_{\mathbf{r}\in Z_{N_i}} P_{\mathbf{r}}^{(i)} = 1$.

(iii) $P_{\mathbf{r}}^{(i)} P_{\mathbf{s}}^{(i)} = \delta_{\mathbf{r},\mathbf{s}} P_{\mathbf{r}}^{(i)}$ $(\mathbf{r}, \mathbf{s} \in Z_{N_i})$.

(2) *The $\mathbf{P}_{\mathbf{r}}^{(i)}$ satisfy the above containment and equalities with the $P_{\mathbf{r}}^{(i)}$ replaced by the $\mathbf{P}_{\mathbf{r}}^{(i)}$.*

Proof. Let $\bar{\ }: \mathcal{B} \to \mathcal{B}_{N_1,\dots,N_n}$ be the quotient map. Clearly the composite map $\mathcal{A}(\gamma/q^i) \xrightarrow{\widetilde{\varphi}_i} \mathcal{B} \xrightarrow{-} \mathcal{B}_{N_1,\dots,N_n}$ induces a homomorphism $\mathcal{A}_{N_i}(\gamma/q^i) \to \mathcal{B}_{N_1,\dots,N_n}$, which we denote by μ_i. This homomorphism μ_i and Lemma 1 prove (1)-(ii), (1)-(iii) and $P_{\mathbf{r}}^{(i)} \in Z(\operatorname{Im}\mu_i)$. By Proposition 5 and (5.2), $\operatorname{Im}\mu_i$ is the subalgebra of $\mathcal{B}_{N_1,\dots,N_n}$ generated by the images of $\mathcal{B}^i$ and $\mathcal{B}^{i+1}$ in $\mathcal{B}_{N_1,\dots,N_n}$. Therefore part (1)-(i) follows from Proposition 12. As for part (2), the claim follows from (1) since $\mathbf{P}_{\mathbf{r}}^{(i)} = \Psi\big(P_{\mathbf{r}}^{(i)}\big)$ by Proposition 10 (2). □

For $(\mathbf{r}_1, \dots, \mathbf{r}_n), (\mathbf{r}'_1, \dots, \mathbf{r}'_n) \in Z_{N_1} \times \cdots \times Z_{N_n}$ set

$$\mathcal{B}_{\mathbf{r}_1,\dots,\mathbf{r}_n;\mathbf{r}'_1,\dots,\mathbf{r}'_n} = \mathcal{B}_{N_1,\dots,N_n} \big/ \big\langle 1 - P_{\mathbf{r}_i}^{(i)}, 1 - \mathbf{P}_{\mathbf{r}'_i}^{(i)} \bigm| 1 \leq i \leq n \big\rangle$$

and let $\mathcal{B}_{\mathbf{r}_1,\dots,\mathbf{r}_n} = \mathcal{B}_{\mathbf{r}_1,\dots,\mathbf{r}_n;\mathbf{r}_1,\dots,\mathbf{r}_n}$.

Lemma 6. *For $(\mathbf{r}_1, \dots, \mathbf{r}_n), (\mathbf{r}'_1, \dots, \mathbf{r}'_n) \in Z_{N_1} \times \cdots \times Z_{N_n}$ the following hold.*

(1) $\mathcal{B}_{\mathbf{r}_1,\dots\mathbf{r}_n\,;\,\mathbf{r}'_1,\dots,\mathbf{r}'_n} = 0$ *unless* $\mathbf{r}'_i = \mathbf{r}_i$ *for any* i.

(2) $\mathcal{B}_{\mathbf{r}_1,\dots,\mathbf{r}_n} = 0$ *unless* $N_2 = \cdots = N_{n-1} = 0$ *and* $(\mathbf{r}_1, \dots, \mathbf{r}_n) = ((N_1, 0), \mathbf{0}, \dots, \mathbf{0}, (0, N_n))$.

Proof. (1) Suppose that $\mathbf{r}'_i \neq \mathbf{r}_i$ for some i. Let $\mu_i : \mathcal{A}_{N_i}(\gamma/q^i) \to \mathcal{B}_{N_1,\dots,N_n}$ be the map in the proof of the previous lemma and $\bar{\ }$ the quotient map $\mathcal{B}_{N_1,\dots,N_n} \to \mathcal{B}_{\mathbf{r}_1,\dots,\mathbf{r}_n\,;\,\mathbf{r}'_1,\dots,\mathbf{r}'_n}$. Clearly the map $\bar{\ } \circ \mu_i$ induces a homomorphism $\mathcal{A}_{\mathbf{r}_i,\mathbf{r}'_i}(\gamma/q^i) \to \mathcal{B}_{\mathbf{r}_1,\dots,\mathbf{r}_n\,;\,\mathbf{r}'_1,\dots,\mathbf{r}'_n}$ $(*)$. Since $\mathcal{A}_{\mathbf{r}_i,\mathbf{r}'_i}(\gamma/q^i) = 0$ by Lemma 1, this implies that the identity element of $\mathcal{B}_{\mathbf{r}_1,\dots,\mathbf{r}_n\,;\,\mathbf{r}'_1,\dots,\mathbf{r}'_n}$ is 0.

(2) Set $\mathbf{r}_j = (r_j, s_j)$ for $1 \leq j \leq n$. Let us denote the induced homomorphism $(*)$ in the case $\mathbf{r}'_j = \mathbf{r}_j$ for all j by ν_i. Then

$$\nu_i(a_{\alpha,r}) = c_{i,\alpha,r} b_{i+\alpha-1,r} \quad \text{and} \quad \nu_i(\mathbf{a}_{\alpha,r}) = c_{i,\alpha,r} \mathbf{b}_{i+\alpha-1,r} \tag{5.4}$$

with $c_{i,\alpha,r} = \varepsilon_\alpha/(\gamma^{2r} q^{-ir} - q^{ir})$ by Proposition 10. Fixing $i \in \{1, \dots, n-1\}$, set $A_r = \nu_i(\Lambda_{2,r})$ and $B_r = \nu_{i+1}(\Lambda_{1,r})$ for $r \geq 0$. Let $A(z) = \sum_{r\geq 0} A_r z^r$ and $B(z) = \sum_{r\geq 0} B_r z^r$.

Then

$$A\big(\gamma^2 z/q^i\big)A\big(q^i z\big)^{-1} = \exp\left(\sum_{r>0}\frac{b_{i+1,r}}{[r]}z^r\right) = B\big(q^{i+1}z\big)B\big(\gamma^2 z/q^{i+1}\big)^{-1}.$$

Therefore

$$A\big(q^i z\big)B\big(q^{i+1}z\big) = A\big(\gamma^2 z/q^i\big)B\big(\gamma^2 z/q^{i+1}\big). \tag{5.5}$$

By Proposition 6

$$A_r = 0 \quad (r > s_i), \qquad B_r = 0 \quad (r > r_{i+1}),$$

and A_{s_i} and $B_{r_{i+1}}$ are invertible in $\mathcal{B}_{\mathbf{r}_1,\dots,\mathbf{r}_n}$. Hence the coefficient of $z^{s_i+r_{i+1}}$ in (5.5) gives the equality $(\gamma/q^i)^{2s_i}(\gamma/q^{i+1})^{2r_{i+1}} = 1$. This implies that $1 = 0$ in $\mathcal{B}_{\mathbf{r}_1,\dots,\mathbf{r}_n}$ unless $s_i = r_{i+1} = 0$. Therefore $\mathcal{B}_{\mathbf{r}_1,\dots,\mathbf{r}_n} = 0$ unless $s_i = r_{i+1} = 0$ for $1 \le i \le n-1$. □

Now we can give the following proof.

Proof of Theorem 1 except the injectivity of f_{N_1,N_n}. By Lemma 5

$$1_{\mathcal{B}_{N_1,\dots,N_n}} = \sum_{(\mathbf{r}_1,\dots,\mathbf{r}_n),(\mathbf{r}'_1,\dots,\mathbf{r}'_n)\in Z_{N_1}\times\cdots\times Z_{N_n}} P^{(1)}_{\mathbf{r}_1}\mathbf{P}^{(1)}_{\mathbf{r}'_1}\cdots P^{(n)}_{\mathbf{r}_n}\mathbf{P}^{(n)}_{\mathbf{r}'_n}$$

is a decomposition of 1 into a sum of orthogonal central idempotents if we allow some of the summands to be 0. This implies

$$\mathcal{B}_{\mathbf{r}_1,\dots,\mathbf{r}_n\,;\,\mathbf{r}'_1,\dots,\mathbf{r}'_n} \simeq P^{(1)}_{\mathbf{r}_1}\mathbf{P}^{(1)}_{\mathbf{r}'_1}\cdots P^{(n)}_{\mathbf{r}_n}\mathbf{P}^{(n)}_{\mathbf{r}'_n}\mathcal{B}_{N_1,\dots,N_n}$$

and

$$\mathcal{B}_{N_1,\dots,N_n} \simeq \bigoplus \mathcal{B}_{\mathbf{r}_1,\dots,\mathbf{r}_n\,;\,\mathbf{r}'_1,\dots,\mathbf{r}'_n}$$

where the sum is taken over $(\mathbf{r}_1,\dots,\mathbf{r}_n),(\mathbf{r}'_1,\dots,\mathbf{r}'_n) \in Z_{N_1}\times\cdots\times Z_{N_n}$. So Lemma 6 proves (1) and

$$\mathcal{B}_{N_1,0,\dots,0,N_n} \simeq \mathcal{B}_{(N_1,0),\mathbf{0},\dots,\mathbf{0},(0,N_n)}. \tag{5.6}$$

Set $A_1 = \mathcal{A}_{(N_1,0)}(\gamma/q)$, $A_i = \mathcal{A}_{\mathbf{0}}(\gamma/q^i)$ $(2 \le i \le n-1)$, $A_n = \mathcal{A}_{(0,N_n)}(\gamma/q^n)$ and $B = \mathcal{B}_{(N_1,0),\mathbf{0},\dots,\mathbf{0},(0,N_n)}$. For $1 \le i \le n$ let $\nu_i : A_i \to B$ be the homomorphism in the proof of Lemma 6. These homomorphisms satisfy (5.4) and $\nu_i(\Lambda_{\alpha,r}) = \Gamma_{i+\alpha-1,r}$, $\nu_i(\mathbf{\Lambda}_{\alpha,r}) = \mathbf{\Gamma}_{i+\alpha-1,r}$ for $(i,\alpha) = (1,1),(n,2)$. Therefore (2)-(i) through (2)-(iii) follow from (5.6) and Proposition 6. Part (2)-(i) and Proposition 12 (3) prove the fact that $\mathcal{B}_{N_1,0,\dots,0,N_n}$ is generated by $\Gamma_{l,r}$ and $\mathbf{\Gamma}_{l,r}$ $(l = 1, n+1, r \in \mathbf{Z})$.

The algebas A_1 and A_n are generated by the elements $a_{1,r}$, $\mathbf{a}_{1,r}$ $(r \in \mathbf{Z} \setminus \{0\})$ and the elements $a_{2,r}$, $\mathbf{a}_{2,r}$ $(r \in \mathbf{Z} \setminus \{0\})$, respectively, by Proposition 7. This, (5.2) and Proposition 12 (2) imply that $[\operatorname{Im}\nu_1, \operatorname{Im}\nu_n] = 0$. Therefore the linear map $A_1 \otimes A_n \to B$ $(a \otimes b \mapsto \nu_1(a)\nu_n(b))$ is an algebra homomorphism. Now the existence of the homomorphism f_{N_1,N_n} follows from Proposition 7 and (5.6). Part (2) proves that this homomorphism is surjective. □

6. Proof of the injectivity of f_{N_1,N_n}

In this section we fix nonnegative integers M and N and denote the algebra $\mathcal{B}_{N_1,\dots,N_n}$ and the ideal $\mathcal{I}_{N_1,\dots,N_n}$ with $(N_1,\dots,N_n) = (M, 0, \dots, 0, N)$ by $\mathcal{B}_{M,N}$ and $\mathcal{I}_{M,N}$, respectively.

6.1. Representation of $\mathcal{U}$. Recalling that $p_+ = \gamma^2$ and $p_- = q^{2(n+1)}/\gamma^2$, set $W = V_{\mathbf{p},M+N,n+1}$ with $\mathbf{p} = (\underbrace{p_+,\dots,p_+}_{M}, \underbrace{p_-,\dots,p_-}_{N})$. For $1 \leq l \leq M+N$ define $\mathcal{E}_i^{(l)}$, $\mathcal{F}_i^{(l)}$, $H_i^{(l)} \in \operatorname{End} W$ $(0 \leq i \leq n)$, $\sigma_l \in \{1,-1\}$ and $p_l \in F^\times$ by

$$\mathcal{E}_i^{(l)} = E^{(l)}_{\bar{i},i+1}, \quad \mathcal{F}_i^{(l)} = E^{(l)}_{i+1,\bar{i}}, \quad H_i^{(l)} = E^{(l)}_{\bar{i},\bar{i}} - E^{(l)}_{i+1,i+1}, \quad \sigma_l = 1, \quad p_l = p_+$$

for $1 \leq l \leq M$ and

$$\mathcal{E}_i^{(l)} = E^{(l)}_{i+1,\bar{i}}, \quad \mathcal{F}_i^{(l)} = E^{(l)}_{\bar{i},i+1}, \quad H_i^{(l)} = E^{(l)}_{i+1,i+1} - E^{(l)}_{\bar{i},\bar{i}}, \quad \sigma_l = -1, \quad p_l = p_-$$

for $M < l \leq M+N$.

Proposition 13. *There exists a homomorphism $\rho\colon \mathcal{U} \to \operatorname{End} W$ determined by*

$$x_{i,r}^+ \mapsto \sum_{l=1}^{M+N} \prod_{l<m} \frac{q^{H_i^{(m)}+\sigma_l i} y_l - q^{-H_i^{(m)}+\sigma_m i} y_m}{q^{\sigma_l i} y_l - q^{\sigma_m i} y_m} q^{\sigma_l i r} y_l^r \mathcal{E}_i^{(l)},$$

$$x_{i,r}^- \mapsto \sum_{l=1}^{M+N} \prod_{m<l} \frac{q^{H_i^{(m)}+\sigma_l i} y_l - q^{-H_i^{(m)}+\sigma_m i} y_m}{q^{\sigma_l i} y_l - q^{\sigma_m i} y_m} q^{\sigma_l i r} y_l^r \mathcal{F}_i^{(l)},$$

$$h_{i,r} \mapsto \frac{[r]}{r}\left(\sum_{l=1}^{M} q^{ir} y_l^r \left(q^{-r} E^{(l)}_{i,i} - q^r E^{(l)}_{i+1,i+1}\right) + \sum_{l=M+1}^{M+N} q^{-ir} y_l^r \left(q^{-r} E^{(l)}_{i+1,i+1} - q^r E^{(l)}_{i,i}\right)\right),$$

$$k_i \mapsto q^{\sum_{l=1}^{M+N} H_i^{(l)}},$$

$$x_{0,r}^+ \mapsto \sum_{l=1}^{M+N} \prod_{l<m} \frac{q^{H_0^{(m)}} p_l^{(1+\sigma_l)/2} y_l - q^{-H_0^{(m)}} p_m^{(H_0^{(m)}+\sigma_m)/2} y_m}{p_l^{(1+\sigma_l)/2} y_l - p_m^{(H_0^{(m)}+\sigma_m)/2} y_m} p_l^{(1+\sigma_l)r/2} y_l^r x_l \mathcal{E}_0^{(l)},$$

$$x_{0,r}^- \mapsto \sum_{l=1}^{M+N} \prod_{m<l} \frac{q^{H_0^{(m)}} p_l^{(-1+\sigma_l)/2} y_l - q^{-H_0^{(m)}} p_m^{(H_0^{(m)}+\sigma_m)/2} y_m}{p_l^{(-1+\sigma_l)/2} y_l - p_m^{(H_0^{(m)}+\sigma_m)/2} y_m} p_l^{(-1+\sigma_l)r/2} y_l^r x_l^{-1} \mathcal{F}_0^{(l)},$$

$$h_{0,r} \mapsto \frac{[r]}{r}\left(\sum_{l=1}^{M} y_l^r\left(q^{-r}p_l^r E_{n+1,n+1}^{(l)} - q^r E_{1,1}^{(l)}\right) + \sum_{l=M+1}^{M+N} y_l^r\left(q^{-r}E_{1,1}^{(l)} - q^r p_l^{-r} E_{n+1,n+1}^{(l)}\right)\right)$$

where $1 \le i \le n$.

Proof. Let S be the endomorphism of W determined by sending $f(y_1,\dots,y_{M+N}) \times \prod_{l=1}^{M+N} x_l^{m_l} \otimes v_{i_1} \otimes \cdots \otimes v_{i_{M+N}}$ to

$$f(q^{\sigma_1}y_1,\dots,q^{\sigma_{M+N}}y_{M+N}) \prod_{l=1}^{M+N} x_l^{m_l - \sigma_l \delta_{i_l,n+1}} \otimes v_{\overline{i_1+1}} \otimes \cdots \otimes v_{\overline{i_{M+N}+1}}.$$

Let $X_{i,m}^{\pm}$, $H_{i,r}$ and K_j be the images of $x_{i,m}^{\pm}$, $h_{i,r}$ and k_j under the assignment ρ in the proposition. Then, for $0 \le i \le n$,

$$SX_{i,r}^{\pm}S^{-1} = \kappa^{-\delta_{i,n}r} X_{\overline{i+1},r}^{\pm}, \quad SH_{i,r}^{\pm}S^{-1} = \kappa^{-\delta_{i,n}r} H_{\overline{i+1},r}^{\pm}, \quad SK_iS^{-1} = K_{\overline{i+1}}$$

where $\kappa = \gamma^2/q^{n+1}$ and $K_0 = (K_1 \cdots K_n)^{-1}$.

The claim can be proven by checking the relations by direct calculations except for (3.10). The use of the above linear map S simplifies the calculations. The relations (3.10) follow from (3.7) and (3.10) with $m_1 = m_2 = 0$, i.e., $\left[x_{i,0}^{\pm}, \left[x_{i,0}^{\pm}, x_{j,l}^{\pm}\right]_{q^{-1}}\right]_q = 0$ ($a_{ij} = -1$), which are proved as in the proof of [12, Proposition 4.5]. □

6.2. The homomorphism $\mathcal{B}_{N_1,0,\dots,0,N_n} \to \mathrm{End}(C_{p_+,N_1} \otimes C_{p_-,N_n})$. For $1 \le l \le n+1$ set $\epsilon_l = (0,\dots,0,\overset{l\text{th}}{1},0,\dots,0) \in \mathbf{Z}^{n+1}$. We identify Q_n with a subgroup of $\mathbf{Z}^{n+1}$ via the correspondence $\alpha_i \leftrightarrow \epsilon_i - \epsilon_{i+1}$ ($1 \le i \le n$). The vector space W is endowed with a structure of $\mathbf{Z}^{n+1}$ graded vector spaces by defining the homogeneous subspace of degree β, W_β, to be

$$W_\beta = \sum_{\sum_{l=1}^{M}\epsilon_{i_l} - \sum_{l=M+1}^{M+N}\epsilon_{i_l} = \beta} C_{\mathbf{p},M+N} \otimes v_{i_1} \otimes \cdots \otimes v_{i_{M+N}}$$

for any $\beta \in \mathbf{Z}^{n+1}$. This structure satisfies $\rho(\mathcal{U}_\alpha)W_\beta \subset W_{\beta+\alpha}$ for $\alpha \in Q_n$ and $\beta \in \mathbf{Z}^{n+1}$.

Let $\lambda = M\epsilon_1 - N\epsilon_{n+1}$. Then $W_\lambda = C_{\mathbf{p},M+N} \otimes v_1^{\otimes M} \otimes v_{n+1}^{\otimes N}$. Set $Q_n^+ = \bigoplus_{j=1}^n \mathbf{Z}_{\ge 0}\alpha_j$ and let $W[i] = \sum_{m\in\mathbf{Z}} W_{\lambda - m\alpha_i}$ for $1 \le i \le n$.

Lemma 7. (1) $W = \sum_{\beta \in Q_n^+} W_{\lambda-\beta}$.
(2) $W[i] = W_\lambda$ *if* $1 < i < n$ *and*

$$W[1] = \sum_{m=0}^{M} W_{\lambda - m\alpha_1} = \sum_{i_1,\dots,i_M=1,2} C_{\mathbf{p},M+N} \otimes v_{i_1} \otimes \cdots \otimes v_{i_M} \otimes v_{n+1}^{\otimes N},$$

$$W[n] = \sum_{m=0}^{N} W_{\lambda - m\alpha_n} = \sum_{j_1,\dots,j_N=n,n+1} C_{\mathbf{p},M+N} \otimes v_1^{\otimes M} \otimes v_{j_1} \otimes \cdots \otimes v_{j_N}.$$

(3) $\rho(\mathcal{I}_{M,N})W_\lambda = 0$.

Proof. Parts (1) and (2) are immediate from the definitions and part (3) follows from (1) and (2). □

We shall identify $C_{p_+,M} \otimes C_{p_-,N}$ with a subspace of $C_{\mathbf{p},M+N}$ via the correspondence

$$\left(f \otimes \prod_{j=1}^{M} x_j^{l_j}\right) \otimes \left(g \otimes \prod_{j=1}^{N} x_j^{m_j}\right) \mapsto f(y_1,\dots,y_M)g(y_{M+1},\dots,y_{M+N}) \otimes \prod_{j=1}^{M} x_j^{l_j} \prod_{j=1}^{N} x_{M+j}^{m_j}.$$

This enables us to identify $V_{p_+,M,n+1} \otimes V_{p_-,N,n+1}$ with a subspace of W since $V_{p_+,M,n+1} \otimes V_{p_-,N,n+1} \simeq (C_{p_+,M} \otimes C_{p_-,N}) \otimes (F^{n+1})^{\otimes M+N}$.

Let $\theta_1 \colon V_{p_+,M,2} \to V_{p_+,M,n+1}$ and $\theta_2 \colon V_{p_-,N,2} \to V_{p_-,N,n+1}$ be the linear maps determined by

$$\theta_1(f_1 \otimes v_{i_1} \otimes \cdots \otimes v_{i_M}) = f_1 \otimes v_{i_1} \otimes \cdots \otimes v_{i_M},$$
$$\theta_2(f_2 \otimes v_{i_1} \otimes \cdots \otimes v_{i_N}) = f_2 \otimes v_{n+2-i_1} \otimes \cdots \otimes v_{n+2-i_N}$$

where $f_1 \in C_{p_+,M}$, $f_2 \in C_{p_-,N}$ and the i_j are in $\{1, 2\}$.

For $1 \leq i \leq n$ $\rho(\mathcal{U}[i])$ clearly preserves $W[i]$ and $\rho(\mathcal{I}[i])$ annihilates $W[i]$ thanks to Lemma 7 (1). Therefore ρ defines a homomorphism $\rho_i \colon \mathcal{U}[i]/\mathcal{I}[i] \to \operatorname{End} W[i]$ ($\overline{u} \mapsto \rho(u)|_{W[i]}$). For the homomorphisms ρ_1 and ρ_n the following lemma holds.

Lemma 8. *Set $\tau_1 = \tau_{1,\gamma/q,M} \circ \iota_{q,1,1}$ and $\tau_2 = \tau_{2,\gamma/q^n,N} \circ \iota_{q^{-n},(-1)^{n-1},1}$. If we regard $V_{p_+,M,n+1} \otimes V_{p_-,N,n+1}$ as a subspace of W, the following hold.*
(1) *For $u \in V_{p_+,M,2}$ and $w \in C_{p_-,N} \otimes v_{n+1}^{\otimes N}$ $(\subset V_{p_-,N,n+1})$,*

$$(\rho_1 \circ \varphi_1)(x)(\theta_1(u) \otimes w) = \theta_1(\tau_1(x)u) \otimes w \quad (x \in U(\gamma/q)).$$

(2) *For $u \in C_{p_+,M} \otimes v_1^{\otimes M}$ $(\subset V_{p_+,M,n+1})$ and $w \in V_{p_-,N,2}$,*

$$(\rho_n \circ \varphi_n)(x)(u \otimes \theta_2(w)) = u \otimes \theta_2(\tau_2(x)w) \quad (x \in U(\gamma/q^n)).$$

Proof. It is sufficient to check the equalities for $x = x_{i,l}^{\pm}$, $h_{i,r}$ and $k_1^{\pm 1}$. Here we show part (1) for $x = x_{0,l}^{+}$ as an example.

Set $v_{i_1,\dots,i_M} = v_{i_1} \otimes \cdots \otimes v_{i_M} \otimes v_{n+1}^{\otimes N} \in (F^{n+1})^{\otimes M+N}$. By Propositions 9 and 13, we find that for $f \in C_{\mathbf{p},M+N}$ and $i_1, \dots, i_M \in \{1, 2\}$

$$(\rho_1 \circ \varphi_1)\big(x_{0,l}^{+}\big)(f \otimes v_{i_1,\dots,i_M})$$

$$= (q/\gamma)^l \rho_1\bigl(x^+_{2,0}\cdots x^+_{n,0}x^+_{0,l}\bigr)(f\otimes v_{i_1,\dots,i_M})$$
$$= (q/\gamma)^l \sum_{j=1}^{M} \delta_{i_j,1} p_+^l y_j^l \prod_{j<m\le M}\left(\frac{q^{-1}p_+y_j - qy_m}{p_+y_j - y_m}\right)^{\delta_{i_m,1}} \prod_{M<m\le M+N} \frac{q^{-1}p_+y_j - qp_-^{-1}y_m}{p_+y_j - p_-^{-1}y_m} w_j$$

where $w_j = \rho\bigl(x^+_{2,0}\cdots x^+_{n,0}\bigr)(x_j f\otimes v_{i_1,\dots,i_{j-1},n+1,i_{j+1},\dots,i_M})$. Since

$$w_j = \prod_{j<m\le M}\left(\frac{qy_j - q^{-1}y_m}{y_j - y_m}\right)^{\delta_{i_m,2}} \prod_{M<m\le M+N}\frac{q^{n+1}y_j - q^{-n-1}y_m}{q^ny_j - q^{-n}y_m} x_j f\otimes v_{i_1,\dots,i_{j-1},2,i_{j+1},\dots,i_M},$$

$p_+ = \gamma^2$ and $p_+p_- = q^{2(n+1)}$, we can see that the above expression is equal to

$$q^l \sum_{j=1}^{M} \delta_{i_j,1}\gamma^l y_j^l \prod_{j<m\le M}\left(\frac{q^{-1}\gamma y_j - q\gamma^{-1}y_m}{\gamma y_j - \gamma^{-1}y_m}\right)^{\delta_{i_m,1}}$$
$$\times\left(\frac{qy_j - q^{-1}y_m}{y_j - y_m}\right)^{\delta_{i_m,2}} x_j f\otimes v_{i_1,\dots,i_{j-1},2,i_{j+1},\dots,i_M}.$$

For $f = f_1\otimes f_2 \in C_{p_+,M}\otimes C_{p_-,N}$, this coincides with

$$q^l\theta_1\bigl(\tau_{1,\gamma/q,M}\bigl(x^+_{0,l}\bigr)(f_1\otimes v_{i_1}\otimes\cdots\otimes v_{i_M})\bigr)\otimes\bigl(f_2\otimes v_{n+1}^{\otimes N}\bigr)$$

if we identify $V_{p_+,M,n+1}\otimes V_{p_-,N,n+1}$ with a subspace of W. □

Proposition 14. *There exists a homomorphism* $\varpi : \mathcal{B}_{M,N}\to \mathrm{End}(C_{p_+,M}\otimes C_{p_-,N})$ *determined by*

$$\varpi(\Gamma_{1,r})(f\otimes g) = e_r f\otimes g,\quad \varpi(\boldsymbol{\Gamma}_{1,r})(f\otimes g) = D_r f\otimes g,$$
$$\varpi(\Gamma_{n+1,r})(f\otimes g) = f\otimes e_r g,\quad \varpi(\boldsymbol{\Gamma}_{n+1,r})(f\otimes g) = f\otimes D_r g$$

where $f\otimes g\in C_{p_+,M}\otimes C_{p_-,N}$.

Proof. Since $\rho(\mathcal{U}_0)$ preserves W_λ, we can see by Lemma 7 (3) that there exists a homomorphism $\widetilde{\rho}\colon \mathcal{B}_{M,N} = \mathcal{U}_0/\mathcal{I}_{M,N}\to \mathrm{End}\,W_\lambda$ ($\overline{u}\mapsto \rho(u)|_{W_\lambda}$). Let j be the quotient map $\mathcal{B}\to\mathcal{B}_{M,N}$. It is easy to show that the homomorphism $\widetilde{\rho}$ satisfies

$$\bigl(\widetilde{\rho}\circ j\circ\widetilde{\varphi}_i\bigr)(\overline{u}) = (\rho_i\circ\varphi_i)(u)|_{W_\lambda}\quad (u\in U(\gamma/q^i)_0) \tag{6.1}$$

for $1\le i\le n$.

For $u = f\otimes v_1^{\otimes M}\in V_{p_+,M,2}$ and $w\in C_{p_-,N}\otimes v_{n+1}^{\otimes N}$ ($\subset V_{p_-,N,n+1}$), we find that

$$\widetilde{\rho}(\Gamma_{1,r})(\theta_1(u)\otimes w) = (\rho_1\circ\varphi_1)(\Lambda_{1,r})(\theta_1(u)\otimes w)\quad \text{(by (5.3) and (6.1))}$$
$$= q^r\theta_1(\tau_{1,\gamma/q,M}(\Lambda_{1,r})u)\otimes w\quad \text{(by Lemma 8)}$$

$$= q^r \left(G_{p_+,M} e_r G^{-1}_{p_+,M} f \otimes v_1^{\otimes M}\right) \otimes w \quad \text{(by Proposition 8 (2)).}$$

This and similar calculations prove that $\widetilde{\rho}(\Gamma_{l,r})$ and $\widetilde{\rho}(\boldsymbol{\Gamma}_{l,r})$ ($l = 1, n+1, r \in \mathbf{Z}$) preserve $W'_\lambda := (C_{p_+,M} \otimes C_{p_-,N}) \otimes \left(v_1^{\otimes M} \otimes v_{n+1}^{\otimes N}\right)(\subset W_\lambda)$. Therefore $\widetilde{\rho}(\mathcal{B}_{M,N})W'_\lambda \subset W'_\lambda$ by Theorem 1 (2).

Let us identify W'_λ with $C_{p_+,M} \otimes C_{p_-,N}$ and define $\widetilde{\rho}' : \mathcal{B}_{M,N} \to \mathrm{End}(C_{p_+,M} \otimes C_{p_-,N})$ by $u \mapsto G^{-1} \circ \widetilde{\rho}(u)|_{W'_\lambda} \circ G$ where $G = G_{p_+,M} \otimes G_{p_-,N}$. Then the above calculations show that

$$\widetilde{\rho}'(\Gamma_{1,r})(f \otimes g) = q^r (e_r f \otimes g), \quad \widetilde{\rho}'(\boldsymbol{\Gamma}_{1,r})(f \otimes g) = D_r f \otimes g,$$
$$\widetilde{\rho}'(\Gamma_{n+1,r})(f \otimes g) = q^{-nr} (f \otimes e_r g), \quad \widetilde{\rho}'(\boldsymbol{\Gamma}_{n+1,r})(f \otimes g) = (-1)^{(n-1)r} (f \otimes D_r g)$$

for $f \otimes g \in C_{p_+,M} \otimes C_{p_-,N}$. This proves the claim. □

6.3. Proof of the injectivity of f_{N_1,N_n}. Now, to complete the proof of Theorem 1, we give the proof of the injectivity of f_{N_1,N_n}. Proposition 14 implies that there exists a homomorphism $g_{N_1,N_n} : \mathcal{B}_{N_1,N_n} \to \widetilde{C}_{p_+,N_1} \otimes \widetilde{C}_{p_-,N_n}$ determined by

$$\Gamma_{1,r} \mapsto e_r \otimes 1, \quad \boldsymbol{\Gamma}_{1,r} \mapsto D_r \otimes 1, \quad \Gamma_{n+1,r} \mapsto 1 \otimes e_r, \quad \boldsymbol{\Gamma}_{n+1,r} \mapsto 1 \otimes D_r.$$

The injectivity of f_{N_1,N_n} follows from $g_{N_1,N_n} \circ f_{N_1,N_n} = \mathrm{id}$. □

7. Proof of Lemma 4

In this section we shall prove Lemma 4. Before doing so, we prepare several equalities in $U_q(\widehat{sl_{n+1}})$ and $\mathcal{U}_q\left(\widehat{sl_{n+1}(\mathcal{C}_\gamma)}\right)$ $(n \geq 2)$ in the first two subsections.

7.1. Some equalities in $U_q(\widehat{sl_{n+1}})$ $(n \geq 2)$. Set

$$J[i] = \sum_{\substack{m_1,\dots,m_n,m'_i \in \mathbf{Z} \\ m_j > 0 \text{ for some } j \in \{1,\dots,n\}\setminus\{i\}}} U_q\left(\widehat{sl_{n+1}}\right)_{m'_i\alpha_i - \sum_{1 \leq j \neq i \leq n} m_j\alpha_j} U_q\left(\widehat{sl_{n+1}}\right)_{m_i\alpha_i + \sum_{1 \leq j \neq i \leq n} m_j\alpha_j}$$

for $1 \leq i \leq n$ and

$$J = \sum_{\substack{m_1,\dots,m_n \in \mathbf{Z} \\ m_j > 0 \text{ for some } j}} U_q\left(\widehat{sl_{n+1}}\right)_{-\sum_{j=1}^n m_j\alpha_j} U_q\left(\widehat{sl_{n+1}}\right)_{\sum_{j=1}^n m_j\alpha_j}.$$

First we note the following result, which follows from [23].

Lemma 9. *In $U_q\left(\widehat{sl_{n+1}}\right)$ the following hold for $1 \leq i, j \leq n$.*
(1) $T_i\left(x^\pm_{j,l}\right) = x^\pm_{j,l}$ *if* $|i-j| > 1$.
(2) $T_i\left(x^+_{j,l}\right) = \left[x^+_{i,0}, x^+_{j,l}\right]_{q^{-1}}$, $T_i\left(x^-_{j,l}\right) = \left[x^-_{j,l}, x^-_{i,0}\right]_q$ *if* $|i-j| = 1$.

Using this lemma, we can prove the following lemma.

Lemma 10. *In $U_q\left(\widehat{sl_{n+1}}\right)$ the following hold for $1 \le i \le n$.*
(1) (i) $x^-_{i,1} \equiv (-1)^{i-1}(-q)^{-(n-1)}C^{-1}t_i(e_{i+1}\cdots e_n)(e_{i-1}\cdots e_1)e_0 \mod J[i]$,
(ii) $x^+_{i,-1} \equiv (-1)^{i-1}(-q)^{n-1} f_0(f_1\cdots f_{i-1})(f_n\cdots f_{i+1})t_i^{-1}C \mod J[i]$,
(iii) $h_{i,1} \equiv (-1)^{i-1}(-q)^{-(n+1)}e_i(e_{i+1}\cdots e_n)(e_{i-1}\cdots e_1)e_0 \mod J$.
(2) $(T_n\cdots T_1)(h_{1,1}) = -\left[[e_n,\dots,e_1]_{q^{-1}}, e_0\right]_{q^{-2}}$.
(3) (i) $(\eta\circ\sigma)(x^-_{i,1}) \equiv (-q)^{-(n-1)}k_i x^-_{i,-1}k_i C^2 \mod J[i]$,
(ii) $(\eta\circ\sigma)(x^+_{i,-1}) \equiv (-q)^{n-1}C^{-2}k_i^{-1}x^+_{i,1}k_i^{-1} \mod J[i]$.

Proof. (1) By Lemma 9 and the definition of the T_j, (3.9) with $(m,l)=(0,0)$ and (3.25) (for the $x^-_{i,r}$) are rewritten as

$$T_i\left(x^-_{j,1}\right) + T_j\left(x^-_{i,1}\right) = 0 \quad \text{if} \quad |i-j| = 1, \tag{7.1}$$

$$x^-_{1,1} = \left(T_2^{-1}\cdots T_n^{-1}\right)\left(t_0^{-1}e_0\right). \tag{7.2}$$

From these we obtain

$$x^-_{i,1} = (-1)^{i-1}\left(T_{i+1}^{-1}\cdots T_n^{-1}\right)\left(T_{i-1}^{-1}\cdots T_1^{-1}\right)\left(t_0^{-1}e_0\right)$$

for $1 \le i \le n$. This proves the expression for $x^-_{i,1}$. The claim for $x^+_{i,-1}$ can be shown similarly and the claim for $h_{i,1}$ follows from the equality

$$h_{i,1} = Ck_i^{-1}\left[x^+_{i,0}, x^-_{i,1}\right]. \tag{7.3}$$

(2) By (7.2) and (7.3)

$$h_{1,1} = -\left[T_2^{-1}\cdots T_n^{-1}e_0, e_1\right]_{q^{-2}}.$$

Set $X = T_3^{-1}\cdots T_n^{-1}e_0$. Then $X = [e_0, e_n, \cdots, e_3]_{q^{-1}}$ and $T_2^{-1}\cdots T_n^{-1}e_0 = [X, e_2]_{q^{-1}}$. Therefore

$$\begin{aligned} T_1h_{1,1} &= \left[\left[[e_1, X]_{q^{-1}}, [e_1, e_2]_{q^{-1}}\right]_{q^{-1}},\ f_1\right]t_1, \\ &= -\left[e_1, [X, e_2]_{q^{-1}}\right]_{q^{-2}}. \end{aligned}$$

Applying $T_n\cdots T_2$ to the above, we obtain the claim.

(3) The automorphism $\eta\circ\sigma$ preserves $J[i]$ and

$$(\eta\circ\sigma)(e_0) = \eta(e_0) = \left[x^-_{n,0}, \dots, x^-_{2,0}, x^-_{1,-1}\right]_q k_1\cdots k_nC$$

by (3.23). Noting these, apply $\eta\circ\sigma$ to the first equality of (1). Then we obtain the claim for $x^-_{i,1}$. The proof for $x^+_{i,-1}$ is similar. □

7.2. Some equalities in $U_q\big(\widehat{sl_{n+1}(\mathcal{C}_\gamma)}\big)$ ($n \geq 2$). Recall that we fixed $n \geq 2$ and set $\widehat{\mathcal{U}} = \widehat{\mathcal{U}}(n, \gamma)$. We further set $\kappa = \gamma^2/q^{n+1}$. First note the following simple fact, which follows from the relation (3.9).

Lemma 11. *If $a_{ij}^{(n)} = -1$, the following hold in $\widehat{\mathcal{U}}$.*
(1) $\kappa_{ij} x^+_{i,l+1} x^+_{j,m} \equiv q^{-1} x^+_{i,l} x^+_{j,m+1} \mod \sum_{r\in\mathbf{Z}} \widehat{\mathcal{U}} x^+_{i,r}$.
(2) $\kappa_{ij} x^-_{i,l+1} x^-_{j,m} \equiv q x^-_{i,l} x^-_{j,m+1} \mod \sum_{r\in\mathbf{Z}} x^-_{j,r} \widehat{\mathcal{U}}$.

Using the above lemma and the results in Section 3.3, we can show the following lemma.

Lemma 12. *Set $\mathbf{h}_{i,r} = \psi(h_{i,r}) \in \widehat{\mathcal{U}}$ for $0 \leq i \leq n$ and $r \in \mathbf{Z}\setminus\{0\}$. Define an ideal $\widehat{\mathcal{I}}$ of $\widehat{\mathcal{U}}$ as we did $\mathcal{I}$* for *$\mathcal{U}$. Let $\equiv$ denote the equality modulo $\widehat{\mathcal{I}}$. Then the following hold in $\widehat{\mathcal{U}}$.*
(1) $\mathbf{h}_{i,1} \equiv (-1)^{i-1}(-q)^{-(n+1)} x^+_{i,0}\big(x^+_{i+1,0}\cdots x^+_{n,0}\big)\big(x^+_{i-1,0}\cdots x^+_{1,0}\big)x^+_{0,0}$ $(1 \leq i \leq n)$.
(2) $\mathbf{h}_{0,1} = -\kappa\Big[\big[x^+_{n,1}, x^+_{n-1,0}, \ldots, x^+_{1,0}\big]_{q^{-1}}, x^+_{0,-1}\Big]_{q^{-2}} = (-1)^n\kappa\Big[\big[x^+_{1,1}, x^+_{2,0}, \ldots, x^+_{n,0}\big]_{q^{-1}}, x^+_{0,-1}\Big]_{q^{-2}}$.
(3) $\sum_{i=0}^{n-1} q^i \mathbf{h}_{i,1} \equiv -\kappa x^+_{n,1} x^+_{n-1,0}\cdots x^+_{1,0} x^+_{0,-1}$.
(4) $\gamma^{-2}\mathbf{h}_{0,1} + \sum_{i=2}^{n} q^{-i}\mathbf{h}_{i,1} \equiv (-1)^n q^{-(n+1)} x^+_{1,1} x^+_{2,0}\cdots x^+_{n,0} x^+_{0,-1}$.

Proof. (1) By Proposition 1 (1), $\psi \circ \varrho_v = \varrho_h$. Applying this to the third equality of Lemma 10 (1), we obtain the claim since ϱ_h is a homomorphism of Q_n graded algebras.

(2) By Propositions 1, 2 and 3,

$$\begin{aligned}\mathbf{h}_{0,1} &= -\widetilde{\mathcal{S}}^{-1}\mathbf{h}_{1,1},\\ &= -\mathcal{Y}_{n+1}(\psi\circ\varrho_v)(T_n\cdots T_1 h_{1,1}),\\ &= -\mathcal{Y}_{n+1}(\varrho_h(T_n\cdots T_1 h_{1,1})).\end{aligned}$$

This and part (2) of Lemma 10 prove the first equality. The second equality follows from this and (3.9).

(3) Using the first equality of part (2) and Lemma 11, we can show

$$\begin{aligned}\mathbf{h}_{0,1} \equiv & -\kappa\big[x^+_{n,1}, x^+_{n-1,0}, \ldots, x^+_{l,0}\big]_{q^{-1}} x^+_{l-1,0}\cdots x^+_{1,0} x^+_{0,-1}\\ &+ \sum_{i=1}^{l-1}(-q)^{i-n-1} x^+_{i,0}\big(x^+_{i+1,0}\cdots x^+_{n,0}\big)\big(x^+_{i-1,0}\cdots x^+_{1,0}\big)x^+_{0,0}\end{aligned}$$

for $1 \leq l \leq n$ by induction on l. The case $l = n$ and part (1) give the claim.

(4) Using the second equality of part (2), the claim can be shown similarly to (3). □

The following lemma is proven by direct calculations.

Lemma 13. *In $\widehat{\mathcal{U}}$ the following equalities hold*:

$$\big[b_{l,r}, x^{\pm}_{i,m}\big] = 0 \quad \textit{if} \quad i \neq 0, l-1, l,$$
$$\big[b_{1,r}, x^{\pm}_{0,m}\big] = \pm\frac{[r]}{r}C^{(r\mp|r|)/2} \times q^{-r}\big(1-\gamma^{2r}\big)x^{\pm}_{0,m+r},$$
$$\big[b_{l,r}, x^{\pm}_{0,m}\big] = \pm\frac{[r]}{r}C^{(r\mp|r|)/2} \times (q^{-r}-q^{r})x^{\pm}_{0,m+r} \quad \textit{for} \quad 2 \le l \le n,$$
$$\big[b_{n+1,r}, x^{\pm}_{0,m}\big] = \pm\frac{[r]}{r}C^{(r\mp|r|)/2} \times \big(q^{-r}-q^{(2n+1)r}\big/\gamma^{2r}\big)x^{\pm}_{0,m+r},$$
$$\big[b_{l,r}, x^{\pm}_{l-1,m}\big] = \pm\frac{[r]}{r}C^{(r\mp|r|)/2} \times \big(q^{lr}-\gamma^{2r}q^{-lr}\big)x^{\pm}_{l-1,m+r} \quad \textit{for} \quad 2 \le l \le n+1,$$
$$\big[b_{l,r}, x^{\pm}_{l,m}\big] = \pm\frac{[r]}{r}C^{(r\mp|r|)/2} \times \big(\gamma^{2r}q^{(1-l)r}-q^{(l-1)r}\big)x^{\pm}_{l,m+r} \quad \textit{for} \quad 1 \le l \le n.$$

REMARK 1. We can also show the following equality in $\widehat{\mathcal{U}}$:

$$[b_{l,r}, b_{m,s}] = \delta_{l,m}\delta_{r+s,0}\frac{[r]}{r}\frac{C^{r}-C^{-r}}{q-q^{-1}}\big(q^{r}+q^{-r}-\big(\gamma^{2}q^{1-2l}\big)^{r}-\big(\gamma^{2}q^{1-2l}\big)^{-r}\big).$$

7.3. Proof of Lemma 4. Now we can give the proof of Lemma 4. Thanks to part (1) of Proposition 11, it is sufficient to show the claim for $r = s = 1$. Here we shall prove $[b_{1,1}, \mathbf{b}_{l,1}] = 0$ for $2 \le l \le n+1$ $(*)$ as an example.

By Lemmas 12 and 13 the elements $\mathbf{h}_{i,1}$ $(2 \le i \le n)$ and $\sum_{i=0}^{n-1} q^{i}\mathbf{h}_{i,1}$ in $\mathcal{B} = \mathcal{U}_0/\mathcal{I}$ have the form $\overline{Xx^{+}_{1,0}x^{+}_{0,m}}$ where X is an element of $\mathcal{U}$ which commutes with $b_{1,1}$ and m is an integer. Lemmas 11 and 13 imply that $\big[b_{1,1}, x^{+}_{1,0}x^{+}_{0,m}\big] = (\gamma^{2}-1)\big(x^{+}_{1,1}x^{+}_{0,m} - q^{-1}x^{+}_{1,0}x^{+}_{0,m+1}\big) \equiv 0 \mod \mathcal{I}$. Therefore

$$[b_{1,1}, \mathbf{h}_{l,1}] = 0 \quad (2 \le l \le n) \quad \text{and} \quad [b_{1,1}, \mathbf{h}_{0,1} + q\mathbf{h}_{1,1}] = 0$$

in $\mathcal{B}$. This and (5.1) prove $(*)$ since $\mathbf{b}_{l,1} = (-q)^{n-1}\psi(b_{l,1})$. □

8. Proof of Proposition 9

The purpose of this section is to prove Proposition 15 below, from which Proposition 9 follows. We keep the notation of Section 5 and further set $\widehat{U}(\xi) = \widehat{\mathcal{U}}(1, \xi)$ for $\xi \in F^{\times}$.

For $1 \le i \le n$ define a subalgebra $\widehat{\mathcal{U}}[i]$ of $\widehat{\mathcal{U}}$ and an ideal $\widehat{\mathcal{I}}[i]$ of $\widehat{\mathcal{U}}[i]$ in the same way that we defined $\mathcal{U}[i]$ and $\mathcal{I}[i]$ for $\mathcal{U}$ in Section 5.1. The automorphisms $\mathcal{X}_j$ and Ψ induce automorphisms of the quotient algebra $\widehat{\mathcal{U}}[i]\big/\widehat{\mathcal{I}}[i]$ as in the case $\mathcal{U}[i]/\mathcal{I}[i]$, which we denote by the same letters.

Proposition 15. (1) *For $1 \leq i \leq n$ there exists a homomorphism $\widehat{\varphi}_i : \widehat{U}(\gamma/q^i) \to \widehat{\mathcal{U}}[i]\big/\widehat{\mathcal{I}}[i]$ determined by*

$$\begin{aligned}
&x^{\pm}_{1,l} \mapsto x^{\pm}_{i,l}, \quad h_{1,r} \mapsto h_{i,r}, \quad k_1 \mapsto k_i,\\
&k_0 \mapsto (k_0 \cdots k_n)k_i^{-1}, \quad C \mapsto C,\\
&x^{+}_{0,l} \mapsto (-1)^{i-1}(q/\gamma)^l \big(x^{+}_{i+1,0} \cdots x^{+}_{n,0}\big)\big(x^{+}_{i-1,0} \cdots x^{+}_{1,0}\big)x^{+}_{0,l},\\
&x^{-}_{0,l} \mapsto (-1)^{i-1}(q/\gamma)^l x^{-}_{0,l}\big(x^{-}_{1,0} \cdots x^{-}_{i-1,0}\big)\big(x^{-}_{n,0} \cdots x^{-}_{i+1,0}\big),\\
&h_{0,r} \mapsto (q/\gamma)^r \left(\sum_{l=0}^{i-1} q^{lr} h_{l,r} + \gamma^{2r} \sum_{l=i+1}^{n} q^{-lr} h_{l,r}\right).
\end{aligned}$$

(2) *The homomorphism $\widehat{\varphi}_i$ satisfies the following equalities*:

(i) $\Psi \circ \widehat{\varphi}_i = \widehat{\varphi}_i \circ \Phi$.

(ii) $$\mathcal{X}_j \circ \widehat{\varphi}_i = \begin{cases} \widehat{\varphi}_i \circ \mathcal{X}_0 \circ \iota_{1,(-1)^j q^{1+j}\gamma^{-1},1} & \textit{if} \quad 0 \leq j < i,\\ \widehat{\varphi}_i \circ \mathcal{X}_1 \circ \iota_{1,1,(-1)^{i-1}} & \textit{if} \quad j = i,\\ \widehat{\varphi}_i \circ \mathcal{X}_0 \circ \iota_{1,(-1)^j q^{1-j}\gamma,1} & \textit{if} \quad i < j \leq n+1. \end{cases}$$

To prove Proposition 15, we need the following lemma, which can be proven by checking the relations.

For $m \geq 2$ set $\widehat{\mathcal{U}}(m) = \widehat{\mathcal{U}}(m, \gamma)$, $\widehat{\mathcal{V}}(m) = \sum_{\beta \in Q'_m} \widehat{\mathcal{U}}(m)_\beta$ and

$$\widehat{\mathcal{J}}(m) = \sum_{\substack{\beta, \beta' \in Q'_m\\ r>0}} \widehat{\mathcal{U}}(m)_{\beta - r\alpha_m} \widehat{\mathcal{U}}(m)_{\beta' + r\alpha_m}$$

where $Q'_m = \bigoplus_{1 \leq i \leq m-1} \mathbf{Z}\alpha_i \subset Q_m$. Then $\widehat{\mathcal{V}}(m)$ is a subalgebra of $\widehat{\mathcal{U}}(m)$ and $\widehat{\mathcal{J}}(m)$ is an ideal of $\widehat{\mathcal{V}}(m)$.

Lemma 14. *For $m \geq 3$ there exists a homomorphism $\phi_m : \widehat{\mathcal{U}}(m-1) \to \widehat{\mathcal{V}}(m)\big/\widehat{\mathcal{J}}(m)$ determined by*

$$\begin{aligned}
&x^{\pm}_{j,l} \mapsto x^{\pm}_{j,l}, \quad h_{j,r} \mapsto h_{j,r}, \quad k_j \mapsto k_j \quad (1 \leq j \leq m-1),\\
&x^{+}_{0,l} \mapsto x^{+}_{m,0} x^{+}_{0,l}, \quad x^{-}_{0,l} \mapsto x^{-}_{0,l} x^{-}_{m,0},\\
&h_{0,r} \mapsto h_{0,r} + \big(\gamma^2/q^m\big)^r h_{m,r}, \quad k_0 \mapsto k_0 k_m, \quad C \mapsto C.
\end{aligned}$$

Proof of Proposition 15. (1) First we show that the case $i = n$ follows from the case $i = 1$. Set $\widehat{\mathcal{U}}' = \widehat{\mathcal{U}}(n, q^{n+1}/\gamma)$ and denote $\widehat{\mathcal{I}}[i]$ defined for $\widehat{\mathcal{U}}'$ (instead of $\widehat{\mathcal{U}}$) by $\widehat{\mathcal{I}}'[i]$ for $1 \leq i \leq n$. Set $\kappa = \gamma^2/q^{n+1}$ and let ω be the isomorphism $\widehat{\mathcal{U}} \to \widehat{\mathcal{U}}'$ determined by

$$\begin{aligned}
&x^{\pm}_{j,l} \mapsto x^{\pm}_{n+1-j,l}, \quad h_{j,r} \mapsto h_{n+1-j,r}, \quad k_j \mapsto k_{n+1-j} \quad (1 \leq j \leq n),\\
&x^{\pm}_{0,l} \mapsto (-1)^{n-1}\kappa^l x^{\pm}_{0,l}, \quad h_{0,r} \mapsto \kappa^r h_{0,r}, \quad k_0 \mapsto k_0, \quad C \mapsto C.
\end{aligned}$$

Since $\omega\big(\widehat{\mathcal{U}}[1]\big) = \widehat{\mathcal{U}}'[n]$ and $\omega\big(\widehat{\mathcal{I}}[1]\big) = \widehat{\mathcal{I}}'[n]$, ω induces an isomorphism $\tilde{\omega}\colon \widehat{\mathcal{U}}[1]\big/\widehat{\mathcal{I}}[1] \to \widehat{\mathcal{U}}'[n]\big/\widehat{\mathcal{I}}'[n]$. Noting that $\widehat{U}(\xi) = \widehat{U}(\xi^{-1})$ for $\xi \in F^{\times}$, we obtain the desired homomorphism $\widehat{\varphi}_n$ from the composite map $\tilde{\omega} \circ \widehat{\varphi}_1\colon \widehat{U}(\gamma/q) \to \widehat{\mathcal{U}}'[n]\big/\widehat{\mathcal{I}}'[n]$ by letting $\gamma \to q^{n+1}/\gamma$.

Next we show the claim by induction on n. The case $n = 2$ and $i = 1$ was proven in [12]. This and the argument in the previous paragraph prove the case $n = 2$. Now, supposing that we have shown the case $n = m - 1$ ($m \geq 3$), we shall prove the case $n = m$ and $1 \leq i \leq m - 1$, from which the case $i = m$ follows as before.

For any integer $l \geq 2$ and $1 \leq i \leq l$ we define $\widehat{\mathcal{U}}(l)[i]$ and $\widehat{\mathcal{I}}(l)[i]$ for $\widehat{\mathcal{U}}(l)$ as we did $\widehat{\mathcal{U}}[i]$ and $\widehat{\mathcal{I}}[i]$ for $\widehat{\mathcal{U}}$. In the case $n = l$, we denote the homomorphism $\widehat{\varphi}_i\colon \widehat{U}(\gamma/q^i) \to \widehat{\mathcal{U}}(l)[i]\big/\widehat{\mathcal{I}}(l)[i]$ by $\widehat{\varphi}_{i,l}$. Let $^{-}\colon \widehat{\mathcal{V}}(m) \to \widehat{\mathcal{V}}(m)\big/\widehat{\mathcal{J}}(m)$ be the canonical map. Then for $1 \leq i \leq m - 1$, since $\widehat{\mathcal{J}}(m) \cap \widehat{\mathcal{U}}(m)[i] \subset \widehat{\mathcal{I}}(m)[i]$, we obtain the following composite map:

$$\widehat{\mathcal{U}}(m-1)[i] \xrightarrow{\phi_m} \overline{\widehat{\mathcal{U}}(m)[i]} \simeq \widehat{\mathcal{U}}(m)[i]\big/\big(\widehat{\mathcal{J}}(m) \cap \widehat{\mathcal{U}}(m)[i]\big) \to \widehat{\mathcal{U}}(m)[i]\big/\widehat{\mathcal{I}}(m)[i]$$

where the last map is defined by $u + \widehat{\mathcal{J}}(m) \cap \widehat{\mathcal{U}}(m)[i] \mapsto u + \widehat{\mathcal{I}}(m)[i]$. The above map induces a homomorphism $\widehat{\mathcal{U}}(m-1)[i]\big/\widehat{\mathcal{I}}(m-1)[i] \to \widehat{\mathcal{U}}(m)[i]\big/\widehat{\mathcal{I}}(m)[i]$ and the composition of this homomorphism and $\widehat{\varphi}_{i,m-1}$ yields $\widehat{\varphi}_{i,m}$.

(2) The claims are proven by checking the equalities on the generators $x^{\pm}_{l,0}$, $x^{\pm}_{1,\mp1}$, $k_l^{\pm1}$ ($l = 0, 1$) and $C^{\pm1}$ of $\widehat{U}(\gamma/q^i)$. Here we show

$$(\Psi \circ \widehat{\varphi}_i)\big(x^{+}_{0,0}\big) = (\widehat{\varphi}_i \circ \Phi)\big(x^{+}_{0,0}\big) \tag{8.1}$$

and (ii) on $x^{+}_{0,0}$ as examples.

Let $^{-}\colon \widehat{\mathcal{U}}[i] \to \widehat{\mathcal{U}}[i]\big/\widehat{\mathcal{I}}[i]$ be the quotient map. Noting that Ψ, ϱ_h and ϱ_v are homomorphisms of Q_n graded algebras, we find that

$$\begin{aligned}
\text{the l.h.s. of (8.1)} &= (-q)^{n-1}\overline{(\psi \circ \varrho_h)\big(k_i^{-1}Cx^{-}_{i,1}\big)} \quad \text{(by Lemma 10 (1))}\\
&= (-q)^{n-1}\overline{(k_iC)^{-1}(\varrho_v \circ \eta \circ \sigma)\big(x^{-}_{i,1}\big)} \quad \text{(by Proposition 1)}\\
&= \overline{x^{-}_{i,-1}k_iC} \quad \text{(by Lemma 10 (3))}.
\end{aligned}$$

By part (2) of Proposition 1, $\Phi\big(x^{+}_{0,0}\big) = x^{-}_{1,-1}k_1C$. This enables us to see that the r.h.s. coincides with the above.

Next we consider (ii). Fixing $i \in \{1, 2, \ldots, n\}$, set

$$z_l = \overline{\big(x^{+}_{i+1,0} \cdots x^{+}_{n,0}\big)\big(x^{+}_{i-1,0} \cdots x^{+}_{1,0}\big)x^{+}_{0,l}} \in \widehat{\mathcal{U}}[i]\big/\widehat{\mathcal{I}}[i]$$

for $l \in \mathbf{Z}$. Then by Lemma 11

$$\mathcal{X}_j(z_l) = \begin{cases} (-q)^j z_{l-1} & \text{if} \quad 0 \leq j < i, \\ z_l & \text{if} \quad j = i, \\ \gamma^2(-q)^{-j} z_{l-1} & \text{if} \quad i < j \leq n+1. \end{cases}$$

Using the above, it is easy to check the equality (ii) on $x^{+}_{0,0}$. □

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