

# REIDEMEISTER TORSION AND LENS SURGERIES ON KNOTS IN HOMOLOGY 3-SPHERES I

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(Received April 25, 2005, revised December 15, 2005)

## Abstract

Let  $\Sigma(K; p/q)$  be the result of  $p/q$ -surgery along a knot  $K$  in a homology 3-sphere  $\Sigma$ . We investigate the Reidemeister torsion of  $\Sigma(K; p/q)$ . Firstly, when the Alexander polynomial of  $K$  is the same as that of a torus knot, we give a necessary and sufficient condition for the Reidemeister torsion of  $\Sigma(K; p/q)$  to be that of a lens space. Secondly, when the Alexander polynomial of  $K$  is of degree 2, we show that if the Reidemeister torsion of  $\Sigma(K; p/q)$  is the same as that of a lens space, then  $\Delta_K(t) = t^2 - t + 1$ .

## 1. Introduction

We investigate when the result of Dehn surgery along a knot in a homology 3-sphere is a lens space. In this paper, we call such a surgery *lens surgery* not only for hyperbolic knots but also for any knots in homology 3-spheres. Many authors have studied lens surgery by geometric method (see [1, 2, 5, 7, 9, 10, 17, 22, 23, 24, 29, 31]). We approach the problem by algebraic method (see [13, 14, 15, 18, 19]).

K. Reidemeister [20] and W. Franz [8] classified lens spaces completely by using the Reidemeister torsion. Franz provided a useful lemma called *Franz's lemma* (see Theorem 3.1) which is deduced by a result of  $L$ -function (see [30]) from Number Theory. We apply the lemma in the present paper. J. Milnor [16] pointed out that the Reidemeister torsion was closely related to the Alexander polynomial. Our method is based on the surgery formula for Reidemeister torsions due to V.G. Turaev (see [21, 25, 26, 27] and Section 2.2). Our results are mentioned in terms of the Alexander polynomials.

We define that an oriented closed 3-manifold  $M$  is of *lens type* (or of  $(p, q)$ -*lens type*) if it has the same Reidemeister torsion as the lens space (or as  $L(p, q)$ ) (for a precise definition, see Section 2.3). By algebraic and number theoretic study, we obtain necessary conditions for the Alexander polynomial of a knot having a lens type surgery. The multiplicative group  $(\mathbf{Z}/n\mathbf{Z})^\times$  plays important roles in our study because it is the Galois group of the  $n$ -th cyclotomic field, and the second term  $q$  of a lens space  $L(p, q)$  is an element of  $(\mathbf{Z}/p\mathbf{Z})^\times$ .

We point out the following lemma which states a property of the Alexander polynomial of a knot having a lens type surgery. We prove it in Section 3.2. Our two main theorems are obtained by using this lemma. Let  $\Sigma(K; p/q)$  be the result of  $p/q$ -surgery along a knot  $K$  in a homology 3-sphere  $\Sigma$ . If  $\Sigma$  is the 3-sphere  $S^3$ , then we denote  $\Sigma(K; p/q)$  by  $(K; p/q)$ .

**Lemma 1.1.** *Let  $K$  be a knot in a homology 3-sphere  $\Sigma$ , and  $\Delta_K(t)$  the Alexander polynomial of  $K$ . Let  $d$  ( $\geq 2$ ) be a divisor of  $p$ , and  $\zeta$  a primitive  $d$ -th root of unity. If the  $p/q$ -surgery  $\Sigma(K; p/q)$  is of lens type for  $p \geq 2$  and  $q \neq 0$ , then*

$$N(\Delta_K(\zeta)) = \begin{cases} \pm 1 & (d = 2), \\ 1 & (d \geq 3), \end{cases}$$

where  $N(\alpha)$  is the norm of  $\alpha$  in the  $d$ -th cyclotomic field  $\mathbf{Q}(\zeta)$  over  $\mathbf{Q}$ .

For the *norm* in a Galois extension, see Section 3.1. In [17], L. Moser firstly showed that non-trivial knots can yield a lens space by Dehn surgery. In fact, Moser determined all rational surgery along all torus knots, and proved that every torus knot yields a lens space as below.

**Theorem 1.2** (Moser [17]). *Let  $K$  be the  $(r, s)$ -torus knot in  $S^3$ , and  $M = (K; p/q)$  the result of  $p/q$ -surgery along  $K$ , where  $p, r, |s| \geq 2$  and  $q \neq 0$ . Then there are three cases:*

- (1) *If  $|p - qrs| \neq 0, 1$ , then  $M$  is a Seifert fibered space with three singular fibers of multiplicities  $r, |s|$  and  $|p - qrs|$ .*
- (2) *If  $|p - qrs| = 1$ , then  $M$  is the lens space  $L(p, qr^2)$ .*
- (3) *If  $|p - qrs| = 0$  (i.e.,  $p/q = rs$ ), then  $M$  is the connected sum of two lens spaces,  $L(r, s) \sharp L(s, r)$ .*

Our first main theorem is an algebraic “translation” of the theorem. The following is the first main theorem of this paper.

**Main Theorem 1.** *Let  $K$  be a knot in a homology 3-sphere  $\Sigma$  whose Alexander polynomial is the same as the  $(r, s)$ -torus knot, and  $M = \Sigma(K; p/q)$ , where  $p, r, s \geq 2$  and  $q \neq 0$ . Then  $M$  is of lens type if and only if the following (1) and (2) hold.*

- (1)  $\gcd(p, r) = 1$  and  $\gcd(p, s) = 1$ ,
- (2)  $r \equiv \pm 1 \pmod{p}$  or  $s \equiv \pm 1 \pmod{p}$  or  $qrs \equiv \pm 1 \pmod{p}$ .

We prove Main Theorem 1 by using Franz’s lemma (see [6, 8, 27]) in Section 3.

H. Goda and M. Teragaito [9] showed that if a genus one knot in  $S^3$  yields a lens space, then the knot is the trefoil. Our second main theorem is included by Goda-Teragaito’s theorem in the restricted case  $\Sigma = S^3$ , but extends theirs to the case of

knots in any homology 3-spheres from the algebraic view point. When we say “the degree of  $\Delta_K(t)$ ,” we take a regularization that  $\Delta_K(t)$  is in  $\mathbf{Z}[t]$  and  $\Delta_K(0) \neq 0$ . We prove Main Theorem 2 by using norm of an algebraic number (see [3, 30]) in Section 4.

**Main Theorem 2.** *Let  $K$  be a knot in a homology 3-sphere  $\Sigma$  whose Alexander polynomial  $\Delta_K(t)$  is of degree 2. If a Dehn surgery  $\Sigma(K; p/q)$  is of lens type for  $p \geq 2$  and  $q \neq 0$ , then  $\Delta_K(t) = t^2 - t + 1$ .*

Recently, P. Ozsváth and Z. Szabó [18] obtained a necessary condition on the Alexander polynomial of a knot in  $S^3$  which yields a lens space by using the knot Floer homology (see [13, 19], and Appendix). Note that Main Theorem 2 extends a special case ( $m = 1, s_1 = 1$ ) of Ozsváth-Szabó’s result to a rational surgery along a knot in a homology 3-sphere.

In Section 2, we recall Turaev’s definition [26] of the Reidemeister torsion, prepare the surgery formula due to Turaev [25, 26] and Sakai [21], and give a precise definition of an oriented closed 3-manifold of lens type. In Section 3, we prove Lemma 1.1 and Main Theorem 1. In Section 4, we prove Main Theorem 2. In Appendix, we show that the Alexander polynomial of  $(p, q)$ -torus knot satisfies Ozsváth-Szabó’s condition only by deformations of the polynomials from the well-known expression.

Lemma 1.1 may have many applications. We applied it in the papers [11, 12] which are joint works with Yuichi Yamada.

**2. Surgery formula of Reidemeister torsion**

**2.1. Torsion of chain complex.** We review the *torsion* of a chain complex.

Let  $V$  be an  $n$ -dimensional vector space over a field  $F$ , and  $\mathbf{b} = (b_1, \dots, b_n)$  and  $\mathbf{c} = (c_1, \dots, c_n)$  two bases of  $V$ . Then  $b_i$  can be expressed by a linear combination of  $c_1, \dots, c_n$ .

$$b_i = \sum_{j=1}^n a_{ij}c_j \quad (i = 1, \dots, n)$$

The matrix  $A = (a_{ij})$  is the transition matrix from  $\mathbf{c}$  to  $\mathbf{b}$ . We denote the determinant of  $A$  by

$$[\mathbf{b}/\mathbf{c}].$$

It is a non-zero element of  $F$ . If  $n = 0$ , then  $[\emptyset/\emptyset] = 1$ .

Let  $\mathbf{C}_*$  be a finitely generated free chain complex over a field  $F$ :

$$\mathbf{C}_* : 0 \rightarrow C_m \xrightarrow{\partial_{m-1}} C_{m-1} \xrightarrow{\partial_{m-2}} \dots \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0 \rightarrow 0$$

In the case that  $C_*$  is *acyclic*, we define the *torsion* of  $C_*$  as follows: Let  $\mathbf{c}_i = (c_i^{(1)}, \dots, c_i^{(p_i)})$  ( $i = 0, \dots, m$ ) be a basis of  $C_i$ . We denote the kernel of  $\partial_{i-1}$  (resp. the image of  $\partial_i$ ) by  $Z_i$  (resp.  $B_i$ ). Then  $Z_i = B_i$  and

$$C_i \cong Z_i \oplus B_{i-1} = B_i \oplus B_{i-1}.$$

We take bases of  $B_i$  as  $\mathbf{b}_i = (b_i^{(1)}, \dots, b_i^{(q_i)})$ . A lift of  $\mathbf{b}_{i-1}$  in  $C_i$  is denoted by  $\tilde{\mathbf{b}}_{i-1}$ . Then  $\mathbf{b}_i \tilde{\mathbf{b}}_{i-1}$  is a basis of  $C_i$ . The *torsion* of  $C_*$  with respect to  $\mathbf{c} = (\mathbf{c}_0, \dots, \mathbf{c}_m)$  is defined by

$$\tau(C_*; \mathbf{c}) = \prod_{i=0}^m [\mathbf{b}_i \tilde{\mathbf{b}}_{i-1} / \mathbf{c}_i]^{(-1)^{i+1}}.$$

The torsion  $\tau(C_*; \mathbf{c})$  is a non-zero element of  $F$ , and does not depend on the choices of  $\mathbf{b}_i$  and  $\tilde{\mathbf{b}}_{i-1}$ . When  $\mathbf{c}$  is clear, we denote  $\tau(C_*; \mathbf{c})$  by  $\tau(C_*)$ . In the other case, i.e., if  $C_*$  is not acyclic, then we define  $\tau(C_*) = 0$  formally. When  $C_*$  is a finitely generated free chain complex over an integral domain  $R$ , we define  $\tau(C_*)$  by

$$\tau(C_*) = \tau(C_* \otimes Q(R)),$$

where  $Q(R)$  is the quotient field of  $R$ .

**2.2. Reidemeister torsion of CW-complex and surgery formula.** We define the *Reidemeister torsion* of a CW-complex, and prepare a surgery formula, which is a main tool of the present paper.

Let  $X$  be a connected finite CW-complex,  $H$  the first homology group  $H_1(X; \mathbf{Z})$ , and  $\mathbf{Z}[H]$  the group ring generated by  $H$  over  $\mathbf{Z}$ . Let  $p: \hat{X} \rightarrow X$  be the maximal abelian covering whose covering transformation group is  $H$ . Then  $\hat{X}$  is also a connected CW-complex whose CW-structure is naturally induced by  $X$ . Let  $C_*(\hat{X})$  be the cellular chain complex of  $\hat{X}$ , where  $C_i(\hat{X})$  is the set of formal linear combinations of oriented  $i$ -cells of  $\hat{X}$  with integer coefficients. Since  $H$  acts on  $\hat{X}$  as the covering transformation group,  $H$  also acts on  $C_*(\hat{X})$ . We may regard  $C_*(\hat{X})$  as a module over  $\mathbf{Z}[H]$ . This module is free. A *fundamental family* of cells is a family of cells in  $\hat{X}$  such that over each cell of  $X$  lies exactly one cell of this family. We can observe that each ordered fundamental family of ordered cells determines basis in  $C_*(\hat{X})$  over  $\mathbf{Z}[H]$ . Let  $F$  be a field, and  $\varphi: \mathbf{Z}[H] \rightarrow F$  a ring homomorphism. We define

$$C_*^\varphi(X) = C_*(\hat{X}) \otimes_{\mathbf{Z}[H]} F,$$

and the *Reidemeister torsion* of  $X$ ,  $\tau^\varphi(X)$ , associated to a ring homomorphism  $\varphi: \mathbf{Z}[H] \rightarrow F$ .

$$\tau^\varphi(X) = \begin{cases} \tau(C_*^\varphi(X)) \in F - \{0\} & \text{if } H_*(C_*^\varphi(X)) = 0, \\ 0 \in F & \text{if } H_*(C_*^\varphi(X)) \neq 0. \end{cases}$$

The invariant  $\tau^\varphi(X)$  is a simple-homotopy invariant determined up to a multiplication of an element in  $\pm\varphi(H)$ . A simple-homotopy invariant is a topological invariant by A. Chapman [4]. For a finite CW-pair  $(X, Y)$ , we can also define  $\tau^\varphi(X, Y)$  associated to  $C_*^\varphi(X, Y) = C_*(\hat{X}, p^{-1}(Y))$ .

The following theorems are fundamental to compute the Reidemeister torsion. We denote the first homology group  $H_1(X; \mathbf{Z})$  of  $X$  over  $\mathbf{Z}$  by  $H_1(X)$  for short.

**Theorem 2.1** (Turaev [26]; the excision theorem). *Let  $X_1$  and  $X_2$  be sub-complexes of  $X$  whose union is  $X$ , and whose intersection is  $Y$ . Let  $j: \mathbf{Z}[H_1(Y)] \rightarrow \mathbf{Z}[H_1(X)]$  and  $j_i: \mathbf{Z}[H_1(X_i)] \rightarrow \mathbf{Z}[H_1(X)]$  ( $i = 1, 2$ ) be homomorphisms induced by the natural inclusions. If  $\tau^{\varphi \circ j}(Y) \neq 0$ , then*

$$\tau^\varphi(X) = \tau^{\varphi \circ j_1}(X_1)\tau^{\varphi \circ j_2}(X_2)[\tau^{\varphi \circ j}(Y)]^{-1}.$$

For example, if  $t$  is a generator of  $H_1(S^1) \cong \mathbf{Z}$ , then  $\tau(S^1) = (t - 1)^{-1}$  and  $\tau(S^1 \times S^1) = 1$ . The following theorem is a special case of more general result [26].

**Theorem 2.2** (Milnor [16], Turaev [26]). *Let  $K$  be a knot in a homology 3-sphere  $\Sigma$ ,  $t$  a generator of  $H_1(\overline{\Sigma - N(K)}) \cong \mathbf{Z}$  where  $N(K)$  is a tubular neighborhood of  $K$ , and  $\Delta_K(t)$  the Alexander polynomial of  $K$ . Then*

$$\tau(\overline{\Sigma - N(K)}) = \Delta_K(t)(t - 1)^{-1}.$$

Since any homology lens space is obtained by a  $p/q$ -surgery along a knot  $K$  in a homology 3-sphere  $\Sigma$ , where  $p \geq 2$  and  $q \neq 0$ , we can compute the Reidemeister torsion of it by Theorem 2.1 and Theorem 2.2. By  $(\mathbf{Z}/n\mathbf{Z})^\times$  we denote the multiplicative group of invertible elements in the ring  $\mathbf{Z}/n\mathbf{Z}$  with respect to the multiplicity. For an element  $x$  of  $(\mathbf{Z}/n\mathbf{Z})^\times$ , we denote the inverse element of  $x$  by  $\bar{x}$ .

**Theorem 2.3** (Turaev [25, 26, 27]; Sakai [21]). *Let  $K$  be a knot in a homology 3-sphere  $\Sigma$ ,  $\Delta_K(t)$  the Alexander polynomial of  $K$ , and  $M = \Sigma(K; p/q)$ , where  $p \geq 2$  and  $q \neq 0$ . Let  $d$  ( $\geq 2$ ) be a divisor of  $p$ ,  $\zeta = \zeta_d$  a primitive  $d$ -th root of unity, and  $\varphi_d: \mathbf{Z}[t, t^{-1}]/(t^p - 1) \rightarrow \mathbf{Q}(\zeta)$  a homomorphism such that  $\varphi_d(t) = \zeta$ . Then the Reidemeister torsion of  $M$ ,  $\tau^{\varphi_d}(M)$ , associated to  $\varphi_d$  is*

$$\tau^{\varphi_d}(M) = \Delta_K(\zeta)(\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1}.$$

**Theorem 2.4** (Reidemeister [20]; Franz [8]). *Let  $L(p, q)$  be the lens space of type  $(p, q)$ ,  $t$  a generator of the first homology group  $H_1(L(p, q))$ ,  $d$  ( $\geq 2$ ) a divisor of  $p$ ,  $\zeta = \zeta_d$  a primitive  $d$ -th root of unity, and  $\varphi_d: \mathbf{Z}[t, t^{-1}]/(t^p - 1) \rightarrow \mathbf{Q}(\zeta)$  a homomorphism such that  $\varphi_d(t) = \zeta$ . Then the Reidemeister torsion of  $L(p, q)$ ,  $\tau^{\varphi_d}(L(p, q))$ , associated to  $\varphi_d$  is*

$$\tau^{\varphi_d}(L(p, q)) = (\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1}.$$

Lens spaces are completely classified by using Theorem 2.4 and Franz’s lemma (see [8] and Section 3). We apply Franz’s lemma to show Main Theorem 1 in Section 3.

**2.3. Closed 3-manifold of lens type.** Let  $M$  be an oriented closed 3-manifold whose first homology group  $H_1(M)$  is a finite cyclic group of order  $p$  (i.e.,  $M$  is a *homology lens space*), and  $t$  a generator of  $H_1(M)$ . Let  $d (\geq 2)$  be a divisor of  $p$ ,  $\zeta$  a primitive  $d$ -th root of unity, and  $\varphi_d: \mathbf{Z}[H_1(M)] \rightarrow \mathbf{Q}(\zeta)$  a ring homomorphism such that  $\varphi_d(t) = \zeta$ . A homology lens space  $M$  is of *lens type* if its Reidemeister torsion  $\tau^{\varphi_d}(M)$  has the form  $(\zeta^i - 1)^{-1}(\zeta^j - 1)^{-1}$  for every  $d$  where  $i$  and  $j$  are coprime to  $p$ , and do not depend on  $d$ . In particular, a homology lens space  $M$  is of  $(p, q)$ -*lens type* if  $\bar{i}j \equiv \pm q$  or  $\pm \bar{q}$  (mod  $p$ ).

It is clear that the lens space  $L(p, q)$  is of  $(p, q)$ -lens type. If a homology lens space of  $(p, q)$ -lens type is a lens space, then it is homeomorphic to  $L(p, \pm q)$  or  $L(p, \pm \bar{q})$ . If a Dehn surgery along a knot in a homology 3-sphere yields a 3-manifold of lens type, then we call it *lens type surgery*. It is clear that a lens surgery is a lens type surgery.

**3. Proof of Main Theorem 1**

In this section we show Main Theorem 1, which states a necessary and sufficient condition for the Reidemeister torsion of  $\Sigma(K; p/q)$  to be of lens type in the case that the Alexander polynomial  $\Delta_K(t)$  of  $K$  is equal to that of the  $(r, s)$ -torus knot.

**3.1. Franz’s lemma and norm of an algebraic number.** We prepare Franz’s lemma and some results about algebraic numbers.

**Theorem 3.1** (Franz [8]). *Let  $\zeta$  be a primitive  $n$ -th root of unity, and  $\{a_i (i \in (\mathbf{Z}/n\mathbf{Z})^\times)\}$  the set of integers satisfying the following conditions:*

- (1)  $a_{-i} = a_i$ ,
- (2)  $\sum_{i \in (\mathbf{Z}/n\mathbf{Z})^\times} a_i = 0$ ,
- (3)  $\prod_{i \in (\mathbf{Z}/n\mathbf{Z})^\times} (\zeta^i - 1)^{a_i} = 1$ .

*Then  $a_i = 0$  for all  $i \in (\mathbf{Z}/n\mathbf{Z})^\times$ .*

Let  $F$  be a finite Galois extension over  $\mathbf{Q}$ , and  $\alpha$  an element of  $F$ . We denote the norm of  $\alpha$  over  $\mathbf{Q}$  by  $N_{F/\mathbf{Q}}(\alpha)$ , or simply  $N(\alpha)$ .

$$N_{F/\mathbf{Q}}(\alpha) = \prod_{\sigma \in \text{Gal}(F/\mathbf{Q})} \sigma(\alpha)$$

The followings are fundamental facts in Number Theory (see [3, p.89], [30]).

**Proposition 3.2.** *In the situation above, we have the followings.*

- (1)  $N(\alpha)$  is a rational number, and  $N(\alpha) = 0$  if and only if  $\alpha = 0$ .
- (2) If  $\alpha$  is an algebraic integer, then  $N(\alpha)$  is an integer.
- (3) An algebraic integer  $\alpha$  is a unit in the ring of algebraic integers if and only if  $N(\alpha) = \pm 1$ .

**3.2. Proof of Lemma 1.1.** If  $\Sigma(K; p/q)$  is of lens type, then there are integers  $i, j$  and  $m$  such that

$$\Delta_K(\zeta)(\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1} = \pm \zeta^m(\zeta^i - 1)^{-1}(\zeta^j - 1)^{-1},$$

where  $i$  and  $j$  are coprime to  $p$ .

By taking the norms of both sides, we have

$$N(\Delta_K(\zeta)) = N(\pm \zeta^m),$$

because

$$N(\zeta - 1) = N(\zeta^{\bar{q}} - 1) = N(\zeta^i - 1) = N(\zeta^j - 1) \neq 0.$$

Since

$$N(\pm \zeta^m) = \begin{cases} \pm 1 & (d = 2), \\ 1 & (d \geq 3), \end{cases}$$

we have the result. □

**3.3. Proof of Main Theorem 1.** Let  $\Delta_{r,s}(t)$  be the Alexander polynomial of the  $(r, s)$ -torus knot

$$\Delta_{r,s}(t) = \frac{(t^{rs} - 1)(t - 1)}{(t^r - 1)(t^s - 1)},$$

$d (\geq 2)$  a divisor of  $p$ ,  $\zeta$  a primitive  $d$ -th root of unity, and  $\varphi_d: \mathbf{Z}[t]/(t^p - 1) \rightarrow \mathbf{Q}(\zeta)$  a ring homomorphism such that  $\varphi_d(t) = \zeta$ . Since  $M = \Sigma(K; p/q)$  is the  $p/q$ -surgery along a knot  $K$  whose Alexander polynomial is  $\Delta_{r,s}(t)$ , we have

$$\tau^{\varphi_d}(M) = \Delta_{r,s}(\zeta)(\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1}$$

by Theorem 2.3. Suppose  $M$  is of lens type, then there are integers  $i, j$  and  $m$  such that

$$\Delta_{r,s}(\zeta)(\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1} = \pm \zeta^m(\zeta^i - 1)^{-1}(\zeta^j - 1)^{-1},$$

where  $i$  and  $j$  are coprime to  $p$ .

Suppose  $\gcd(p, r) \geq 2$ , we take  $d = \gcd(p, r)$ . Then  $\gcd(d, s) = 1$  because  $\gcd(r, s) = 1$ . We set  $p = p'd$  and  $r = r'd$ . Then

$$\begin{aligned} \Delta_{r,s}(t) &= \frac{(t^{rs} - 1)(t - 1)}{(t^r - 1)(t^s - 1)} = \frac{(t^{r'sd} - 1)(t - 1)}{(t^{r'd} - 1)(t^s - 1)} \\ &= (t^{(s-1)r'd} + t^{(s-2)r'd} + \dots + t^{2r'd} + t^{r'd} + 1) \cdot \frac{t - 1}{t^s - 1}, \end{aligned}$$

and therefore

$$N(\Delta_{r,s}(\zeta)) = s^{\varphi(d)}.$$

By Lemma 1.1,  $M$  is not of lens type. Thus we have the conclusion (1).

We assume  $\gcd(p, r) = 1$  and  $\gcd(p, s) = 1$ , and take any divisor  $d (\geq 2)$  of  $p$ . Then

$$\Delta_{r,s}(\zeta) = \frac{(\zeta^{rs} - 1)(\zeta - 1)}{(\zeta^r - 1)(\zeta^s - 1)}.$$

If  $M$  is of lens type, then

$$(\zeta^{rs} - 1)(\zeta^i - 1)(\zeta^j - 1) = \pm \zeta^m (\zeta^{\bar{q}} - 1)(\zeta^r - 1)(\zeta^s - 1).$$

Multiplying the complex conjugates to both sides, we have

$$\begin{aligned} &(\zeta^{rs} - 1)(\zeta^i - 1)(\zeta^j - 1)(\zeta^{-rs} - 1)(\zeta^{-i} - 1)(\zeta^{-j} - 1) \\ &= (\zeta^{\bar{q}} - 1)(\zeta^r - 1)(\zeta^s - 1)(\zeta^{-\bar{q}} - 1)(\zeta^{-r} - 1)(\zeta^{-s} - 1). \end{aligned}$$

By the same argument as the proof of the classification of lens spaces (see [6, 8, 20, 27], and Theorem 3.1),

$$\{rs, i, j\} = \{\bar{q}, r, s\} \quad \text{in } (\mathbf{Z}/d\mathbf{Z})/\{\pm 1\}.$$

There are two cases.

- (i)  $rs \equiv \pm \bar{q} \pmod{d}$ .

This is equivalent to  $qrs \equiv \pm 1 \pmod{d}$ .

- (ii)  $rs \equiv \pm s \pmod{d}$  or  $rs \equiv \pm r \pmod{d}$ .

This is equivalent to  $r \equiv \pm 1 \pmod{d}$  or  $s \equiv \pm 1 \pmod{d}$ .

If  $qrs \equiv \pm 1$  (resp.  $r \equiv \pm 1, s \equiv \pm 1$ )  $\pmod{p}$  holds, then  $qrs \equiv \pm 1$  (resp.  $r \equiv \pm 1, s \equiv \pm 1$ )  $\pmod{d}$  holds for any  $d$ . So we state only the case of  $d = p$ .

The converse is obvious. This completes the proof. □

#### 4. Proof of Main Theorem 2

In this section we show Main Theorem 2, which states a necessary and sufficient condition for the Reidemeister torsion of  $\Sigma(K; p/q)$  to be of lens type in the case that the Alexander polynomial  $\Delta_K(t)$  of  $K$  is of degree 2.



We prepare some lemmas.

Let  $n$  be a positive integer,  $\varphi(n)$  the Euler function,  $\zeta$  a primitive  $n$ -th root of unity, and

$$\Phi_n(x) = \prod_{i \in (\mathbf{Z}/n\mathbf{Z})^\times} (x - \zeta^i)$$

the  $n$ -th cyclotomic polynomial. Then  $\Phi_n(x)$  is an irreducible monic symmetric polynomial over  $\mathbf{Z}$  with degree  $\varphi(n)$ , and

$$\Phi_n(1) = \begin{cases} 0 & (n = 1), \\ p & (n = p^r, p: \text{prime}), \\ 1 & (\text{otherwise}). \end{cases}$$

The Alexander polynomial of a knot in a homology sphere with degree 2 has the following form for some integer  $n \neq 0$ :

$$(4.1) \quad \Delta_n(t) = nt^2 - (2n - 1)t + n = t + n(t - 1)^2 \quad (n \neq 0)$$

Let  $\zeta$  be a primitive  $p$ -th root of unity, and  $\alpha_1$  and  $\alpha_2$  the roots of  $\Delta_n(t) = 0$ . Then we have

$$(4.2) \quad N(\Delta_n(\zeta)) = \prod_{i \in (\mathbf{Z}/p\mathbf{Z})^\times} n(\zeta^i - \alpha_1)(\zeta^i - \alpha_2) = n^{\varphi(p)} \Phi_p(\alpha_1) \Phi_p(\alpha_2)$$

We regard the right-hand side of (4.2) as a polynomial of  $n$  over  $\mathbf{Z}$  depending on  $p$ , denote it by  $f_p(n)$  (i.e.,  $f_p(n) \in \mathbf{Z}[n]$ ), and call it *the  $p$ -th norm polynomial* or simply *the norm polynomial*.

- Lemma 4.1.** (1) *If  $n \leq -1$ , then  $f_p(n) \neq \pm 1$ .*  
 (2) *If  $|n| \geq 2$  and  $p$  is a prime number, then  $f_p(n) \neq \pm 1$ .*

Main Theorem 2 is proved by Lemma 1.1 and Lemma 4.1: By the assumption that  $\Sigma(K; p/q)$  is of lens type, by definition,  $N(\Delta_K(\zeta)) = \pm 1$  holds not only in the  $p$ -th cyclotomic field but also in the  $d$ -th cyclotomic field for any divisor  $d$  of  $p$ . In the case that  $n \geq 2$  and  $p$  is not prime, we study  $f_d(n)$  for a prime divisor  $d$  of  $p$  in Lemma 4.1 (2).

In the case that  $p = 2$ , Lemma 4.1 holds because  $f_2(n) = 4n - 1$ . From now on, we assume  $p \geq 3$ . To show Lemma 4.1, we study properties of  $f_p(n)$ .

- Proposition 4.2.** (1) *The degree of  $f_p(n)$  is  $\varphi(p)$ .*  
 (2) *If  $p \geq 3$ , then there exists a polynomial of  $n$ ,  $g_p(n)$ , over  $\mathbf{Z}$  with degree  $\varphi(p)/2$  such that  $f_p(n) = \{g_p(n)\}^2$ .*

Proof. (1) Since  $\Delta_n(\zeta) = (1 - \zeta)^2 n + \zeta$ , the degree of  $f_p(n)$  is  $\varphi(p)$ .

(2) Firstly we note

$$\Delta_n(\zeta) = \zeta^2 \Delta_n(\zeta^{-1}).$$

From this equation,

$$\delta(\zeta) = \frac{\Delta_n(\zeta)}{\zeta}$$

satisfies  $\delta(\zeta^{-1}) = \delta(\zeta)$ , and  $\delta(\zeta)$  is an element of  $\mathbf{Q}(\zeta + \zeta^{-1})$ . Since  $\zeta \neq \zeta^{-1}$ , we have  $[\mathbf{Q}(\zeta) : \mathbf{Q}(\zeta + \zeta^{-1})] = 2$  and  $[\mathbf{Q}(\zeta + \zeta^{-1}) : \mathbf{Q}] = \varphi(p)/2$ . If we set

$$g_p(n) = N_{\mathbf{Q}(\zeta + \zeta^{-1})/\mathbf{Q}}(\delta(\zeta)),$$

then  $g_p(n)$  is a polynomial of  $n$  over  $\mathbf{Z}$  with degree  $\varphi(p)/2$  such that  $f_p(n) = \{g_p(n)\}^2$ . □

We write down  $f_p(n)$  and  $g_p(n)$  in the following form:

$$f_p(n) = \sum_{i=0}^{\varphi(p)} a_i n^i, \quad g_p(n) = \sum_{j=0}^{\varphi(p)/2} b_j n^j.$$

Let  $F(n) = s_0 n^m + s_1 n^{m+1} + s_2 n^{m+2} + \dots + s_d n^{m+d}$  ( $s_0 \neq 0, s_d \neq 0$ ) be a polynomial of  $n$  over  $\mathbf{R}$ . If (i)  $d = 0$  or (ii)  $d \geq 1$  and  $s_{i-1} s_i < 0$  ( $i = 1, 2, \dots, d$ ), then we say that  $F(n)$  is an *alternating polynomial*. We note that if all roots of  $F(n) = 0$  are positive real numbers or 0, then  $F(n)$  is an alternating polynomial.

**Lemma 4.3.** (1) *The polynomials  $f_p(n)$  and  $g_p(n)$  are alternating polynomials.*

(2)  $a_{\varphi(p)} = \{\Phi_p(1)\}^2$  and  $a_0 = 1$ .

(3)  $b_{\varphi(p)/2} = (-1)^{\varphi(p)/2} \Phi_p(1)$  and  $b_0 = 1$ .

(4)  $a_1 = 2T_{\mathbf{Q}(\zeta)/\mathbf{Q}}(\zeta) - 2\varphi(p)$ , where  $T_{\mathbf{Q}(\zeta)/\mathbf{Q}}(\zeta)$  is the trace of  $\zeta$  in  $\mathbf{Q}(\zeta)/\mathbf{Q}$ . In particular, if  $p$  is an odd prime number, then  $a_1 = -2p$  and  $b_1 = -p$ .

Proof. (1) Firstly we note

$$\delta(\zeta) = \frac{\Delta_n(\zeta)}{\zeta} = 1 - \{2 - (\zeta + \zeta^{-1})\}n.$$

Since  $2 - (\zeta + \zeta^{-1}) > 0$ , the polynomials  $f_p(n)$  and  $g_p(n)$  are alternating polynomials.

(2)  $a_{\varphi(p)} = N_{\mathbf{Q}(\zeta)/\mathbf{Q}}((1 - \zeta)^2) = \{\Phi_p(1)\}^2$  and  $a_0 = N_{\mathbf{Q}(\zeta)/\mathbf{Q}}(\zeta) = 1$ .

(3) It is clear by (1) and (2).

(4) By the definition,

$$\begin{aligned} a_1 &= \sum_{i \in (\mathbf{Z}/p\mathbf{Z})^\times} (1 - \zeta^i)^2 \cdot \frac{N_{\mathbf{Q}(\zeta)/\mathbf{Q}}(\zeta)}{\zeta^i} = \sum_{i \in (\mathbf{Z}/p\mathbf{Z})^\times} (\zeta^i + \zeta^{-i} - 2) \\ &= 2 \sum_{i \in (\mathbf{Z}/p\mathbf{Z})^\times} \zeta^i - 2\varphi(p) = 2T_{\mathbf{Q}(\zeta)/\mathbf{Q}}(\zeta) - 2\varphi(p), \end{aligned}$$

If  $p$  is a prime number, then  $T_{\mathbf{Q}(\zeta)/\mathbf{Q}}(\zeta) = -1$  and  $\varphi(p) = p - 1$ . Therefore  $a_1 = -2p$ . This completes the proof. □

Proof of Lemma 4.1 (1). Assume  $n \leq -1$ . From Lemma 4.3 (1) and (2), we see

$$f_p(n) \geq \{\Phi_p(1)\}^2 + 1 \geq 2.$$

This completes the proof. □

To prove Lemma 4.1 (2), we need the following lemma.

**Lemma 4.4.** (1) *If  $p$  is a prime number, then*

$$f_p(n) = n^p (\alpha_1^p - 1)(\alpha_2^p - 1),$$

where  $\alpha_1$  and  $\alpha_2$  are roots of  $\Delta_n(t)$  in (4.1).

(2) *If  $p$  is an odd prime number, then  $b_j \equiv 0 \pmod{p}$  for  $j = 1, 2, \dots, \varphi(p)/2$ .*

Proof. (1) From  $\varphi(p) = p - 1$  and  $t^p - 1 = (t - 1)\Phi_p(t)$  if  $p$  is a prime, we have

$$\frac{n^p(\alpha_1^p - 1)(\alpha_2^p - 1)}{n^{\varphi(p)}\Phi_p(\alpha_1)\Phi_p(\alpha_2)} = n(\alpha_1 - 1)(\alpha_2 - 1) = \Delta_n(1) = 1.$$

By (4.2), we have the equality.

(2) Let  $(p)$  be an ideal in the polynomial ring  $\mathbf{Z}[n]$  generated by  $p$ , and  $\alpha_1$  and  $\alpha_2$  roots of the Alexander polynomial  $\Delta_n(t)$  with degree 2 in (4.2). Since  $\Delta_n(t)$  is a polynomial over  $\mathbf{Z}$  and  $p$  is an odd prime number, we have

$$n^p(\alpha_1 - 1)^p(\alpha_2 - 1)^p \equiv n^p(\alpha_1^p - 1)(\alpha_2^p - 1) \pmod{(p)}.$$

By (1),

$$f_p(n) \equiv n^p(\alpha_1 - 1)^p(\alpha_2 - 1)^p = \{n(\alpha_1 - 1)(\alpha_2 - 1)\}^p = 1 \pmod{(p)}.$$

Since  $\mathbf{Z}[n]/(p) = (\mathbf{Z}/p\mathbf{Z})[n]$  is a unique factorization domain,

$$g_p(n) \equiv b_0 = 1 \pmod{(p)}.$$

This means  $b_j \equiv 0 \pmod{p}$  for  $j = 1, 2, \dots, \varphi(p)/2$ . □

Proof of Lemma 4.1 (2). Let  $p$  be an odd prime number, and  $h_p(n)$  a polynomial of  $n$  satisfying

$$g_p(n) = pn h_p(n) + 1.$$

Then  $h_p(n)$  is a polynomial over  $\mathbf{Z}$  by Lemma 4.4 (2), and  $f_p(n) = 1$  if and only if  $h_p(n) = 0$ . We write down

$$h_p(n) = \sum_{k=0}^{\varphi(p)/2-1} c_k n^k.$$

By Lemma 4.3 (3) and (4),  $c_{\varphi(p)/2-1} = \pm 1$  and  $c_0 = -1$ . From this, if  $h_p(n) = 0$ , then  $n = \pm 1$ . Therefore if  $|n| \geq 2$ , then  $h_p(n) \neq 0$ . This completes the proof of Lemma 4.1 (2).  $\square$

For example,  $h_3(n) = -1$ ,  $h_5(n) = n - 1$ ,  $h_7(n) = -(n - 1)^2$ ,  $h_{11}(n) = -(n - 1)(n^3 - 4n^2 + 3n - 1)$ .

**Corollary 4.5.** *Let  $K$  be a knot in a homology 3-sphere  $\Sigma$ ,  $\Delta_K(t)$  the Alexander polynomial of  $K$ , and  $M = \Sigma(K; p/q)$  the  $p/q$ -surgery for  $p \geq 2$  and  $q \neq 0$ . If  $\Delta_K(t)$  is divisible by  $nt^2 - (2n - 1)t + n$  ( $n \in \mathbf{Z}; n \neq 0, 1$ ), then  $M$  is not of lens type.*

**Appendix**

We introduce a result in [18] due to Ozsváth and Szabó, which is a necessary condition on the Alexander polynomial of a knot in  $S^3$  which yields a lens space. They show it by using knot Floer homology ([13, 18, 19]).

**Theorem** [Ozsváth-Szabó [18]]. *Let  $K$  be a knot in  $S^3$ , and  $M = (K; p)$ , where  $p$  is an integer. If  $M$  is a lens space, then the Alexander polynomial of  $K$  is of the following form*

$$\Delta_K(t) = (-1)^m + \sum_{j=1}^m (-1)^{m-j} (t^{s_j} + t^{-s_j}),$$

where  $0 < s_1 < s_2 < \dots < s_m$ .

By Moser’s theorem (Theorem 1.2), the Alexander polynomial of a torus knot satisfies the condition above. We can check it easily as follows.

**Proposition.** *The Alexander polynomial of a torus knot has the form in Ozsváth-Szabó’s theorem.*

Proof. Let  $\Delta_{p,q}(t)$  be the Alexander polynomial of  $(p, q)$ -torus knot. We may assume  $2 \leq q \leq p$ . There are integers  $c_k$  and  $d_k$  ( $k = 0, 1, \dots, q - 1$ ) such that  $pk = qc_k + d_k$  ( $0 \leq d_k \leq q - 1$ ). The integers  $d_0, \dots, d_{q-1}$  are mutually distinct, because  $p$  and  $q$  are coprime integers. It is clear that  $c_0 = d_0 = 0$ .

We list the following formulas which are proved easily.

$$(1) \quad \Delta_{p,q}(t) = \frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)} = \frac{(t^{p(q-1)} + t^{p(q-2)} + \dots + t^p + 1)(t - 1)}{t^q - 1}$$

$$(2) \quad t^{pk} = t^{d_k}(t^{qc_k} - 1) + t^{d_k}$$

$$(3) \quad \frac{t^{pk}(t - 1)}{t^q - 1} = t^{d_k}(t^{q(c_k-1)} + t^{q(c_k-2)} + \dots + t^q + 1)(t - 1) + \frac{t^{d_k}(t - 1)}{t^q - 1}$$

$$(4) \quad \sum_{k=0}^{q-1} \frac{t^{d_k}(t - 1)}{t^q - 1} = \frac{(t^{q-1} + t^{q-2} + \dots + t + 1)(t - 1)}{t^q - 1} = 1$$

$$(5) \quad \Delta_{p,q}(t) = 1 + \sum_{k=1}^{q-1} \sum_{l=0}^{c_k-1} (t^{ql+d_k+1} - t^{ql+d_k})$$

Equation (5) is obtained from equations (1), (2), (3) and (4). If two pairs  $(k, l)$  and  $(k', l')$  are distinct, then two numbers  $ql + d_k$  and  $q'l' + d_{k'}$  are distinct. Therefore  $\Delta_{p,q}(t)$  has the form in Ozsváth-Szabó's theorem.  $\square$

In [12], we characterized the Alexander polynomial of a knot in any homology 3-sphere having a lens type surgery. Infinitely many knots in  $S^3$  having lens surgery appear as certain families or sequences (see [1]), and no counterexample is discovered. The author thinks such a deformation of the Alexander polynomial above is related to the structure of each family or sequence not only in  $S^3$  but also in any homology 3-sphere.

ACKNOWLEDGEMENT. The author would like to thank Akio Kawauchi for suggesting him to study the Reidemeister torsion theory, Hitoshi Murakami for giving him a chance to make a lecture notes about the lecture of V.G. Turaev at RIMS in 2000 [28], Makoto Sakuma and Ikuo Tayama for giving him useful advices at private seminars, Tsuyoshi Sakai for motivating him the present result, Hiroshi Goda and Kazuhiro Ichihara for informing him about a recent work of P. Kronheimer, T. Mrowka, P. Ozsváth and Z. Szabó [13], Taizo Kanenobu for informing him about C. Gordon's result [10], Yuichi Yamada for giving him useful advices, and Kunio Murasugi for reading carefully a draft version.

This paper is supported by the 21 COE program "Constitution of wide-angle mathematical basis focused on knots."

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