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LOWER BOUNDS OF THE LIFESPAN OF SMALL DATA SOLUTIONS TO THE NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract

Let T_ε be the lifespan of solutions to the initial value problem for the one dimensional, derivative nonlinear Schrödinger equations with small initial data of size $O(\varepsilon)$. If the nonlinear term is cubic and gauge invariant, it is known that $\liminf_{\varepsilon\to+0}\varepsilon^2\log T_\varepsilon$ is positive. In this paper we obtain a sharp estimate of this lower limit, which is explicitly computed from the initial data and the nonlinear term.

1. Introduction

We consider the initial value problem for the nonlinear Schrödinger equation of the following type:

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\partial_x^2 u = F(u, \partial_x u), & t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = \varepsilon\varphi(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where u is a complex-valued unknown function of $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, $i = \sqrt{-1}$, $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, $\varepsilon \in\,]0, 1]$ and φ is a complex-valued smooth function which decays sufficiently fast as $|x| \to \infty$. We will occasionally write u_x for $\partial_x u$, and $\overline{u}$ denotes the complex conjugate of u. The nonlinear term $F(u, u_x)$ is a cubic homogeneous polynomial in $(u, \overline{u}, u_x, \overline{u}_x)$ with complex coefficients, and it satisfies so-called gauge invariance, that is,

$$F\left(e^{i\theta}u, e^{i\theta}q\right) = e^{i\theta}F(u, q) \quad \text{for} \quad u, q \in \mathbb{C} \quad \text{and} \quad \theta \in \mathbb{R}.$$

This paper is devoted to the study of the lifespan of solutions to (1.1). Roughly speaking, lifespan is the maximal existence time of solutions. Precise estimates of the lifespan sometimes enable us to know how the nonlinearity affects large time behavior of solutions to nonlinear evolution equations. As is well known, for quasilinear wave equations, there are surprising connections between the lifespan and the “null condi-

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tion" (see [1], [2], [3], [4], [5], [8], [11], [19], [20], [21], [22], [23], [24], [28], etc.). What we intend to do for (1.1) is an analogue of them.

Now, let us give a precise definition of the lifespan $T_\varepsilon = T_\varepsilon(\varphi, F)$ which we discuss. T_ε is the supremum of all $T > 0$ such that there exists a unique solution $u \in C([0, T[; H^3)$ of (1.1), where H^m denotes the standard Sobolev space of order m. Since the local existence is well known for H^3 data (see e.g., [26], [6], [18] and Appendix of [25]), we see that $T_\varepsilon > 0$ for any $\varepsilon > 0$. In [25], S. Katayama and Y. Tsutsumi proved that $T_\varepsilon \geq \exp(C/\varepsilon^2)$ with some positive constant C which is independent of ε, provided that ε is small enough (see also Section 5 of [12]). In other words, we know that

$$\liminf_{\varepsilon \to +0} \varepsilon^2 \log T_\varepsilon > 0.$$

Note that in the case of higher space dimensions, small data global existence (i.e. $T_\varepsilon = \infty$ for sufficiently small ε) was established by H. Chihara [7].

The aim of this paper is to obtain a more precise estimate of the lower bound of the lifespan, which is explicitly computed from φ and F. Before stating our main result, let us define $A \in \mathbb{R} \cup \{+\infty\}$ by

$$\frac{1}{A} = \sup_{\xi \in \mathbb{R}} \bigl(2|\hat{\varphi}(\xi)|^2 \operatorname{Im} F(1, i\xi)\bigr) \tag{1.2}$$

(we associate $1/A = 0$ with $A = +\infty$), where $\hat{\varphi}(\xi)$ denotes the Fourier transform of $\varphi(x)$:

$$\hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iy\xi} \varphi(y)\, dy.$$

Remark that A is strictly positive or $+\infty$ if $\varphi(x)$ is sufficiently smooth (equivalently, if $\hat{\varphi}(\xi)$ decays sufficiently fast as $|\xi| \to \infty$). The main result is as follows:

Theorem 1.1. *Let F be a cubic homogeneous polynomial in $(u, \overline{u}, u_x, \overline{u}_x)$ with gauge invariance and let $\varphi \in L^2$ such that $\sum_{j+k \leq m} \|x^j \partial_x^k \varphi\|_{L^2} < \infty$ for sufficiently large $m \in \mathbb{N}$. Denote by T_ε the lifespan of solutions to* (1.1). *Then we have*

$$\liminf_{\varepsilon \to +0} \varepsilon^2 \log T_\varepsilon \geq A,$$

where $A \in]0, \infty]$ is given by (1.2).

REMARK 1.1. This is an NLS analogue of F. John and L. Hörmander's result concerning quasilinear wave equations (see Theorem 1 of [22] and Theorem 2.4.4 of [19]).

REMARK 1.2. In the above assertion, if φ belongs to the Schwartz class $\mathcal{S}$, i.e.

$$\sup_{x\in\mathbb{R}}\left|x^j \partial_x^k \varphi(x)\right| < \infty$$

for any $j, k \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, then we can show that $u(t, \cdot\,) \in \mathcal{S}$ for all $t \in \left[0, e^{A/\varepsilon^2}\right[$.

REMARK 1.3. It is troublesome to specify the minimal value of m in Theorem 1.1 because our proof is based on construction of an approximate solution which requires regularity and decay of φ in several steps (see Section 3). Actually, we can check that $m \geq 7$ is enough for our proof, but it is not our main purpose here.

As a consequence of Theorem 1.1 we see that, if $\operatorname{Im} F(1, i\xi)$ vanishes identically, the lifespan must be much longer than that we can expect in general. Here we focus our attentions on this case. Without loss of generality, a gauge-invariant cubic term F can be written as

$$F(u, q) = \lambda_1|u|^2 u + \lambda_2|u|^2 q + \lambda_3 u^2\overline{q} + \lambda_4 u|q|^2 + \lambda_5\overline{u}q^2 + \lambda_6|q|^2 q \tag{1.3}$$

with constants $\lambda_1, \ldots, \lambda_6 \in \mathbb{C}$. For this F, it holds that

$$F(1, i\xi) = \lambda_1 + i(\lambda_2 - \lambda_3)\xi + (\lambda_4 - \lambda_5)\xi^2 + i\lambda_6\xi^3,$$

whence

$$\operatorname{Im} F(1, i\xi) = \operatorname{Im}\lambda_1 + \operatorname{Re}(\lambda_2 - \lambda_3)\xi + \operatorname{Im}(\lambda_4 - \lambda_5)\xi^2 + \operatorname{Re}\lambda_6\xi^3$$

and

$$\operatorname{Re} F(1, i\xi) = \operatorname{Re}\lambda_1 - \operatorname{Im}(\lambda_2 - \lambda_3)\xi + \operatorname{Re}(\lambda_4 - \lambda_5)\xi^2 - \operatorname{Im}\lambda_6\xi^3$$

for $\xi \in \mathbb{R}$. We see from this expression that $\operatorname{Im} F(1, i\xi)$ vanishes identically if and only if

$$\lambda_1,\ i(\lambda_2 - \lambda_3),\ (\lambda_4 - \lambda_5) \quad \text{and} \quad i\lambda_6 \quad \text{are real.} \tag{1.4}$$

It should be remarkable that the same condition as (1.4) is found in the work of N. Hayashi, P.I. Naumkin and H. Uchida [16]. Under the assumption (1.4), they proved that the small amplitude solution exists globally in time and it behaves like

$$\frac{1}{\sqrt{it}}W\left(\frac{x}{t}\right)\exp\left(\frac{ix^2}{2t} + i\Lambda\left(\frac{x}{t}\right)\left|W\left(\frac{x}{t}\right)\right|^2\log t\right) + o\big(t^{-1/2}\big)$$

as $t \to \infty$ unifomly in $x \in \mathbb{R}$, where $\Lambda(\xi) = -\operatorname{Re} F(1, i\xi)$ and $W \in L^\infty$ (see also [29], [17], [30], [13], [15], etc.).

We also note that the "null gauge condition of order 3" studied in [25], [34] is equivalent to

$$F(1, i\xi) = 0 \quad \text{for any} \quad \xi \in \mathbb{R}.$$

Indeed, $F(1, i\xi)$ vanishes identically if and only if

$$\lambda_1 = \lambda_6 = 0, \lambda_2 = \lambda_3 \quad \text{and} \quad \lambda_4 = \lambda_5,$$

which is nothing but the condition that $F(u, \partial_x u)$ is of the form $(\lambda u + \mu\partial_x u)\partial_x(|u|^2)$, where $\lambda, \mu \in \mathbb{C}$. Remark that

$$(\lambda u + \mu\partial_x u)\partial_x\big(|u|^2\big) = \frac{1}{it}(\lambda u + \mu\partial_x u)\big(\overline{u}Ju - u\overline{Ju}\big),$$

where $J = x + it\partial_x$. Using this extra time-decay property, S. Katayama and Y. Tsutsumi [25] succeeded in proving the small solution exists globally and it behaves like a free solution.

It will be also interesting to consider what happens after $t = \exp(A/\varepsilon^2)$ if $A < \infty$, whose typical example is the case where $F = \lambda|u|^2 u$ with $\operatorname{Im}\lambda > 0$ and φ does not identically vanish. Some remarks on this case will be given in Section 5.

We close this section by summarizing our strategy. We first construct a suitable approximate solution and get an a priori estimate for the difference between exact and approximate solutions instead of the solution itself. The constant A, our lower bound, comes from the blow-up time of the approximate solution which is obtained by solving a simple ODE (see §3.2). Such an approach is originated by Hörmander [19] and John [22] cited before (see also [1], [9], [20], [23], etc.).

2. Preliminaries

In this section, we prepare several lemmas which are useful in the proof of Theorem 1.1. In what follows, we will denote several positive constants by C, which may be different line by line. If we want to emphasize that C depends on some parameter γ, we may write it C_γ.

Lemma 2.1. *Let $P : \mathbb{C}^2 \to \mathbb{C}$. If P satisfies*

$$P(rz, rw) = r^3 P(z, w) \quad \text{and} \quad P\big(e^{i\theta}z, e^{i\theta}w\big) = e^{i\theta}P(z, w)$$

for any $z, w \in \mathbb{C}$, $r > 0$ and $\theta \in \mathbb{R}$, then we have

$$P(z, zw) = P(1, w)|z|^2 z$$

for any $z, w \in \mathbb{C}$.

Proof. For $z \in \mathbb{C}\backslash\{0\}$, we put $z = re^{i\theta}$ ($r > 0$, $\theta \in \mathbb{R}$). Then

$$\begin{aligned} P(z, zw) &= P\left(re^{i\theta}, re^{i\theta}w\right) \\ &= P(r, rw)e^{i\theta} \\ &= P(1, w)r^3e^{i\theta} \\ &= P(1, w)|z|^2 z. \end{aligned}$$

When $z = 0$, the conclusion is trivial since $P(0, 0) = 0$. □

Lemma 2.2. *Let $J = x + it\partial_x$. We have*

$$[\partial_x, J] = 1 \quad \text{and} \quad \left[i\partial_t + \frac{1}{2}\partial_x^2, J\right] = 0, \tag{2.1}$$

where $[\,\cdot\,,\,\cdot\,]$ denotes the commutator, i.e. $[L, M] = LM - ML$ for linear operators L and M. Also we have

$$\|f(t,\,\cdot\,)\|_{L^\infty} \le \frac{C}{\sqrt{t}}\|f(t,\,\cdot\,)\|_{L^2}^{1/2}\|Jf(t,\,\cdot\,)\|_{L^2}^{1/2}, \tag{2.2}$$

$$\|f(t,\cdot)\|_{L^2} \le C\sqrt{1+t}\sum_{j+k\le 1}\left\|\partial_x^j J^k f(t,\,\cdot\,)\right\|_{L^\infty} \tag{2.3}$$

and

$$J\left(fg\overline{h}\right) = (Jf)g\overline{h} + f(Jg)\overline{h} - fg\left(\overline{Jh}\right) \tag{2.4}$$

for smooth functions $f(t, x)$, $g(t, x)$ and $h(t, x)$.

Proof. We prove only (2.3) since others are standard (see e.g., [25], [34]). We observe that

$$\frac{x}{t} = -i\partial_x + \frac{1}{t}J,$$

which leads to

$$\begin{aligned} \|f(t,\,\cdot\,)\|_{L^2}^2 &\le \left(\int_{-\infty}^{\infty}\frac{dx}{1+(x/t)^2}\right)\left(\|f(t,\,\cdot\,)\|_{L^\infty}^2 + \left\|\left(-i\partial_x + \frac{1}{t}J\right)f(t,\,\cdot\,)\right\|_{L^\infty}^2\right) \\ &\le \pi t\|f(t,\,\cdot\,)\|_{L^\infty}^2 + 2\pi t\|\partial_x f(t,\,\cdot\,)\|_{L^\infty}^2 + 2\pi\|Jf(t,\,\cdot\,)\|_{L^\infty}^2 \end{aligned}$$

for $t \ge 1$. When $t \in [0, 1]$, use the relation $x = J - it\partial_x$ instead. □

REMARK 2.1. Using (2.2) and the standard Gagliardo-Nirenberg-Sobolev inequality $\|f\|_{L^\infty} \le C\|f\|_{L^2}^{1/2}\|\partial_x f\|_{L^2}^{1/2}$, we have

$$\|f(t,\cdot)\|_{L^\infty} \le \frac{C}{\sqrt{1+t}} \sum_{j+k\le 1} \left\|\partial_x^j J^k f(t,\cdot)\right\|_{L^2}. \tag{2.5}$$

REMARK 2.2. The identity (2.4) is the Leibniz rule for the operator J acting on cubic gauge-invariant terms. This should be compared with

$$J(fg) = \frac{1}{2}\{(Jf)g + f(Jg)\} + \frac{it}{2}\{(\partial_x f)g + f(\partial_x g)\},$$

which implies action of J causes loss of time-decay in general (cf. [12], [14], [15], etc.).

Lemma 2.3 (Energy inequality). *Let $\psi(t,x)$ be a smooth function satisfying*

$$i\partial_t\psi + \frac{1}{2}\partial_x^2\psi + b_1(t,x)\partial_x\psi + b_2(t,x)\partial_x\overline{\psi} = f(t,x)$$

for $(t,x)\in[0,T]\times\mathbb{R}$ with some smooth functions $b_1(t,x)$, $b_2(t,x)$, $f(t,x)$ and some $T>0$. Then we have

$$\frac{d}{dt}\left\|e^{p(t,\cdot)}\psi(t,\cdot)\right\|_{L^2} \le C\beta(t)\left\|e^{p(t,\cdot)}\psi(t,\cdot)\right\|_{L^2} + \left\|e^{p(t,\cdot)}f(t,\cdot)\right\|_{L^2}$$

for $t\in[0,T]$, where

$$p(t,x) = \int_{-\infty}^{x} \mathrm{Re}\, b_1(t,y)\,dy, \tag{2.6}$$

$$\beta(t) = \sum_{j=1}^{2}\left(\|\partial_x b_j(t,\cdot)\|_{L^\infty} + \|b_j(t,\cdot)\|_{L^\infty}^2\right) + \sup_{x\in\mathbb{R}}\left|\partial_t\int_{-\infty}^{x}\mathrm{Re}\, b_1(t,y)\,dy\right|. \tag{2.7}$$

Proof of Lemma 2.3 can be found in Section 2 of [25] (see also [6] and [18]).

Lemma 2.4. *Let $\psi_1(t,x)$, $\psi_2(t,x)$ be smooth functions decaying sufficiently fast as $|x|\to\infty$. Then we have*

$$\begin{aligned}\sup_{x\in\mathbb{R}}\left|\partial_t\int_{-\infty}^{x}\psi_1(t,y)\overline{\psi_2(t,y)}\,dy\right| &\le \frac{C}{1+t}\sum_{l=1}^{2}\sum_{j+k\le 2}\left\|\partial_x^j J^k\psi_l(t,\cdot)\right\|_{L^2}^2\\ &\quad + C\left(\sum_{l=1}^{2}\|\psi_l(t,\cdot)\|_{L^2}\right)\left(\sum_{l=1}^{2}\left\|\left(i\partial_t + \frac{1}{2}\partial_x^2\right)\psi_l(t,\cdot)\right\|_{L^2}\right).\end{aligned}$$

Proof. Lemma 2.4 follows from the identity

$$\partial_t\big(\psi_1\overline{\psi_2}\big) = \frac{i}{2}\partial_x\big(\psi_2\overline{\partial_x\psi_1} - \psi_1\overline{\partial_x\psi_2}\big) + i\,\psi_1\overline{\left(i\partial_t + \frac{1}{2}\partial_x^2\right)\psi_2} - i\,\overline{\psi_2}\left(i\partial_t + \frac{1}{2}\partial_x^2\right)\psi_1$$

combined with (2.5) (cf. Lemma 2.6 of [25]) . □

3. Construction of an approximate solution

In what follows, we shall use the following notations: $Z = (\partial_x, J)$, $Z^\alpha = \partial_x^{\alpha_1}J^{\alpha_2}$ for a multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$. Also, we suppose $\varphi \in \mathcal{S}$ for simplicity. The goal of this section is to construct a smooth function $u_a(t,x)$ defined on $\big[0, e^{A/\varepsilon^2}\big[\times \mathbb{R}$, which satisfies

$$u_a(0,x) = \varepsilon\varphi(x), \tag{3.1}$$

$$\sup_{t\in[0,e^{B/\varepsilon^2}]} \|Z^\alpha u_a(t,\,\cdot\,)\|_{L^2} \le C_{\alpha,B}\,\varepsilon \tag{3.2}$$

and

$$\int_0^{e^{B/\varepsilon^2}} \|Z^\alpha R(t,\,\cdot\,)\|_{L^2}\,dt \le C_{\alpha,B}\,\varepsilon^2 \tag{3.3}$$

for any $\alpha \in \mathbb{N}_0^2$ and $B \in\,]0, A[$, where

$$R(t,x) = \left(i\partial_t + \frac{1}{2}\partial_x^2\right)u_a - F(u_a, \partial_x u_a).$$

The construction is divided into three steps.

REMARK 3.1. In view of (2.3) and (2.5), we see that (3.2) is equivalent to

$$\sup_{t\in[0,e^{B/\varepsilon^2}]} \sqrt{1+t}\,\|Z^\alpha u_a(t,\,\cdot\,)\|_{L^\infty} \le C_{\alpha,B}\,\varepsilon \tag{3.4}$$

for any $\alpha \in \mathbb{N}_0^2$ and $B \in\,]0, A[$.

REMARK 3.2. As we shall see below, our approximate solution $u_a(t,x)$ is equal to

$$\frac{\varepsilon\hat{\varphi}(x/t)}{\sqrt{it}}\,\frac{\exp(ix^2/2t + iG(\varepsilon^2\log t, x/t))}{\sqrt{1 - 2\,\mathrm{Im}\,F(1, ix/t)|\hat{\varphi}(x/t)|^2\varepsilon^2\log t}} \tag{3.5}$$

when $2/\varepsilon \le t < \exp(A/\varepsilon^2)$, where

$$G(s,\xi) = -\,\mathrm{Re}\,F(1, i\xi)|\hat{\varphi}(\xi)|^2\int_0^s \frac{d\sigma}{1 - 2\,\mathrm{Im}\,F(1, i\xi)|\hat{\varphi}(\xi)|^2\sigma}. \tag{3.6}$$

Note that (3.5) blows up as $t \to \exp(A/\varepsilon^2)$ if $A < \infty$. That is the reason why A is given by (1.2). We also remark that, if $\operatorname{Im} F(1, i\xi)$ vanishes identically, (3.5) reduces

$$\frac{1}{\sqrt{it}} W\left(\frac{x}{t}\right) \exp\left(\frac{ix^2}{2t} + i\Lambda\left(\frac{x}{t}\right)\left|W\left(\frac{x}{t}\right)\right|^2 \log t\right),$$

where $\Lambda(\xi) = -\operatorname{Re} F(1, i\xi)$ and $W(\xi) = \varepsilon\hat{\varphi}(\xi)$.

3.1. First step: free evolution. First we consider the free Schrödinger equation:

$$\begin{cases} \left(i\partial_t + \dfrac{1}{2}\partial_x^2\right) u_0 = 0, & (t, x) \in [0, \infty[\times \mathbb{R}, \\ u_0(0, x) = \varepsilon\varphi(x), & x \in \mathbb{R}. \end{cases}$$

It follows from the commutation relations (2.1) and L^2-conservation that

$$\left\|\partial_x^j J^k u_0(t, \cdot)\right\|_{L^2} = \varepsilon\left\|\partial_x^j x^k \varphi\right\|_{L^2}$$

for any $j, k \in \mathbb{N}_0$. Also, since $u_0(t, x)$ is explicitly written as

$$\begin{aligned} u_0(t, x) &= \frac{1}{\sqrt{2\pi it}} \int_{-\infty}^{\infty} e^{i(x-y)^2/2t} \varepsilon\varphi(y)\, dy \\ &= \frac{\varepsilon e^{ix^2/2t}}{\sqrt{2\pi it}} \int_{-\infty}^{\infty} e^{-i\frac{x}{t}\cdot y} e^{iy^2/2t} \varphi(y)\, dy, \end{aligned}$$

we see that

$$\begin{aligned} \left|u_0(t, x) - \frac{\varepsilon e^{ix^2/2t}}{\sqrt{it}}\hat{\varphi}\left(\frac{x}{t}\right)\right| &= \frac{\varepsilon}{\sqrt{2\pi t}}\left|\int_{-\infty}^{\infty} e^{-ixy/t}\left(e^{iy^2/2t} - 1\right)\varphi(y)\, dy\right| \\ &\le \frac{\varepsilon}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \frac{y^2}{2t}|\varphi(y)|\, dy \\ &\le C\varepsilon t^{-3/2}, \end{aligned}$$

which implies the free solution $u_0(t, x)$ behaves like

$$\frac{\varepsilon e^{ix^2/2t}}{\sqrt{t}} e^{-i\pi/4}\hat{\varphi}\left(\frac{x}{t}\right)$$

in the large time. Similarly we have

$$\left|\partial_x^j J^k u_0(t, x) - \frac{\varepsilon e^{ix^2/2t}}{\sqrt{it}}\hat{\varphi}_{j,k}\left(\frac{x}{t}\right)\right| \le C_{j,k}\varepsilon t^{-3/2}$$

for any $j, k \in \mathbb{N}_0$, where $\hat{\varphi}_{j,k}(\xi) = i^{j+k}\xi^j \partial_\xi^k \hat{\varphi}(\xi)$.

3.2. Second step: nonlinear correction in the large time. Let $V(s,\xi)$ be the solution of

$$\begin{cases} i\partial_s V = F(1,i\xi)|V|^2V, & (s,\xi)\in[0,A[\times\mathbb{R}, \\ V(0,\xi)=e^{-i\pi/4}\hat{\varphi}(\xi), & \xi\in\mathbb{R} \end{cases} \tag{3.7}$$

and set

$$U(t,x)=\frac{\varepsilon e^{ix^2/2t}}{\sqrt{t}}V\left(\varepsilon^2\log t,\frac{x}{t}\right),$$

$$Q(t,x)=\left(i\partial_t+\frac{1}{2}\partial_x^2\right)U(t,x)-F(U(t,x),\partial_x U(t,x))$$

for $(t,x)\in\left[1,e^{A/\varepsilon^2}\right[\times\mathbb{R}$. Note that (3.7) is solved explicitly for $s\in[0,A[$:

$$V(s,\xi)=\frac{\exp\left(iG(s,\xi)-i\pi/4\right)}{\sqrt{1-2\,\mathrm{Im}\,F(1,i\xi)\left|\hat{\varphi}(\xi)\right|^2 s}}\hat{\varphi}(\xi),$$

where $G(s,\xi)$ is given by (3.6). In particular, we can check that

$$\sup_{(s,\xi)\in[0,B]\times\mathbb{R}}\left|\xi^j\partial_\xi^k V(s,\xi)\right|<\infty$$

for any $B\in\,]0,A[$ and $j,k\in\mathbb{N}_0$, while

$$\sup_{\xi\in\mathbb{R}}|V(s,\xi)|\to\infty \quad \text{as} \quad s\to A$$

if $A<\infty$. Also, since

$$\partial_x^j J^k U(t,x)=\frac{\varepsilon e^{ix^2/2t}}{\sqrt{t}}i^{j+k}\left(\xi+\frac{1}{it}\partial_\xi\right)^j\partial_\xi^k V|_{(s,\xi)=(\varepsilon^2\log t,x/t)},$$

we have

$$\sup_{t\in[1,e^{B/\varepsilon^2}]}\sqrt{1+t}\,\|Z^\alpha U(t,\cdot)\|_{L^\infty}\le C_{\alpha,B}\,\varepsilon$$

for any $B\in\,]0,A[$ and $\alpha\in\mathbb{N}_0^2$. Moreover, it follows from (2.3) that

$$\sup_{t\in[1,e^{B/\varepsilon^2}]}\|Z^\alpha U(t,\cdot)\|_{L^2}\le C_{\alpha,B}\,\varepsilon.$$

Next we calculate

$$\begin{aligned}\left(i\partial_t+\frac{1}{2}\partial_x^2\right)U(t,x)&=i\frac{\varepsilon e^{ix^2/2t}}{\sqrt{t}}\left(-\frac{ix^2}{2t^2}-\frac{1}{2t}-\frac{x}{t^2}\partial_\xi+\frac{\varepsilon^2}{t}\partial_s\right)V\left(\varepsilon^2\log t,\frac{x}{t}\right)\\&\quad+\frac{1}{2}\frac{\varepsilon e^{ix^2/2t}}{\sqrt{t}}\left(-\frac{x^2}{t^2}+\frac{i}{t}+\frac{2ix}{t^2}\partial_\xi+\frac{1}{t^2}\partial_\xi^2\right)V\left(\varepsilon^2\log t,\frac{x}{t}\right)\end{aligned}$$

$$= \frac{\varepsilon^3 e^{ix^2/2t}}{t^{3/2}} i\partial_s V\left(\varepsilon^2 \log t, \frac{x}{t}\right) + \frac{\varepsilon e^{ix^2/2t}}{2t^{5/2}} \partial_\xi^2 V\left(\varepsilon^2 \log t, \frac{x}{t}\right)$$

and

$$\begin{aligned}\frac{\varepsilon^3 e^{ix^2/2t}}{t^{3/2}} i\partial_s V\left(\varepsilon^2 \log t, \frac{x}{t}\right) &= \frac{\varepsilon^3 e^{ix^2/2t}}{t^{3/2}} F\left(1, i\frac{x}{t}\right)|V|^2 V \\ &= F\left(1, i\frac{x}{t}\right)|U|^2 U \\ &= F\left(U, i\frac{x}{t}U\right) \\ &= F\left(U, \partial_x U + \frac{i}{t}JU\right).\end{aligned}$$

Here we have used Lemma 2.1. Substituting them into the definition of Q, we have

$$Q(t,x) = \left\{F\left(U, \partial_x U + \frac{i}{t}JU\right) - F(U, \partial_x U)\right\} + \frac{\varepsilon e^{ix^2/2t}}{2t^{5/2}} \partial_\xi^2 V\left(\varepsilon^2 \log t, \frac{x}{t}\right),$$

whence

$$\|Z^\alpha Q(t,\cdot)\|_{L^\infty} \le C_{\alpha,B}\left(\frac{\varepsilon}{t^{1/2}}\right)^2 \frac{\varepsilon}{t^{3/2}} + C_{\alpha,B}\frac{\varepsilon}{t^{5/2}} \le C_{\alpha,B}\frac{\varepsilon}{t^{5/2}}, \quad t \in \left[1, e^{B/\varepsilon^2}\right]$$

for any $B \in]0, A[$ and $\alpha \in \mathbb{N}_0^2$.

3.3. Final step: piecing together. Let $\chi \in C^\infty(\mathbb{R})$ be a decreasing function satisfying $\chi = 1$ on $]-\infty, 1]$ and $\chi = 0$ on $[2, \infty[$. With this cut-off function χ, we set

$$u_a(t,x) = \chi(\varepsilon t) u_0(t,x) + (1 - \chi(\varepsilon t)) U(t,x).$$

Then it is easy to check that (3.1) and (3.2) hold. To verify (3.3), it suffices to show that

$$\|Z^\alpha R(t,\cdot)\|_{L^\infty} \le C_{\alpha,B}\frac{\varepsilon^{5/2}}{1+t} 1_{\{\varepsilon t < 2\}} + C_{\alpha,B}\frac{\varepsilon}{t^{5/2}} 1_{\{\varepsilon t > 1\}}, \quad t \in \left[0, e^{B/\varepsilon^2}\right] \tag{3.8}$$

for any $B \in]0, A[$ and $\alpha \in \mathbb{N}_0^2$, where

$$1_{\{t \in S\}} = \begin{cases} 1, & t \in S, \\ 0, & t \notin S. \end{cases}$$

Indeed, (2.3) and (3.8) leads to

$$\begin{aligned}\int_0^{e^{B/\varepsilon^2}} \|Z^\alpha R(t,\cdot)\|_{L^2}\, dt &\le C_{\alpha,B}\int_0^{2/\varepsilon} \frac{\varepsilon^{5/2}}{(1+t)^{1/2}}\, dt + C_{\alpha,B}\int_{1/\varepsilon}^\infty \frac{\varepsilon}{t^2}\, dt \\ &\le C_{\alpha,B}\varepsilon^2.\end{aligned}$$

The rest part of this section is devoted to the proof of (3.8). First we consider the case where $0 \le t \le 1/\varepsilon$ or $2/\varepsilon \le t \le \exp(B/\varepsilon^2)$. In this case it is easy to see that (3.8) holds since $R = -\chi(\varepsilon t)F(u_0, \partial_x u_0)$ or $R = Q$. Next let $1/\varepsilon \le t \le 2/\varepsilon$. Noting that

$$\left(i\partial_t + \frac{1}{2}\partial_x^2\right) u_a(t,x) = i\varepsilon\chi'(\varepsilon t)u_0 + \chi(\varepsilon t)\left(i\partial_t + \frac{1}{2}\partial_x^2\right) u_0 - i\varepsilon\chi'(\varepsilon t)U + (1-\chi(\varepsilon t))\left(i\partial_t + \frac{1}{2}\partial_x^2\right) U = i\varepsilon\chi'(\varepsilon t)(u_0 - U) + (1-\chi(\varepsilon t))\{F(U, \partial_x U) + Q\},$$

we have

$$R = i\varepsilon\chi'(\varepsilon t)(u_0 - U) + (1-\chi(\varepsilon t))\{F(U, \partial_x U) - F(u_a, \partial_x u_a)\} - \chi(\varepsilon t)F(u_a, \partial_x u_a) + (1-\chi(\varepsilon t))Q.$$

The last two terms in the right hand side can be handled in the same way as the previous cases. To estimate the first two terms, we observe that

$$u_0(t,x) - U(t,x) = \frac{\varepsilon e^{ix^2/2t}}{\sqrt{t}}\left\{V\left(0, \frac{x}{t}\right) - V\left(\varepsilon^2 \log t, \frac{x}{t}\right)\right\} + \frac{\varepsilon e^{ix^2/2t}}{\sqrt{2\pi i t}}\int_{-\infty}^{\infty} e^{-ixy/t}\big(e^{iy^2/2t} - 1\big)\varphi(y)\,dy = \frac{\varepsilon e^{ix^2/2t}}{\sqrt{t}}\left\{-\int_0^1 \partial_s V\left(\theta\varepsilon^2\log t, \frac{x}{t}\right) d\theta\right\}\varepsilon^2 \log t + O\big(\varepsilon t^{-3/2}\big).$$

Then, since $t \sim \varepsilon^{-1}$, we have

$$\|Z^\alpha\{i\varepsilon\chi'(\varepsilon t)(u_0 - U)\}\|_{L^\infty} \le C_{\alpha,B}\varepsilon\left(\frac{\varepsilon^3\log t}{t^{1/2}} + \frac{\varepsilon}{t^{3/2}}\right) \le C_{\alpha,B}\varepsilon\left(\frac{\varepsilon^{(3-1/2)}\log(1/\varepsilon)}{t^{1/2+1/2}} + \frac{\varepsilon^{1+1/2}}{t^{3/2-1/2}}\right) \le C_{\alpha,B}\frac{\varepsilon^{5/2}}{1+t}.$$

Similarly, when we note that

$$U(t,x) - u_a(t,x) = \chi(\varepsilon t)(U(t,x) - u_0(t,x)),$$

we obtain

$$\|Z^\alpha(1-\chi(\varepsilon t))\{F(U, \partial_x U) - F(u_a, \partial_x u_a)\}\|_{L^\infty} \le C_{\alpha,B}\left(\frac{\varepsilon}{t^{1/2}}\right)^2\left(\frac{\varepsilon^3\log(1/\varepsilon)}{t^{1/2}} + \frac{\varepsilon}{t^{3/2}}\right) \le C_{\alpha,B}\frac{\varepsilon^{5/2}}{1+t}.$$

Summing up, we arrive at (3.8). □

4. Proof of the main theorem

This section is devoted to the proof of Theorem 1.1. For this purpose, let us define

$$E_m(t) = \sum_{|\alpha|\le m} \| Z^\alpha (u - u_a)(t, \cdot)\|_{L^2}$$

for the solution u of (1.1) and the approximate solution u_a obtained in the previous section. First we shall see that Theorem 1.1 is reduced to the following lemma:

Lemma 4.1. *Let $B \in]0, A[$ and let $m \in \mathbb{N}$ with $m \ge 3$. There exists $\varepsilon_0 \in]0, 1]$, which depends only on B and m, such that the following holds true: Under the assumptions*

$$0 < T < \min\left\{T_\varepsilon, e^{B/\varepsilon^2}\right\} \quad \textit{and} \quad \sup_{0\le t\le T} E_m(t) \le \varepsilon,$$

we have

$$\sup_{0\le t\le T} E_m(t) \le \frac{\varepsilon}{2}$$

for any $\varepsilon \in]0, \varepsilon_0]$.

Proof of Theorem 1.1 via Lemma 4.1. Fix $B \in]0, A[$, $m \ge 3$ and $\varepsilon \in]0, \varepsilon_0]$ arbitrarily. Since $E_m(0) = 0$, we can choose some $T^* > 0$ such that

$$\sup_{0\le t\le T^*} E_m(t) < \varepsilon.$$

If $T^* \ge \exp(B/\varepsilon^2)$, then we see that $T_\varepsilon \ge \exp(B/\varepsilon^2)$ because of the estimate

$$\begin{aligned}\sup_{0\le t\le e^{B/\varepsilon^2}} \sum_{|\alpha|\le m} \| Z^\alpha u(t, \cdot)\|_{L^2} &\le \sup_{0\le t\le e^{B/\varepsilon^2}} \sum_{|\alpha|\le m} \| Z^\alpha u_a(t, \cdot)\|_{L^2} + \sup_{0\le t\le T^*} E_m(t)\\ &\le C_{B,m}\varepsilon + \varepsilon\end{aligned}$$

combined with the local existence theorem. In the other case, there exists some $0 < T^{**} < \min\left\{T_\varepsilon, e^{B/\varepsilon^2}\right\}$ such that

$$E_m(t) < \varepsilon \quad \text{for} \quad 0 \le t < T^{**} \quad \text{and} \quad E_m(T^{**}) = \varepsilon. \tag{4.1}$$

Then it follows from Lemma 4.1 that

$$E_m(T^{**}) \le \frac{\varepsilon}{2} < \varepsilon,$$

which contradicts (4.1). Therefore we must have $T_\varepsilon \geq \exp(B/\varepsilon^2)$ i.e. $\varepsilon^2 \log T_\varepsilon \geq B$ for any $\varepsilon \in]0, \varepsilon_0]$. Since $B \in]0, A[$ is arbitrary, the proof is completed. $\square$

We turn to the proof of Lemma 4.1. In what follows, we use the following notations:

$$|w(t,x)|_m = \sum_{|\alpha|\leq m} |Z^\alpha w(t,x)|$$

and

$$\|w(t)\|_{m,2} = \big\| |w(t,\,\cdot\,)|_m \big\|_{L^2}, \quad \|w(t)\|_{m,\infty} = \big\| |w(t,\,\cdot\,)|_m \big\|_{L^\infty}$$

for smooth function $w(t,x)$ decaying fast as $|x| \to \infty$. Note that $E_m(t)$ is equivalent to $\|(u-u_a)(t)\|_{m,2}$.

From now on, we fix $0 < B < A$, $0 < T < \min\{T_\varepsilon, e^{B/\varepsilon^2}\}$ and put $v = u - u_a$ for $t \in [0,T]$. Then we see that v satisfies

$$\begin{cases} \left(i\partial_t + \dfrac{1}{2}\partial_x^2\right) Z^\alpha v = Z^\alpha (P_1 - R), & (t,x) \in [0,T] \times \mathbb{R}, \\ Z^\alpha v(0,x) = 0, & x \in \mathbb{R}, \end{cases}$$

where $P_1(t,x) = F(u_a + v, \partial_x u_a + \partial_x v) - F(u_a, \partial_x u_a)$. Note that P_1 can be rewritten as

$$\begin{aligned} P_1 = &\left(\int_0^1 \frac{\partial F}{\partial u}(u_a + \theta v, \partial_x u_a + \theta \partial_x v)\,d\theta\right) v + \left(\int_0^1 \frac{\partial F}{\partial \overline{u}}(u_a + \theta v, \partial_x u_a + \theta \partial_x v)\,d\theta\right) \overline{v} \\ &+ \left(\int_0^1 \frac{\partial F}{\partial q}(u_a + \theta v, \partial_x u_a + \theta \partial_x v)\,d\theta\right) \partial_x v + \left(\int_0^1 \frac{\partial F}{\partial \overline{q}}(u_a + \theta v, \partial_x u_a + \theta \partial_x v)\,d\theta\right) \partial_x \overline{v}, \end{aligned}$$

where $\partial F/\partial u$, $\partial F/\partial \overline{u}$, $\partial F/\partial q$ and $\partial F/\partial \overline{q}$ are of the form

$$\begin{aligned} \frac{\partial F}{\partial u}(u,q) &= 2\lambda_1 |u|^2 + \lambda_2 \overline{u} q + 2\lambda_3 u \overline{q} + \lambda_4 |q|^2, \\ \frac{\partial F}{\partial \overline{u}}(u,q) &= \lambda_1 u^2 + \lambda_2 u q + \lambda_5 q^2, \\ \frac{\partial F}{\partial q}(u,q) &= \lambda_2 |u|^2 + \lambda_4 u \overline{q} + 2\lambda_5 \overline{u} q + 2\lambda_6 |q|^2 \end{aligned}$$

and

$$\frac{\partial F}{\partial \overline{q}}(u,q) = \lambda_3 u^2 + \lambda_4 u q + \lambda_6 q^2$$

when F is given by (1.3). In particular, we have

$$\begin{aligned}\|Z^\alpha P_1(t,\ \cdot\ )\|_{L^2} &\le C_\alpha\Big(\|v(t)\|^2_{[|\alpha|/2]+1,\infty}+\|u_a(t)\|^2_{[|\alpha|/2]+1,\infty}\Big)\|v(t)\|_{|\alpha|+1,2}\\ &\quad+C_\alpha\Big(\|v(t)\|_{[|\alpha|/2]+1,\infty}+\|u_a(t)\|_{[|\alpha|/2]+1,\infty}\Big)\\ &\quad\times\Big(\|v(t)\|_{[|\alpha|/2]+1,\infty}\|v(t)\|_{|\alpha|+1,2}+\|u_a(t)\|_{|\alpha|+1,\infty}\|v(t)\|_{[|\alpha|/2]+1,2}\Big)\\ &\le C_m\Big(\|v(t)\|^2_{[(m-1)/2]+1,\infty}+\|u_a(t)\|^2_{m,\infty}\Big)\|v(t)\|_{m,2}\\ &\le C_{B,m}\frac{\varepsilon^2}{1+t}\|v(t)\|_{m,2}\end{aligned}$$

for $|\alpha|\le m-1$, provided that $[(m-1)/2]+2\le m$, i.e. $m\ge 2$. Here $[\sigma]$ denotes the largest integer which does not exceed $\sigma\in\mathbb{R}$. Therefore the standard energy inequality yields

$$\begin{aligned}\sum_{|\alpha|\le m-1}\|Z^\alpha v(t,\ \cdot\ )\|_{L^2} &\le \sum_{|\alpha|\le m-1}\|Z^\alpha v\|_{L^2}\Big|_{t=0}\\ &\quad+\sum_{|\alpha|\le m-1}\int_0^t\|Z^\alpha P_1(\tau,\ \cdot\ )\|_{L^2}+\|Z^\alpha R(\tau,\ \cdot\ )\|_{L^2}\,d\tau\\ &\le C_{B,m}\varepsilon^2\int_0^t\frac{E_m(\tau)}{1+\tau}\,d\tau+C_{B,m}\varepsilon^2\end{aligned}\tag{4.2}$$

for $t\in[0,T]$.

Next we consider the case where $|\alpha|=m$. Set

$$b_1(t,x)=-\frac{\partial F}{\partial q}(u,\partial_x u),\qquad b_2(t,x)=-(-1)^{\alpha_2}\frac{\partial F}{\partial\overline{q}}(u,\partial_x u)$$

and

$$P_2(t,x)=Z^\alpha P_1-\frac{\partial F}{\partial q}(u_a+v,\ \partial_x u_a+\partial_x v)\partial_x(Z^\alpha v)-(-1)^{\alpha_2}\frac{\partial F}{\partial\overline{q}}(u_a+v,\ \partial_x u_a+\partial_x v)\partial_x\Big(\overline{Z^\alpha v}\Big)$$

so that $Z^\alpha v$ satisfies

$$i\partial_t(Z^\alpha v)+\frac{1}{2}\partial_x^2(Z^\alpha v)+b_1\partial_x(Z^\alpha v)+b_2\partial_x\Big(\overline{Z^\alpha v}\Big)=P_2-Z^\alpha R$$

(remember that $u=u_a+v$). Then we can apply Lemma 2.3 with $\psi=Z^\alpha v$ to obtain

$$\frac{d}{dt}\Big\|e^{p(t,\cdot)}Z^\alpha v(t,\ \cdot\ )\Big\|_{L^2}\le C\beta(t)\Big\|e^{p(t,\cdot)}Z^\alpha v(t,\ \cdot\ )\Big\|_{L^2}+\Big\|e^{p(t,\cdot)}(P_2-Z^\alpha R)\Big\|_{L^2},$$

where $p(t,x)$ and $\beta(t)$ are given by (2.6) and (2.7) with above b_1, b_2. Note that

$$e^{-C_B\varepsilon^2}\le e^{p(t,x)}\le e^{C_B\varepsilon^2}\tag{4.3}$$

because

$$\sup_{x\in\mathbb{R}} |p(t,x)| \le C\|u(t)\|_{H^1}^2 \le C\left(\|u_a(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2\right) \le C_B\varepsilon^2.$$

To estimate $\beta(t)$, we shall use Lemma 2.4 to obtain

$$\begin{aligned}\sup_{x\in\mathbb{R}}\left|\partial_t\int_{-\infty}^x \operatorname{Re} b_1(t,y)\,dy\right| &= \sup_{x\in\mathbb{R}}\left|\partial_t\int_{-\infty}^x \operatorname{Re}\frac{\partial F}{\partial q}(u,\partial_x u)(t,y)\,dy\right|\\ &\le C\sum_{l_1,l_2=0}^{1}\sup_{x\in\mathbb{R}}\left|\partial_t\int_{-\infty}^x \partial_x^{l_1}u(t,y)\overline{\partial_x^{l_2}u(t,y)}\,dy\right|\\ &\le \frac{C}{1+t}\|u(t)\|_{3,2}^2 + C\|u(t)\|_{H^1}\|F(u,\partial_x u)\|_{H^1}\\ &\le C_B\frac{\varepsilon^2}{1+t} + C_B\frac{\varepsilon^4}{1+t}.\end{aligned}$$

Also we have

$$\begin{aligned}\sum_{l=1}^{2}\left(\|\partial_x b_l(t,\cdot)\|_{L^\infty} + \|b_l(t,\cdot)\|_{L^\infty}^2\right) &\le C\left(\|u(t)\|_{2,\infty}^2 + \|u(t)\|_{1,\infty}^4\right)\\ &\le C\left(\frac{\|u(t)\|_{3,2}^2}{1+t} + \frac{\|u(t)\|_{2,2}^4}{(1+t)^2}\right)\\ &\le C_B\frac{\varepsilon^2}{1+t} + C_B\frac{\varepsilon^4}{(1+t)^2}.\end{aligned}$$

Therefore we have

$$\beta(t) \le C_B\frac{\varepsilon^2}{1+t}.$$

As for the estimate of $P_2(t,x)$, in the same way as that of $P_1(t,x)$, we have

$$\begin{aligned}\|P_2(t,\cdot)\|_{L^2} &\le C_m\left(\|v(t)\|_{[m/2]+1,\infty}^2 + \|u_a(t)\|_{[m/2]+1,\infty}^2\right)\|v(t)\|_{m,2}\\ &\quad + C_m\left(\|v(t)\|_{[m/2]+1,\infty} + \|u_a(t)\|_{[m/2]+1,\infty}\right)\|u_a(t)\|_{m+1,\infty}\|v(t)\|_{[m/2]+1,2}\\ &\le C_m\left(\|v(t)\|_{[m/2]+1,\infty}^2 + \|u_a(t)\|_{m+1,\infty}^2\right)\|v(t)\|_{m,2}\\ &\le C_{B,m}\frac{\varepsilon^2}{1+t}\|v(t)\|_{m,2},\end{aligned}$$

provided that $[m/2]+2 \le m$, i.e. $m \ge 3$. Summing up, we have

$$\frac{d}{dt}\left\|e^{p(t,\cdot)}Z^\alpha v(t,\cdot)\right\|_{L^2} \le C_{B,m}\frac{\varepsilon^2}{1+t}\|v(t)\|_{m,2} + C_{B,m}\|Z^\alpha R\|_{L^2},$$

which leads to

$$\sum_{|\alpha|=m} \left\| e^{p(t,\cdot)} Z^\alpha v(t,\ \cdot)\right\|_{L^2} \le C_{B,m}\varepsilon^2 \int_0^t \frac{E_m(\tau)}{1+\tau}\, d\tau + C_{B,m}\varepsilon^2 \tag{4.4}$$

for $t \in [0, T]$.

From (4.2), (4.3) and (4.4), it follows that

$$\begin{aligned} E_m(t) &\le \sum_{|\alpha|\le m-1} \| Z^\alpha v(t,\ \cdot)\|_{L^2} + e^{C_{B,m}\varepsilon^2} \sum_{|\alpha|=m} \left\| e^{p(t,\cdot)} Z^\alpha v(t,\ \cdot)\right\|_{L^2} \\ &\le C_{B,m}\varepsilon^2 \int_0^t \frac{E_m(\tau)}{1+\tau}\, d\tau + C_{B,m}\varepsilon^2. \end{aligned}$$

Therefore the Gronwall lemma implies

$$E_m(t) \le C_{B,m} e^{C_{B,m}\varepsilon^2 \log(1+t)} \varepsilon^2 \le C_{B,m} e^{C_{B,m} B} \varepsilon^2$$

for $t \in [0, T]$. Finally, choosing $\varepsilon_0 \in\ ]0, 1]$ so that $2C_{B,m} e^{C_{B,m} B} \varepsilon_0 \le 1$, we obtain

$$\sup_{t\in[0,T]} E_m(t) \le \frac{\varepsilon}{2}$$

for $\varepsilon \in\ ]0, \varepsilon_0]$. This completes the proof of Lemma 4.1. □

REMARK 4.1. Roughly speaking, what we can expect without using the approximate solution is

$$\sum_{|\alpha|\le m} \| Z^\alpha u(t,\ \cdot)\|_{L^2} \le K\varepsilon + f(\varepsilon^2 \log t)\varepsilon$$

under the assumption that the left hand side is dominated by $2K\varepsilon$, where K is a positive constant depending (less explicitly) on φ, and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, increasing function satisfying $f(0) = 0$ (see [25] for detail). On the other hand, the above argument yields

$$\sum_{|\alpha|\le m} \| Z^\alpha (u - u_a)(t,\ \cdot)\|_{L^2} = O(\varepsilon^2)$$

as long as $t \le \exp(B/\varepsilon^2)$, $B \in\ ]0, A[$. This is the main difference between the previous approach and ours.

5. Concluding remarks

We conclude this paper with the following three remarks:

REMARK 5.1. In the case of $A < \infty$, the fact that $u_a(t,x)$ blows up as $t \to \exp(A/\varepsilon^2)$ suggests the upper bound of $\varepsilon^2 \log T_\varepsilon$ cannot be larger than A. However, as far as the author knows, there are no results on *small data* blow-up (which lead to upper bounds of T_ε) nor global existence results (which imply $T_\varepsilon = \infty$) when $A < \infty$. Concerning quasilinear wave equations, the corresponding problems have been studied extensively by S. Alinhac [1]–[5], etc. Note that analogous problems for nonlinear Klein-Gordon equations are also left unsolved (see [9], [10], [33] for recent progress on NLKG).

REMARK 5.2. If $\mathbb{R}$ is replaced by $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ in (1.1), we can find the following example on small data blow-up:

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\partial_x^2 u = \lambda|u|^2 u, & t > 0,\ \ x \in \mathbb{T}, \\ u(0,x) = \varphi(x), & x \in \mathbb{T}, \end{cases}$$

where $\lambda \in \mathbb{C}$, $\operatorname{Im}\lambda > 0$ and $\varphi \in L^2(\mathbb{T})$, $\varphi \not\equiv 0$. Indeed, from the equality

$$\frac{d}{dt}\|u(t)\|_{L^2(\mathbb{T})}^2 = 2\operatorname{Im}\lambda\|u(t)\|_{L^4(\mathbb{T})}^4$$

and the Cauchy-Schwarz inequality, we deduce that

$$\frac{d}{dt}\|u(t)\|_{L^2(\mathbb{T})}^2 \geq \frac{\operatorname{Im}\lambda}{\pi}\|u(t)\|_{L^2(\mathbb{T})}^4$$

with $\|u(0)\|_{L^2(\mathbb{T})} = \|\varphi\|_{L^2(\mathbb{T})} > 0$, which imply $L^2(\mathbb{T})$-norm of the solution goes to infinity before $t = \pi\big/\big(\operatorname{Im}\lambda\|\varphi\|_{L^2(\mathbb{T})}^2\big)$. (Related observation may be found in [31], [27], etc.) Unfortunately, however, this argument fails in the case of $\mathbb{R}$ because $\int_{\mathbb{R}} 1\,dx = \infty$.

REMARK 5.3. In the case of $\sup_{\xi\in\mathbb{R}} \operatorname{Im} F(1, i\xi) \leq 0$, we can easily check that $1/A = 0$, whence Theorem 1.1 leads to $\lim_{\varepsilon\to+0} \varepsilon^2 \log T_\varepsilon = +\infty$. This suggests that the restriction (1.4) in the result of Hayashi–Naumkin–Uchida may be weakened when we consider the *forward* Cauchy problem (i.e. evolution in the *positive* time direction). Moreover, in view of the approximate solution $u_a(t,x)$, it would be quite natural to expect the solution decays like $O(t^{-1/2}(\log t)^{-1/2})$ as $t \to +\infty$ uniformly in $x \in \mathbb{R}$ if $\sup_{\xi\in\mathbb{R}} \operatorname{Im} F(1, i\xi) < 0$. However, we do not have any proof except A.Shimomura's recent work [32]. He considered the case where $F = \lambda|u|^2 u$ with $\lambda \in \mathbb{C}$, $\operatorname{Im}\lambda < 0$ and found that the solution does decay like $O(t^{-1/2}(\log t)^{-1/2})$ as $t \to +\infty$ uniformly in $x \in \mathbb{R}$, but his method does not seem to be directly applicable for the derivative nonlinear Schrödinger equations. Finally we mention that the author [33] has succeeded in obtaining this kind of additional time-decay result for a class of nonlinear Klein-Gordon equations.

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