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THE CONVERSE OF ISOVARIANT BORSUK-ULAM RESULTS FOR SOME ABELIAN GROUPS

Dedicated to Professor Yasuhiko Kitada on the occasion of his sixtieth birthday

IKUMITSU NAGASAKI

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Abstract
The isovariant Borsuk-Ulam theorem provides nonexistence results on isovariant maps between representations. In this paper we shall deal with the existence problem of isovariant maps as a converse to the isovariant Borsuk-Ulam theorem, and show that the converse holds for representations of an abelian p-group or a cyclic groups of order $p^n q^m$ or pqr, where p, q, r are distinct primes.

0. Introduction

A map $f: X \to Y$ between G-spaces is called *G-isovariant* if it is G-equivariant and preserves the isotropy groups, i.e., $G_{f(x)} = G_x$ for all $x \in X$. Throughout this paper all maps are understood to be continuous. Isovariant maps often play important roles in equivariant topology, see, for example, [2], [5], [8]. The existence problem of isovariant maps is, therefore, fundamental and important, as well as that of equivariant maps.

We shall study isovariant maps between representations, especially the existence problem of isovariant maps between representations of some abelian groups. A starting point of this study is the isovariant Borsuk-Ulam theorem [10], which provides nonexistence results on isovariant maps between representations.

Theorem 0.1 (Isovariant Borsuk-Ulam theorem). *Let G be a finite solvable group. If there exists a G-isovariant map $f: V \to W$ between representations, then the following inequality holds*:

$$\dim V - \dim V^G \leq \dim W - \dim W^G.$$

We say that G has the *IB-property* (isovariant Borsuk-Ulam property) if it holds that $\dim V - \dim V^G \leq \dim W - \dim W^G$ for every pair (V, W) of G-representations such

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that there is a G-isovariant map from V to W. As a result, every finite solvable group has the IB-property.

REMARK. It is known [10] that some kind of nonsolvable groups have the IB-property, and [6] that a weaker version of the isovariant Borsuk-Ulam theorem holds for an arbitrary compact Lie group; the author, however, does not know whether an arbitrary compact Lie group has the IB-property.

Let G be a finite solvable group, and let V and W be G-representations. Suppose that there exists a G-isovariant map $f\colon V \to W$. For any pair of subgroups $H \triangleleft K$ (H is normal in K), the restriction of f to the H-fixed point sets yields a K/H-isovariant map $f^H\colon V^H \to W^H$. Since K/H is also solvable, it follows from Theorem 0.1 that

$$(C_{V,W})\colon \dim V^H - \dim V^K \leq \dim W^H - \dim W^K \text{ for any pair } H \triangleleft K.$$

Moreover the pair (V, W) obviously satisfies

$$(I_{V,W})\colon \operatorname{Iso} V \subset \operatorname{Iso} W,$$

where $\operatorname{Iso} V$ denotes the set of isotropy subgroups of V. For the converse of these facts, we shall give the following definition and question.

DEFINITION. We say that a finite solvable group G has the *complete IB-property* if for every pair (V, W) of G-representations satisfying conditions $(C_{V,W})$ and $(I_{V,W})$, there exists a G-isovariant map from V to W.

QUESTION. Which finite solvable groups have the complete IB-property?

REMARK. As seen in §1, if G is nilpotent, $(C_{V,W})$ implies $(I_{V,W})$. In the case, $(I_{V,W})$ can be removed from the above definition.

Concerning this question, we shall show in this paper that certain abelian groups have the complete IB-property; precisely,

Theorem A. *Let p, q, r be distinct primes. The following groups have the complete IB-property*:
(1) *abelian p-groups,*
(2) *$C_{p^m q^n}$, the cyclic group of order $p^m q^n$ ($m \geq 1$, $n \geq 1$),*
(3) *C_{pqr}, the cyclic group of order pqr.*

REMARK. S. Kôno announces that $C_{p^m q^n}$ has an analogous property for complex representations.

Since conditions ($C_{V,W}$) and ($C_{V\oplus U,V\oplus U}$) are equivalent, as a consequence, one can see the following.

Corollary B. *Let G be one of abelian groups listed in* Theorem A. *Then there exists a G-isovariant map from V to W if and only if there exists a G-isovariant map from $V \oplus U$ to $W \oplus U$ for some representation U.*

This paper is organized as follows. In §1 we shall recollect basic properties of isovariant maps. In §2 we shall show that an arbitrary abelian p-group has the complete IB-property after recalling some facts from representation theory. §§3 and 4 will be devoted to showing Theorem A (2); in §3 we shall introduce the notion of an elementary isovariant map, and in §4, construct an isovariant map combining elementary isovariant maps. In §5 we shall show Theorem A (3). To do that, in addition to elementary isovariant maps, we need another kind of isovariant map. We shall show the existence of such a map using equivariant obstruction theory discussed in [7].

1. Basic properties of isovariant maps

We first recall basic notations and facts on isovariant maps, which are freely used throughout this paper.

Let G be a finite group. We write $H \leq K$ when H is a subgroup of K, and $H < K$ when H is a proper subgroup of K. Let X, Y be G-spaces and $f\colon X \to Y$ a G-map. Let H be a subgroup of G. Restricting the action, we obtain an H-map $\mathrm{Res}_H f\colon \mathrm{Res}_H X \to \mathrm{Res}_H Y$, and restricting f to the H-fixed point set X^H, we obtain an $N_G(H)/H$-map $f^H\colon X^H \to Y^H$, where $N_G(H)$ denotes the normalizer of H in G. Suppose next that H is normal. Let X, Y be G/H-spaces and $f\colon X \to Y$ a G/H-map. Via the projection $p\colon G \to G/H$, X and Y are thought of as G-spaces, denoted by $\mathrm{Inf}^G_{G/H} X$ and $\mathrm{Inf}^G_{G/H} Y$ respectively, and f is thought of as a G-map, denoted by $\mathrm{Inf}^G_{G/H} f$. These are called the *inflation* of a G/H-space or a G/H-map. We often omit the symbols Res_H and $\mathrm{Inf}^G_{G/H}$ for simplicity if there is no misunderstanding in context. We first note

Lemma 1.1. *The following hold.*

(1) *If f is G-isovariant, then* $\mathrm{Res}_H f$ *is H-isovariant for any $H \leq G$.*

(2) *Let H be a normal subgroup. If f is G-isovariant, then f^H is G/H-isovariant.*

(3) *Let H be a normal subgroup. If $f\colon X \to Y$ is G/H-isovariant, then* $\mathrm{Inf}^G_{G/H} f$ *is G-isovariant.*

(4) *If $f\colon X_1 \to Y_1$ and $g\colon X_2 \to Y_2$ are G-isovariant, then so is $f \times g\colon X_1 \times X_2 \to Y_1 \times Y_2$.*

(5) *If $f\colon X_1 \to Y_1$ and $g\colon X_2 \to Y_2$ are G-isovariant, then so is $f * g\colon X_1 * X_2 \to Y_1 * Y_2$, where $*$ means join, in particular, the cone of f, $Cf\colon CX_1 \to CY_1$, is G-isovariant.*

(6) *If* $f\colon X \to Y$ *and* $g\colon Y \to Z$ *are* G*-isovariant, then so is* $g \circ f\colon X \to Z$.
(7) *If* $f\colon X \to Y$ *is* H*-isovariant, then* $G \times_H f\colon G \times_H X \to G \times_H Y$ *is* G*-isovariant.*

REMARK. This lemma still holds for topological group actions.

Proof. It is clear that all maps are equivariant. It suffices to show that the maps preserve the isotropy groups.

(1): This follows from $H_x = G_x \cap H$.

(2) and (3): These follow from $(G/H)_x = G_x/H$.

(4): This follows from $G_{(x,y)} = G_x \cap G_y$.

(5): For any $z = tx \oplus sy \in X_1 * X_2$, $t + s = 1$, $t \geq 0$, $s \geq 0$, one can see that $G_z = G_x \cap G_y$ when $t \neq 0$ and $s \neq 0$, $G_z = G_x$ when $s = 0$, and $G_z = G_y$ when $t = 0$. This leads to the G-isovariance of $f * g$.

(6): This follows from $G_{g \circ f(x)} = G_{f(x)} = G_x$.

(7): This follows from $G_{[g,x]} = gH_xg^{-1}$. □

By definition, a real representation of G is a homomorphism $\rho\colon G \to GL(V)$, where $GL(V)$ is the general linear group of a (finite dimensional) real vector space V. Via this homomorphism, V becomes a G-space with linear action, called a G-representation space, or simply G-representation. By representation theory, cf. [9], any real representation is isomorphic to an orthogonal representation, i.e., a homomorphism from G to $O(V)$ the orthogonal group of V with inner product. In particular any G-representation is G-diffeomorphic to some orthogonal G-representation; hence for our purpose it is sufficient to treat only orthogonal representations, and a G-representation hereafter means an orthogonal G-representation. Since the action of G is orthogonal, the unit sphere $S(V)$ and the unit disk $D(V)$ of V are G-invariant, called the *representation sphere* and the *representation disk* of V, respectively. Let $V^{G\perp}$ denote the subrepresentation defined by the orthogonal complement $V - V^G$ of V^G in V. The following lemma says that the existence of an isovariant map between representations is equivalent to that of an isovariant map between the representation spheres or disks.

Lemma 1.2. *Let* V, W *be* G*-representations. The following statements are equivalent.*
(1) *There exists a* G*-isovariant map* $f\colon V \to W$.
(2) *There exists a* G*-isovariant map* $f\colon V^{G\perp} \to W^{G\perp}$.
(3) *There exists a* G*-isovariant map* $f\colon S(V) \to S(W)$.
(4) *There exists a* G*-isovariant map* $f\colon S(V^{G\perp}) \to S(W^{G\perp})$.
(5) *There exists a* G*-isovariant map* $f\colon D(V) \to D(W)$.
(6) *There exists a* G*-isovariant map* $f\colon D(V^{G\perp}) \to D(W^{G\perp})$.

REMARK. This lemma still holds for representations of a compact Lie group.

Proof. (1) $\Rightarrow$ (2): The inclusion $i\colon V^{G\perp} \to V$ is clearly G-isovariant, and the projection $p\colon W = W^{G\perp} \oplus W^G \to W^{G\perp}$ is also G-isovariant, since G acts trivially on W^G. Hence the composite map $p \circ f \circ i\colon V^{G\perp} \to W^{G\perp}$ is G-isovariant.

(2) $\Rightarrow$ (4): Since $\left(V^{G\perp}\right)^G = \left(W^{G\perp}\right)^G = 0$, we have $f^{-1}(0) = \{0\}$, and hence a G-isovariant map $g\colon S\left(V^{G\perp}\right) \to S\left(W^{G\perp}\right)$ can be defined by $g(x) = f(x)/\|f(x)\|$.

(4) $\Rightarrow$ (3): Since G act trivially on V^G and W^G, any map $g\colon S(V^G) \to S(W^G)$ is G-isovariant. Taking join, we obtain a G-isovariant map

$$f * g\colon S(V) \cong S\left(V^{G\perp}\right) * S\left(V^G\right) \to S\left(W^{G\perp}\right) * S\left(W^G\right) \cong S(W).$$

(3) $\Rightarrow$ (1): Taking the open cone of $f\colon S(V) \to S(W)$, we obtain a G-isovariant map $\tilde{f}\colon V \cong \operatorname{Int} D(V) \to W \cong \operatorname{Int} D(W)$. Thus (1)–(4) are equivalent.

(4) $\Rightarrow$ (6): Taking the cone of $f\colon S\left(V^{G\perp}\right) \to S\left(W^{G\perp}\right)$, we obtain a G-isovariant map $\tilde{f}\colon D\left(V^{G\perp}\right) \to D\left(W^{G\perp}\right)$.

(6) $\Rightarrow$ (5): Since any map $g\colon D(V^G) \to D(W^G)$ is G-isovariant, taking product, we obtain a G-isovariant map

$$f \times g\colon D(V) \cong D\left(V^{G\perp}\right) \times D\left(V^G\right) \to D(W) \cong D\left(W^{G\perp}\right) \times D\left(W^G\right).$$

(5) $\Rightarrow$ (1): Let $f\colon D(V) \to D(W)$ be a G-isovariant map. We define $g\colon D(V) \to D(W)$ by $g(x) = f(x)/2$, then g is G-isovariant and maps $D(V)$ to the interior $\operatorname{Int} D(W)$ of $D(W)$. Hence we obtain a G-isovariant map $g|_{\operatorname{Int} D(V)}\colon V \cong \operatorname{Int} D(V) \to W \cong \operatorname{Int} D(W)$.

Thus the proof is complete. $\square$

In the rest of this section, we shall give some remarks related to condition $(C_{V,W})$. Consider the following condition

$(C'_{V,W})$: $\dim V^H - \dim W^K \le \dim W^H - \dim W^K$ for any pair $H \lhd K$ with K/H of prime order.

As the first Remark,

Proposition 1.3. *Let G be a solvable group. Conditions $(C_{V,W})$ and $(C'_{V,W})$ are equivalent. Moreover if G is nilpotent, then these conditions are equivalent to the following condition*

$(C''_{V,W})$: $\dim V^H - \dim W^K \le \dim W^H - \dim W^K$ *for any pair $H < K$ of subgroups.*

Proof. It is trivial that $(C_{V,W})$ implies $(C'_{V,W})$. For any pair $H \lhd K$ of subgroups, since K/H is solvable, one can take subgroups H_i, $i = 0, \dots, r$, such that

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_r = K,$$

and H_i/H_{i-1} is of prime order for each i. By $(C'_{V,W})$ we have

$$\begin{aligned}\dim V^H - \dim V^K &= \sum_{i=1}^{r}\bigl(\dim V^{H_{i-1}} - \dim V^{H_i}\bigr)\\ &\leq \sum_{i=1}^{r}\bigl(\dim W^{H_{i-1}} - \dim W^{H_i}\bigr)\\ &= \dim W^H - \dim W^K.\end{aligned}$$

Thus $(C'_{V,W})$ implies $(C_{V,W})$.

If G is nilpotent, then every subgroup is also nilpotent. One may assume that $K = G$ by restricting the action. It is known from group theory that the normalizer $N_G(H)$ of every proper subgroup H is strictly larger than H, i.e., $H < N_G(H)$. Using this fact repeatedly, we have a sequence of subgroups:

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_r = G.$$

From the same argument as above, $(C_{V,W})$ implies that

$$\dim V^H - \dim V^G \leq \dim W^H - \dim W^G. \qquad \square$$

The following proposition shows that, if G is nilpotent, condition $(I_{V,W})$ can be removed from the definition of the IB-property.

Proposition 1.4. *Let G be a nilpotent group. Then $(C_{V,W})$ implies $(I_{V,W})$.*

Proof. Note that, for any representation U, $H \in \operatorname{Iso} U$ if and only if $\dim U^H > \dim U^K$ for every K with $H < K$. By Proposition 1.3, we have

$$\dim V^H - \dim V^K \leq \dim W^H - \dim W^K, \quad H < K.$$

If $H \in \operatorname{Iso} V$, then $\dim V^H - \dim V^K > 0$, and hence $\dim W^H - \dim W^K > 0$. Thus it follows that $H \in \operatorname{Iso} W$. $\square$

Finally we list some properties of $(C_{V,W})$, which are easily verified.

Proposition 1.5. (1) $(C_{V,W})$ *implies* $(C_{\operatorname{Res}_H V, \operatorname{Res}_H W})$ *for any subgroup* H.
(2) $(C_{V,W})$ *implies* (C_{V^H,W^H}) *for any normal subgroup* H.
(3) $(C_{\overline{V},\overline{W}})$ *for* G/H*-representations,* $H \lhd G$, *implies* $(C_{\operatorname{Inf}^G_{G/H}\overline{V}, \operatorname{Inf}^G_{G/H}\overline{W}})$.
(4) $(C_{V,W})$ *and* $(C_{V',W'})$ *imply* $(C_{V\oplus V', W\oplus W'})$.
(5) $(C_{V,U})$ *and* $(C_{U,W})$ *imply* $(C_{V,W})$.
(6) $(C_{V\oplus U, W\oplus U})$ *implies* $(C_{V,W})$.

2. The case of abelian p-groups

In this section G is a finite abelian group. We shall recall several facts from representation theory and transformation group theory. Let V be a G-representation and $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ the irreducible decomposition. Since G is abelian, each V_i is (real) 1- or 2-dimensional. For any subgroup H of G, we set $V(H) = \bigoplus_{i:\ \mathrm{Ker}\, V_i = H} V_i$, where $\mathrm{Ker}\, V_i$ denotes the kernel of the representation homomorphism $\rho_{V_i}: G \to O(k)$, $k = 1$ or 2; if there are no irreducible representations with kernel H, we set $V(H) = 0$. Thus V is decomposed into $\bigoplus_H V(H)$.

A representation with trivial kernel is called *faithful*. Let V be an irreducible G-representation with kernel K; then V^K $(= V)$ is a faithful irreducible G/K-representation. Conversely if U is a faithful irreducible G/K-representation, then $\mathrm{Inf}_{G/K}^G U$ is an irreducible G-representation with kernel K. Since $\left(\mathrm{Inf}_{G/K}^G U\right)^K = U$ and $\mathrm{Inf}_{G/K}^G(V^K) = V$, the irreducible G-representations with kernel K stand in one-to-one correspondence with the faithful irreducible G/K-representations.

Lemma 2.1. *If K is the kernel of an irreducible G-representation V, then G/K is cyclic.*

Proof. The representation homomorphism $\rho_V: G \to O(k)$, $k = 1, 2$, induces the injective homomorphism $\bar{\rho}_V: G/K \to O(k)$, which is the representation homomorphism of the irreducible G/K-representation V^K $(= V)$. Hence G/K is cyclic, or isomorphic to $C_2 \times C_2$, but it does not happen that G/K is isomorphic to $C_2 \times C_2$. In fact every irreducible real representation of $C_2 \times C_2$ is 1-dimensional, and hence G/K $(\cong C_2 \times C_2)$ must be a subgroup of $O(1) = C_2$; this is a contradiction. □

Let $\mathcal{D}$ denote the set of subgroups H such that G/H is cyclic. Note that $\mathcal{D}$ is a closed family in the sense of [3], i.e., if $H \leq K$ and $H \in \mathcal{D}$, then $K \in \mathcal{D}$. For a G-representation V, we set $\mathcal{D}(V) = \{H \mid V(H) \neq 0\}$, and then V is expressed as $V = \bigoplus_{H \in \mathcal{D}(V)} V(H)$. By Lemma 2.1, $V(H) = 0$ for $H \notin \mathcal{D}$, and hence $\mathcal{D}(V) \subset \mathcal{D}$. Thus the representations of an abelian group are essentially reduced to those of cyclic groups. We here recall the irreducible representations of the cyclic group C_n of order n. Let g be a generator of C_n. The unitary C_n-representation t_i with underlying space $\mathbb{C}$ is defined by setting $gz = \xi_n^i z$, where $z \in \mathbb{C}$ and $\xi_n = \exp(2\pi\sqrt{-1}/n)$. Representation theory shows that t_i, $0 \leq i \leq n-1$, represent all irreducible unitary C_n-representations. Over the real number field, t_i turns to an orthogonal representation (not necessarily irreducible), denoted by T_i. Then T_i for $1 \leq i \leq [(n-1)/2]$ represent all 2-dimensional irreducible representations, where $[m]$ denotes the greatest integer not larger than m, and $T_i \cong T_{n-i}$ as orthogonal representations. The 1-dimensional irreducible C_n-representations are $\mathbb{R}$, the trivial 1-dimensional representation, and $\mathbb{R}^-$, the nontrivial 1-dimensional representation (i.e., g acts on $\mathbb{R}^-$ by $gx = -x$), where n must be even in the latter case. Note that $T_0, T_{n/2}, n$ is even in the latter case, are not irreducible and isomorphic to twice as many as a 1-dimensional irreducible representation; $T_0 \cong 2\mathbb{R} := \mathbb{R} \oplus \mathbb{R}$, $T_{n/2} \cong 2\mathbb{R}^- := \mathbb{R}^- \oplus \mathbb{R}^-$. Note also that $\operatorname{Ker} T_i \cong C_{(i,n)}$, and in particular T_i is faithful if and only if i is prime to n, which is equivalent to that C_n acts freely on $S(T_i)$.

We next show the following.

Proposition 2.2. *Let V be a representation of an abelian group G.*
(1) *For any nonempty subset $\mathcal{F} \subset \mathcal{D}(V)$, $\bigcap_{H\in\mathcal{F}} H \in \operatorname{Iso} V$. Conversely, for any $v \in V$, there is a subset $\mathcal{F} \subset \mathcal{D}(V)$ such that $G_v = \bigcap_{H\in\mathcal{F}} H$.*
(2) *V is faithful if and only if $1 \in \operatorname{Iso} V$.*

Proof. (1): For any $H \in \mathcal{D}(V)$, since G/H acts freely on $V(H)\setminus\{0\}$, it follows that $\operatorname{Iso}(V(H)\setminus\{0\}) = \{H\}$, i.e., for any nonzero $v_H \in V(H)$, $G_{v_H} = H$. Take $v = (v_H) \in V = \bigoplus_{H\in\mathcal{D}(V)} V(H)$ such that $v_H \neq 0$ for $H \in \mathcal{F}$ and $v_H = 0$ for $H \in \mathcal{D}(V)\setminus\mathcal{F}$. Then $G_v = \bigcap_{H\in\mathcal{D}(V)} G_{v_H} = \bigcap_{H\in\mathcal{F}} H \in \operatorname{Iso} V$. Conversely, for any $v = (v_H) \in V = \bigoplus_{H\in\mathcal{D}(V)} V(H)$, set $\mathcal{F} = \{H \mid v_H \neq 0\}$. Then $G_v = \bigcap_{H\in\mathcal{F}} H$.

(2): Since $\operatorname{Ker} V = \bigcap_{v\in V} G_v$, by (1) $\operatorname{Ker} V$ is expressed as an intersection of some elements of $\mathcal{D}(V)$; this shows that if V is faithful, then $1 \in \operatorname{Iso} V$. Conversely, if $1 \in \operatorname{Iso} V$, then 1 is expressed as an intersection of some elements of $\mathcal{D}(V)$; this shows that $\operatorname{Ker} V = 1$. □

In order to prove Theorem A (1), we shall prepare the following.

Lemma 2.3. *Let G be an abelian group. If V and W are irreducible G-representations with the same kernel, then there exists a G-isovariant map $f : V \to W$.*

Proof. If V, U are trivial, this is obvious. Suppose that V, W are nontrivial. By Lemmas 1.1 (3) and 2.1, it suffices to show this in the case where V and W are faithful C_n-representations. One may set $V = T_i$, $W = T_j$ (i, j are prime to n) when $n \neq 2$, and $V = W = \mathbb{R}^-$ when $n = 2$. In the first case a C_n-isovariant map is constructed as follows: Choose a positive integer k with $ik \equiv 1 \mod n$, and define $f : T_i \to T_j$ by setting $f(z) = z^{kj}$. Then f is equivariant, in fact, for a generator g of C_n,

$$f(gz) = \left(\xi_n^i z\right)^{kj} = \xi_n^{ikj} z^{kj} = \xi_n^j z^{kj} = g f(z).$$

Moreover $f^{-1}(0) = \{0\}$, and C_n acts freely on $T_i \setminus \{0\}$ and $T_j \setminus \{0\}$; hence f preserves the isotropy groups. In the second case the identity map can be taken as an isovariant map. □

As a consequence of Lemma 2.3, one can see

Proposition 2.4. *Let V and W be representations of an abelian group G. If $\dim V(H) \leq \dim W(H)$ for every $H \in \mathcal{D} \setminus \{G\}$, then there exists a G-isovariant map $f : V \to W$.*

Proof. It suffices to show that there is a G-isovariant map between $V(H)$ and $W(H)$ for every $H \in \mathcal{D} \setminus \{G\}$. Let $V(H) = \bigoplus_{i=1}^{r} V_i$ and $W(H) = \bigoplus_{i=1}^{s} W_i$, where V_i and W_i are irreducible representations with kernel H. Since $r \leq s$, by Lemma 2.3 there is an isovariant map from V_i to W_i for every $1 \leq i \leq r$, and hence an isovariant map $f : V(H) \to \bigoplus_{i=1}^{r} W_i \subset W(H)$. □

We now show Theorem A (1).

Theorem 2.5. *An arbitrary abelian p-group G has the complete IB-property; namely, for any pair (V, W) of representations satisfying condition $(C_{V,W})$, there exists a G-isovariant map $f : V \to W$.*

Proof. By Proposition 2.4 it suffices to show that $\dim V(H) \leq \dim W(H)$ for any $H \in \mathcal{D} \setminus \{G\}$. Since G is an abelian p-group, for any $H \in \mathcal{D} \setminus \{G\}$, there is a unique minimal subgroup K in $\mathcal{D}$ strictly containing H. In fact, suppose that $K_1, K_2 \in \mathcal{D}$ are minimal subgroups strictly containing H. Since K_i/H, $i = 1, 2$, are subgroups of a cyclic p-group G/H, it follows that $K_1 \leq K_2$ or $K_1 \geq K_2$, and the minimality shows $K_1 = K_2$.

Let (V, W) be a pair of representations satisfying condition $(C_{V,W})$. We may set $V = \bigoplus_{L \in \mathcal{D}} V(L)$ and $W = \bigoplus_{L \in \mathcal{D}} W(L)$. Let $H \in \mathcal{D} \setminus \{G\}$ and $K \in \mathcal{D}$ a unique minimal subgroup strictly containing H. Then $V^H = \bigoplus_{H \leq L \in \mathcal{D}} V(L)$, and

$$V^K = \bigoplus_{K \leq L \in \mathcal{D}} V(L) = \bigoplus_{H < L \in \mathcal{D}} V(L)$$

by the minimality of K. Consequently we obtain that

$$\dim V^H - \dim V^K = \dim V(H),$$

and similarly

$$\dim W^H - \dim W^K = \dim W(H).$$

Thus we have $\dim V(H) \leq \dim W(H)$ by $(C_{V,W})$. □

3. Elementary isovariant maps

Throughout this section G is an abelian group not of prime power order. We shall introduce a special kind of isovariant map, called an elementary isovariant map, between certain G-representations.

DEFINITION. Let p, q be distinct prime divisors of $|G|$. A sequence of subgroups of G: $\{H_1, \dots, H_r\,; K_1, \dots, K_{r+1}\}$, $r \geq 1$, is called a *W-sequence* of type (p, q) (with length r) if the following conditions are satisfied:
(1) $H_i, K_j \in \mathcal{D} \setminus \{G\}$ for any i, j,
(2) $H_i < K_i$ and $H_i < K_{i+1}$ for any $1 \leq i \leq r$,
(3) K_i/H_i is of p-power order and K_{i+1}/H_i is of q-power order for any $1 \leq i \leq r$.

For $H \in \mathcal{D} \setminus \{G\}$, let T_H denote the G-representation inflated from the G/H-representation T_1 of the cyclic group G/H, i.e., $T_H = \operatorname{Inf}_{G/H}^{G} T_1$. Note that $\operatorname{Ker} T_H = H$. If $G/H \not\cong C_2$, then T_H is irreducible as an orthogonal G-representation, and if $G/H \cong C_2$, then T_H is twice as many as the nontrivial 1-dimensional representation $\mathbb{R}_H^- = \operatorname{Inf}_{G/H}^{G} \mathbb{R}^-$; $T_H \cong \mathbb{R}_H^- \oplus \mathbb{R}_H^-$.

DEFINITION. Let $\{H_1, \dots, H_r\,; K_1, \dots, K_{r+1}\}$ be a W-sequence. A G-isovariant map from $T_{H_1} \oplus \cdots \oplus T_{H_r}$ to $T_{K_1} \oplus \cdots \oplus T_{K_{r+1}}$ is called an *elementary* G-isovariant map (with respect to the W-sequence).

Proposition 3.1. *For any W-sequence $\{H_1, \dots, H_r\,; K_1, \dots, K_{r+1}\}$ of type (p, q), there exists an elementary G-isovariant map*

$$f\colon T_{H_1} \oplus \cdots \oplus T_{H_r} \to T_{K_1} \oplus \cdots \oplus T_{K_{r+1}}.$$

In order to prove this proposition, we shall first show basic properties of a W-sequence.

Lemma 3.2. *Let $\{H_1, \dots, H_r\,; K_1, \dots, K_{r+1}\}$ be a W-sequence of type (p, q).*
(1) *$H_i \not\leq H_j$ and $H_i \not\geq H_j$ for $i \neq j$, and similarly $K_i \not\leq K_j$ and $K_i \not\geq K_j$ for $i \neq j$.*
(2) *For any $H_{i_1}, \dots, H_{i_k}$ $(1 \leq i_1 < \cdots < i_k \leq r)$, $\bigcap_{s=1}^{k} H_{i_s} = H_{i_1} \cap H_{i_k}$. Similarly for any $K_{i_1}, \dots, K_{i_k}$ $(1 \leq i_1 < \cdots < i_k \leq r+1)$, $\bigcap_{s=1}^{k} K_{i_s} = K_{i_1} \cap K_{i_k}$.*
(3) *$H_i \cap H_j \in \mathcal{D}$ $(i \neq j)$; namely, $G/(H_i \cap H_j)$ is cyclic.*
(4) *$K_i \cap K_j = H_i \cap H_{j-1}$ $(i < j)$, in particular, $K_i \cap K_{i+1} = H_i$.*

Proof. For each H, decompose H into the form of $H_p \times H_q \times H'$, where H_l denotes a Sylow l-group of H, $l = p, q$, and $H' = \prod_{l \neq p,q} H_l$.

(1): Let $H_i = H_{i,p} \times H_{i,q} \times H_i'$ and $K_i = K_{i,p} \times K_{i,q} \times K_i'$. Since K_i/H_i is of p-power order and K_{i+1}/H_i is of q-power order, we obtain

$$\text{(a)}\quad H_{i,p} < K_{i,p}, \qquad \text{(b)}\quad H_{i,q} = K_{i,q}, \qquad \text{(c)}\quad H_{i,p} = K_{i+1,p},$$
$$\text{(d)}\quad H_{i,q} < K_{i+1,q}, \qquad \text{(e)}\quad K_i' = H_i' = K_{i+1}'$$

for every i. It follows from (e) that $H_1' = \cdots = H_r' = K_1' = \cdots = K_{r+1}'$. We denote by L this common subgroup. Moreover we obtain

$$\text{(f)}\quad H_{i,p} > H_{i+1,p}, \qquad \text{(g)}\quad H_{i,q} < H_{i+1,q}, \qquad \text{(h)}\quad K_{i,p} > K_{i+1,p}, \qquad \text{(i)}\quad K_{i,q} < K_{i+1,q}.$$

In fact (f) follows from (c) and (a), and (g) follows from (b) and (d); the others are similar. The inclusions (f)–(i) show (1).

(2): The inclusions (f)–(i) imply that

$$\begin{aligned}\bigcap_s H_{i_s} &= \bigcap_s H_{i_s,p} \times H_{i_s,q} \times L \\ &= H_{i_k,p} \times H_{i_1,q} \times L \\ &= H_{i_k} \cap H_{i_1},\end{aligned}$$

and similarly $\bigcap_s K_{i_s} = K_{i_1} \cap K_{i_k}$.

(3): Suppose $i < j$. The above inclusions show that $H_i \cap H_j = H_{j,p} \times H_{i,q} \times L$. Since $G/H_i \cong G_p/H_{i,p} \times G_q/H_{i,q} \times G'/L$ and $G/H_j \cong G_p/H_{j,p} \times G_q/H_{j,q} \times G'/L$ are cyclic, $G_p/H_{j,p}, G_q/H_{i,q}$ and G'/L are also cyclic, and their orders are pairwise coprime. Hence $G/(H_i \cap H_j) \cong G_p/H_{j,p} \times G_q/H_{i.q} \times G'/L$ is cyclic.

(4): Similarly we obtain that $K_i \cap K_j = K_{j,p} \times K_{i,q} \times L$ and $H_i \cap H_{j-1} = H_{j-1,p} \times H_{i,q} \times L$. By (c) and (b), $K_{j,p} = H_{j-1,p}$ and $K_{i,q} = H_{i,q}$. Hence $K_i \cap K_j = H_i \cap H_{j-1}$. □

Lemma 3.3. *Let $U = T_{L_1} \oplus \cdots \oplus T_{L_r}$, $L_i \in \mathcal{D} \setminus \{G\}$. Then for any nonzero $z = (z_1, \ldots, z_r) \in U$, the isotropy group G_z is equal to $\bigcap_{i\,:\,z_i \neq 0} L_i$.*

Proof. This follows from Proposition 2.2 (1). □

We now prove Proposition 3.1.

Proof of Proposition 3.1. Set $V = T_{H_1} \oplus \cdots \oplus T_{H_r}$ and $W = T_{K_1} \oplus \cdots \oplus T_{K_{r+1}}$, and set $a_i = |K_i/H_i|$ and $b_i = |K_{i+1}/H_i|$. We define a map $f: V \to W$ by setting

$$f(z_1, \ldots, z_r) = \left(z_1^{a_1}, z_1^{b_1} + z_2^{a_2}, \ldots, z_{r-1}^{b_{r-1}} + z_r^{a_r}, z_r^{b_r}\right).$$

We claim that this map is G-isovariant. Since $h_k: T_H \to T_K$; $h_k(z) = z^k$, $k = |K/H|$, is G-equivariant for any pair $H < K$ in $\mathcal{D} \setminus \{G\}$, it follows that f is G-equivariant. Let $z = (z_1, \ldots, z_r)$ be any nonzero vector of V. Let $s = \min\{i \mid z_i \neq 0\}$ and $t = \max\{i \mid z_i \neq 0\}$. Then $f(z)$ is expressed as

$$f(z) = \left(0, \ldots, 0, z_s^{a_s}, z_s^{b_s} + z_{s+1}^{a_{s+1}}, \ldots, z_{t-1}^{b_{t-1}} + z_t^{a_t}, z_t^{b_t}, 0, \ldots, 0\right).$$

By Lemmas 3.2 (2) and 3.3, it follows that $G_z = H_s \cap H_t$ and $G_{f(z)} = K_s \cap K_{t+1}$; hence $G_z = G_{f(z)}$ by Lemma 3.2 (4). If $z = 0$, then $f(z) = 0$, and so $G_z = G = G_{f(z)}$. Thus f is G-isovariant. □

4. The case of the cyclic group of order $p^n q^m$

The aim of this section is to give a proof of the following result.

Theorem 4.1. *The cyclic group of order $p^n q^m$ has the complete IB-property, where p, q are distinct primes.*

In general, condition $(C_{V,W})$ does not imply that $\dim V(H) \leq \dim W(H)$, and the argument in §3 does not work. For example, consider C_{pq}-representations $V = T_1$ and $W = T_p \oplus T_q$, where p, q are distinct primes. Then the pair (V, W) satisfies $(C_{V,W})$. On the other hand, $\dim V(1) = \dim T_1 = 2 > \dim W(1) = 0$.

Let G be an abelian group not of prime power order. Suppose that a pair (V, W) of G-representations satisfies condition $(C_{V,W})$. In order to show the existence of an isovariant map from V to W, one may assume that $V^G = W^G = 0$ by Lemma 1.2. Set $\alpha_{W,V}(H) = \dim W(H) - \dim V(H)$ for $H < G$. If $\alpha_{W,V}(H) \geq 0$, from Proposition 2.4 there is an isovariant map from $V(H)$ to some subrepresentation W' of $W(H)$ with $\dim V(H) = \dim W'$. Similarly, if $\alpha_{W,V}(H) \leq 0$, then there is an isovariant map from some subrepresentation V' of $V(H)$ with $\dim V' = \dim W(H)$ to $W(H)$.

Lemma 4.2. *With the notation above, a pair of $\overline{V} := V - V(H)$, $\overline{W} := W - W'$ satisfies $(C_{\overline{V},\overline{W}})$ when $\alpha_{W,V}(H) \geq 0$. Similarly a pair of $\overline{V} := V - V'$, $\overline{W} := W - W(H)$ satisfies $(C_{\overline{V},\overline{W}})$ when $\alpha_{W,V}(H) \leq 0$.*

Proof. Note first that for any G-representation U and for any subgroups L, M, it holds that $U(L)^M = U(L)$ if $M \leq L$ and that $U(L)^M = 0$ if $M \not\leq L$. When $\alpha_{W,V}(H) \geq 0$, we obtain that

$$\dim \overline{V}^S = \begin{cases} \dim V^S & \text{if} \quad S \not\leq H \\ \dim V^S - \dim V(H) & \text{if} \quad S \leq H, \end{cases}$$

and

$$\dim \overline{W}^S = \begin{cases} \dim W^S & \text{if} \quad S \not\leq H \\ \dim W^S - \dim W' & \text{if} \quad S \leq H. \end{cases}$$

Since $\dim V(H) = \dim W'$,

$$\dim \overline{V}^S - \dim \overline{W}^S = \dim V^S - \dim W^S$$

for any subgroup S. Noting that $(C_{V,W})$ is equivalent to the following condition

$$\dim V^S - \dim W^S \leq \dim V^T - \dim W^T \quad \text{for every pair} \quad S \leq T,$$

one can see that $(C_{V,W})$ implies $(C_{\overline{V},\overline{W}})$. The other case is similar. □

By this lemma and Lemma 1.1 (4), the existence problem of an isovariant map is reduced to a simpler case; namely, it suffices to consider the problem for any pair (V, W) of representations satisfying the following condition

$(D_{V,W})$: (1) For each $H \in \mathcal{D} \setminus \{G\}$, (a) $V(H) = 0$, $W(H) \neq 0$, (b) $V(H) \neq 0$, $W(H) = 0$, or (c) $V(H) = 0$, $W(H) = 0$,
(2) $V(G) = W(G) = 0$.

Set

$$\mathcal{E}_+(V, W) = \{H \mid \alpha_{W,V}(H) > 0, H \neq G\},$$
$$\mathcal{E}_-(V, W) = \{H \mid \alpha_{W,V}(H) < 0, H \neq G\}.$$

For simplicity we denote $\mathcal{E}_+(V, W)$ by $\mathcal{E}_+$ and $\mathcal{E}_-(V, W)$ by $\mathcal{E}_-$. If (V, W) satisfies condition $(D_{V,W})$, then $\mathcal{E}_+$ [resp. $\mathcal{E}_-$] coincides with the set of subgroups satisfying (a) [resp. (b)] of $(D_{V,W})$. Note also that $\mathcal{E}_+, \mathcal{E}_- \subset \mathcal{D} \setminus \{G\}$.

REMARK. Condition $(D_{V,W})$ is equivalent to that $V = \bigoplus_{H \in \mathcal{E}_-} V(H)$ and $W = \bigoplus_{H \in \mathcal{E}_+} W(H)$.

Lemma 4.3. *If (V, W) satisfies conditions $(C_{V,W})$ and $(D_{V,W})$, then $G/H \not\cong C_2$ for any $H \in \mathcal{E}_-$, in particular, $\dim V(H)$ is even for any $H \in \mathcal{E}_-$.*

Proof. If $G/H \cong C_2$, then $(C_{V,W})$ for the pair (H, G) implies $\alpha_{W,V}(H) \geq 0$. □

One can further reduce the problem as follows.

Lemma 4.4. *If (V, W) satisfies conditions $(C_{V,W})$ and $(D_{V,W})$, then the existence problem is reduced to the case* (1): *$V(H)$ is isomorphic to a direct sum of copies of T_H for every $H \in \mathcal{E}_-$. In addition, if G is cyclic, it is also reduced to the case* (2): *$W(H)$ is a direct sum of copies of T_H for every $H \in \mathcal{E}_+$.*

Proof. (1): From Lemma 4.3 there is no 1-dimensional irreducible subrepresentation of $V(H)$. Using Lemma 2.3, one may assume that $V(H)$ is a direct sum of copies of T_H.

(2): When $G/H \not\cong C_2$, in the same way, one may assume that $W(H)$ is a direct sum of copies of T_H. Suppose $G/H \cong C_2$. Then $W(H) \cong b\mathbb{R}_H^-$, $b = \dim W(H)$. If b is even, then $W(H) \cong (b/2)T_H$ since $T_H \cong 2\mathbb{R}_H^-$. When b is odd, we set $W' = W - \mathbb{R}_H^- \subset W$. Then the pair (V, W') satisfies $(C_{V,W'})$, in fact, for any pair of $L < K$, we have

$$\dim V^L - \dim V^K \leq \dim W^L - \dim W^K$$

using $(C_{V,W})$. If $K \leq H$ or $L \not\leq H$, then it can be seen that

$$\dim W^L - \dim W^K = \dim W'^L - \dim W'^K.$$

Hence it follows that

$$\dim V^L - \dim V^K \leq \dim W'^L - \dim W'^K.$$

If $K \not\leq H$ and $L \leq H$, then $\dim W(H)^L - \dim W(H)^K = b$ is odd, and $\dim W(S)^L - \dim W(S)^K$ is even for every $S \neq H$, since G is cyclic. Consequently we obtain that $\dim W^L - \dim W^K$ is odd. Moreover we have

$$\dim W^L - \dim W^K = \dim W'^L - \dim W'^K + 1.$$

Since $\dim V^L - \dim V^K$ is even by Lemma 4.3, it turns out that

$$\dim V^L - \dim V^K < \dim W^L - \dim W^K,$$

and hence

$$\dim V^L - \dim V^K \leq \dim W'^L - \dim W'^K.$$

Thus $(C_{V,W'})$ is satisfied. Since $\dim W'(H)$ is even and $W'(H) \cong ((b-1)/2)T_H$, the problem is reduced to the case where $W(H)$ is a direct sum of copies of T_H. $\square$

We shall give the following definition.

DEFINITION. A pair (V, W) of representations is called *reduced* if
(1) (V, W) satisfies condition $(D_{V,W})$,
(2) $V(H) = a_H T_H$ for $H \in \mathcal{E}_-$ and $W(H) = b_H T_H$ for $H \in \mathcal{E}_+$, where a_H, b_H are some positive integers.

From the argument above we conclude the following.

Proposition 4.5. *Let G be a cyclic group. If there exists a G-isovariant map from V to W for every reduced pair (V, W) satisfying condition $(C_{V,W})$, then G has the complete IB-property.*

We hereafter focus on the case of the cyclic group $G = C_{p^n q^m}$ of order $p^n q^m$ (p, q: distinct primes and $m, n \geq 1$).

Lemma 4.6. *Let $G = C_{p^n q^m}$. Suppose that a pair (V, W) of G-representations satisfies condition $(C_{V,W})$ and $(D_{V,W})$. For any $H \in \mathcal{E}_-$, there exist subgroups K, K' in $\mathcal{E}_+$ containing H such that $K \cap K' = H$. In the case, K/H is a cyclic l-group and K'/H a cyclic l'-group, where l is one of p and q, and l' the other one.*

Proof. Since $W = \bigoplus_{K \in \mathcal{E}_+} W(K)$ [resp. $V = \bigoplus_{K \in \mathcal{E}_-} V(K)$], every isotropy group G_x of W [resp. V] is described as an intersection of some subgroups $K \in \mathcal{E}_+$ [resp. $\mathcal{E}_-$], and vice versa, cf. Proposition 2.2. Since $H \in \operatorname{Iso} V$, it follows from Proposition 1.4 that H is in $\operatorname{Iso} W$, and that H is described as an intersection of some subgroups $K \in \mathcal{E}_+$, say, $H = \bigcap_{i=1}^{r} K_i$, $K_i \in \mathcal{E}_+$. Since $H \notin \mathcal{E}_+$, each K_i is strictly larger than H. Let $H = H_p \times H_q$ and $K_i = K_{i,p} \times K_{i,q}$ be the decompositions into product of Sylow subgroups. Since each $K_{i,l}$, $l = p, q$, is a cyclic l-group, there are the minima $K_{i_0,p}$ and $K_{i_1,q}$ of $\{K_{i,p}\}$ and $\{K_{i,q}\}$, respectively. Therefore

$$H = \bigcap_i K_{i,p} \times \bigcap_i K_{i,p} = K_{i_0,p} \times K_{i_1,q} = K_{i_0} \cap K_{i_1}.$$

In the case, since $K/H \cap K'/H = 1$, $|K/H|$ and $|K'/H|$ are coprime; hence $K/H, K'/H$ are of prime power order. $\square$

Now we prove Theorem 4.1.

Proof of Theorem 4.1. We show the theorem by induction on $\dim V$. If $V = 0$, then the theorem is trivial. Suppose $\dim V > 0$. By Proposition 4.5, we may assume that (V, W) is a reduced pair satisfying $(C_{V,W})$. Take a subgroup $H \in \mathcal{E}_-$. By Lemma 4.6, there exist $K, K' \in \mathcal{E}_+$ such that K/H is a cyclic p-group and K'/H is a cyclic q-group. Then $S_1 = \{H\,; K, K'\}$ is a W-sequence of type (p, q). Take a maximal W-sequence $S = \{H_1, \ldots, H_r\,; K_1, \ldots, K_{r+1}\}$ of type (p, q) in the following sense:

(1) $\{H_1, \ldots, H_r\} \subset \mathcal{E}_-$ and $\{K_1, \ldots, K_{r+1}\} \subset \mathcal{E}_+$,

(2) there is no W-sequence strictly containing S with property (1).

Set $V' := \bigoplus_i T_{H_i}$ and $W' := \bigoplus_i T_{K_i}$. By Proposition 3.1 there is an isovariant map $f'\colon V' \to W'$. Set $\overline{V} = V - V'$ and $\overline{W} = W - W'$. Then the next lemma says that the pair $\left(\overline{V}, \overline{W}\right)$ satisfies $(C_{\overline{V},\overline{W}})$, and hence there is an isovariant map $\overline{f}\colon \overline{V} \to \overline{W}$ by the inductive assumption. Thus we obtain an isovariant map $f := \overline{f} \oplus f'\colon V \to W$. $\square$

The remainder of proof is to show the following:

Lemma 4.7. *The pair $\left(\overline{V}, \overline{W}\right)$ satisfies $(C_{\overline{V},\overline{W}})$.*

Proof. It suffices to check $(C'_{\overline{V},\overline{W}})$ by Proposition 1.3. Let $H < K$ with $K/H \cong C_l$, $l = p, q$. One may suppose $l = p$ without loss of generality. We set

$$S_q(H) = \{L \mid H \le L \le G \quad \text{and} \quad L/H \quad \text{is of } q\text{-power order}\}.$$

Let $\mathcal{H} = \{H_1, \dots, H_r\}$ and $\mathcal{K} = \{K_1, \dots, K_{r+1}\}$. Note first that

$$\dim V^H - \dim V^K = \sum_{L \in \mathcal{S}_q(H) \cap \mathcal{E}_-} \dim V(L),$$
$$\dim W^H - \dim W^K = \sum_{L \in \mathcal{S}_q(H) \cap \mathcal{E}_+} \dim V(L),$$
$$\dim V'^H - \dim V'^K = \sum_{L \in \mathcal{S}_q(H) \cap \mathcal{H}} \dim T_L,$$
$$\dim W'^H - \dim W'^K = \sum_{L \in \mathcal{S}_q(H) \cap \mathcal{K}} \dim T_L.$$

From condition $(C_{V,W})$, it holds that

$$\dim V^H - \dim V^K \le \dim W^H - \dim W^K.$$

Looking at the diagram of the subgroup lattice of $C_{p^n q^m}$, one can see from Lemma 3.2 (1) that there are the following three possibilities:
(1) $S_q(H) \cap \mathcal{H} = \{H_i\}$, $S_q(H) \cap \mathcal{K} = \{K_{i+1}\}$ for some i,
(2) $S_q(H) \cap \mathcal{H} = \emptyset$, $S_q(H) \cap \mathcal{K} = \{K_1\}$,
(3) $S_q(H) \cap \mathcal{H} = \mathcal{S}_q(H) \cap \mathcal{K} = \emptyset$.

In case (1), it follows that

$$\dim V'^H - \dim V'^K = \dim W'^H - \dim W'^K (= 2),$$

and hence

$$\dim \overline{V}^H - \dim \overline{V}^K \le \dim \overline{W}^H - \dim \overline{W}^K.$$

In case (2), one can see that $\mathcal{S}_q(H) \cap \mathcal{E}_-$ is empty, in fact if there is $H_0 \in \mathcal{S}_q(H) \cap \mathcal{E}_-$, then there is $K_0 \in \mathcal{E}_+$ such that K_0/H_0 is a cyclic p-group by Lemma 4.6. Then $\{H_0, H_1, \dots, H_r; K_0, K_1, \dots, K_{r+1}\}$ is a larger W-sequence containing $S = \{H_1, \dots, H_r; K_1, \dots, K_{r+1}\}$; this contradicts the maximality of S. Thus we see that $\dim V^H - \dim V^K = 0$. On the other hand $\dim W^H - \dim W^K \ge 2$, since $K_1 \in \mathcal{S}_q(H) \cap \mathcal{E}_+$. Moreover, since $\dim V'^H - \dim V'^K = 0$ and $\dim W'^H - \dim W'^K = 2$, it follows that

$$0 = \dim \overline{V}^H - \dim \overline{V}^K \le \dim \overline{W}^H - \dim \overline{W}^K.$$

In case (3), obviously

$$\dim V'^H - \dim V'^K = 0, \quad \dim W'^H - \dim W'^K = 0,$$

and hence

$$\dim \overline{V}^H - \dim \overline{V}^K \le \dim \overline{W}^H - \dim \overline{W}^K.$$

Thus the proof is complete. □

REMARK. From the proof of Theorem 4.1, we see that for any reduced pair (V, W) satisfying $(C_{V,W})$, an isovariant map from V to W is constructed as a direct sum of elementary isovariant maps.

5. The case of the cyclic group of order *pqr*

Let $G = C_{pqr}$. Generally an isovariant map between G-representations is not constructed by using only elementary isovariant maps as described in §3 For example, a pair of G-representations $V = T_p \oplus T_q \oplus T_r$ and $W = T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{rp}$ satisfies condition $(C_{V,W})$, but an isovariant map from V to W cannot be constructed by using only elementary isovariant maps. We shall show the existence of an isovariant map using equivariant obstruction theory.

Proposition 5.1. *Let $G = C_{pqr}$, where p, q, r are distinct primes. Then there exists a G-isovariant map from $V = T_p \oplus T_q \oplus T_r$ to $W = T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{rp}$.*

Proof. Note that each G-representation T_i is obtained by restricting an S^1-representation. We regard V, W as S^1-representations. By Lemmas 1.1 (1) and 1.2, it suffices to show that there exists an S^1-isovariant map from $S(V)$ to $S(W)$. The singular set $S(V)^{>1} := \bigcup_{H\neq 1} S(V)^H$ of $S(V)$ consists of disjoint three circles $S(T_p)$, $S(T_q)$, $S(T_r)$, which are exceptional orbits (in the sense of [1]) isomorphic to S^1/C_p, S^1/C_q and S^1/C_r, respectively. Let N_i, $i = p, q$ or r, be a closed S^1-tubular neighborhood of $S(T_i)$ in $S(V)$ such that N_i are disjoint. The slice theorem (cf. [1], [4]) says that N_i is equivariantly diffeomorphic to $S^1 \times_{C_i} D(T_j \oplus T_k)$, where $i, j, k \in \{p, q, r\}$ are distinct. Similarly take an orbit in $S(W)$ isomorphic to S^1/C_i and its closed S^1-tubular neighborhood A_i, equivariantly diffeomorphic to $S^1 \times_{C_i} D(W_i)$ for some C_i-representation W_i, such that A_i are disjoint. There is an S^1-isovariant map $\tilde{f}_i : N_i \to A_i$ such that $\tilde{f}_i(\partial N_i) \subset \partial A_i$, in fact, since C_i acts freely on $S(T_j \oplus T_k)$ and $S(W_i) \setminus S(W_i)^{>1}$, and since

$$\dim S(T_j \oplus T_k) = 3 \leq \dim S(W_i) - \dim S(W_i)^{>1} = 4,$$

it follows that the pair $(T_j \oplus T_k, W_i)$ of C_i-representations satisfies $(C_{T_i \oplus T_j, W_i})$, and from Theorem 2.5 that there is a C_i-isovariant map $\bar{f}_i : S(T_j \oplus T_k) \to S(W_i)$. Taking cone, we have a C_i-isovariant map $C\bar{f}_i : D(T_j \oplus T_k) \to D(W_i)$, and hence an S^1-isovariant map $\tilde{f}_i = S^1 \times_{C_i} C\bar{f}_i : N_i \to A_i$ such that $\tilde{f}_i(\partial N_i) \subset \partial A_i$.

Next set $Y = S(W) \backslash S(W)^{>1}$, $X = S(V) \backslash \mathrm{Int}\big(N_p \coprod N_q \coprod N_r\big)$, and $f_i = \tilde{f}_i|_{\partial N_i} : \partial N_i \to \partial A_i \subset Y$. Since S^1 acts freely on X and Y, it suffices to see that there is an S^1-map from X to Y extending $f := \coprod_i f_i : \coprod_i \partial N_i \to \coprod_i \partial A_i \subset Y$. Note that $\dim X/S^1 = 4$ and Y is 2-connected by an argument of general position. Note also that $\pi_3(Y) \cong H_3(Y) \cong \mathbb{Z}^3$. The obstruction to an extension of f lies in $\mathfrak{H}^4_{S^1}(X, \partial X; \pi_3(Y)) \cong$

$H^4(X/S^1, \partial X/S^1; \pi_3(Y)) \cong \pi_3(Y)$, see [3, II §3]. One can detect this obstruction using notion of the multidegree [7]. Here we shall recall necessary facts from [7]. The multidegree of an S^1-map $h\colon \partial N_i \to Y$ is defined by setting

$$\text{m-Deg}\, h = \overline{h}_*([S(T_j \oplus T_k)]) \in H_3(Y) \cong \mathbb{Z}^3,$$

where $\overline{h} = h|_{S(T_j \oplus T_k)}\colon S(T_j \oplus T_k) \to Y$, and $[S(T_j \oplus T_k)]$ is the fundamental class of $S(T_j \oplus T_k)$. We identify $H_3(Y)$ with $\mathbb{Z}^3$ via the isomorphisms induced by the inclusions:

$$H_3(Y) \underset{\cong}{\to} \bigoplus_i H_3(SW \setminus S(T_i)) \underset{\cong}{\leftarrow} \bigoplus_i H_3(S(T_j \oplus T_k)) = \mathbb{Z}^3.$$

Let $d_i(h) \in \mathbb{Z} = H_3(S(T_j \oplus T_k))$ denote the i-component of m-Deg h for $i = p, q, r$; namely, m-Deg $h = (d_p(h), d_q(h), d_r(h)) \in \mathbb{Z}^3$. Note that there exists an S^1-map $F_0\colon X \to Y$ (not necessarily extending f), since the obstruction group $\mathfrak{H}^*_{S^1}(X, \pi_{*-1}(Y)) \cong H^*(X/S^1; \pi_{*-1}(Y))$ vanishes. We fix such a map F_0 and set $f_{0,i} = F_0|_{\partial N_i}$. The following facts are derived from [7, §3].
(1) $d_i(f_j) = 0$ for $i \neq j$.
(2) $\text{m-Deg}(f_i) - \text{m-Deg}(f_{0,i}) \in i\mathbb{Z}^3$.
(3) For any $a \in i\mathbb{Z}$ there exists an S^1-isovariant map $\tilde{f}_i'\colon N_i \to A_i \subset SW$ such that $\tilde{f}_i'(\partial N_i) \subset \partial A_i$ and such that $d_i(f_i') = d_i(f_i) + a$ and $d_j(f_i') = 0$ for $j \neq i$, where $f_i' = \tilde{f}_i'|_{\partial N_i}$.
(4) Under identifying the obstruction group $\mathfrak{H}^4_{S^1}(X, \partial X; \pi_3(Y))$ with $\mathbb{Z}^3$, the obstruction class $\gamma_{S^1}(f)$ to an extension of f is described as

$$\gamma_{S^1}(f) = \sum_{i=p,q,r} \frac{\text{m-Deg}\, f_i - \text{m-Deg}\, f_{0,i}}{i}.$$

Using the facts (3) and (4), one can take suitable S^1-isovariant maps $\tilde{f}_i'\colon N_i \to A_i \subset S(W)$ such that $\gamma_{S^1}(f') = 0$, where $f' = \coprod_i f_i'$, $f_i' = \tilde{f}'|_{\partial N_i}$. Hence there exists an S^1-map $F\colon X \to Y$ extending f'. Attaching the boundaries, we obtain an S^1-isovariant map $F \cup \coprod_i \tilde{f}_i'\colon S(V) \to S(W)$. □

The main result of this section is the following:

Theorem 5.2. *C_{pqr} has the complete IB-property, where p, q, r are distinct primes.*

We first show the following.

Lemma 5.3. *Let (V, W) be a reduced pair of C_{pqr}-representations. Let i, j, k denote distinct primes in $\{p, q, r\}$. Then*
(1) $C_{ij} \notin \mathcal{E}_-$.

(2) *If* $C_i \in \mathcal{E}_-$, *then* $\dim V(C_i) \le \dim W(C_{ij})$, *in particular*, $C_{ij} \in \mathcal{E}_+$.

Proof. (1): By $(C_{V,W})$ for the pair (C_{ij}, G),

$$\dim V(C_{ij}) = \dim V^{C_{ij}} \le \dim W^{C_{ij}} = \dim W(C_{ij}).$$

This implies $C_{ij} \notin \mathcal{E}_-$.

(2): $(C_{V,W})$ for the pair (C_i, C_{ik}) says that

$$\dim V^{C_i} - \dim V^{C_{ik}} \le \dim W^{C_i} - \dim W^{C_{ik}}.$$

It is seen by (1) that $\dim V^{C_i} = \dim V(C_i)$ and $\dim V^{C_{ik}} = 0$. Noting that $C_i \notin \mathcal{E}_+$, we have $\dim W^{C_i} = \dim W(C_{ij}) + \dim W(C_{ik})$. Since $W^{C_{ik}} = W(C_{ik})$, it follows from the above inequality that $\dim V(C_i) \le \dim W(C_{ij})$. In particular $\dim W(C_{ij}) > 0$, and hence $C_{ij} \in \mathcal{E}_+$. □

Proof of Theorem 5.2. By Proposition 4.5, one may assume that a pair (V, W) of representations satisfying $(C_{V,W})$ is a reduced pair; namely,

$$V = \bigoplus_{H \in \mathcal{E}_-} V(H), \quad V(H) = a_H T_H,$$

$$W = \bigoplus_{H \in \mathcal{E}_+} W(H), \quad W(H) = b_H T_H,$$

where a_H, b_H are positive integers. The proof of Theorem 5.2 is divided into several cases. From Lemma 5.3 and a symmetrical role of p, q, r, it suffices to consider the following seven cases: (1) $\mathcal{E}_- = \{1\}$, (2) $\mathcal{E}_- = \{C_p\}$, (3) $\mathcal{E}_- = \{1, C_p\}$, (4) $\mathcal{E}_- = \{C_p, C_q\}$, (5) $\mathcal{E}_- = \{1, C_p, C_q\}$, (6) $\mathcal{E}_- = \{C_p, C_q, C_r\}$, (7) $\mathcal{E}_- = \{1, C_p, C_q, C_r\}$.

CASE (1): In this case, G acts freely on $S(V)$. By $(C_{V,W})$ of a pair $(1, H)$, we see

$$\dim S(V) + 1 \le \dim S(W) - \dim S(W)^H$$

for any subgroup H, and hence

$$\dim S(V) + 1 \le \dim S(W) - \dim S(W)^{>1}.$$

Set $d = \dim S(W) - \dim S(W)^{>1}$ and $Y = S(W) \setminus S(W)^{>1}$. Since Y is $(d-2)$-connected by an argument of general position, the obstruction to the existence of a G-map $f\colon S(V) \to S(W)$ lies in $\mathfrak{H}^*_G(S(V); \pi_{*-1}(Y)) \cong H^*(S(V)/G; \pi_{*-1}(Y))$, $* \ge d$. The above inequality, however, shows that the cohomology groups vanish. Hence there is a G-map $f\colon S(V) \to Y$, which is G-isovariant since G acts freely on $S(V)$ and Y. Composing f with the inclusion $Y \subset S(W)$, we obtain a G-isovariant map from $S(V)$ to $S(W)$, which induces a G-isovariant map from V to W.

CASE (2): Note that the kernel of V is C_p. Since $(C_{V^{C_p},W^{C_p}})$ is satisfied and $G/C_p \cong C_{qr}$, there is a G/C_p-isovariant map $f\colon V = V^{C_p} \to W^{C_p}$ by Theorem 4.1. Thus we obtain a G-isovariant map $\mathrm{Inf}_{G/C_p}^G f\colon V \to W^{C_p} \subset W$.

CASE (3): By Lemma 5.3, we have $\dim V(C_p) \leq \dim W(C_{pj})$, $j = q, r$. Take a subrepresentation $W'(C_{pj}) \subset W(C_{pj})$ with $\dim W'(C_{pj}) = \dim V(C_p)$. Using Proposition 3.1, we obtain a G-isovariant map $f_1\colon V(C_p) \to W'(C_{pq}) \oplus W'(C_{pr})$. Set $\overline{V} = V - V(C_p)$ and $\overline{W} = W - W'(C_{pq}) \oplus W'(C_{pr})$. One can easily verify that $(C_{\overline{V},\overline{W}})$ are satisfied. Since $\overline{V}$ is of case (1), there exists a G-isovariant map $f_2\colon \overline{V} \to \overline{W}$, and hence a G-isovariant map $f_1 \oplus f_2\colon V \to W$.

CASE (4): One may suppose that $\dim V(C_p) \geq \dim V(C_q) =: m$ without loss of generality. Since $\dim V(C_i) \leq \dim W(C_{ij})$ for $i \neq j$ $(i \in \{p, q\},\ j \in \{p, q, r\})$ by Lemma 5.3, one can take m-dimensional subrepresentations $V'(C_p) \subset V(C_p)$, $W'(C_{ij}) \subset W(C_{ij})$. Using Proposition 3.1, we have a G-isovariant map

$$f_1\colon V'(C_p) \oplus V(C_q) \to W'(C_{pr}) \oplus W'(C_{pq}) \oplus W'(C_{qr}).$$

Set $\overline{V} = V - V'(C_p) \oplus V(C_q)$ and $\overline{W} = W - W'(C_{pr}) \oplus W'(C_{pq}) \oplus W'(C_{qr})$. Then one can see that $(C_{\overline{V},\overline{W}})$ are satisfied. Since $\overline{V}$ is of case (2), there exists a G-isovariant map $f_2\colon \overline{V} \to \overline{W}$, and hence a G-isovariant map $f_1 \oplus f_2\colon V \to W$.

CASE (5): With the same notation and argument as in case (4), one can see that there is a G-isovariant map

$$f_1\colon V'(C_p) \oplus V(C_q) \to W'(C_{pr}) \oplus W'(C_{pq}) \oplus W'(C_{qr}).$$

Since $(C_{\overline{V},\overline{W}})$ are satisfied and $\overline{V}$ is of case (3) or (1), there exists a G-isovariant map $f_2\colon \overline{V} \to \overline{W}$, and hence a G-isovariant map $f_1 \oplus f_2\colon V \to W$.

CASE (6): One may suppose that $\dim V(C_p) \geq \dim V(C_q) \geq \dim V(C_r) =: m$ without loss of generality. By Lemma 5.3, $\dim V(C_i) \leq \dim W(C_{ij})$ for $i \neq j$ $(i, j \in \{p, q, r\})$.

SUBCASE (i): $\dim W(1) \geq m$. In this case, one can take m-dimensional subrepresentations $V'(C_s) \subset V(C_s)$ $(s = p, q)$, $W'(C_{ij}) \subset W(C_{ij})$, $i \neq j$ $(i, j \in \{p, q, r\})$, and $W'(1) \subset W(1)$. By Proposition 5.1, we have a G-isovariant map

$$f_1\colon V'(C_p) \oplus V'(C_q) \oplus V(C_r) \to W'(1) \oplus W'(C_{pq}) \oplus W'(C_{qr}) \oplus W'(C_{rp}).$$

Set $\overline{V} = V - V'(C_p) \oplus V'(C_q) \oplus V(C_r)$ and $\overline{W} = W - W'(C_{pq}) \oplus W'(C_{qr}) \oplus W'(C_{rp})$. Then one can verify that $(C_{\overline{V},\overline{W}})$ is satisfied. Since $\overline{V}$ is of case (4) or (2), there exists a G-isovariant map $f_2\colon \overline{V} \to \overline{W}$, and hence a G-isovariant map $f_1 \oplus f_2\colon V \to W$.

SUBCASE (ii): $\dim W(1) < m$. Set $n = \dim W(1)$ and take n-dimensional subrepresentations $V'(C_s) \subset V(C_s)$ $(s = p, q, r)$, $W'(C_{ij}) \subset W(C_{ij})$ $(i \neq j, i,\ j \in \{p, q, r\})$.

By Proposition 5.1, we have a G-isovariant map

$$f_1\colon V'(C_p)\oplus V'(C_q)\oplus V'(C_r)\to W(1)\oplus W'(C_{pq})\oplus W'(C_{qr})\oplus W'(C_{rp}).$$

Set

$$\begin{aligned}\overline{V}&=V-V'(C_p)\oplus V'(C_q)\oplus V'(C_r),\\ \overline{W}&=W-W(1)\oplus W'(C_{pq})\oplus W'(C_{qr})\oplus W'(C_{rp}).\end{aligned}$$

Then one can see that $(C_{\overline{V},\overline{W}})$ is satisfied, and that $\mathcal{E}_-\big(\overline{V},\overline{W}\big)=\{C_p,C_q,C_r\}$ and $\mathcal{E}_+\big(\overline{V},\overline{W}\big)=\{C_{pq},C_{qr},C_{pr}\}$. By assumption,

$$\dim\overline{V}(C_p)\geq\dim\overline{V}(C_q)\geq\dim\overline{V}(C_r).$$

Set $m'=\dim\overline{V}(C_r)$. By Lemma 5.3, we have $\dim\overline{V}(C_i)\leq\dim\overline{W}(C_{ij})$ for $i\neq j$ $(i,j\in\{p,q,r\})$. Take m'-dimensional subrepresentations $\overline{V}'(C_s)\subset\overline{V}(C_s)$, $s=q,r$, and $\overline{W}'(C_{ij})\subset\overline{W}(C_{ij})$, $i\neq j$ $(i,j\in\{p,q,r\})$. By Proposition 3.1 there exists a G-isovariant map

$$\overline{f_1}\colon\overline{V}'(C_q)\oplus\overline{V}(C_r)\to\overline{W}'(C_{pq})\oplus\overline{W}'(C_{qr})\oplus\overline{W}'(C_{pr}).$$

Set $\underline{V}=\overline{V}-\overline{V}'(C_q)\oplus\overline{V}(C_r)$ and $\underline{W}=\overline{W}-\overline{W}'(C_{pq})\oplus\overline{W}'(C_{qr})\oplus\overline{W}'(C_{pr})$. Then one can see that $(C_{\underline{V},\underline{W}})$ is satisfied, for example, $(C_{\underline{V},\underline{W}})$ for a pair (C_p,C_{pr}), i.e., $\dim\underline{V}(C_p)\leq\dim\underline{W}(C_{pq})$, can be verified as follows (other cases are easier): $(C_{\overline{V},\overline{W}})$ for the pair $(1,C_r)$ implies that

$$\dim\overline{V}(C_p)+\dim\overline{V}(C_q)\leq\overline{W}(C_{pq}),$$

and hence

$$\begin{aligned}\dim\underline{V}(C_p)=\dim\overline{V}(C_p)&\leq\dim\overline{W}(C_{pq})-\dim\overline{V}(C_q)\\ &\leq\dim\overline{W}(C_{pq})-\dim\overline{V}(C_r)\\ &=\dim\overline{W}(C_{pq})-\dim\overline{W}'(C_{pq})\\ &=\dim\underline{W}(C_{pq}).\end{aligned}$$

Since $\underline{V}$ is of case (4) or (2), there exists a G-isovariant map from $\underline{V}$ to $\underline{W}$. By the same argument as before, one can see that there is a G-isovariant map from V to W.

CASE (7): Suppose that $\dim V(C_p)\geq\dim V(C_q)\geq\dim V(C_r)=:m$. By Lemma 5.3, $\dim V(C_i)\leq\dim W(C_{ij})$, $i\neq j$ $(i,j\in\{p,q,r\})$. Take m-dimensional subrepresentations $V'(C_i)\subset V(C_i)$ and $W'(C_{ij})\subset W(C_{ij})$. Then there exists a G-isovariant map

$$f_1\colon V'(C_q)\oplus V(C_r)\to W'(C_{pq})\oplus W'(C_{qr})\oplus W'(C_{pr}).$$

Set $\overline{V} = V - V'(C_q) \oplus V(C_r)$ and $\overline{W} = W - W'(C_{pq}) \oplus W'(C_{qr}) \oplus W'(C_{pr})$. By a similar argument as Case (6), one can verify that $(C_{\overline{V},\overline{W}})$ is satisfied. Since $\overline{V}$ is of Case (5) or (3), there exists a G-isovariant map from $\overline{V}$ to $\overline{W}$, and hence from V to W.

Thus the proof is complete. $\square$

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Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka 560-0043
Japan

Current address:
Department of Mathematics
Graduate School of Medical Science
Kyoto Prefectural University of Medicine
13 Nishitakatsukasa-cho, Taishogun
Kita-ku, Kyoto 603-8334
Japan
e-mail: nagasaki@koto.kpu-m.ac.jp