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SPACELIKE CONSTANT MEAN CURVATURE 1 TRINOIDS IN DE SITTER THREE-SPACE

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Abstract
We derive a holomorphic spinor representation formula for spacelike surfaces of constant mean curvature 1 in de Sitter 3-space, and use it to construct examples of spacelike catenoids and trinoids with constant mean curvature 1.

1. Introduction

Spacelike surfaces of constant mean curvature (CMC) in pseudo-Riemannian space forms share many interesting properties in common with CMC surfaces in Riemannian space forms. In particular, there exist representation theorems by null holomorphic maps for minimal surfaces in Euclidean 3-space $\mathbb{E}^3$ [15], CMC 1 surfaces in hyperbolic 3-space $\mathbb{H}^3(-1)$ [5] [20], spacelike maximal surfaces in Lorentzian 3-space $\mathbb{L}^3$ [10] [14], and spacelike CMC 1 surfaces in de Sitter 3-space $\mathbb{S}^3_1(1)$ [1] [12], which enable us to use the powerful complex function theory for studying those surfaces.

Even though it is invaluable to have a large collection of examples for a well-developed surface theory, not many examples of *global* spacelike surfaces of CMC 1 in $\mathbb{S}^3_1(1)$ are known to this date. A reason might be that, unlike the Riemannian counterparts, spacelike CMC 1 surfaces in $\mathbb{S}^3_1(1)$ are not complete in general, and people have not paid much attention. The only complete spacelike surfaces of CMC 1 in $\mathbb{S}^3_1(1)$ are totally umbilic flat surfaces [2] [17].

If we allow some sort of singularities, however, for CMC surfaces in pseudo-Riemannian space forms we may expect to have many interesting examples. For example, Umehara and Yamada recently studied maximal surfaces with singularities in $\mathbb{L}^3$ and showed that there are interesting examples of such surfaces [24]. For spacelike surfaces in $\mathbb{S}^3_1(1)$, R. Aiyama and K. Akutagawa noted in [1] that the same null holomorphic map produces both CMC 1 surfaces in $\mathbb{H}^3(-1)$ and spacelike CMC 1 surfaces in $\mathbb{S}^3_1(1)$, hence there is a local one-to-one correspondence between them. The first named author further developed local theories of CMC 1 spacelike surfaces in $\mathbb{S}^3_1(1)$ [12] [13] in comparison with the CMC 1 surfaces in $\mathbb{H}^3(-1)$. Through their study it is naturally expected that *global* CMC 1 spacelike surfaces *with some sort of singularities* may be obtained by transferring the data for CMC 1 surfaces in hyperbolic

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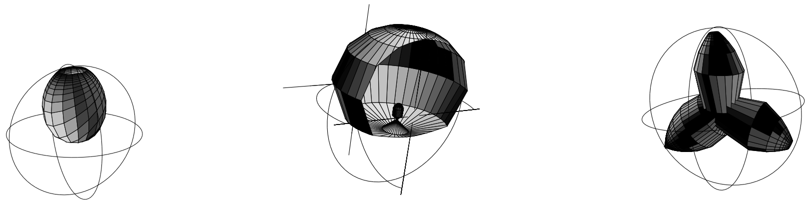

Fig. 1. Horosphere, catenoid, trinoid in the hollow ball model of $\mathbb{S}^3_1(1)$. See the last paragraph in Section 2 for the description of the hollow ball model. The three circles represent the future ideal boundary.

3-space if the period problem can be solved.

One of the interesting classes of CMC surfaces are trinoids. Umehara and Yamada gave a full classification of irreducible CMC 1 trinoids in $\mathbb{H}^3(-1)$ [23]. Then, Bobenko, Pavlyukevich, and Springborn developed a representation formula for CMC 1 surfaces in $\mathbb{H}^3(-1)$ in terms of holomorphic spinors and derived explicit formulae for CMC 1 catenoids and trinoids in $\mathbb{H}^3(-1)$ in [4]. Since the pioneering work of Bryant in 1987 [5], the main technique of constructing CMC 1 surfaces in $\mathbb{H}^3(-1)$ has been Bryant's representation theorem, or variants of it, which consists of finding a null holomorphic Ψ by integrating $\Psi^{-1}d\Psi$ given in terms of a holomorphic function and a holomorphic one-form. Bobenko, Pavlyukevich, and Springborn noted that it is more efficient to integrate $(d\Psi)\Psi^{-1}$ given in terms of spinors when one wants to find an explicit formula of Ψ. The basic reason for this phenomenon is that the data $(d\Psi)\Psi^{-1}$ is geometric, and is well defined on the same Riemann surface as the conformal immersion that Ψ represents. Rossman, Umehara, and Yamada already knew and used the equation $(d\Psi)\Psi^{-1}$ to construct CMC surfaces in $\mathbb{H}^3_1(-1)$ [18], but they interpreted it as the data for the dual surface. The second named author also integrates $(d\Psi)\Psi^{-1}$ to construct Björling representation formulae for CMC 1 surfaces in $\mathbb{H}^3(-1)$ and in $\mathbb{S}^3_1(1)$ [25].

Motivated by the results of [4], we develop in this article a representation formula for CMC 1 spacelike surfaces in $\mathbb{S}^3_1(1)$ in terms of holomorphic spinors, and use it to derive explicit formulae for CMC 1 spacelike surfaces of two-noid or trinoid type in $\mathbb{S}^3_1(1)$. In the process, we rediscover the horosphere type surfaces as degenerate catenoids, which already appear in [2] [17]. In our work, we were able to use without significant modifications many ideas and complicated computational results carried out in [4]. We are certainly indebted to them for their work.

A substantial amount of our work is to determine when the explicit solutions of the spinorial equation derived in [4] parameterized on the universal cover of $\hat{\mathbb{C}}\setminus\{0,\infty\}$ or of $\hat{\mathbb{C}}\setminus\{0,1,\infty\}$ can produce CMC 1 surfaces in $\mathbb{S}^3_1(1)$. We could completely characterize when they do.[1] It turns out that the period problem for CMC 1 two-noids and trinoids in $\mathbb{S}^3_1(1)$ can be solved in much more cases than for CMC 1 two-noids and trinoids in $\mathbb{H}^3(-1)$. Furthermore, we found that, in contrast to $\mathbb{H}^3(-1)$, there appear to be catenoidal ends in $\mathbb{S}^3_1(1)$ with abnormal behavior. It oscillates between the future and past ideal boundaries of $\mathbb{S}^3_1(1)$ infinitely many times. See Fig. 2 and comments before Definition 10. So there exists a much richer structure for two-noids and trinoids in $\mathbb{S}^3_1(1)$ than in $\mathbb{H}^3(-1)$.

Note that a spacelike surface has a natural orientation. In computing the mean curvature in this article, we choose the future, or past, pointing unit normal vector field if a secondary Gauß map g satisfies $|g| < 1$, or $|g| > 1$, respectively. The precise definition of g is given in Section 4. If $|g| = 1$, we get singularities. The existence of singularities distinguishes the theory of global spacelike surfaces in pseudo-Riemannian spaces from the theory of complete surfaces in Riemannian spaces. See Fig. 1, where a catenoid is clipped so that the conic singularity is visible. Fernández, López, and Souam studied maximal surfaces with isolated singularities in $\mathbb{L}^3$ [6]. Umehara and Yamada gave full criteria for a singularity of maximal surfaces in $\mathbb{L}^3$ to be a cuspidal edge or a swallowtail [24]. Clarifying the nature of the singularities for CMC 1 surfaces in $\mathbb{S}^3_1(1)$ requires further study. Recently, Fujimori developed in [7] a theory of spacelike CMC 1 surfaces with singularities in $\mathbb{S}^3_1(1)$, and constructed numerous examples by transferring the CMC 1 surfaces in $\mathbb{H}^3(-1)$ with a holomorphic null lift with the monodromy representation in U(1).

Construction and classification of trinoids in de Sitter three-space are not complete yet, since in this article we assume above all things that the eigenvalues of the monodromy matrices are not half integers as in [4]. Further cases as well as the study of singularities will be studied in [8] with a different method.

2. Preliminaries

Let $\mathbb{L}^4$ be the Minkowski 4-space with rectangular coordinates x_0, x_1, x_2, x_3 and the standard Lorentzian metric $\langle\ ,\ \rangle$ of signature $(-,+,+,+)$ given by the quadratic form $-(x_0)^2 + (x_1)^2 + (x_2)^2 + (x_3)^2$. The de Sitter 3-space $\mathbb{S}^3_1(1)$ is a complete timelike pseudo-Riemannian 3-manifold of sectional curvature 1 that can be realized as the hyperboloid of one sheet in $\mathbb{L}^4$:

$$\mathbb{S}^3_1(1) = \left\{(x_0, x_1, x_2, x_3) \in \mathbb{L}^4 \colon -(x_0)^2 + (x_1)^2 + (x_2)^2 + (x_3)^2 = 1\right\}.$$

Let $\mathrm{SO}(3,1)^+$ be the identity component of the special Lorentz group

$$\mathrm{SO}(3,1) = \left\{\mathcal{A} \in \mathrm{GL}(4;\mathbb{R}) \colon \det\mathcal{A} = 1,\ \ \langle\mathcal{A}\mathbf{v}, \mathcal{A}\mathbf{w}\rangle = \langle\mathbf{v}, \mathbf{w}\rangle \quad \text{for any} \quad \mathbf{v}, \mathbf{w} \in \mathbb{L}^4\right\}.$$

[1]Recently, S. Fujimori found out that we missed one case for two-noids [8].

$\mathbf{v} = (x_0, x_1, x_2, x_3) \in \mathbb{L}^4$ can be identified with the 2×2 Hermitian matrix

$$\begin{pmatrix} x_0 + x_3 & x_1 + i x_2 \\ x_1 - i x_2 & x_0 - x_3 \end{pmatrix} = \sum_{\alpha=0}^{3} x_\alpha \sigma_\alpha ,$$

where σ_α are the Pauli spin matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In terms of the corresponding matrices, the inner product $\langle\ ,\ \rangle$ of $\mathbb{L}^4$ satisfies

$$\langle \mathbf{v}, \mathbf{v} \rangle = -\det \mathbf{v}.$$

Under this identification, de Sitter 3-space $\mathbb{S}^3_1(1)$ is represented as

$$\mathbb{S}^3_1(1) = \{g\sigma_3 g^* : g \in \mathrm{SL}(2;\mathbb{C})\}, \quad \text{where} \quad g^* := \overline{g}^t .$$

The complex special linear group $\mathrm{SL}(2;\mathbb{C})$ acts isometrically on $\mathbb{L}^4$ via the C^∞ action:

$$\mathrm{SL}(2;\mathbb{C}) \times \mathbb{L}^4 \to \mathbb{L}^4; \quad (g, \mathbf{v}) \mapsto g\mathbf{v}g^*, \quad g \in \mathrm{SL}(2;\mathbb{C}), \quad \mathbf{v} \in \mathbb{L}^4.$$

This action induces a double covering $\mathrm{SL}(2;\mathbb{C}) \to SO(3,1)^+$ of the identity component of the special Lorentz group $SO(3,1)$.

Any smooth spacelike surface in $\mathbb{S}^3_1(1)$ has a natural orientation. Given a smooth spacelike surface $f : M \to \mathbb{S}^3_1(1)$, we choose a conformal structure with local coordinates $z = x + iy$ such that the future pointing unit normal vector field N satisfies $\det(N, f_x, f_y, f) > 0$. The first and second fundamental forms are

$$\mathrm{I} = \langle df, df \rangle = e^u\, dz\, d\overline{z} = e^u |dz|^2, \quad \mathrm{II} = -\langle df, dN \rangle = Q\, dz^2 + He^u\, dz\, d\overline{z} + \overline{Q}\, d\overline{z}^2,$$

where the quadratic 1-form $Q\,dz^2 := \langle f_{zz}, N \rangle\, dz^2$ is the *Hopf differential* and $H := 2e^{-u}\langle f_{z\overline{z}}, N \rangle$ is the mean curvature. The Gauß-Weingarten equations are

$$f_{zz} = u_z f_z - QN, \quad f_{z\overline{z}} = -\frac{1}{2}e^u f - \frac{1}{2}He^u N, \quad N_z = -Hf_z - 2e^{-u}Qf_{\overline{z}}. \tag{1}$$

For visualization, we identify $\mathbb{S}^3_1(1)$ with the hollow ball $e^{-\pi/2} < \sqrt{y_1^2 + y_2^2 + y_3^2} < e^{\pi/2}$ via the following formula [25]:

$$y_k = \frac{e^{\tan^{-1} x_0}}{\sqrt{1 + x_0^2}} x_k, \quad k = 1, 2, 3.$$

3. An adapted spinor frame representation

Let $f: M \to \mathbb{S}^3_1(1)$ be as in the previous section. Then, by the doubling covering, there exists a lift $F: U \to \mathrm{SL}(2;\mathbb{C})$, called a local adapted framing of f, of the orthonormal frame field

$$\mathcal{F} = \left(N, e^{-u/2} f_x, e^{-u/2} f_y, f\right): M \to \mathrm{SO}(3,1)^+,$$

where U is an oriented and simply-connected open set in M, such that

$$F\sigma_0 F^* = FF^* = N, \quad F\sigma_1 F^* = e^{-u/2} f_x, \quad F\sigma_2 F^* = e^{-u/2} f_y, \quad F\sigma_3 F^* = f. \tag{2}$$

Note that $\operatorname{tr} FF^* > 0$ for any $F \in \mathrm{SL}(2;\mathbb{C})$, hence FF^* is future pointing.

Let $\Omega = F^{-1} dF = F^{-1}F_z\,dz + F^{-1}F_{\bar z}\,d\bar z: TU \to \mathfrak{sl}(2;\mathbb{C})$ be the Maurer-Cartan form. By calculating $(f_z)_z, (f_z)_{\bar z}, N_z$ from (2) and comparing the results with (1), we see

(3)

$$F^{-1}F_z = \begin{pmatrix} \dfrac{u_z}{4} & -\dfrac{1}{2}(H+1)e^{u/2} \\ -Qe^{-u/2} & -\dfrac{u_z}{4} \end{pmatrix}, \quad F^{-1}F_{\bar z} = \begin{pmatrix} -\dfrac{u_{\bar z}}{4} & -\bar{Q}e^{-u/2} \\ -\dfrac{1}{2}(H-1)e^{u/2} & \dfrac{u_{\bar z}}{4} \end{pmatrix}.$$

The Maurer-Cartan equation $d\Omega + \Omega \wedge \Omega = 0$ and the Gauß-Codazzi equations

$$u_{z\bar z} - \frac{1}{2}(H^2-1)e^u + 2e^{-u}Q\bar{Q} = 0, \quad Q_{\bar z} = \frac{1}{2}e^u H_z \tag{4}$$

are equivalent. We immediately see that $f: M \to \mathbb{S}^3_1(1)$ has constant mean curvature if and only if the Hopf differential is holomorphic, i.e., $Q_{\bar z} = 0$.

We observe from (2) that

$$df = f_z\,dz + f_{\bar z}\,d\bar z = e^{u/4}F \begin{pmatrix} 0 & dz \\ d\bar z & 0 \end{pmatrix} \left(e^{u/4}F\right)^*.$$

Let $\Phi := e^{u/4}F$. Then the spinor

$$\Phi \begin{pmatrix} \sqrt{dz} & 0 \\ 0 & \sqrt{d\bar z} \end{pmatrix} \tag{5}$$

is well-defined globally on the Riemann surface M, while F is not. (For more details, see, for example, [12].) Note that $\det \Phi = e^{u/2}$. In sum, we have

Theorem 1. *A smooth conformal spacelike immersion $f: M \to \mathbb{S}^3_1(1)$ defines, uniquely up to sign, a spinor* (5) *on M such that locally*

$$f = e^{-u/2}\Phi\sigma_3\Phi^*, \quad df = \Phi \begin{pmatrix} 0 & dz \\ d\bar z & 0 \end{pmatrix} \Phi^*, \quad N = e^{-u/2}\Phi\Phi^*. \tag{6}$$

Furthermore, $\det\Phi = e^{u/2}$ *and* Φ *satisfies the following Lax equations*

(7)
$$\Phi^{-1}\Phi_z = \begin{pmatrix} \frac{u_z}{2} & -\frac{1}{2}(H+1)e^{u/2} \\ -Qe^{-u/2} & 0 \end{pmatrix}, \quad \Phi^{-1}\Phi_{\bar z} = \begin{pmatrix} 0 & -\bar{Q}e^{-u/2} \\ -\frac{1}{2}(H-1)e^{u/2} & \frac{u_{\bar z}}{2} \end{pmatrix}.$$

Conversely, consider a spinor on M *given locally by* (5)*. Suppose that* Φ *satisfies* (7) *where* $e^{u/2} := \det\Phi$*. Then the formulae* (6) *describe a conformally parameterized spacelike surface in* $\mathbb{S}^3_1(1)$*.*

4. A holomorphic spinor representation for spacelike surfaces of CMC 1

Let $f\colon M \to \mathbb{S}^3_1(1)$ be as in Theorem 1, and suppose further that f has CMC 1 with respect to the (future pointing) unit normal N.

Let $\Phi = \left(\begin{smallmatrix} \mathsf{P} & \mathsf{R} \\ \mathsf{Q} & \mathsf{S} \end{smallmatrix}\right)$ be the one given by the theorem. From the second equation in (7), we see that the entries P and Q are holomorphic spinors on M. From the first equation in (7), we see

$$\mathsf{P}_z = \frac{u_z}{2}\mathsf{P} - Qe^{-u/2}\mathsf{R}, \quad \mathsf{Q}_z = \frac{u_z}{2}\mathsf{Q} - Qe^{-u/2}\mathsf{S},$$

hence the holomorphic spinors are related to Hopf differential by the equation

$$\mathsf{P}_z\mathsf{Q} - \mathsf{P}\mathsf{Q}_z = Q.$$

The Gauß-Codazzi equations (4) with $H = 1$ are invariant under the transformation

(8)
$$Q \to \lambda Q, \quad e^u \to |\lambda|^2 e^u \quad \text{for any} \quad \lambda \in \mathbb{C}\setminus\{0\}.$$

Thus, every CMC 1 spacelike surface f in $\mathbb{S}^3_1(1)$ has a two-parameter family f_λ of deformations (8) within the CMC 1 class. Let $F_\lambda\colon U \to \mathrm{SL}(2;\mathbb{C})$ be the corresponding lift and let $\Phi_\lambda := e^{u/4}F_\lambda$. Then, $\det\Phi_\lambda = e^{u/2}$ and

(9)
$$\Phi_\lambda^{-1}(\Phi_\lambda)_z = \begin{pmatrix} \frac{u_z}{2} & -|\lambda|e^{u/2} \\ -\frac{\lambda}{|\lambda|}e^{-u/2}Q & 0 \end{pmatrix}, \quad \Phi_\lambda^{-1}(\Phi_\lambda)_{\bar z} = \begin{pmatrix} 0 & -\frac{\bar\lambda}{|\lambda|}e^{-u/2}\bar{Q} \\ 0 & \frac{u_{\bar z}}{2} \end{pmatrix}.$$

Note that

(10)
$$\Phi_1 = \Phi = \begin{pmatrix} \mathsf{P} & \mathsf{R} \\ \mathsf{Q} & \mathsf{S} \end{pmatrix}.$$

Now let $\lambda \to 0$ while $\lambda > 0$. The corresponding equations

$$\det\Phi_0 = e^{u/2}, \quad \Phi_0^{-1}(\Phi_0)_z = \begin{pmatrix} \frac{u_z}{2} & 0 \\ -e^{-u/2}Q & 0 \end{pmatrix}, \quad \Phi_0^{-1}(\Phi_0)_{\bar z} = \begin{pmatrix} 0 & -e^{-u/2}\bar{Q} \\ 0 & \frac{u_{\bar z}}{2} \end{pmatrix}$$

have solutions of the form

$$\Phi_0 = \begin{pmatrix} \mathsf{p} & \overline{\mathsf{q}} \\ \mathsf{q} & \overline{\mathsf{p}} \end{pmatrix}, \tag{11}$$

where p and q are holomorphic spinors on the universal cover $\widetilde{M}$ of M, and we see that

$$e^{u/2} = |\mathsf{p}|^2 - |\mathsf{q}|^2, \quad Q = \mathsf{p}_z\mathsf{q} - \mathsf{p}\mathsf{q}_z. \tag{12}$$

Note that this implies $|\mathsf{p}/\mathsf{q}| > 1$.

Proposition 2. $\Psi := \Phi_1\Phi_0^{-1} \colon \widetilde{M} \to \mathrm{SL}(2;\mathbb{C})$ *is holomorphic, and satisfies*:

$$\Psi_z = \Psi \begin{pmatrix} \mathsf{pq} & -\mathsf{p}^2 \\ \mathsf{q}^2 & -\mathsf{pq} \end{pmatrix}, \tag{13}$$

$$\Psi_z = \begin{pmatrix} \mathsf{PQ} & -\mathsf{P}^2 \\ \mathsf{Q}^2 & -\mathsf{PQ} \end{pmatrix} \Psi, \tag{14}$$

where p, q *are the holomorphic spinors on* $\widetilde{M}$ *given by* (11), *and* P, Q *are the holomorphic spinors on* M *given by* (10). *Furthermore,* $f = \Psi\sigma_3\Psi^*$.

Proof. Ψ is holomorphic since $\Psi_{\bar z} = \left(\Phi_1\Phi_0^{-1}\right)_{\bar z} = \Phi_1\left(\Phi_1^{-1}(\Phi_1)_{\bar z} - \Phi_0^{-1}(\Phi_0)_{\bar z}\right)\Phi_0^{-1} = 0$. Now we observe that

$$\Psi_z = \Phi_1\left(\Phi_1^{-1}(\Phi_1)_z - \Phi_0^{-1}(\Phi_0)_z\right)\Phi_0^{-1} = e^{u/2}\Phi_1 \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \Phi_0^{-1}.$$

Thus, using $\det\Phi_0 = \det\Phi_1 = e^{u/2}$,

$$\Psi^{-1}\Psi_z = e^{u/2}\Phi_0 \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \Phi_0^{-1}, \quad \Psi_z\Psi^{-1} = e^{u/2}\Phi_1 \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \Phi_1^{-1},$$

from which (13), (14) follow. Since $\Phi_0^{-1}\sigma_3(\Phi_0^*)^{-1} = e^{-u/2}\sigma_3$, we have $\Psi\sigma_3\Psi^* = e^{-u/2}\Phi_1\sigma_3\Phi_1^* = f$. □

Consider the maximal analytic extension of Ψ, and call it again Ψ. Recall that the metric of $f = \Psi\sigma_3\Psi^*$ is $\mathrm{I} = e^u|dz|^2 = (|\mathsf{p}|^2 - |\mathsf{q}|^2)^2|dz|^2$. Since (12) implies $|\mathsf{p}/\mathsf{q}| > 1$, Ψ must be restricted on an appropriate domain in $\widetilde{M}$ in order for $\Psi\sigma_3\Psi^*$ to be smooth (and connected). However, note that

$$e^{-u/2}\Phi_0\Phi_0^* = \frac{1}{|\mathsf{p}/\mathsf{q}|^2 - 1} \begin{pmatrix} 1 + \left|\dfrac{\mathsf{p}}{\mathsf{q}}\right|^2 & 2\dfrac{\mathsf{p}}{\mathsf{q}} \\ 2\overline{\left(\dfrac{\mathsf{p}}{\mathsf{q}}\right)} & 1 + \left|\dfrac{\mathsf{p}}{\mathsf{q}}\right|^2 \end{pmatrix},$$

hence $N = e^{-u/2}\Phi_1\Phi_1^* = \Psi(e^{-u/2}\Phi_0\Phi_0^*)\Psi^*$ is past pointing if $|\mathsf{p}/\mathsf{q}| < 1$.

It turns out that still on other regions of $\widetilde{M}$, $\Psi\sigma_3\Psi^*$ is of CMC 1 with respect to the future pointing unit normal if $|\mathsf{p}/\mathsf{q}| > 1$, but with respect to the past pointing unit normal if $|\mathsf{p}/\mathsf{q}| < 1$. That is, $\Psi\sigma_3\Psi^*$ has singularities on regions of $\widetilde{M}$ where $|\mathsf{p}/\mathsf{q}| = 1$ but is smooth and of CMC 1 elsewhere. Combining all these we can state a global representation theorem. Recall that

$$\mathrm{SU}(1,1) := \{\mathrm{U} \in \mathrm{SL}(2;\mathbb{C}) \colon \mathrm{U}\sigma_3\mathrm{U}^* = \sigma_3\} = \left\{ \begin{pmatrix} \bar{b} & a \\ \bar{a} & b \end{pmatrix} : a, b \in \mathbb{C},\ b\bar{b} - a\bar{a} = 1 \right\}.$$

Theorem 3. *Let* $f \colon M \to \mathbb{S}^3_1(1)$ *be a smooth spacelike surface in* $\mathbb{S}^3_1(1)$ *of CMC* 1 *with respect to the future pointing unit normal, and let* P, Q *be the holomorphic spinors on M given in* Proposition 2. *Then* $f = \Psi\sigma_3\Psi^*$, *where* Ψ *is a solution of the equation* (14). Ψ *is unique up to right multiplication by* $\mathrm{SU}(1,1)$.

Conversely, let P *and* Q *be two holomorphic spinors with the same spin structure on a Riemann surface* M. *Suppose that* $\Psi \colon \widetilde{M} \to \mathrm{SL}(2;\mathbb{C})$ *is a solution to the equation* (14) *where* $\widetilde{M}$ *is the universal cover of* M. *Then* $f := \Psi\sigma_3\Psi^* \colon \widetilde{M} \to \mathbb{S}^3_1(1)$ *defines a smooth spacelike immersion into* $\mathbb{S}^3_1(1)$ *on the region of* $\widetilde{M}$ *where* $|(\Psi^{-1}\,d\Psi)_{12}/(\Psi^{-1}\,d\Psi)_{22}| \neq 1$, *and it is of CMC* 1 *with respect to the*

$$\begin{cases} \textit{future pointing unit normal vector field if} & \left|\dfrac{(\Psi^{-1}\,d\Psi)_{12}}{(\Psi^{-1}\,d\Psi)_{22}}\right| > 1, \\ \textit{past pointing unit normal vector field if} & \left|\dfrac{(\Psi^{-1}\,d\Psi)_{12}}{(\Psi^{-1}\,d\Psi)_{22}}\right| < 1. \end{cases}$$

The equation (13) can be rewritten, by letting $g = (\Psi^{-1}d\Psi)_{12}/(\Psi^{-1}\,d\Psi)_{22} = \mathsf{p}/\mathsf{q}$ and $\omega = (\Psi^{-1}\,d\Psi)_{21} = \mathsf{q}^2\,dz$, as

$$\Psi^{-1}\,d\Psi = \begin{pmatrix} 1 & -g \\ g^{-1} & -1 \end{pmatrix} g\omega.$$

The map g, or its inverse in some articles, is called the *secondary Gauß map*. In fact, there is a 1 : 1 correspondence, so-called *Lawson type correspondence*, between spacelike CMC 1 surfaces in $\mathbb{S}^3_1(1)$ and spacelike maximal surfaces in $\mathbb{L}^3$, and the map g coincides with the projected Gauß map of corresponding spacelike maximal surface in $\mathbb{L}^3$. See [12] for more details. The ordered pair (g, ω) of a holomorphic map g and a holomorphic 1-form ω is used as the *Weierstraß data* for the CMC 1 spacelike surface f in [12]. However, g and ω are not well-defined on the same Riemann surface M on which the conformal spacelike immersion f is defined (they are well-defined on the universal cover $\widetilde{M}$ of M). On the other hand, the hyperbolic Gauß map $G = (d\Psi\Psi^{-1})_{12}/(d\Psi\Psi^{-1})_{22} = \mathsf{P}/\mathsf{Q}$ and the holomorphic 1-form $\Omega = (d\Psi\Psi^{-1})_{21} = \mathsf{Q}^2\,dz$ are well-defined on the Riemann surface M itself. (In some articles, the hyperbolic

Gauß map is defined to be Q/P.) In terms of these, the equation (14) can be written

$$(d\Psi)\Psi^{-1} = \begin{pmatrix} 1 & -G \\ G^{-1} & -1 \end{pmatrix} G\Omega.$$

In this paper, the ordered pair (G, Ω), or equivalently (P, Q), is used as the Weierstraß data for CMC 1 spacelike surface f. Note that in [12], (G, Ω) is used as the Weierstraß data not for the immersion f itself but for the *dual* CMC 1 spacelike surface $f^\sharp$.

Note that if Ψ is a solution of (14) and $\Psi_i := \sigma_i \Psi$, then

$$\Psi_1'\Psi_1^{-1} = \begin{pmatrix} -\mathsf{PQ} & \mathsf{Q}^2 \\ -\mathsf{P}^2 & \mathsf{PQ} \end{pmatrix}, \qquad \Psi_1\sigma_3\Psi_1^* = \sigma_1\left(\Psi\sigma_3\Psi^*\right)\sigma_1^*,$$
$$\Psi_2'\Psi_2^{-1} = \begin{pmatrix} -\mathsf{PQ} & -\mathsf{Q}^2 \\ \mathsf{P}^2 & \mathsf{PQ} \end{pmatrix}, \qquad \Psi_2\sigma_3\Psi_2^* = \sigma_2\left(\Psi\sigma_3\Psi^*\right)\sigma_2^*,$$
$$\Psi_3'\Psi_3^{-1} = \begin{pmatrix} \mathsf{PQ} & \mathsf{P}^2 \\ -\mathsf{Q}^2 & -\mathsf{PQ} \end{pmatrix}, \qquad \Psi_3\sigma_3\Psi_3^* = \sigma_3\left(\Psi\sigma_3\Psi^*\right)\sigma_3^*.$$

Since we will use the techniques and results of [4], we will consider the following form of the spinor equation in the rest of this paper:

$$\Psi'\Psi^{-1} = \begin{pmatrix} \mathsf{PQ} & \mathsf{P}^2 \\ -\mathsf{Q}^2 & -\mathsf{PQ} \end{pmatrix}. \tag{15}$$

Note that $\sigma_3 \begin{pmatrix} x_0+x_3 & x_1+ix_2 \\ x_1-ix_2 & x_0-x_3 \end{pmatrix} \sigma_3^* = \begin{pmatrix} x_0+x_3 & -x_1-ix_2 \\ -x_1+ix_2 & x_0-x_3 \end{pmatrix}$. That is, the action of σ_3 on $\mathbb{L}^4$ maps (x_0, x_1, x_2, x_3) to $(x_0, -x_1, -x_2, x_3)$.

5. Catenoids

In this section, we describe some CMC 1 surfaces which we call catenoids, motivated by [4]. They are the images of

$$\Psi\sigma_3\Psi^*\colon \hat{\mathbb{C}} \setminus \{0, \infty\} \to \mathbb{S}^3_1(1)$$

where Ψ satisfies (15) with

$$\mathsf{P} = \frac{p_0}{z} + p_\infty, \quad \mathsf{Q} = \frac{q_0}{z} + q_\infty, \quad \text{where} \quad p_0, p_\infty, q_0, q_\infty \in \mathbb{C} \quad \text{and} \quad p_0 q_\infty - p_\infty q_0 \neq 0.$$

It should be remarked that p_0 or q_0 may be 0. The case of $p_0 q_\infty - p_\infty q_0 = 0$ will be treated in the next section.

A particular solution of (15) with these data is (cf. [4])

$$\Psi(z) := cB(z)\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2}\end{pmatrix} C \begin{pmatrix} z^{\lambda} & 0 \\ 0 & z^{-\lambda}\end{pmatrix}$$

where

$$\lambda = \frac{1}{2}\sqrt{1+4(p_0q_\infty - p_\infty q_0)}, \quad c = \sqrt{\frac{p_0q_\infty - p_\infty q_0}{2\lambda}},$$

$$B(z) = \begin{pmatrix} \dfrac{p_0}{z} + p_\infty & \dfrac{p_0}{p_0q_\infty - p_\infty q_0} \\ -\left(\dfrac{q_0}{z} + q_\infty\right) & \dfrac{-q_0}{p_0q_\infty - p_\infty q_0}\end{pmatrix}, \quad C = \begin{pmatrix} \dfrac{2\lambda - 1}{2(p_0q_\infty - p_\infty q_0)} & \dfrac{-(2\lambda+1)}{2(p_0q_\infty - p_\infty q_0)} \\ 1 & 1\end{pmatrix}.$$

The general solution is $\Psi_R(z) := \Psi(z)R$ with $R \in \mathrm{SL}(2;\mathbb{C})$. This Ψ_R is in general not well defined on $\hat{\mathbb{C}} \setminus \{0,\infty\}$. Suppose Ψ_R transforms to $\widetilde{\Psi}_R$ as z traverses once around 0 counterclockwise. Then, $\widetilde{\Psi}_R = \Psi_R M_R$, where the monodromy matrix M_R is $-R^{-1}\begin{pmatrix} e^{2\pi i\lambda} & 0 \\ 0 & e^{-2\pi i\lambda}\end{pmatrix} R$. And, $\widetilde{\Psi}_R\sigma_3\widetilde{\Psi}_R^* = \Psi_R\sigma_3\Psi_R^*$ if and only if $M_R\sigma_3M_R^* = \sigma_3$. Note that $p_0q_\infty - p_\infty q_0 \neq 0$ implies $\lambda \neq \pm 1/2$. Now we classify R such that $\Psi_R\sigma_3\Psi_R^*$ is well defined on $\hat{\mathbb{C}} \setminus \{0,\infty\}$. In the following, $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$.

Theorem 4. (i) *If $\lambda \in (1/2)\mathbb{Z}$ but $\lambda \neq \pm 1/2$, then $M_R\sigma_3M_R^* = \sigma_3$ for any $R \in \mathrm{SL}(2,\mathbb{C})$.*
(ii) *If $\lambda \in \mathbb{R}\setminus(1/2)\mathbb{Z}$, then $M_R\sigma_3M_R^* = \sigma_3$ if and only if $R = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s}\end{pmatrix} S$ or $R = \begin{pmatrix} 0 & e^s \\ -e^{-s} & 0\end{pmatrix} S$ for some $s \in \mathbb{R}$ and $S \in \mathrm{SU}(1,1)$.*
(iii) *If $\lambda \in (1/2)\mathbb{Z} \oplus i\mathbb{R}^*$, then $M_R\sigma_3M_R^* = \sigma_3$ if and only if $R = 1/\sqrt{2}\begin{pmatrix} e^{\phi i} & -e^{\phi i} \\ e^{-\phi i} & e^{-\phi i}\end{pmatrix} S$ for some $\phi \in \mathbb{R}$ and $S \in \mathrm{SU}(1,1)$.*
(iv) *If $\lambda \in \mathbb{R}\setminus(1/2)\mathbb{Z} \oplus i\mathbb{R}^*$, then $M_R\sigma_3M_R^* \neq \sigma_3$ for any $R \in \mathrm{SL}(2;\mathbb{C})$.*

Proof. From the definition of M_R, we have that $M_R\sigma_3M_R^* = \sigma_3$ if and only if

$$R\sigma_3R^* = \begin{pmatrix} e^{2\pi i\lambda} & 0 \\ 0 & e^{-2\pi i\lambda}\end{pmatrix} R\sigma_3R^* \begin{pmatrix} e^{2\pi i\lambda} & 0 \\ 0 & e^{-2\pi i\lambda}\end{pmatrix}^*. \tag{16}$$

(i) follows immediately. Now suppose $\lambda \in \mathbb{C}\setminus(1/2)\mathbb{Z}$ and $R \in \mathrm{SL}(2;\mathbb{C})$ satisfies (16). Since $R\sigma_3R^*$ is Hermitian, we may write $R\sigma_3R^* = \begin{pmatrix} r_{11} & r_{12} \\ \overline{r_{12}} & r_{22}\end{pmatrix}$ for some $r_{11}, r_{22} \in \mathbb{R}$ and $r_{12} \in \mathbb{C}$. Then (16) is equal to

$$\begin{pmatrix} r_{11} & r_{12} \\ \overline{r_{12}} & r_{22}\end{pmatrix} = \begin{pmatrix} e^{2\pi i(\lambda-\bar\lambda)}r_{11} & e^{2\pi i(\lambda+\bar\lambda)}r_{12} \\ e^{-2\pi i(\lambda+\bar\lambda)}\overline{r_{12}} & e^{-2\pi i(\lambda-\bar\lambda)}r_{22}\end{pmatrix}. \tag{17}$$

Since $\det R\sigma_3R^* = -1$, at least one of r_{11} and r_{12} is nonzero. If both of them are nonzero, then $(1,1)$-components imply that $\lambda - \bar\lambda \in \mathbb{Z}$ and $(1,2)$-components imply

Fig. 2. Catenoid with $p_0 = -q_\infty = 1$ and $p_\infty = q_0 = 0$, and $\phi = \pi/2$. The figure is the image of $z = re^{i\theta}$ where $e^{-3} < r < e^3$, $0 \le \theta \le \pi$ and $1 < r < e^3$, $0 \le \theta \le \pi$, respectively.

that $\lambda + \bar{\lambda} \in \mathbb{Z}$, hence $2\lambda \in \mathbb{Z}$, which is not under consideration. So we may assume without loss of generality that exactly one of them is nonzero.

Suppose $r_{11} \neq 0$. Then $r_{12} = 0$, and $(1, 1)$-components of (17) imply that $\lambda \in \mathbb{R}$. Since $\det R\sigma_3 R^* = -1$, we have $R\sigma_3 R^* = \pm \begin{pmatrix} e^{2s} & 0 \\ 0 & -e^{-2s} \end{pmatrix}$ for some $s \in \mathbb{R}$. (ii) follows.

Suppose $r_{11} = 0$. Then, $r_{12} \neq 0$, which imply that $Re\lambda \in (1/2)\mathbb{Z}$. ((iv) follows.) If $r_{22} \neq 0$ in this case, then $\lambda \in \mathbb{R}$, hence $\lambda = Re\lambda \in (1/2)\mathbb{Z}$, which is not a case under consideration. Therefore, $R\sigma_3 R^* = \begin{pmatrix} 0 & e^{2\phi i} \\ e^{-2\phi i} & 0 \end{pmatrix}$ for some $\phi \in \mathbb{R}$, and (iii) follows. □

It is clear that for any $\lambda \in \mathbb{C}$ there are $p_0, p_\infty, q_0, q_\infty$ with $\lambda = (1/2) \times \sqrt{1 + 4(p_0 q_\infty - p_\infty q_0)}$.

Through computer graphics we see only conic singularities occur. At this moment we are not completely sure if they are the only kind of singularities that are allowed for catenoids, though we believe that is the case.

The most interesting phenomenon is the behavior of the ends when $\lambda \in (1/2)\mathbb{Z} \oplus i\mathbb{R}^*$. The ends oscillate between the future and past boundaries of $\mathbb{S}^3_1(1)$. The right picture in Fig. 2 shows half of the end at $z = \infty$. See Section 7 where we provide a complete analysis of the behavior of arbitrary catenoidal ends.

6. When catenoids degenerate

In this section, we consider solutions Ψ of (15) with $\mathsf{P} = p_0/z + p_\infty$, $\mathsf{Q} = q_0/z + q_\infty$ with $p_0 q_\infty - p_\infty q_0 = 0$. Note that the condition implies that $\mathsf{Q} = \alpha \mathsf{P}$ or $\mathsf{P} = \alpha \mathsf{Q}$ for some $\alpha \in \mathbb{C}$. This again implies that the hyperbolic Gauß map is constant, (which characterizes the horospheres in $\mathbb{H}^3(-1)$). They arise as the catenoids constructed in the previous section degenerate as $p_0 q_\infty - p_\infty q_0 \to 0$.

Consider (15) with

$$\mathsf{P} = \frac{p_0}{z} + p_\infty, \quad \mathsf{Q} = \alpha \mathsf{P}, \quad p_0, p_\infty, \alpha \in \mathbb{C}. \tag{18}$$

By changing z to $1/z$ if necessary, we may assume without loss of generality that $p_0 \neq 0$. Following [9, pp.79–81] and [4, Lemma 1], we see that if $p_\infty = 0$ then $\Psi = ABPz^\Lambda$ is a particular solution of (15) with (18) and that if $p_\infty \neq 0$ then

$$\Psi = ABP \begin{pmatrix} \sqrt{z} & 0 \\ 0 & \dfrac{1}{\sqrt{z}} \end{pmatrix} \begin{pmatrix} \sqrt{2\dfrac{p_\infty}{p_0}} & 0 \\ 0 & \sqrt{\dfrac{p_0}{2p_\infty}} \end{pmatrix} \begin{pmatrix} 1 & \dfrac{1}{2}\dfrac{p_\infty}{p_0}z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \ln z \\ 0 & 1 \end{pmatrix}$$

is a particular solution of (15) with (18), where

$$A := \begin{pmatrix} p_0 & 0 \\ -\alpha p_0 & \dfrac{1}{p_0} \end{pmatrix}, \quad B := \begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix}, \quad P = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \dfrac{1}{2} & 0 \\ 0 & -\dfrac{1}{2} \end{pmatrix}.$$

The general solution of (15) with (18) is $\Psi_R := \Psi R$ with $R \in \mathrm{SL}(2;\mathbb{C})$. Now we classify R with which $\Psi_R \sigma_3 \Psi_R^*$ is well defined on $\hat{\mathbb{C}} \setminus \{0\}$.

Theorem 5. (i) *If* $p_\infty = 0$, *then* $\Psi_R\sigma_3\Psi_R^*$ *is well defined on* $\hat{\mathbb{C}} \setminus \{0\}$ *for any* $R = \left(\begin{smallmatrix} r_1 & r_2 \\ r_3 & r_4 \end{smallmatrix}\right) \in \mathrm{SL}(2;\mathbb{C})$. *If* $|r_3| \neq |r_4|$, *then it is spacelike everywhere on* $\hat{\mathbb{C}} \setminus \{0\}$, *complete, totally umbilic, flat, unique up to an isometry of* $\mathbb{S}^3_1(1)$ *and a coordinate change* $z \to az$, *and has constant hyperbolic and secondary Gauß maps. If* $|r_3| = |r_4|$, *the image of the map is a lightlike line.*
(ii) *Suppose* $p_\infty \in \mathbb{C} \setminus \{0\}$. *Then* $\Psi_R\sigma_3\Psi_R^*$ *is well defined on* $\hat{\mathbb{C}} \setminus \{0, \infty\}$ *if and only if* $R = \left(\begin{smallmatrix} (r+1)/2 & (r-1)/2 \\ 1 & 1 \end{smallmatrix}\right) S$ *or* $R = \left(\begin{smallmatrix} (1-r)/2 & (1+r)/2 \\ -1 & 1 \end{smallmatrix}\right) S$ *for some* $r \in \mathbb{R}$ *and* $S \in \mathrm{SU}(1,1)$. *In this case, the image of the map is a lightlike line.*

Proof. When $p_\infty = 0$ and $R = \left(\begin{smallmatrix} r_1 & r_2 \\ r_3 & r_4 \end{smallmatrix}\right)$, $A^{-1}\Psi_R\sigma_3(A^{-1}\Psi_R)^*$ is

$$\begin{pmatrix} \left|r_1 - \dfrac{r_3}{z}\right|^2 - \left|r_2 - \dfrac{r_4}{z}\right|^2 & (r_1\overline{r_3} - r_2\overline{r_4}) + \dfrac{|r_4|^2 - |r_3|^2}{z} \\ (\overline{r_1}r_3 - \overline{r_2}r_4) + \dfrac{|r_4|^2 - |r_3|^2}{\overline{z}} & |r_3|^2 - |r_4|^2 \end{pmatrix}.$$

Its image is a lightlike line if $|r_3| = |r_4|$. Otherwise, it is complete and spacelike everywhere on $\hat{\mathbb{C}} \setminus \{0\}$, and the claims follows from [17] or [2]. The secondary Gauß map is $-r_4/r_3$.

Now assume $p_\infty \neq 0$. The monodromy matrix M of Ψ is $M = \left(\begin{smallmatrix} 1 & 2\pi i \\ 0 & 1 \end{smallmatrix}\right)$, hence $M_R\sigma_3M_R^* = \sigma_3$ if and only if $R\sigma_3R^* = MR\sigma_3R^*M^*$ if and only if $R\sigma_3R^* = \pm\left(\begin{smallmatrix} r & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

The formula for R follows. In this case, we see that

$$A^{-1}\Psi R\sigma_3 R^*\Psi^*(A^{-1})^* = \begin{pmatrix} ** & \dfrac{\sqrt{2p_\infty/p_0}}{\overline{\sqrt{2p_\infty/p_0}}} \\ \dfrac{\overline{\sqrt{2p_\infty/p_0}}}{\sqrt{2p_\infty/p_0}} & 0 \end{pmatrix}$$

where $**$ is a certain real valued function of z. Therefore the image is a lightlike line. $\square$

The case where $\mathsf{P} = \alpha\mathsf{Q}$ can be treated similarly.

7. Catenoidal and horospherical ends

Motivated by the previous examples, we define the following:

DEFINITION 6. Suppose that for a local coordinate z,

$$\mathsf{P} = \frac{a_{-1}}{z} + a_0 + o(1), \quad \mathsf{Q} = \frac{b_{-1}}{z} + b_0 + o(1) \tag{19}$$

and that a solution of (15) with these data provides a well defined map from a neighborhood of $z = 0$ into $\mathbb{S}^3_1(1)$. We call the image of the neighborhood a catenoidal end if $a_{-1}b_0 - a_0b_{-1} \neq 0$, or a horospherical end if $a_{-1}b_0 - a_0b_{-1} = 0$ and $a_{-1}b_{-1} \neq 0$.

Note that we do *not* require $a_{-1}b_{-1} \neq 0$ for catenoidal ends.

Let Ψ be a solution of (15) with (19). Then $\widetilde{\Psi} := B^{-1}A^{-1}\Psi$, where $A = \left(\begin{smallmatrix} a_{-1} & 0 \\ -b_{-1} & 1/a_{-1} \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1/\sqrt{z} & 0 \\ 0 & \sqrt{z} \end{smallmatrix}\right)$, satisfy (cf. [4, Lemma 1])

$$\widetilde{\Psi}'\widetilde{\Psi}^{-1} = \frac{1}{z}\begin{pmatrix} \dfrac{1}{2}+r & 1 \\ -r^2 & -\dfrac{1}{2}-r \end{pmatrix} + O(1) \quad \text{for} \quad z \to 0, \quad \text{where} \quad r = a_{-1}b_0 - a_0b_{-1}. \tag{20}$$

There are two cases we need to consider in finding $\widetilde{\Psi}$. Let $\lambda := \sqrt{(1/4)+r}$. Note that $\left(\begin{smallmatrix} (1/2)+r & 1 \\ -r^2 & -(1/2)-r \end{smallmatrix}\right) = P\Lambda P^{-1}$ where $\Lambda = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & -\lambda \end{smallmatrix}\right)$ and

$$P = \frac{1}{\sqrt[4]{1+4r}}\begin{pmatrix} \dfrac{1}{2}+\sqrt{\dfrac{1}{4}+r} & \dfrac{1}{r}\left(\dfrac{1}{2}-\sqrt{\dfrac{1}{4}+r}\right) \\ r\left(\dfrac{1}{2}-\sqrt{\dfrac{1}{4}+r}\right) & \dfrac{1}{2}+\sqrt{\dfrac{1}{4}+r} \end{pmatrix}.$$

Lemma 7 ([9, pp.79–81]). (i) *If $\lambda \in \mathbb{C} \setminus (1/2)\mathbb{Z}$, then there is a particular solution of* (20) *in the form*

$$\widetilde{\Psi}(z) = P\Phi(z)z^{\Lambda}$$

where $\Phi(z)$ is holomorphic at $z = 0$ with $\Phi(0) = I$.
(ii) *If $\lambda \in (1/2)\mathbb{Z}$, then there is a particular solution of* (20) *in the form*

$$\widetilde{\Psi}(z) = P_1 \begin{pmatrix} \sqrt{z} & 0 \\ 0 & \dfrac{1}{\sqrt{z}} \end{pmatrix} P_2 \begin{pmatrix} \sqrt{z} & 0 \\ 0 & \dfrac{1}{\sqrt{z}} \end{pmatrix} \cdots P_{|2\lambda|} \begin{pmatrix} \sqrt{z} & 0 \\ 0 & \dfrac{1}{\sqrt{z}} \end{pmatrix} \Phi(z) \begin{pmatrix} 1 & f(z) \\ 0 & 1 \end{pmatrix}$$

where $P_i \in \mathrm{SL}(2;\mathbb{C})$, $\Phi(z)$ is holomorphic at $z = 0$, and $f(z)$ is either 0 or $\ln z$ depending upon the coefficients a_i, b_j. (*When $\lambda = 0$, only the last two terms survive.*)

Corollary 8. (i) [4, Corollary 2] *If $\lambda \in \mathbb{C} \setminus (1/2)\mathbb{Z}$, then there is a particular solution Ψ of* (15) *with* (19) *whose monodromy matrix M around $z = 0$ is $-\begin{pmatrix} e^{2\pi i\lambda} & 0 \\ 0 & e^{-2\pi i\lambda} \end{pmatrix}$.*
(ii) *If $\lambda \in (1/2)\mathbb{Z}$, then there is a particular solution Ψ of* (15) *with* (19) *whose monodromy matrix M around $z = 0$ is $(-1)^{1+|2\lambda|} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ where $*$ is 0 or $2\pi i$.*

Note that both cases in (ii) have appeared in the previous two sections.

The general solution is $\Psi_R := \Psi R$ for $R \in \mathrm{SL}(2;\mathbb{C})$. Now we classify R with which the catenoidal end is well defined on a punctured neighborhood of $z = 0$. Arguing exactly as in the proof of Theorems 4 and 5, we have the following proposition, where we assume $\lambda \in \mathbb{C} \setminus (1/2)\mathbb{Z}$ to make the situation simple.

Proposition 9. (i) *If $\lambda \in \mathbb{R} \setminus (1/2)\mathbb{Z}$, then there is no period if and only if $R = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} S$ or $R = \begin{pmatrix} 0 & e^s \\ -e^{-s} & 0 \end{pmatrix} S$ for some $s \in \mathbb{R}$ and $S \in \mathrm{SU}(1,1)$.*
(ii) *If $\lambda \in (1/2)\mathbb{Z} \oplus i\mathbb{R}^*$, then there is no period if and only if $R = (1/\sqrt{2}) \begin{pmatrix} e^{\phi i} & -e^{\phi i} \\ e^{-\phi i} & e^{-\phi i} \end{pmatrix} S$ for some $\phi \in \mathbb{R}$ and $S \in \mathrm{SU}(1,1)$.*
(iii) *If $\lambda \in \mathbb{R} \setminus (1/2)\mathbb{Z} \oplus i\mathbb{R}^*$, then the period problem cannot be solved for any $R \in \mathrm{SL}(2;\mathbb{C})$.*

When $\lambda \in (1/2)\mathbb{Z} \oplus i\mathbb{R}^*$, the catenoidal ends behave in an interesting way as $z \to 0$. To observe it, we note that if $z \sim 0$, then $\Phi(z) \sim I$ hence

$$\begin{aligned} A^{-1}\Psi_R\sigma_3\Psi_R^*(A^{-1})^* &\sim (BP)z^{\Lambda} \begin{pmatrix} 0 & e^{2\phi i} \\ e^{-2\phi i} & 0 \end{pmatrix} \bar{z}^{\bar{\Lambda}}(BP)^* \\ &= \begin{pmatrix} \dfrac{P_{11}}{\sqrt{z}} & \dfrac{P_{12}}{\sqrt{z}} \\ P_{21}\sqrt{z} & P_{22}\sqrt{z} \end{pmatrix} \begin{pmatrix} 0 & e^{2\phi i}\dfrac{z^{\lambda}}{\bar{z}^{\bar{\lambda}}} \\ e^{-2\phi i}\dfrac{\bar{z}^{\bar{\lambda}}}{z^{\lambda}} & 0 \end{pmatrix} \begin{pmatrix} \overline{\dfrac{P_{11}}{\sqrt{z}}} & \overline{P_{21}\sqrt{z}} \\ \overline{\dfrac{P_{12}}{\sqrt{z}}} & \overline{P_{22}\sqrt{z}} \end{pmatrix}, \end{aligned}$$

where P_{ij} is the (i, j) entry of P. Therefore

$$\operatorname{tr}\left[A^{-1}\Psi_R\sigma_3\Psi_R^*(A^{-1})^*\right] \sim 2\,Re\left\{\frac{\overline{P_{11}}P_{12}}{|z|e^{2\phi i}}\frac{\overline{z^{\lambda}}}{z^{\lambda}} + P_{21}\overline{P_{22}}|z|e^{2\phi i}\frac{z^{\lambda}}{\overline{z^{\lambda}}}\right\}$$
$$= 2\,Re\left\{r\overline{P_{11}}P_{12}\left[\frac{\overline{z^{\lambda}}}{r|z|e^{2\phi i}z^{\lambda}} + \frac{r|z|e^{2\phi i}z^{\lambda}}{\overline{z^{\lambda}}}\right]\right\}$$

where we have used $r\overline{P_{11}}P_{12} = (1/r)P_{21}\overline{P_{22}}$. Let us restrict z to be real and positive. If we write $r = e^{A+iB}$, $\lambda = \alpha + i\beta$, $z = e^s > 0$ for $A, B, \alpha, \beta, s \in \mathbb{R}$, then

$$\frac{\overline{z^{\lambda}}}{r|z|e^{2\phi i}z^{\lambda}} + \frac{r|z|e^{2\phi i}z^{\lambda}}{\overline{z^{\lambda}}} = 2\cosh(A+s)\cos(B+2\phi+2s\beta)$$
$$+ 2i\sinh(A+s)\sin(B+2\phi+2s\beta).$$

We immediately see that if $\beta \neq 0$, then $\operatorname{tr}\left[A^{-1}\Psi_R\sigma_3\Psi_R^*(A^{-1})^*\right]$, hence the time component of $\Psi_R\sigma_3\Psi_R^*$, oscillates between ∞ and $-\infty$ as s approaches $-\infty$. This means that the end oscillates between the future and past ideal boundaries of $\mathbb{S}^3_1(1)$ indefinitely, and the singularities accumulate at the end.

Therefore we define the following:

DEFINITION 10. (i) A catenoidal end is "normal" if $\lambda \in \mathbb{R} \setminus (1/2)\mathbb{Z}$.
(ii) A catenoidal end is "abnormal" if $\lambda \in (1/2)\mathbb{Z} \oplus i\mathbb{R}^*$.

It is clear that this definition does not depend upon the representation of P and Q.

8. Trinoids

In this section, we describe some CMC 1 surfaces which we call trinoids, motivated by [4]. They are the images of

$$\Psi\sigma_3\Psi^* : \hat{\mathbb{C}} \setminus \{0, 1, \infty\} \to \mathbb{S}^3_1(1)$$

where Ψ satisfies (15) with

$$\mathsf{P} = \frac{p_0}{z} + \frac{p_1}{z-1} + p_\infty, \quad \mathsf{Q} = \frac{q_0}{z} + \frac{q_1}{z-1} + q_\infty, \tag{21}$$

where $p_0, p_1, p_\infty, q_0, q_1, q_\infty \in \mathbb{C}$, and there are no periods for $z = 0, 1, \infty$.

The existence and properties of the solutions of the equation (15) with data (21) is presented in [4] in detail, which we summarize here for the convenience of the reader: *Let*

$$\alpha = \frac{1}{2}\left(1 - \sqrt{1 + 4\langle p, q\rangle_{0\infty} + 4\langle p, q\rangle_{10}}\right),$$

$$
\begin{aligned}
\beta &= \frac{1}{2}\frac{\langle p,q\rangle_{10}(1-2\alpha)-\langle p,q\rangle_{0\infty}}{\langle p,q\rangle_{0\infty}+\langle p,q\rangle_{10}},\\
\gamma &= \langle p,q\rangle_{0\infty}\left(\frac{\langle p,q\rangle_{1\infty}}{\Delta}+\frac{1}{\alpha}\right),\\
\delta &= \frac{\Delta}{\langle p,q\rangle_{0\infty}}\frac{\Delta+\langle p,q\rangle_{0\infty}\alpha}{\Delta-\langle p,q\rangle_{0\infty}\langle p,q\rangle_{10}+(\Delta+\langle p,q\rangle_{1\infty})\alpha},\\
\mu &= 2\langle p,q\rangle_{0\infty}\left(1-k\frac{\langle p,q\rangle_{1\infty}}{\Delta}\right),\\
k &= \Delta\frac{\langle p,q\rangle_{0\infty}\langle p,q\rangle_{10}-\Delta\alpha}{\Delta^2+\langle p,q\rangle_{10}\langle p,q\rangle_{0\infty}\langle p,q\rangle_{1\infty}},\\
\alpha_1 &= -\frac{p_\infty\langle p,q\rangle_{10}}{\Delta},\quad \alpha_2=\frac{q_\infty\langle p,q\rangle_{10}}{\Delta},\\
\beta_1 &= \frac{p_0\langle p,q\rangle_{1\infty}}{\Delta},\quad \beta_2=-\frac{q_0\langle p,q\rangle_{1\infty}}{\Delta},\\
\Delta &= \langle p,q\rangle_{10}\langle p,q\rangle_{0\infty}+\langle p,q\rangle_{10}\langle p,q\rangle_{1\infty}+\langle p,q\rangle_{0\infty}\langle p,q\rangle_{1\infty},\\
\langle p,q\rangle_{ij} &= p_iq_j-p_jq_i \quad \textit{for} \quad i,j=0,1,\infty,\\
\tau &= \sqrt{\beta^2+\gamma\delta},\quad \rho=\sqrt{(\alpha+\beta)^2+\gamma\delta},\\
a &= \alpha+\tau+\rho,\quad b=\alpha+\tau-\rho,\quad c=2\alpha,\\
D(z) &:= \begin{pmatrix} \mathsf{P} & \alpha_1 z\\ -\mathsf{Q} & \alpha_2 z+\beta_2\end{pmatrix}\begin{pmatrix}\sqrt{z-1} & 0\\ \dfrac{k}{z\sqrt{z-1}} & \dfrac{1}{\sqrt{z-1}}\end{pmatrix}\begin{pmatrix}\dfrac{2\alpha}{\mu} & 0\\ 1 & 1\end{pmatrix},
\end{aligned}
$$

and

$$
\Phi^{(0)}(z) := \begin{pmatrix} -\dfrac{2\alpha+1}{\delta}z^{\alpha}(z-1)^{\tau}\,{}_2\mathrm{F}_1(a,b;c;z) & z^{1-\alpha}(z-1)^{\tau}\,{}_2\mathrm{F}_1(a-c+1,b-c+1;2-c;z)\\ z^{1+\alpha}(z-1)^{\tau}\,{}_2\mathrm{F}_1(a+1,b+1;c+2;z) & \dfrac{2\alpha-1}{\gamma}z^{-\alpha}(z-1)^{\tau}\,{}_2\mathrm{F}_1(a-c,b-c;-c;z)\end{pmatrix}
$$

where ${}_2\mathrm{F}_1(a,b;c;z)$ *is the hypergeometric function.* $\Phi^{(0)}$ *has branch points at* $0,1,\infty$. *We choose the branch cuts from* 1 *to* ∞ *along the positive real axis and from* 0 *to* ∞ *along the negative real axis. If*

$$\alpha,\tau,\rho\in\mathbb{C}\setminus\frac{1}{2}\mathbb{Z}, \tag{22}$$

then $\Psi(z):=D(z)\Phi^{(0)}(z)$ *is a particular solution of the differential equation* (15) *with* P,Q *as in* (21), *and the monodromy matrices of* $\Psi(z)$ *as* z *traverses once around*

$0, 1, \infty$, *respectively, are as follows*:

$$\mathcal{M}_0 = \begin{pmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\alpha} \end{pmatrix}, \quad \mathcal{M}_\infty = \mathcal{M}_0^{-1}\mathcal{M}_1^{-1},$$

$$\mathcal{M}_1 = \begin{pmatrix} e^{2\pi i\tau} - 2i\dfrac{\sin\pi a \sin\pi b}{\sin\pi c} & \dfrac{2\pi i}{\gamma}\dfrac{2\alpha-1}{2\alpha+1}\dfrac{\Gamma^2(-c)}{\Gamma(-a)\Gamma(-b)\Gamma(a-c)\Gamma(b-c)} \\ \dfrac{2\pi i}{\delta}\dfrac{2\alpha+1}{2\alpha-1}\dfrac{\Gamma^2(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} & e^{2\pi i\tau} + 2i\dfrac{\sin\pi(c-a)\sin\pi(c-b)}{\sin\pi c} \end{pmatrix}.$$

We assume (22) in the rest of this article. The general solution of (15) with (21) is $\Psi_R = D(z)\Phi^{(0)}(z)R$ with $R \in \mathrm{SL}(2;\mathbb{C})$. Now we want to classify R with which $\Psi_R\sigma_3\Psi_R^*$ is a well defined map from $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ into $\mathbb{S}^3_1(1)$. We first observe that $\Psi_R = D\Phi^{(0)}R$ transforms to $\widetilde{\Psi}_R := D\Phi^{(0)}\mathcal{M}_\nu R$ as z traverses once around z_ν counter-clockwise. Then

$$\begin{aligned} \widetilde{\Psi}_R\sigma_3\widetilde{\Psi}_R^* &= D\Phi^{(0)}\mathcal{M}_\nu R\sigma_3\left(D\Phi^{(0)}\mathcal{M}_\nu R\right)^* \\ &= D\Phi^{(0)}\mathcal{M}_\nu R\sigma_3 R^*\mathcal{M}_\nu^*\left(\Phi^{(0)}\right)^* D^*, \\ \Psi_R\sigma_3\Psi_R^* &= D\Phi^{(0)}R\sigma_3 R^*\left(\Phi^{(0)}\right)^* D^*. \end{aligned}$$

Therefore, $\Psi_R\sigma_3\Psi_R^*$ is well defined on $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ if and only if

$$R\sigma_3 R^* = \mathcal{M}_\nu R\sigma_3 R^*\mathcal{M}_\nu^* \quad \text{for} \quad \nu = 0, 1, \infty. \tag{23}$$

Now we classify R which satisfies (23). It is best done in terms of the α, τ, ρ. We first state a nonexistence result, which follows immediately from Proposition 9.

Lemma 11. *If at least one of* α, τ, ρ *belongs to* $(\mathbb{R} \setminus (1/2)\mathbb{Z}) \oplus i\mathbb{R}^*$, *the period problem cannot be solved*.

Since we are assuming (22), there remain only the following four cases, after a suitable change of coordinates if necessary:

(1) (eee case) $\alpha, \tau, \rho \in \mathbb{R} \setminus (1/2)\mathbb{Z}$.
(2) (eeh case) $\alpha, \tau \in \mathbb{R} \setminus (1/2)\mathbb{Z}$ and $\rho \in (1/2)\mathbb{Z} \oplus i\mathbb{R}^*$.
(3) (hhe case) $\alpha, \tau \in (1/2)\mathbb{Z} \oplus i\mathbb{R}^*$ and $\rho \in \mathbb{R} \setminus (1/2)\mathbb{Z}$.
(4) (hhh case) $\alpha, \tau, \rho \in (1/2)\mathbb{Z} \oplus i\mathbb{R}^*$.

"e" and "h" stand for elliptic and hyperbolic, respectively [8]. An end $\Psi\sigma_3\Psi$ is called elliptic, parabolic, or hyperbolic if the monodromy matrix of the lift Ψ is conjugate in $\mathrm{SU}(1, 1)$ to an elliptic, parabolic, hyperbolic matrix, respectively.

In each case, the trinoid has three normal ends, two normal and one abnormal ends, two abnormal and one normal ends, or three abnormal ends, respectively.

We first prove an auxiliary lemma. Recall that $a = \alpha + \tau + \rho$, $b = \alpha + \tau - \rho$, $c = 2\alpha$.

Lemma 12. *Consider* $(\sin\pi a \sin\pi b)/\sin\pi c$ *and* $(\sin\pi(a-c)\sin\pi(b-c))/\sin\pi c$.

(1) *In the eee and eeh case, both of them are real.*

(2) *In the ehh and hhh case, both of them are purely imaginary.*

Proof. Note that if $n \in \mathbb{Z}$ and $y \in \mathbb{R}\setminus\{0\}$, then

$$\cos 2\pi\left(\frac{1}{2}n + yi\right) \in \mathbb{R}, \quad \sin 2\pi\left(\frac{1}{2}n + yi\right) \in i\mathbb{R}.$$

Now we observe

$$\begin{aligned} \sin\pi a \sin\pi b &= \frac{1}{2}\{\cos 2\pi\rho - \cos 2\pi(\alpha+\tau)\}, \\ \sin\pi(a-c)\sin\pi(b-c) &= \frac{1}{2}\{\cos 2\pi\rho - \cos 2\pi\tau\}. \end{aligned}$$

These are real in all the four cases. On the other hand, $\sin\pi c = \sin 2\pi\alpha$ is real in the eee and eeh cases, but is purely imaginary in the hhe and hhh cases. Therefore the conclusion follows. □

Theorem 13. (i) *In the eee and eeh cases, the period problem* (23) *can be solved if and only if*

$$\sin\pi a \sin\pi b \sin\pi(a-c)\sin\pi(b-c) > 0. \tag{24}$$

When (24) *holds,* $R \in \mathrm{SL}(2;\mathbb{C})$ *solves* (23) *if and only if*

$$R = \begin{pmatrix} r & 0 \\ 0 & r^{-1}\end{pmatrix} S \quad \textit{or} \quad R = \begin{pmatrix} 0 & r \\ -r^{-1} & 0\end{pmatrix} S$$

for some $S \in \mathrm{SU}(1,1)$ *and*

$$r = \left(\left|\frac{2\alpha-1}{2\alpha+1}\right|^2 \frac{\pi^2}{|\gamma|^2}\frac{1}{4\alpha^2}\frac{|\Gamma(-a)\Gamma(-b)\Gamma(a-c)\Gamma(b-c)|^{-2}|\Gamma(-c)|^4\overline{(c\sin\pi c)^2}}{\sin\pi a\sin\pi b\sin\pi(c-a)\sin\pi(c-b)}\right)^{1/4} \in \mathbb{R}^+.$$

(ii) *In the hhe and hhh cases, the period problem* (23) *can be solved for any* α, τ, ρ. R *solves* (23) *if and only if*

$$R = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\phi i} & -e^{\phi i} \\ e^{-\phi i} & e^{-\phi i}\end{pmatrix} S \tag{25}$$

for some $\phi \in \mathbb{R}$ *as in the proof and an arbitrary* $S \in \mathrm{SU}(1,1)$.

REMARK 14. Note that the sign of $\sin\pi a \sin\pi b \sin\pi(a-c)\sin\pi(b-c)$ in (24) above is different from the sign of that in [4, Theorem 6 (ii)].

Proof. It is enough to find all R's such that $R\sigma_3 R^* = \mathcal{M}_\nu R\sigma_3 R^* \mathcal{M}_\nu^*$ for $\nu = 0, 1$.

Let's first consider the eee and eeh cases. Proposition 9 applied to the end $z = 0$ implies that

$$R = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} S \quad \text{or} \quad R = \begin{pmatrix} 0 & r \\ -r^{-1} & 0 \end{pmatrix} S$$

for some $r > 0$ and $S \in \mathrm{SU}(1,1)$. Now consider the end at $z = 1$. If we write $\mathcal{M}_1 = \left(\begin{smallmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{smallmatrix}\right)$, then the above two equations and $R\sigma_3 R^* = \mathcal{M}_\nu R\sigma_3 R^* \mathcal{M}_\nu^*$ imply

$$\begin{aligned} \begin{pmatrix} r^2 & 0 \\ 0 & -r^{-2} \end{pmatrix} &= \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} r^2 & 0 \\ 0 & -r^{-2} \end{pmatrix} \begin{pmatrix} \overline{M_{11}} & \overline{M_{21}} \\ \overline{M_{12}} & \overline{M_{22}} \end{pmatrix} \\ &= \begin{pmatrix} r^2 M_{11}\overline{M_{11}} - r^{-2} M_{12}\overline{M_{12}} & r^2 M_{11}\overline{M_{21}} - r^{-2} M_{12}\overline{M_{22}} \\ r^2 M_{21}\overline{M_{11}} - r^{-2} M_{22}\overline{M_{12}} & r^2 M_{21}\overline{M_{21}} - r^{-2} M_{22}\overline{M_{22}} \end{pmatrix}. \end{aligned} \tag{26}$$

By comparing the $(1,2)$-components, we conclude that

$$r^4 = \frac{M_{12}\overline{M_{22}}}{M_{11}\overline{M_{21}}}.$$

We have from Lemma 12 that

$$M_{11} - \overline{M_{22}} = e^{2\pi i\tau} - 2i \frac{\sin \pi a \sin \pi b}{\sin \pi c} - e^{-2\pi i \tau} + 2i \frac{\sin \pi(c-a) \sin \pi(c-b)}{\sin \pi c},$$

hence

$$\begin{aligned} M_{11} - \overline{M_{22}} &= 2i \sin 2\pi\tau + \frac{2i}{\sin \pi c} (\sin(\pi c - \pi a) \sin(\pi c - \pi b) - \sin \pi a \sin \pi b) \\ &= 2i \sin 2\pi\tau - 2i \sin \pi(a + b - c). \end{aligned}$$

Since $a + b - c = 2\tau$, we have $M_{11} = \overline{M_{22}}$. On the other hand, since $\overline{\Gamma(x)}/\Gamma(-x) = 1/|\Gamma(-x)|^2 - \pi/\left(\overline{x \sin \pi x}\right)$, we have

$$\begin{aligned} \frac{M_{12}}{\overline{M_{21}}} = \left| \frac{2\alpha - 1}{2\alpha + 1} \right|^2 \frac{\pi^2}{|\gamma|^2} \overline{\left(\frac{-\gamma\delta}{ab(c-a)(c-b)} \right)} \\ \times \frac{|\Gamma(-a)\Gamma(-b)\Gamma(a-c)\Gamma(b-c)|^{-2} |\Gamma(-c)|^4 \overline{(c \sin \pi c)^2}}{\overline{\sin \pi a \sin \pi b \sin \pi(c-a) \sin \pi(c-b)}}. \end{aligned}$$

We have

$$\gamma\delta = \frac{ab(c-a)(c-b)}{-4\alpha^2}. \tag{27}$$

from $\gamma\delta = \tau^2 - \beta^2$, $\beta = (\alpha^2 + \tau^2 - \rho^2)/(-2\alpha)$, and the definitions of a, b, c. So $-\gamma\delta/(ab(c-a)(c-b)) > 0$ from (27) and the fact that $\alpha \in \mathbb{R} \setminus (1/2)\mathbb{Z}$. Also,

$(c\sin\pi c)^2 > 0$ since $c = 2\alpha$. Therefore, $M_{12}/\overline{M_{21}} > 0$ if and only if (24) is satisfied. Conversely, we see that if r as in the statement of the Theorem, then (26) is satisfied. By combining all the above arguments, we complete the proof for eee and eeh case.

Now we consider the hhe and hhh cases. Proposition 9 applied to the end $z = 0$ implies that the period problem at $z = 0$ can be solved if and only if

$$R\sigma_3 R^* = \begin{pmatrix} 0 & e^{2\phi i} \\ e^{-2\phi i} & 0 \end{pmatrix} \quad \text{for some} \quad \phi \in \mathbb{R}. \tag{28}$$

If we write $\mathcal{M}_1 = \left(\begin{smallmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{smallmatrix}\right)$ as before, then (23) with $z = 1$ is equal to

$$\begin{pmatrix} 0 & e^{2\phi i} \\ e^{-2\phi i} & 0 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 0 & e^{2\phi i} \\ e^{-2\phi i} & 0 \end{pmatrix} \begin{pmatrix} \overline{M_{11}} & \overline{M_{21}} \\ \overline{M_{12}} & \overline{M_{22}} \end{pmatrix}.$$

By comparing the four components of this matrix equation, we conclude that the period is solvable at $z = 1$ if and only if

$$-\frac{M_{12}}{\overline{M_{12}}}\frac{\overline{M_{11}}}{M_{11}} = -\frac{\overline{M_{21}}}{M_{21}}\frac{M_{22}}{\overline{M_{22}}} = \frac{M_{12}\overline{M_{21}}}{1 - M_{11}\overline{M_{22}}} = \frac{1 - M_{22}\overline{M_{11}}}{M_{21}\overline{M_{12}}} = e^{4\phi i}. \tag{29}$$

Note that the first term in (29) has modulus 1. We first show that the first three equalities of (29) are true in any hhh and hhe case.

We immediately see from Lemma 12 that

$$M_{11}, M_{22} \in \mathbb{R},$$

and, from Lemma 12, formulas (27) and $\Gamma(x)\Gamma(-x) = -\pi/(\sin\pi x)$, that

$$\begin{aligned} M_{12}M_{21} &= \frac{-4\pi^2}{\gamma\delta}\frac{(\Gamma(c)\Gamma(-c))^2}{\Gamma(a)\Gamma(-a)\Gamma(b)\Gamma(-b)\Gamma(c-a)\Gamma(a-c)\Gamma(b-c)\Gamma(c-b)} \\ &= 4\frac{\sin\pi a\sin\pi b}{\sin\pi c}\frac{\sin\pi(a-c)\sin\pi(b-c)}{\sin\pi c} \\ &\in \mathbb{R}. \end{aligned}$$

The first equality of (29) is now obvious. Both the left and the right hand sides of the second equality in (29) are equal to $-\overline{M_{21}}/M_{21}$ since

$$1 - M_{11}\overline{M_{22}} = 1 - M_{11}M_{22} = -M_{12}M_{21}.$$

Finally,

$$\begin{aligned} \text{RHS of the third equality in (29)} &= \frac{-M_{12}M_{21}}{M_{21}\overline{M_{12}}} = -\frac{M_{12}}{\overline{M_{12}}} = -\frac{\overline{M_{21}}}{M_{21}} \\ &= \text{RHS of the second equality} \\ &= \text{LHS of the third equality}. \end{aligned}$$

Since the first three equalities always hold in the hhe and hhh cases, the matrix $(1/\sqrt{2})\begin{pmatrix} e^{\phi i} & -e^{\phi i} \\ e^{-\phi i} & e^{-\phi i} \end{pmatrix}$ where ϕ makes the fourth equality true solves the period problem. $\square$

Now we want to classify the values of p_0, p_1, p_∞ and q_0, q_1, q_∞ which yield given α, τ, ρ. It is convenient to use the following quantities

$$
\begin{aligned}
c_0 &= (p_1q_0 - p_0q_1) + (p_0q_\infty - p_\infty q_0),\\
c_1 &= (p_1q_0 - p_0q_1) + (p_1q_\infty - p_\infty q_1),\\
c_\infty &= (p_0q_\infty - p_\infty q_0) + (p_1q_\infty - p_\infty q_1),
\end{aligned} \tag{30}
$$

which are related to α, τ, ρ (from [4, p.80]) by

$$
\alpha = \frac{1}{2} - \sqrt{c_0 + \frac{1}{4}}, \quad \tau = \sqrt{c_1 + \frac{1}{4}}, \quad \rho = \sqrt{c_\infty + \frac{1}{4}}. \tag{31}
$$

Lemma 15. *For any $c = (c_0, c_1, c_\infty)^T \in \mathbb{C}^3 \setminus \{\vec{0}\}$ there are $p = (p_0, p_1, p_\infty)^T$ and $q = (q_0, q_1, q_\infty)^T$ in $\mathbb{C}^3$ which solve* (30). *$\tilde{p}, \tilde{q}$ is also a solution if and only if $\begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = A \begin{pmatrix} p \\ q \end{pmatrix}$ for some $A \in \mathrm{SL}(2;\mathbb{C})$.*

Proof. Given a nonzero vector $x = (x_0, x_1, x_\infty)^T \in \mathbb{C}^3 \setminus \{\vec{0}\}$, define

$$
A_x = \begin{pmatrix} -x_1 + x_\infty & x_0 & -x_0 \\ -x_1 & x_0 + x_\infty & -x_1 \\ x_\infty & x_\infty & -x_0 - x_1 \end{pmatrix}.
$$

Then,
(1) the rank of A_x is 2,
(2) x is a basis of the null space of A_x,
(3) $N_x = (x_0 + x_1 + x_\infty, -x_0 - x_1 + x_\infty, -x_0 + x_1 - x_\infty)^T$ is normal to the column space of A_x,
(4) for any x, y we have $A_x y = -A_y x$.
Now we see that (30) is equal to $c = A_q p$, which has a solution if and only if c is perpendicular to N_q. This is equivalent to saying that q is in the following plane $\subset \mathbb{C}^3$

$$
(c_0 - c_1 - c_\infty)x + (c_0 - c_1 + c_\infty)y + (c_0 + c_1 - c_\infty)z = 0.
$$

So, we just choose a nonzero q from this plane. Then there must exist p which satisfies the equation.

By switching the roles of q and p, we see that p must lie in this plane also. Furthermore, if $p = tq$ for some $t \in \mathbb{C}$, then $A_q p = tA_q q = 0 \neq c$. Therefore p and q are linearly independent. Now suppose $\tilde{p}, \tilde{q}$ also solve (30). Then they must be in the above plane, hence $\widetilde{p} = a_1 p + a_2 q$, $\widetilde{q} = a_3 p + a_4 q$ for some $a_1, a_2, a_3, a_4 \in \mathbb{C}$. Now we see that

$$c = A_{a_3 p + a_4 q}(a_1 p + a_2 q) = (a_1 a_4 - a_2 a_3) A_q p = (a_1 a_4 - a_2 a_3) c.$$

So we conclude that $a_1 a_4 - a_2 a_3 = 1$. It is obvious that if $\widetilde{p}, \widetilde{q}$ are of the form mentioned above, then they solve (30). □

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