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BP-THEORETIC INSTABILITIES TO THE MOTION PLANNING PROBLEM IN 4-TORSION LENS SPACES

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Abstract

The Brown-Peterson cohomology for skeleta of the classifying space of the group $\mathbb{Z}_4 \times \mathbb{Z}_4$ is analyzed in order to describe obstructions to the motion planning problem for a particle moving in a 4-torsion lens space. We discuss the relationship of this situation to the Euclidean immersion problem for 2^e-torsion lens spaces, and the way this leads to an alternative approach to the classical immersion problem for real projective spaces.

1. Introduction

Despite the ample bibliography there is on the immersion problem for projective spaces (see [3] for an updated summary), the solution to this still open problem appears to be completely out of hands with present techniques. Here the result that is perhaps the most comprehensive and, at the same time, with an amazingly simple statement is Davis' theorem [4] claiming

$$\mathbb{R}P^{2(n+\alpha(n)-1)} \not\subseteq \mathbb{R}^{4n-2\alpha(n)}, \tag{1}$$

where $\mathbb{R}P^m$ stands for the m-dimensional real projective space, $\alpha(n)$ is the number of ones in the dyadic expansion of n, and the symbol $\not\subseteq$ means "does not admit an immersion in."

In [11] the first author has contextualized such a result within a more general situation by considering the immersion problem for the 2^e-torsion $(2m+1)$-dimensional lens space $L^{2m+1}(2^e)$ and suggesting that for $\delta = \max\{0, \alpha(n) - e\}$

$$L^{2(n+\delta)+1}(2^e) \not\subseteq \mathbb{R}^{4n-2\alpha(n)}. \tag{2}$$

For instance, the case $e = 1$ is essentially Davis' result, whereas the $e \geq \alpha(n)$ case, which is true and picks up nicely the complex situation [10], has been conjectured to

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be close to optimal in [5]. Yet the above generalization has been proved only under certain hypothesis. For instance, (2) is verified in [11] provided

$$2^{\nu(n)} > \alpha(n) - e, \tag{3}$$

where $\nu(n)$ stands for the highest power of 2 dividing n.

However, the fact that no counterexample is known to (2) may just as well be a consequence of our little knowledge of the immersion problem for this manifolds. One of the goals of this work is to partially mend the last situation by studying the problem from a slightly different point of view (see (5) and (6) below), which we now elaborate on.

The immersion problem for $\mathbb{R}P^n$ has several equivalent presentations, one of which has been recently developed in [8], and has to do with the instabilities in the motion planning problem for the system consisting of a line which is revolving in $(n+1)$-dimensional affine space through a fixed point. For our purposes the relevant concepts are as follows: given a space X, let $P(X)$ denote the space of free paths on X, and let $TC(X)$ stand for the smallest number (either a positive integer or ∞) of open sets U that cover $X \times X$ in such a way that the fibration

$$ev\colon P(X) \to X \times X$$

defined by $ev(\gamma) = (\gamma(0), \gamma(1))$, admits a local section on each U. Then the main result in [8] is the fact that there is an *optimal* immersion

$$\mathbb{R}P^n \subseteq \mathbb{R}^{TC(\mathbb{R}P^n)-\epsilon}, \tag{4}$$

where $\epsilon = 1$, except for $n = 1, 3, 7$ where $\epsilon = 0$. In other words, finding optimal Euclidean immersions for $\mathbb{R}P^n$ is equivalent to a full understanding of the instabilities arising in the motion planning for a particle in this manifold (see [6, 7]).

Now, the obvious question of whether $TC(L^{2n+1}(k))$ has to do with the immersion problem for $L^{2n+1}(k)$ has, however, a negative answer [9], and an indication of this fact is given by the formula $TC(\mathbb{C}P^n) = 2n + 1$, which is in tremendous contrast with the subtleties arising in the immersion problem for $\mathbb{C}P^n$. Nonetheless the "TC approach" for lens spaces can still be used as a way to understand the immersion problem for (odd dimensional) real projective spaces (see (5) below), a philosophy developed in [9] by means of the following general result.

Theorem 1.1 ([9]). *For $n \geq 0$ and $e \geq 1$ let $s(n,e)$ denote the integral part of $[TC(L^{2n+1}(2^e)) + 1]/2$, so that*

$$TC(L^{2n+1}(2^e)) = 2s(n,e) - \tau, \quad \text{with} \quad \tau = \tau(n,e) \in \{0,1\}.$$

Then:

a) *$s(n,e)$ equals the smallest positive integer l such that there is a $\mathbb{Z}/2^e$-bi-equivariant map $S^{2n+1} \times S^{2n+1} \to S^{2l-1}$.*

b) *$s(n,e)$ equals the smallest positive integer l such that there is a homotopy commutative diagram*

$$\begin{tikzcd} L^\infty(2^e) \times L^\infty(2^e) \arrow[r, "\mu"] & L^\infty(2^e) \\ L^{2n+1}(2^e) \times L^{2n+1}(2^e) \arrow[r] \arrow[u, hook] & L^{2l-1}(2^e) \arrow[u, hook] \end{tikzcd}$$

where μ is the H-multiplication.

c) *$s(n,e)$ equals the smallest positive integer l such that the iterated l-fold Whitney sum of the exterior tensor product*

$$\eta \otimes_{\mathbb{C}} \eta \to L^{2n+1}(2^e) \times L^{2n+1}(2^e)$$

admits a nowhere zero section, where η is the pullback under the canonical projection $L^{2n+1}(2^e) \to \mathbb{C}P^n$ of the complex Hopf line bundle over $\mathbb{C}P^n$.

It is obvious that the numbers $s(n,e)$ satisfy the relations $s(n,e) \le s(n',e')$ provided $n \le n'$ or $e \le e'$, and in this terms (4) and Theorem 1.1 are just saying that the chain of inequalities

$$s(n,1) \le s(n,2) \le s(n,3) \le \cdots \tag{5}$$

can indeed be considered as a way to understand the difficulties in (roughly) *half* the immersion problem for odd dimensional real projective spaces (the comments after (7) give a more precise statement of what is meant here). In such an approach one has the extra bonus that $s(n,e)$ is described easily enough for “large” e (see [9, Proposition 2.2]):

$$s(n,e) = \begin{cases} 2n+1 & \text{for} \quad e > \alpha(n), \\ 2n & \text{for} \quad e = \alpha(n). \end{cases} \tag{6}$$

On the other hand, it is quite profitable to compare the “s-approach” in (5) with the immersion problem for 2^e-torsion lens spaces. This is done by means of the main result in [2] to obtain (see [9])

$$2s(n,e) - 1 \ge \operatorname{Imm}(n,e). \tag{7}$$

Here $\operatorname{Imm}(n,e)$ is the minimal Euclidean dimension where $L^{2n+1}(2^e)$ can be immersed. It is worth noticing that for $e = 1$, (7) is optimal up to parity, namely $s(n,1) - 1$ is just the integral part of $(1/2)\operatorname{Imm}(n,1)$ (see [1] or [9, (24)]).

We are now in position to introduce the main motivation for this work: in terms of (7), "halving" Davis' theorem (1) yields

$$s(n+\alpha(n)-1,1)\geq 2n-\alpha(n)+1, \tag{8}$$

that is a particular case of the analogue for (2) which, when valid, would read

$$s(n+\alpha(n)-e,e)\geq 2n-\alpha(n)+1, \tag{9}$$

at least for $e\leq\alpha(n)$. But in view of (6), for $e=\alpha(n)$, (9) is optimal only for $\alpha(n)=1$, and in general, the lower bound given by (9) seems to be rather weak for large values of e. The calculations in this paper give evidence to the following (conjectural) improvement of (9):

$$s(n+\alpha(n)-e,e)\geq 2n-\alpha(n)+e. \tag{10}$$

For instance (6) claims that (10) is in fact an equality for $e=\alpha(n)$.

The philosophy behind (10) parallels that in Davis' result (1) and its proposed generalization (2)—namely, that in order to get information on the immersion problem for real projective spaces, one can make use of the interaction dimension/torsion in lens spaces—, having now the advantage that, by using the approach in (5), not only dimension/torsion play a role, but also the actual lower bound for $s(m,e)$ depends on the torsion, something not present in (2) nor (9).

One should remark that the restriction $e\leq\alpha(n)$ suggested in (9) is indeed needed for (10) to hold, and this is closely related to the "stable" values of $s(n,e)$ described in (6). A clean view of this is obtained by comparing with the way one uses the full power of Davis' result: in order to get the best non-immersion for $\mathbb{R}P^{2m}$ coming from (1), one takes out of those n having

$$m\geq n+\alpha(n)-1 \tag{11}$$

the one for which $4n-2\alpha(n)$ is largest possible. In any case such an n must have $n\leq m$. However without the restriction $e\leq\alpha(n)$ the analogous condition $m\geq n+\alpha(n)-e$—to be used when taking full advantage of (10)—may hold even for $n>m$ and, in such a situation, (10) leads to nonsense. For instance take $m=2^{l+1}-2$ so that $\alpha(m)=l$, and take $n=m+2=2^{l+1}$. Then $m\geq n+\alpha(n)-l$ for $l\geq 3$, so that $s(m,l)\geq s(n+\alpha(n)-l,l)$ which would be at least $2n-\alpha(n)+l=2m+l+3$ if (10) applied. But this is in contradiction to (6): $s(m,l)=2m$.

We summarize this motivational section by stressing the geometrical meaning and importance of (10): as suggested by (7), the inequality in (10) gives lower bounds for the topological complexity for 2^e-torsion lens spaces which, towards an understanding of the immersion problem for real projective spaces, improve the lower bounds one could get by studying Euclidean nonimmersions for lens spaces.

2. Main result and outline of proof

Let h^* be a multiplicative complex oriented cohomology theory, let x and y stand for the corresponding (2-dimensional) orientations over the axes in $L^{2n+1}(2^e)\times L^{2n+1}(2^e)$, and let F denote the formal group law for h^*. Then an easy consequence of Theorem 1.1 is:

Corollary 2.1. *$s(n, e) > l$ whenever $(x +_F y)^l \neq 0$.*

We deduce a number of cases of (10) by analyzing $(x +_F y)^l$ for $n = m + \alpha(m) - e$ and $l = 2m - \alpha(m) + e - 1$, when h^* is Brown-Peterson cohomology (BP^*) at the prime 2. In that case F is in fact the universal 2-typical formal group law, whose importance and complexity has put a great challenge for its practical use. Despite this we will only require the obvious observation that F can be written in the form

$$x +_F y = x + uy \tag{12}$$

where $u \in BP^*(L^{2n+1}(2^e) \times L^{2n+1}(2^e))$ is a unit. The real crux of the matter will rather be to perform calculations in this last ring. As a first simplification, we will restrict the computation of powers of (12) to $BP^*(L^{2n}(2^e) \times L^{2n}(2^e))$ which, according to [10, Proposition 3.1], in the relevant dimensions takes the form

$$A_{n,e} = BP_*[x, y] \Big/ \left(x^{n+1}, y^{n+1}, [2^e](x), [2^e](y)\right). \tag{13}$$

Here $[2^e](x)$ stands for the associated (universal 2-typical) 2^e-series, and BP_* is the $\mathbb{Z}_{(2)}$-polynomial algebra on generators $v_1, v_2, \ldots$ with $\deg(v_i) = 2(2^i - 1)$ (as usual we have changed the "cohomological" dimensions of the coefficient ring to "homological" dimensions, so that each v_i acts on Brown-Peterson cohomology by lowering degrees). In order to apply Corollary 2.1, we reduce the l-th power of (12) to a suitable multiple of $x^n y^n \in A_{n,e}$. The next result summarizes the calculations in the rather technical Section 3.

Proposition 2.2. *For $\nu(m) \gg 0$ and $\alpha(m) \geq e$, $(x + uy)^l$ divides $2^{e-1}v_2^{\alpha(m)-e}x^n y^n$ in $A_{n,e}\big/\left(v_1^3 x^n y^n\right)$, provided $e = 2$.*

The required conclusion will then be derived from the next result, which in turn will be deduced in Section 4 from the work of G. Nakos on the generalized Conner-Floyd conjecture for $\mathbb{Z}_4 \times \mathbb{Z}_4$.

Proposition 2.3. *For $e = 2$ the annihilator ideal of $x^n y^n$ in $A_{n,e}$ is contained in $\left(4, 2v_1, v_1^4\right)$.*

We will then have proved our main result in the direction of (10):

Theorem 2.4. *For $\nu(m) \gg 0$ and $\alpha(m) \geq 2$, $s(m+\alpha(m)-2, 2) \geq 2m-\alpha(m)+2$.*

Lemma 3.1 in the next section is more specific on how large $\nu(m)$ is required to be in Proposition 2.2 and Theorem 2.4.

As explained in the introduction, having a proof for an unrestricted (10) not only would yield information on the topological complexity of lens spaces (a problem relevant to the motion planning in these manifolds), but would also push a long way our understanding of the role the 2-torsion plays in the immersion problem for projective spaces. The calculations needed to approach the general case of (10) would depend on having versions for Propositions 2.2 and 2.3 valid for $e \geq 3$. After going through the next section the reader will realize this would require a major computational effort in the case of 2.2 (both for extending the calculations to $e \geq 3$, as well as for removing the extra hypothesis $\nu(m) \gg 0$). Yet a generalization for 2.3 appears to be much less transparent: as explained in the final section, one would need to have a good hold on the annihilator ideal for the toral class in the *BP-homology* of the classifying space for $\mathbb{Z}_{2^e} \times \mathbb{Z}_{2^e}$—a generalized Conner-Floyd conjecture. As observed in [16, 17], such a goal could seem to be far from reach with present knowledge. Nonetheless the authors hope that this paper (Section 3 in particular) encourages and helps researchers in the field to settle the structure of annihilator ideals of the type above. As exemplified by the work on the classical Conner-Floyd conjecture [18] and [13, 14], such a task would contribute to extending the computability—and thus usefulness—for Brown-Peterson theory.

3. Algebraic input

As in the last section, we fix the notation $l = 2m-\alpha(m)+e-1$, $n = m+\alpha(m)-e$, and $A_{n,e}$ to represent the ring in (13). We will also assume $e \leq \alpha(m)$. In terms of (12), the relevant class in Corollary 2.1 takes the form

$$(x +_F y)^l = \sum_{i=0}^{3(\alpha(m)-e)+1} \binom{l}{n-i} x^{n-i} y^{l-n+i} u^{l-n+i}. \tag{14}$$

The extra hypothesis in Proposition 2.2 and Theorem 2.4 represents the analogue of the assumption (3) used in [11]. Here it is only used to make sure all but two of the summands in (14) are trivial.

Lemma 3.1. *Under the two conditions*

- $\alpha(m) - e < 2^{\nu(m)-1}$,
- $\nu(m) > \alpha(m) - e + \nu(\alpha(m)-e) + \nu\binom{\alpha(m)-e-1}{2\alpha(m)-2e-i}$, *for* $\alpha(m)-e+1 \leq i \leq 2(\alpha(m)-e)$,

we have

a) $\nu\binom{l}{n-i} \geq i+e$, *provided* $0 \leq i \leq 2(\alpha(m)-e)$ *and* $i \neq \alpha(m)-e$,

b) $\nu\binom{l}{n-i} = \alpha(m)-1$, *provided* $i = \alpha(m)-e$.

Proof. We will use without further notice the well known relation $\nu\binom{a}{b} = \alpha(b) + \alpha(a-b) - \alpha(a)$; likewise, in view of our hypothesis on m, the fact that $\alpha(2^L - a) = L - \alpha(a-1)$, for $0 < a \leq 2^L$, will be translated below as the formula $\alpha(m-a) = \alpha(m) - 1 + \nu(m) - \alpha(a-1)$, for $0 < a \leq 2^{\nu(m)}$. Thus for b) we have

$$
\begin{aligned}
\nu\binom{l}{n-i} &= \nu\binom{2m-\alpha(m)+e-1}{m} \\
&= \alpha(m) + \alpha(m-\alpha(m)+e-1) - \alpha(2m-\alpha(m)+e-1) \\
&= \alpha(m) + (\alpha(m) - 1 + \nu(m) - \alpha(\alpha(m)-e)) \\
&\quad - (\alpha(m) - 1 + \nu(m) + 1 - \alpha(\alpha(m)-e))
\end{aligned}
$$

which, as claimed, is $\alpha(m) - 1$. For a) we note that $\nu\binom{l}{n-i}$ is equal to

$$\alpha(m + (\alpha(m) - e - i)) + \alpha(m - (2\alpha(m) + 1 - 2e - i)) - \alpha(2m - (\alpha(m) - e + 1)). \tag{15}$$

Note that both $2\alpha(m)+1-2e-i$ and $\alpha(m)-e+1$ are positive. In case that $\alpha(m)-e-i$ is positive too, (15) takes the form

$$
\begin{aligned}
&\alpha(m) + \alpha(\alpha(m)-e-i) + (\alpha(m) - 1 + \nu(m) - \alpha(2\alpha(m) - 2e - i)) \\
&- (\alpha(m) - 1 + \nu(m) + 1 - \alpha(\alpha(m)-e)) \\
&= \alpha(m) - 1 + \alpha(\alpha(m)-e-i) + \alpha(\alpha(m)-e) - \alpha(2\alpha(m)-2e-i) \\
&= \alpha(m) - 1 + \nu\binom{2\alpha(m)-2e-i}{\alpha(m)-e} \\
&\geq \alpha(m) - 1 \\
&\geq i + e.
\end{aligned}
$$

In case that $\alpha(m) - e - i$ is negative, (15) should be thought of as

$$
\begin{aligned}
&\alpha(m - (e + i - \alpha(m))) + \alpha(m - (2\alpha(m) + 1 - 2e - i)) \\
&- \alpha(2m - (\alpha(m) - e + 1)) \\
&= (\alpha(m) - 1 + \nu(m) - \alpha(e + i - \alpha(m) - 1)) \\
&\quad + (\alpha(m) - 1 + \nu(m) - \alpha(2\alpha(m) - 2e - i)) \\
&\quad - (\alpha(m) - 1 + \nu(m) + 1 - \alpha(\alpha(m) - e)).
\end{aligned}
$$

Using the second condition in the hypothesis and the well known relation $\alpha(a-1) = \alpha(a) - 1 + \nu(a)$, the last expression is easily seen to be bounded from below by $i+e$. □

Together with [11, Corollary 2.6], Lemma 3.1 implies that, up to a unit, (14) reduces to

$$2^{\alpha(m)-1}\left(x^{n-(\alpha(m)-e)}y^{n-2(\alpha(m)-e)-1} + x^{n-2(\alpha(m)-e)-1}y^{n-(\alpha(m)-e)}u^{n-(\alpha(m)-e)}\right).$$

Using now [11, Corollary 2.7], the y-multiple of the last expression takes the form

$$2^{e-1} v_1^{\alpha(m)-e} x^n y^{n-2(\alpha(m)-e)}. \tag{16}$$

The remaining of the section is then devoted to proving the following key relation.

Proposition 3.2. *For $e = 2$ and $i \geq 0$ with $n \geq 2i + 2$, the following relation holds up to units in $A_{n,e}\big/\big(v_1^3 x^n y^n\big)$:*

$$2^{e-1} v_1^i x^n y^{n-2i} = 2^{e-1} v_2^i x^n y^n.$$

Write $[4](x) = \sum_{s\geq 0} a_s x^{s+1}$, and recall from [12] that the 2-divisibility properties of the coefficients $a_s \in BP_*$ are given by

$$\nu(a_s) = 2d_0 + d_1 \tag{17}$$

where $s + 1 = d_0 + 2d_1 + 4d_2 + \cdots$ is the 2-adic expansion of $s + 1$.

The rest of the properties we need about the 4-series are contained in the following result whose verification is done by a straight forward calculation left to the reader.

Lemma 3.3. a) $a_3 = w_1 v_1^3 + 2w_2 v_2$, *where w_1 and w_2 are odd numbers.*
b) $a_7 \in (2, v_1)$.
c) *Up to units, $a_0 = 4$, $a_1 = 2v_1$, and $a_2 = 8v_1^2$.*

Proposition 3.2 is deduced below from (17), Lemma 3.3 and the next result whose proof constitutes the technical core of the paper.

Lemma 3.4. *The following relations hold in $A_{n,2}\big/\big(v_1^3 x^n y^n\big)$ for $i \geq 0$:*
(a) $2v_1^{i+2} x^n y^{n-2i} = 0$, *provided $n - 2i \geq 2$.*
(b) $v_1^{i+3} x^n y^{n-2i} = 0$, *provided $n - 2i \geq 4$.*

Proof of Proposition 3.2. We proceed by induction on i, the case $i = 0$ being trivial. Using (13) and (17), we express $2v_1^i x^n y^{n-2i}$ as an element of the ideal

$$\begin{aligned} \mathcal{I} = v_1^{i-1} x^n \big(4y^{n-2i-1}, \quad & \quad 8y^{n-2i+1}, a_3 y^{n-2i+2}, \\ 4y^{n-2i+3},\ & 2y^{n-2i+4}, 8y^{n-2i+5}, a_7 y^{n-2i+6}, \\ 4y^{n-2i+7},\ & 2y^{n-2i+8}, 8y^{n-2i+9}, y^{n-2i+10} \quad \big). \end{aligned}$$

The generators in the first and third columns are trivial since $4x^n = 0$. The generators in the second column are trivial by induction. By Lemma 3.3, $a_7 v_1^{i-1} x^n y^{n-2i+6}$ lies in the ideal generated by $2v_1^{i-1} x^n y^{n-2i+6}$ and $v_1^i x^n y^{n-2i+6}$, both of which are zero in view of Lemma 3.4 (the later is trivial for $i = 3$ in view of the indeterminacy we are imposing; we note however that $v_1^3 x^n y^n \neq 0$ in $A_{n,2}$ [16, 17]). Likewise, the last generator of the ideal $\mathcal{I}$ is trivial by Lemma 3.4 too. It is then clear that up to units

$$2v_1^i x^n y^{n-2i} = a_3 v_1^{i-1} x^n y^{n-2i+2},$$

which once again by Lemmas 3.3 and 3.4 takes the form $2v_1^{i-1} v_2 x^n y^{n-2(i-1)}$. Thus, the result follows from the inductive assumption. □

In proving Lemma 3.4 it turns out to be easier to verify the following more complete statement (parts (a) and (b) of Lemma 3.4 are respectively contained in parts (a'_{i-1}) and (b_i) of Lemma 3.5).

Lemma 3.5. *The following relations hold in* $A_{n,2}\big/\big(v_1^3 x^n y^n\big)$*, for* $i \geq 0$.
(a_i) $2v_1^{i+2} x^{n-u} y^{n-v} = 0$, *for* $u + v = 2i + 1$.
(α_i) $4v_1^{i+1} x^{n-u-1} y^{n-v} = 0$, *for* $u + v = 2i + 1$.
(b_i) $v_1^{i+3} x^{n-u} y^{n-v} = 0$, *for* $u + v = 2i$.
(a'_i) $2v_1^{i+3} x^{n-u} y^{n-v} = 0$, *for* $u + v = 2i + 2$.
(α'_i) $4v_1^{i+2} x^{n-u-1} y^{n-v} = 0$, *for* $u + v = 2i + 2$.
(b'_i) $v_1^{i+4} x^{n-u} y^{n-v} = 0$, *for* $u + v = 2i + 1$.

REMARK 3.6. Although not explicitly noted, the relations in Lemma 3.5 (just as in Lemma 3.4) are claimed to hold provided there are "enough" powers of y (and thus of x). In detail, in parts a, α, a', α' above we require $n - u - v \geq 2$, whereas in parts b and b' we need $n - u - v \geq 4$. For instance, the latter condition will allow us to use the 4-series on y as indicated in (18), (19) and (21). The six relations above will be verified inductively, and the reader can check that the lower bounds just described on the powers of x and y behave well in the induction.

REMARK 3.7. With respect to the statement of Lemma 3.5 it is clear that (a_i), (α_i) and (b_i) imply the corresponding primed versions for $i - 1$. We have however chosen to write the result in such a way to reflect the strategy for the proof, which will depend on a detailed analysis of the interdependency of these six statements. (In retrospect, we will see that the primed versions are logically equivalent to the corresponding non-primed versions.)

Proof. The six relations will be proved by induction on i in the following order

$$(a_0), (\alpha_0) \Rightarrow (b_0) \Rightarrow (a'_0), (\alpha'_0) \Rightarrow (b'_0) \Rightarrow (a'_1), (\alpha'_1) \Rightarrow (b'_1) \Rightarrow (a_2), (\alpha_2) \Rightarrow \cdots.$$

The basic strategy in the proof is to let both relations coming from the 4-series in (13) to interact among each other. In the process we make use (without further notice) of (17) and Lemma 3.3.

For (a_0) it is enough to note that $2v_1x^ny^{n-1} \in x^n\big(4y^{n-2}, 8v_1^2y^n\big) = 0$. For (α_0) we have: $4v_1x^{n-1}y^{n-1} \in v_1(2v_1x^n)y^{n-1}$ which has already been seen to be trivial, whereas $4v_1x^{n-2}y^n$ is trivial since in fact $4y^n = 0$. We have already noted that (b_0) is trivial (just) by indeterminacy. For (a_0') and (α_0') we have: $2v_1^2x^ny^{n-2} \in v_1x^n\big(4y^{n-3}, 8v_1^2y^{n-1}, a_3y^n\big)$. The first two generators of this ideal are certainly trivial, so that $2v_1^2x^ny^{n-2} \in \big(v_1^3, 2\big)v_1x^ny^n$ which has already been shown to be trivial. Now we have $4v_1x^{n-1}y^{n-2} \in v_1(2v_1x^n)y^{n-2} = 0$ and $2v_1^2x^{n-1}y^{n-1} \in v_1x^{n-1}\big(4y^{n-2}, 8v_1^2y^n\big) = 0$ (all the other relations in (a_0') and (α_0') follow by symmetry).

For (b_0'), and in order to complete the start of induction, we first note

$$v_1^3y^{n-1} \in \big(4y^{n-4}, 2v_1y^{n-3}, 8v_1^2y^{n-2}, 2v_2y^{n-1}, 4y^n\big)\,. \tag{18}$$

Then letting $v_1^3y^{n-1} = 2A$ we have $v_1^4x^ny^{n-1} = 2Av_1x^n \in A(4x^{n-1}) = 2x^{n-1}\big(v_1^3y^{n-1}\big)$, which has already been shown to be trivial.

The inductive step is verified in a similar way (all essential ideas have already been used up to here). So assume the six relations have been verified for $j < i$.

In order to prove (a_i) and (α_i) we start by noticing:

$$\begin{aligned}2v_1^{i+2}x^ny^{n-2i-1} \in v_1^{i+1}x^n\big(&4y^{n-2i-2}, 8v_1^2y^{n-2i}, a_3y^{n-2i+1}, 4y^{n-2i+2},\\ &2y^{n-2i+3},\ 8y^{n-2i+4},\ Dy^{n-2i+5},\ y^{n-2i+9}\big).\end{aligned}$$

Here the coefficient D lies in $(2, v_1)$ and includes the terms a_i with $7 \le i \le 10$ (in view of (17), a_i is in fact divisible by 2 for $8 \le i \le 10$), moreover the first, second, fourth and sixth generators in the last ideal are zero by order, so the ideal is contained in

$$\big(v_1^{i+4}x^ny^{n-2i+1}, 2v_1^{i+1}x^ny^{n-2i+1}, v_1^{i+2}x^ny^{n-2i+5}, v_1^{i+1}x^ny^{n-2i+9}\big)\,.$$

This last ideal is zero because all generators are. The first one by (b_{i-1}'), the second one by (a_{i-1}), the third one by (b_{i-3}') and the fourth one by (b_{i-5}').

Suppose now that $2v_1^{i+2}x^{n-u}y^{n-v} = 0$, for $u + v = 2i + 1$, and observe

$$\begin{aligned}4v_1^{i+1}x^{n-u-1}y^{n-v} \in v_1^{i+1}y^{n-v}\big(&2v_1x^{n-u}, 8v_1^2x^{n-u+1}, a_3x^{n-u+2}, 4x^{n-u+3},\\ &2x^{n-u+4},\quad 8x^{n-u+5},\ \ x^{n-u+6}\big)\end{aligned}$$

and

$$\begin{aligned}2v_1^{i+2}x^{n-u-1}y^{n-v+1} \in v_1^{i+1}x^{n-u-1}\big(&4y^{n-v}, 8v_1^2y^{n-v+2}, a_3y^{n-v+3}, 4y^{n-v+4},\\ &2y^{n-v+5},\quad 8y^{n-v+6},\ \ y^{n-v+7}\big).\end{aligned}$$

As before, all generators in this last two ideals are trivial. Indeed, for the first ideal we have that the first and second generators are zero by induction. For the third generator: the term that is multiple of 2 is zero by (a_{i-1}) and the term that is multiple of v_1^3 is zero by (b_{i-1}'). The fourth, the fifth and the sixth generators are zero by (a_{i-2}'). The last generator is zero by (b_{i-3}').

Note that the first, the second, the fourth and the sixth generators of the second ideal have just been shown to vanish, and then we have that this ideal is contained in

$$\left(v_1^{i+4}x^{n-u-1}y^{n-v+3},\, 2v_1^{i+1}x^{n-u-1}y^{n-v+3},\, v_1^{i+1}x^{n-u-1}y^{n-v+7}\right).$$

For this last ideal we have that all generators are zero; the first one by (b'_{i-1}), the second one because (a_{i-1}) and the last one by (b'_{i-3}).

To verify (b_i) we consider $v_1^{i+3}x^{n-u}y^{n-v}$ with $u+v=2i$. Let A, B and C be the elements fitting in the formula

$$v_1^3y^{n-v} = 2A + By^{n-v+4} + Cy^{n-v+8} \tag{19}$$

which comes (analogously to (18)) from the relation $0 = y^{n-v-4}\cdot[4](y)$, where $B \in (2, v_1)$ contains the terms a_i with $7 \le i \le 10$, A contains the $2v_2$-multiple in a_3 as well as the terms a_i with $0 \le i \le 6$, $i \neq 3$, and C contains all other a_i's. Then

$$v_1^{i+3}x^{n-u}y^{n-v} \in \left(2Av_1^ix^{n-u},\, Bv_1^ix^{n-u}y^{n-v+4},\, v_1^ix^{n-u}y^{n-v+8}\right).$$

In this last ideal the second and third generators are zero. The second one because (b_{i-2}) and (a'_{i-3}), for the multiples of v_1 and 2 respectively. The third generator is zero by (b_{i-4}). So we have that the ideal above is contained in

$$\begin{aligned} v_1^{i-1}A\big(&4x^{n-u-1},\, 8v_1^2x^{n-u+1},\, a_3x^{n-u+2},\, 4x^{n-u+3},\ 2x^{n-u+4},\\ &8x^{n-u+5},\ Dx^{n-u+6},\, x^{n-u+10}\big), \end{aligned} \tag{20}$$

where as above $D \in (2, v_1)$. We will see that all generators of (20) are trivial. For the first one we get from (19)

$$4Av_1^{i-1}x^{n-u-1} \in 2v_1^{i-1}x^{n-u-1}\left(v_1^3y^{n-v},\, By^{n-v+4},\, y^{n-v+8}\right)$$

and this ideal is contained in

$$\left(2v_1^{i+2}x^{n-u-1}y^{n-v},\, 2v_1^ix^{n-u-1}y^{n-v+4},\, 4v_1^{i-1}x^{n-u-1}y^{n-v+4},\, 2v_1^{i-1}x^{n-u-1}y^{n-v+8}\right),$$

all of whose generators are trivial. The first one because (a_i), the second one by (a_{i-2}), the third one by (α'_{i-3}), and for the last one we use (a_{i-4}).

The above argument also takes care of the second, fourth and sixth generators in (20). As for the first half of the third generator in (20), namely the term that is multiple of 2, we use again (19) to obtain:

$$2v_1^{i-1}Ax^{n-u+2} \in v_1^{i-1}x^{n-u+2}\left(v_1^3y^{n-v},\, v_1y^{n-v+4},\, 2y^{n-v+4},\, y^{n-v+8}\right).$$

The first generator of this last ideal is zero by (b_{i-1}), the second by (b_{i-3}), the third because (a'_{i-4}) and the last one by (b_{i-5}).

This argument takes care also of the fifth and the first half of the seventh generators in (20). For the second half of the third generator in (20), namely $v_1^{i+2}Ax^{n-u+2}$, we note that from its definition

$$A \in \frac{1}{2}\left(4y^{n-v-3}, 2v_1y^{n-v-2}, 8v_1^2y^{n-v-1}, 2v_2y^{n-v}, 4y^{n-v+1}, 2y^{n-v+2}, 8y^{n-v+3}\right),$$

that is,

$$A \in \left(2y^{n-v-3}, v_1y^{n-v-2}, y^{n-v}\right),$$

so that

$$v_1^{i+2}Ax^{n-u+2} \in v_1^{i+2}x^{n-u+2}\left(2y^{n-v-3}, v_1y^{n-v-2}, y^{n-v}\right).$$

As before, all generators of this last ideal are zero. The first one by (a_i), the second one by an inductive argument (grounded by the relation $x^{n+1}=0$) on u, and the third one because (b_{i-1}).

For the second half of the seventh generator in (20):

$$v_1^{i}Ax^{n-u+6} \in v_1^{i}x^{n-u+6}\left(2y^{n-v-3}, v_1y^{n-v-2}, y^{n-v}\right) = 0.$$

The first generator is zero by (a_{i-2}), the second one because (b_{i-2}) and the last one by (b_{i-3}).

Finally, for the eighth generator in (20):

$$v_1^{i-1}Ax^{n-u+10} \in v_1^{i-1}x^{n-u+10}\left(2y^{n-v-3}, v_1y^{n-v-2}, y^{n-v}\right) = 0.$$

In this case, (a_{i-4}), (b_{i-4}) and (b_{i-5}) take account of the first, the second and the third generators respectively.

We now proceed to prove relations (a'_i) and (α'_i). We start by noticing that the term $2v_1^{i+3}x^ny^{n-2i-2}$ lies in the ideal

$$v_1^{i+2}x^n\left(4y^{n-2i-3}, 8v_1^2y^{n-2i-1}, a_3y^{n-2i}, 4y^{n-2i+1}, 2y^{n-2i+2}, 8y^{n-2i+3}, Dy^{n-2i+4}, y^{n-2i+8}\right),$$

where again $D \in (2, v_1)$. The first, the second, the fourth and the sixth generators are zero by order. The fifth generator, as well as the first halves (multiples of 2) of the third and seventh generators are trivial by (a'_{i-1}), whereas the second half of the third (the multiple of v_1^3) because (b_i). The second half of the seventh generator vanishes because (b_{i-2}), and (b_{i-4}) takes account of the last generator.

Suppose now that

$$2v_1^{i+3}x^{n-u}y^{n-v} = 0, \quad \text{for} \quad u+v = 2i+2,$$

and observe that $4v_1^{i+2}x^{n-u-1}y^{n-v}$ lies in the ideal

$$v_1^{i+2}y^{n-v}\left(2v_1x^{n-u}, 8v_1^2x^{n-u+1}, a_3x^{n-u+2}, 4x^{n-u+3}, 2x^{n-u+4}, 8x^{n-u+5}, x^{n-u+6}\right),$$

whereas $2v_1^{i+3}x^{n-u-1}y^{n-v+1}$ lies in the ideal

$$v_1^{i+2}x^{n-u-1}\left(4y^{n-v}, 8v_1^2y^{n-v+2}, a_3y^{n-v+3}, 4y^{n-v+4}, 2y^{n-v+5}, 8y^{n-v+6}, y^{n-v+7}\right).$$

For the first ideal we have: the first and second generators are zero by induction. The first half of the third generator (the multiple of 2), the fourth, the fifth and the sixth are zero by (a'_{i-1}). The second half of the third generator by (b_i). The last generator by (b_{i-2}).

For the last ideal we note that the first, the second, the fourth and the sixth generators are zero as we have just shown. (a'_{i-1}) takes account of the fifth generator as well as the first half of the third generator. (b_i) does the corresponding for the second half of the third generator, and (b_{i-2}) for the last one.

To prove (b'_i) we consider $v_1^{i+4}x^{n-u}y^{n-v}$ with $u+v=2i+1$ and use the formula—identical to (19)—

$$v_1^3y^{n-v} = 2A + By^{n-v+4} + Cy^{n-v+8} \tag{21}$$

(recall $B \in (2, v_1)$), to observe that $v_1^{i+4}x^{n-u}y^{n-v}$ lies in the ideal

$$\left(2Av_1^{i+1}x^{n-u}, v_1^{i+2}x^{n-u}y^{n-v+4}, 2v_1^{i+1}x^{n-u}y^{n-v+4}, v_1^{i+1}x^{n-u}y^{n-v+8}\right).$$

Now (b'_{i-2}), (a_{i-2}) and (b'_{i-4}) take account of the second, the third and the fourth generators respectively. In this way we have that this last ideal is contained in

$$\begin{aligned} v_1^iA\big(&4x^{n-u-1}, 8v_1^2x^{n-u+1}, a_3x^{n-u+2}, 4x^{n-u+3},\\ &2x^{n-u+4}, 8x^{n-u+5}, \quad Dx^{n-u+6}, x^{n-v+10}\big), \end{aligned} \tag{22}$$

where once again $D \in (2, v_1)$. We will see that all generators of (22) are trivial. For the first generator we get from (21)

$$v_1^iA4x^{n-u-1} \in 2v_1^ix^{n-u-1}\left(v_1^3y^{n-v}, By^{n-v+4}, y^{n-v+8}\right).$$

In this last ideal (a'_i), (α_{i-2}), (a'_{i-2}) and (a'_{i-4}) take care of the first generator, the first (multiple of 2) and the second (multiple of v_1) halves of the second generator, and the third generator respectively. Now note that the same argument takes account of the second, the fourth and the sixth generators of (22).

For the first half of the third generator of (22), namely $2Av_1^ix^{n-u+2}$, we have from (21):

$$2Av_1^ix^{n-u+2} \in v_1^ix^{n-u+2}\left(v_1^3y^{n-v}, By^{n-v+4}, y^{n-v+8}\right) = 0,$$

where (b'_{i-1}), (a_{i-3}), (b'_{i-3}), and (b'_{i-5}) respectively guarantee that the first generator, the first (multiple of 2) and the second (multiple of v_1) halves of the second generator, and the third generator of this last ideal are zero. The fact that this ideal is zero, shows that the fifth and first half (multiple of 2) of the seventh generators of (22) are zero too.

For the second half of the third generator of (22), that is $v_1^{i+3}Ax^{n-u+2}$, we recall that from its definition

$$A \in \frac{1}{2}\left(4y^{n-v-3}, 2v_1y^{n-v-2}, 8v_1^2y^{n-v-1}, 2v_2y^{n-v}, 4y^{n-v+1}, 2y^{n-v+2}, 8y^{n-v+3}\right),$$

that is,

$$A \in \left(2y^{n-v-3}, v_1y^{n-v-2}, y^{n-v}\right),$$

so that

$$v_1^{i+3}Ax^{n-u+2} \in v_1^{i+3}x^{n-u+2}\left(2y^{n-v-3}, v_1y^{n-v-2}, y^{n-v}\right) = 0,$$

where (a_i') and (b_{i-1}') take care of the first and last generators, whereas the second one is zero by an inductive argument (again grounded by the relation $x^{n+1} = 0$) on u.

For the second half of the seventh generator of (22):

$$Av_1^{i+1}x^{n-u+6} \in v_1^{i+1}x^{n-u+6}\left(2y^{n-v-3}, v_1y^{n-v-2}, y^{n-v}\right) = 0.$$

The first, the second and the third generators of the last ideal are zero by (a_{i-2}'), (b_{i-2}') and (b_{i-3}'), respectively.

Finally for the eighth generator of (22):

$$Av_1^{i}x^{n-u+10} \in v_1^{i}x^{n-u+10}\left(2y^{n-v-3}, v_1y^{n-v-2}, y^{n-v}\right) = 0,$$

where (a_{i-4}'), (b_{i-4}') and (b_{i-5}') take care of the first, the second and the third generators of this last ideal, respectively. $\square$

4. Topological feedback

In this final section we prove Proposition 2.3 by making use of G. Nakos' work on the Brown-Peterson homology of the classifying space for $\mathbb{Z}_4 \times \mathbb{Z}_4$, thus completing the proof outlined in Section 2 for our main Theorem 2.4.

We begin by observing that the ring $A_{n,e}$ in Section 3 can alternatively be interpreted as the tensor product $BP^*(L^{2n}(2^e)) \otimes BP^*(L^{2n}(2^e))$ where, as usual, $BP^*(L^{2n}(2^e))$ is given as the BP_*-polynomial algebra on a generator x with relations $x^{n+1} = 0$ and $[2^e](x) = 0$. On the other hand, a standard Gysin sequence argument (see [13, (2.11)] for the $n = \infty$ case) exhibits $\widetilde{BP}_*(L^{2n}(2^e))$ as the BP_*-module with generators $z_i \in \widetilde{BP}_{2i-1}(L^{2n}(2^e))$ for $1 \leq i \leq n$, and relations

$$\sum_{0 \leq s < i} a_s z_{i-s} = 0 \quad (1 \leq i \leq n)$$

where, as in Section 3, $\sum_{s\geq 0} a_s x^{s+1} = [2^e](x)$. It is then clear that the corresponding tensor product $\widetilde{BP}_*(L^{2n}(2^e)) \otimes \widetilde{BP}_*(L^{2n}(2^e))$ can be thought of as a submodule of $A_{n,e}$ in such a way that the bottom "toral" class $z_1 \otimes z_1 \in \widetilde{BP}_*(L^{2n}(2^e)) \otimes \widetilde{BP}_*(L^{2n}(2^e))$ gets identified with the "top" class $x^n y^n \in A_{n,e}$.

Consider now the Künneth-Landweber map [15]

$$\widetilde{BP}_*(\mathrm{B}(\mathbb{Z}_{2^e})) \otimes \widetilde{BP}_*(\mathrm{B}(\mathbb{Z}_{2^e})) \to \widetilde{BP}_*(\mathrm{B}(\mathbb{Z}_{2^e}) \wedge \mathrm{B}(\mathbb{Z}_{2^e})). \tag{23}$$

According to [17] (see also [16]), for $e = 2$, the annihilator for the image of $z_1 \otimes z_1$ under the composite of

$$\widetilde{BP}_*\big(L^{2n}(2^e)\big) \otimes \widetilde{BP}_*\big(L^{2n}(2^e)\big) \to \widetilde{BP}_*(\mathrm{B}\mathbb{Z}_{2^e}) \otimes \widetilde{BP}_*(\mathrm{B}\mathbb{Z}_{2^e})$$

with (23) is given by the ideal $\left(4, 2v_1, v_1^4\right)$. Proposition 2.3 now follows by assembling all these pieces together.

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