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SALEM-BOYD SEQUENCES AND HOPF PLUMBING

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Abstract

Given a fibered link, consider the characteristic polynomial of the monodromy restricted to first homology. This generalizes the notion of the Alexander polynomial of a knot. We define a construction, called iterated plumbing, to create a sequence of fibered links from a given one. The resulting sequence of characteristic polynomials for these links has the same form as those arising in work of Salem and Boyd in their study of distributions of Salem and P-V numbers. From this we deduce information about the asymptotic behavior of the large roots of the generalized Alexander polynomials, and define a new poset structure for Salem fibered links.

1. Introduction

Let (K, Σ) denote a fibered link $K \subset S^3$ with fibering surface Σ. Hopf plumbing defines a natural operation on fibered links that allows one to construct new fibered links from a given one while keeping track of useful information [15] [5]. Furthermore, a theorem of Giroux [6] shows that any fibered link can be obtained from the unknot by a sequence of Hopf plumbings and de-plumbings (see also [7]).

A fibered link (K, Σ) has an associated homeomorphism $h\colon \Sigma \to \Sigma$, called the *monodromy* of (K, Σ), such that the complement in S^3 of a regular neighborhood of K is homeomorphic to a mapping torus for h. Let h_* be the restriction of h to first homology $\mathrm{H}_1(\Sigma, \mathbb{R})$, and let $\Delta_{(K,\Sigma)}(t)$ be the characteristic polynomial of the monodromy h_*. If K is connected, that is, a *fibered knot*, then $\Delta_{(K,\Sigma)}(t)$ is the usual Alexander polynomial of K and the mapping torus structure is unique. We extend this terminology and call $\Delta_{(K,\Sigma)}(t)$ the *Alexander polynomial* of the fibered link (K, Σ).

A polynomial f of degree d is *reciprocal* if $f = f_*$, where $f_*(t) = t^d f(1/t)$. The Alexander polynomials $\Delta_{(K,\Sigma)}(t)$ are monic integer polynomials and reciprocal up to multiples of $(t-1)$. Burde [4] shows that there exists a fibered knot (K, Σ) with $\Delta_{(K,\Sigma)} = f$, if and only if

(i) f is a reciprocal monic integer polynomial; and

(ii) $f(1) = \pm 1$,

Kanenobu [8] shows that (i) is true if and only if $\Delta_{(K,\Sigma)} = f$ up to multiples of $(t-1)$,

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where (K, Σ) is a fibered link. Our goal in this paper is to study how the roots of $\Delta_{(K,\Sigma)}(t)$ are affected by Hopf plumbing.

In Section 2, we define a construction called iterated (trefoil) plumbing, which produces a sequence of fibered links (K_n, Σ_n) from a given fibered link (K, Σ) and a choice of path τ properly embedded on Σ, called the *plumbing locus*.

Our main result is the following.

Theorem 1. *If (K_n, Σ_n) is obtained from (K, Σ) by $\pm$ iterated trefoil plumbing, then there is a polynomial $P = P_{\Sigma,\tau}$ depending only on the location and orientation of the plumbing, such that $\Delta_n = \Delta_{(K_n,\Sigma_n)}$ is given by*

$$\Delta_n(t) = \frac{t^{2n} P(t) \pm (-1)^r P_*(t)}{t+1}, \tag{1}$$

where r is the number of components of K.

We call sequences of polynomials of the form given in Equation (1) *Salem-Boyd sequences*, after work of Salem [12] and Boyd [1] [2].

For a monic integer polynomial $f(t)$, let $\lambda(f)$ be the maximum absolute value among all roots of $f(t)$; $N(f)$, the number of roots with absolute value greater than one; and $M(f)$, the product of absolute values of roots of f whose absolute value is greater than one. The latter invariant $M(f)$ is known as the *Mahler measure of f*. Clearly $N(f)$ is discrete, while $\lambda(f)$ can be made arbitrarily close to but greater than one, for example, by taking $f(t) = t^n - 2$. Whether or not the values of $M(f)$ can also be brought arbitrarily close to one from above is an open problem posed by Lehmer in 1933 [9]. Lehmer originally formulated his question as follows:

QUESTION 2 (Lehmer). For each $\delta > 0$ does there exist a monic integer polynomial f such that $1 < M(f) < 1 + \delta$?

We are still far from answering Lehmer's question, but show in Section 3 how to apply Salem and Boyd's work and Theorem 1 to obtain information about the asymptotic behavior of $N(\Delta_n)$, $\lambda(\Delta_n)$, and $M(\Delta_n)$ from properties of the original fibered link and location of plumbing.

Theorem 3. *The sequences $N(\Delta_n)$, $\lambda(\Delta_n)$ and $M(\Delta_n)$ converge to $N(P)$, $\lambda(P)$, and $M(P)$, respectively, where $P = P_{\Sigma,\tau}$.*

Theorem 3 is useful for finding minimal Mahler measures appearing in particular families of fibered links, since the polynomials $P_{\Sigma,\tau}$ are easy to compute for explicit examples. We give an illustration in Section 5.

Iterated plumbing may be seen as the result of iterating full twists on a pair of strands of K, with some extra conditions on the pair of strands. For the case where

K has one component, the convergence of Mahler measure in Theorem 3 agrees with a result of Silver and Williams, which in general form may be stated as follows. Let L be a link and k an unknot disjoint from L such that L and k have non-zero linking number. Let L_n be obtained from L by doing $1/n$ surgery along k. This amounts to taking the strands of L encircled by k and doing n full-twists to obtain L_n. Silver and Williams show that the multi-variable Mahler measures of the links L_n converge to the multi-variable Mahler measure of $L \cup k$ [14]. Combining our results with that of Silver and Williams, and using the formulas for $P_{\Sigma,\tau}$ given in Section 2 (Equations 2 and 3) gives a new effective way to calculate the multi-variable Mahler measure of $L \cup k$.

It is not hard to see that if one fixes the degree of f, then the answer to Lehmer's question is negative. Theorem 3 makes it possible to study Mahler measures for sequences of fibered links whose fibers have increasing genera, and hence for polynomials of increasing degree. Although, in general, $\lambda(\Delta_n)$ and $M(\Delta_n)$ are not monotone sequences (see Theorem 13), monotonicity can be shown (at least for large enough n) when $P_{\Sigma,\tau}$ has special properties.

In Section 3, we review properties of Salem-Boyd sequences, following work of Salem [12] and Boyd [1], and consider the question of monotonicity. A *Perron polynomial* is a monic integer polynomial f with a real root $\lambda = \lambda(f) > 1$ satisfying $|\alpha| < \lambda$ for all roots α of f not equal to λ.

Theorem 4. *Suppose $P_{\Sigma,\tau}$ is a Perron polynomial. Then $\lambda(\Delta_n)$ is an eventually monotone (increasing or decreasing) sequence converging to $\lambda(P_{\Sigma,\tau})$.*

In the special case when $N(P_{\Sigma,\tau}) = 1$, more can be shown by applying results of Salem [12] and Boyd [1].

Theorem 5. *Suppose $N(P_{\Sigma,\tau}) = 1$. Then $M(\Delta_n) = \lambda(\Delta_n)$ is a monotone (increasing or decreasing) sequence converging to $\lambda(P_{\Sigma,\tau})$.*

Section 4 studies the poset structure on fibered links defined by Hopf plumbing, and the corresponding poset structure on homological dilatations. We also give an example in Section 4 that shows how Theorem 5 can be used to give explicit solutions to Lehmer's problem for restricted families.

2. Iterations of Hopf plumbings

We recall some basic definitions surrounding the Alexander polynomial of an oriented link. A *Seifert surface* for an oriented link K is an oriented surface Σ whose boundary is K For any collection of free loops $\sigma_1, \dots, \sigma_d$ on Σ forming a basis for $H_1(\Sigma; \mathbb{R})$, the associated *Seifert matrix* S is given by

$$S = [lk(\sigma_i^+, \sigma_j)],$$

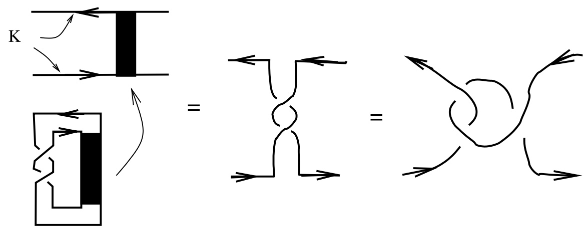

Fig. 1. Positive Hopf plumbing

where σ_i^+ is the push-off of σ_i off Σ into $S^3 \setminus \Sigma$ in the positive direction with respect to the orientation of Σ, and $lk(\ ,\ )$ is the linking form on S^3. Let S^{tr} denote the transpose of S. The polynomial

$$\Delta_K(t) = \left|tS - S^{\mathrm{tr}}\right|$$

is uniquely defined up to units in the Laurent polynomial ring $\Lambda(t) = \mathbb{Z}[t, t^{-1}]$, and is reciprocal (it is the same if S is replaced by S^{tr}). For the purposes of this paper, we will always normalize Δ_K so that $\Delta_K(0) \neq 0$, and the highest degree coefficient of $\Delta_K(t)$ is positive. Then for any nonsingular Seifert matrix for K,

$$\Delta_K(t) = \mathrm{s}(S)\left|tS - S^{\mathrm{tr}}\right|,$$

where $\mathrm{s}(S)$ is the sign of the coefficient of $|tS - S^{\mathrm{tr}}|$ of highest degree.

If K is fibered, and Σ is the fibering surface, then the Seifert matrix S is invertible over the integers, and the monodromy restricted to $\mathrm{H}_1(\Sigma; \mathbb{R})$ satisfies $h_* = S^{\mathrm{tr}}S^{-1}$. In this case $\mathrm{s}(S) = |S|$, and $\Delta_K(t)$ is characteristic polynomial of h_*. Since $|S|$ is invariant under change of basis, and the fiber surface is fixed, we will write $\mathrm{s}(K) = \mathrm{s}(S)$ if K is fibered. If K is a fibered knot, then $\mathrm{s}(K) = \Delta_K(1)$.

A *properly embedded path* on Σ is a smooth embedding

$$\tau\colon [0, 1] \to \Sigma$$

such that $\tau(0), \tau(1) \in \partial\Sigma$. The surface $\Sigma_2^+(\tau)$ (resp., Σ_2^- is obtained from Σ by positive (resp., negative) Hopf plumbing if it is obtained from Σ by gluing on a positive (resp., negative) Hopf band as in Fig. 1. The definition is independent of the orientation of τ.

Set $\Sigma_1^\pm = \Sigma$. For $n \geq 1$, let $\Sigma_{n+1}^\pm$ be the (*positive or negative*) *Hopf n-plumbing* of Σ along τ, which is obtained by Hopf plumbing along n paths as shown in Fig. 2, starting with the vertical path τ.

The positive (resp., negative) Hopf n-plumbings can also be considered as a Murasugi sum of Σ with the fiber surface of the torus link $T(2, n)$ (resp., $T(2, -n)$). Let $K_n^\pm(\Sigma, \tau)$ be the boundary of the surface $\Sigma_n^\pm$. For $n = 1$, we have $K_1^\pm = K$. The local

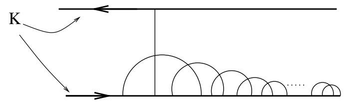

Fig. 2. Base paths for iterated Hopf plumbings

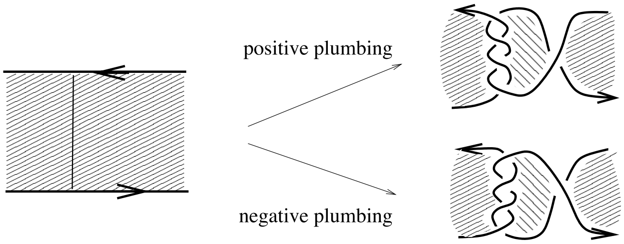

Fig. 3. Result of iterated Hopf 4-plumbing

oriented link diagram for $K_n^{\pm}$ is shown in Fig. 3, and $\Sigma_n^{\pm}$ is the corresponding Seifert surface.

Denote by $\langle\ ,\ \rangle$ the intersection pairing

$$\mathrm{H}_1(\Sigma;\mathbb{Z}) \times \mathrm{H}_1(\Sigma,\partial\Sigma;\mathbb{Z}) \to \mathbb{Z},$$

and let $v \in \mathrm{H}_1(\Sigma;\mathbb{Z})$ be the vector such that v^{tr} represents the vector $[\tau]$ in $\mathrm{H}_1(\Sigma;\partial\Sigma;\mathbb{B}) \simeq \mathrm{H}_1(\Sigma;\mathbb{B})^{\mathrm{dual}}$. Then, in terms of the basis $\sigma_1,\dots,\sigma_d$, v is given by

$$v = [\langle\sigma_1,\tau\rangle,\dots,\langle\sigma_d,\tau\rangle].$$

Set

$$P^{\pm}_{\Sigma,\tau}(t) = \left|tI - (S^{\mathrm{tr}} \mp vv^{\mathrm{tr}})S^{-1}\right|, \tag{2}$$

where I is the identity matrix.

Before proving the Theorem 1, we put $P^{\pm}_{\Sigma,\tau}$ in an alternate form. Let $N(\tau)$ be a regular neighborhood of τ on Σ, and let $\Sigma_0 = \Sigma \setminus N(\tau)$. Let $K_0 = K_0(\Sigma,\tau) = \partial\Sigma_0$. Let $\sigma_1,\dots,\sigma_{d-1}$ be a collection of free loops on Σ_0 forming a basis for $\mathrm{H}_1(\Sigma_0;\mathbb{Z})$. Let σ_d be a free loop on Σ so that $\sigma_1,\dots,\sigma_d$ is a basis for $\mathrm{H}_1(\Sigma;\mathbb{Z})$, and such that $\langle\sigma_d,\tau\rangle = 1$. Let S_1 and S_0 be the corresponding Seifert matrices for K and K_0, respectively.

Lemma 6. *The Seifert matrix S_0 is non-singular.*

Proof. By our definitions, the transpose of the Seifert matrix defines a linear transformation from the first homology of the Seifert surface to its dual. We thus have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_1(\Sigma_0;\mathbb{Z}) & \xrightarrow{S_0^{\mathrm{tr}}} & \mathrm{H}_1(\Sigma_0;\mathbb{Z})^{\mathrm{dual}} \\ \downarrow & & \downarrow \\ \mathrm{H}_1(\Sigma_1;\mathbb{Z}) & \xrightarrow{S_1^{\mathrm{tr}}} & \mathrm{H}_1(\Sigma_1;\mathbb{Z})^{\mathrm{dual}}, \end{array}$$

where vertical arrows are the inclusions determined by the choice of bases. Since S_1 is non-singular, it follows that S_0 must also be non-singular. ☐

Lemma 7. *The polynomial in* Equation (2) *can be rewritten as*

$$P^{\pm}_{\Sigma,\tau}(t) = \Delta_K(t) \pm \mathrm{s}(K)\mathrm{s}(S_0)\Delta_{K_0}(t). \tag{3}$$

Proof. The choice of basis $\sigma_1, \ldots, \sigma_d$ above yields the Seifert matrix

$$S_1 = \left[\begin{array}{c|c} S_0 & x \\ \hline y^{\mathrm{tr}} & s \end{array}\right]$$

for K, where $x, y \in \mathbb{Z}^{d-1}$, and $s \in \mathbb{Z}$. The vector v written with respect to the dual elements of $\sigma_1, \ldots, \sigma_d$ is given by $v = [0, \ldots, 0, 1]^{\mathrm{tr}}$. We thus have

$$\left|tS_1 - \left(S_1^{\mathrm{tr}} \mp vv^{\mathrm{tr}}\right)\right| = \left|\begin{array}{c|c} tS_0 - S_0^{\mathrm{tr}} & tx - y \\ \hline ty^{\mathrm{tr}} - x^{\mathrm{tr}} & s(t-1) \pm 1 \end{array}\right|.$$

Therefore

$$P^{\pm}_{\Sigma,\tau}(t) = \mathrm{s}(K)\left(\left|tS_1 - S_1^{\mathrm{tr}}\right| \pm \left|tS_0 - S_0^{\mathrm{tr}}\right|\right)$$

and the claim follows. ☐

For a polynomial g, define

$$\overline{g}(t) = t^{-m} g(t),$$

where m is the largest power of t dividing g. Then it is easy to check that $g_*(t) = \overline{g}_*(t)$. Also, if g and f are polynomials of degrees d' and d, respectively, then for $h(t) = g(t) \pm f(t)$, we have

$$h_*(t) = g_*(t) \pm t^{d'-d} f_*(t).$$

Lemma 8. *Let r be the number of components of K, and $P(t) = P^{\pm}_{\Sigma,\tau}(t)$. Then*

$$P_*(t) = (-1)^{r+1}\left(\Delta_K(t) \mp \mathrm{s}(K)\mathrm{s}(S_0)t\,\Delta_{K_0}(t)\right).$$

Proof. If d is the rank of $\mathrm{H}_1(\Sigma;\mathbb{R})$, we have

$$P_*(t) = t^d\,\Delta_K\left(\frac{1}{t}\right) \pm \mathrm{s}(K)t^d\left(\left|\frac{1}{t}S_0 - S_0^{\mathrm{tr}}\right|\right).$$

The Alexander polynomial of a link is reciprocal (anti-reciprocal) if the number of components is odd (even). Thus, the first summand equals $(-1)^{r+1}\Delta_K(t)$. Since, by Lemma 6, S_0 is a non-singular matrix, t does not divide $\left|tS_0 - S_0^{\mathrm{tr}}\right|$. It is also not difficult to check that the number of components of K_0 and K_1 have opposite parity, and the degree of $\left|tS_0 - S_0^{\mathrm{tr}}\right|$ is one less than the degree of $\Delta_K(t)$. We thus have

$$t^d\left|\frac{1}{t}S_0 - S_0^{\mathrm{tr}}\right| = (-1)^r t\left|tS_0 - S_0^{\mathrm{tr}}\right| = (-1)^r \mathrm{s}(S_0)t\,\Delta_{K_0}(t). \qquad \square$$

Theorem 1 is implied by the following stronger version.

Theorem 9. *Let (K_n, Σ_n) be obtained by $\pm$ iterated Hopf plumbing on a fibered link (K, Σ) with r-components. Let $\Delta_n = \Delta_{(K_n,\Sigma_n)}$, and let $P = P^{\pm}_{\Sigma,\tau}$. Then*

$$\Delta_n(t) = \frac{t^n P(t) \pm (-1)^{r+n} P_*(t)}{t+1}.$$

Proof. By Lemma 8, we have

$$\begin{aligned} tP(t) + (-1)^{r+1}P_*(t) &= t\,\Delta_K(t) \pm \mathrm{s}(K)\mathrm{s}(S_0)t\,\Delta_{K_0}(t) \\ &\quad + (\Delta_K(t) \mp \mathrm{s}(K)\mathrm{s}(S_0)t\,\Delta_{K_0}(t)) \\ &= (t+1)\Delta_K(t). \end{aligned}$$

For $m \geq 1$, the Seifert matrix for $S^{\pm}_m$ is given by

$$S^{\pm}_m = \left[\begin{array}{c|c} S^{\pm}_{m-1} & 0 \\ \hline w & \pm 1 \end{array}\right],$$

where $w = [0, \ldots, 0, -1]$. Thus, the Alexander polynomial for $K^{\pm}_m$ is given by

$$\Delta_{K^{\pm}_m}(t) = \mathrm{s}\big(K^{\pm}_m\big)\left|\begin{array}{c|c} tS^{\pm}_{m-1} - \big(S^{\pm}_{m-1}\big)^{\mathrm{tr}} & -w^{\mathrm{tr}} \\ \hline tw & \pm(t-1) \end{array}\right|.$$

It follows that for $n \geq 2$, $\Delta_{K^{\pm}_n}(t)$ satisfies

$$(t+1)\Delta_{K^{\pm}_n}(t) = \mathrm{s}\big(K^{\pm}_n\big)(t+1)\left[\pm(t-1)\left|tS^{\pm}_{n-1} - \big(S^{\pm}_{n-1}\big)^{\mathrm{tr}}\right| + t\left|tS^{\pm}_{n-2} - \big(S^{\pm}_{n-2}\big)^{\mathrm{tr}}\right|\right],$$

and $s(K_n^\pm) = \pm s(K_{n-1}^\pm)$. For $n = 2$, using $s(K) = s(K_1) = \pm s(K_2^\pm)$, we have

$$\begin{aligned}(t+1)\Delta_{K_2^\pm}(t) &= s(K_2^\pm)\left[\pm(t^2-1)\left|tS_1 - S_1^{\mathrm{tr}}\right| + (t^2+t)\left|tS_0 - S_0^{\mathrm{tr}}\right|\right]\\ &= \pm s(K_2^\pm)\left((t^2-1)\left|tS_1 - S_1^{\mathrm{tr}}\right| \pm (t^2+t)\left|tS_0 - S_0^{\mathrm{tr}}\right|\right)\\ &= s(K)t^2\left(\left|tS_1 - S_1^{\mathrm{tr}}\right| \pm \left|tS_0 - S_0^{\mathrm{tr}}\right|\right)\\ &\quad - s(K)\left(\left|tS_1 - S_1^{\mathrm{tr}}\right| \mp t\left|tS_0 - S_0^{\mathrm{tr}}\right|\right)\\ &= t^2P(t) + (-1)^r P_*(t)\\ &= t^2P(t) + (-1)^{r+2}P_*(t).\end{aligned}$$

If $n > 2$, we use induction, to obtain

$$\begin{aligned}(t+1)\Delta_{K_n^\pm}(t) &= \pm s(K_n^\pm)\left[(t^2-1)\left|tS_{n-1}^\pm - (S_{n-1}^\pm)^{\mathrm{tr}}\right| \pm t(t+1)\left|tS_{n-2}^\pm - (S_{n-2}^\pm)^{\mathrm{tr}}\right|\right]\\ &= s(K_{n-1}^\pm)\Big[s(K_{n-1}^\pm)(t+1)(t-1)\Delta_{K_{n-1}^\pm}(t)\\ &\qquad \pm s(K_{n-2}^\pm)t(t+1)\Delta_{K_{n-2}^\pm}(t)\Big]\\ &= (t-1)(t+1)\Delta_{K_{n-1}^\pm}(t) + t(t+1)\Delta_{K_{n-2}^\pm}(t)\\ &= (t-1)\left(t^{n-1}P(t) + (-1)^{n-1+r}P_*(t)\right) + t\left(t^{n-2}P(t) + (-1)^{n-2+r}P_*(t)\right)\\ &= t^nP(t) - t^{n-1}P(t) + (-1)^{n+r-1}tP_*(t) + (-1)^{n+r}P_*(t)\\ &\quad + t^{n-1}P(t) + (-1)^{n+r-2}tP_*(t)\\ &= t^nP(t) + (-1)^{n+r}P_*(t) \qquad \square\end{aligned}$$

3. Properties of Salem-Boyd sequences

In this section we review some general properties of roots of polynomials in Salem-Boyd sequences (see also, [12], [1]), and apply them to the Alexander polynomials of iterated plumbings.

3.1. Asymptotic behavior of roots of Salem-Boyd sequences.

Given a monic integer polynomial $P(t)$ define

$$Q_n^\pm(t) = t^nP(t) \pm P_*(t). \tag{4}$$

We will call the sequence of polynomials given in Equation 4 the *Salem-Boyd sequence* associated to P. For all positive integers n, $Q_n^\pm(t)$ is equal to a reciprocal polynomial up to a multiple of $t-1$. We are interested in the asymptotic behavior of roots of $Q_n^\pm(t)$.

S. Williams suggested the use of Rouché's theorem to prove the following.

Lemma 10. *Let P be a monic integer polynomial, and let $R(t)$ be any integer polynomial, and*

$$Q_n(t) = t^n P(t) \pm R(t).$$

Then the roots of $Q_n(t)$ outside C converge to those of $P(t)$ counting multiplicity as n increases.

Proof. Consider the rational function

$$S_n(t) = \frac{Q_n(t)}{t^n} = P(t) \pm \frac{R(t)}{t^n}.$$

Let α be a root of $P(t)$ (counted with multiplicity), and let D_α be any small disk around α that is also strictly outside C and that contains no roots of $P(t)$ other than α. Then $P(t)$ has a lower bound on the boundary ∂D_α, and thus there exists an n_α depending on α and D_α such that

$$\left|\frac{R(t)}{t^n}\right| < |P(t)|$$

on ∂D_α for all $n > n_\alpha$. By Rouché's theorem, it follows that for $n > n_\alpha$, $P(t)$ and $S_n(t)$ (and hence also $Q_n(t)$) have m roots in D_α counted with multiplicity. Since the disks could be made arbitrarily small, and there are only a finite number of roots, the claim follows. □

Lemma 11. *Let P be a monic integer polynomial and let $Q_n(t)$ be the associated Salem-Boyd sequence. Then $N(Q_n) \leq N(P)$ for all n.*

A proof of this Lemma is contained in [1] (p.317), but we include it here for the convenience of the reader.

Proof. We first assume that $P(t)$ has no roots on the unit circle. This does not change the statement's generality. To study the roots of $Q_n(t)$ it suffices to consider the case when $P(t)$ has no reciprocal or anti-reciprocal factors, since such factors will be factors of Q_n for all n. If $P(t)$ has a root on the unit circle, then the minimal polynomial of that root would be necessarily reciprocal or anti-reciprocal, and we can factor the minimal polynomial out of P and the Q_n.

Consider the two variable polynomial

$$Q_n(z, u) = z^n P(z) \pm u P_*(z) \tag{5}$$

where z is any complex number and $u \in [0, 1]$.

Suppose $P(t)$ has roots $\theta_1, \ldots, \theta_s$ outside the unit circle C counted with multiplicity. Then $Q_n^{\pm}(z, u)$ defines an algebraic curve $z = Z(u)$ with branches $z_1(u), \ldots, z_s(u)$

satisfying $z_i(0) = \theta_i$. For $z \in C$ we have $|P(z)| = |P_*(z)|$. Now suppose that $0 < u < 1$ and $1 = |z_i(u)|$. Then

$$1 = |z_i(u)|^n = \frac{u|P_*(z_i(u))|}{|P(z_i(u))|} = u$$

yielding a contradiction. Thus, by continuity

$$|z_i(u)| > 1,$$

for $u \in [0, 1)$. It follows that $Q_n^{\pm}(t)$ has at most s roots outside C. □

Summarizing the contents of Lemma 10 and Lemma 11 we have the following.

Theorem 12. *Let P be a monic integer polynomial, and let*

$$Q_n(t) = t^n P(t) \pm P_*(t).$$

Then

$$\begin{aligned} N(Q_n) &\leq N(P); \\ \lim_{n\to\infty} \lambda(Q_n) &= \lambda(P); \quad \textit{and} \\ \lim_{n\to\infty} M(Q_n) &= M(P). \end{aligned}$$

Theorem 1 and Theorem 12 imply Theorem 3.

A natural question is whether $M(Q_n)$ is a monotone sequence, perhaps on arithmetic progressions, when P has more than one root outside C. The proof of Lemma 10, does not restrict the directions by which the roots of Q_n outside C approach those of P. If a root θ of P is not real, then the root(s) of Q_n approaching θ typically rotate around θ as they converge. More precisely, we have the following. For z a complex number, let $A = \mathrm{Arg}(z)$ be such that $z = |z|e^{2\pi i A}$.

Theorem 13. *Let $\alpha_1, \ldots, \alpha_s$ be the roots of P outside C. Take N_0, so that Q_n has s roots outside C for $n \geq N_0$. Label these roots $\alpha_i^{(n)}$, for $i = 1, \ldots, s$, so that*

$$\lim_{n\to\infty} \alpha_i^{(n)} = \alpha_i.$$

Then, there is a constant c such that for any $\delta > 0$, and $n > N_\delta > N_0$,

$$\mathrm{Arg}\left(\alpha_i^{(n)} - \alpha_i\right) = c + n\,\mathrm{Arg}(\alpha_i) + \delta_n,$$

where the error term δ_n satisfies $|\delta_n| < \delta$.

Proof. Let $P_1(x)$ be the largest degree monic integer factor of $P(x)$ with no roots outside C. For $i = 1, \ldots, s$, we have

$$\alpha_i^{(n)} - \alpha_i = \left(\frac{1}{\alpha_i^{(n)}}\right)^n R_n,$$

where

$$R_n = \frac{P_*\big(\alpha_i^{(n)}\big)}{P_1\big(\alpha_i^{(n)}\big)\big(\alpha_i^{(n)} - \alpha_1\big) \cdots \big[\big(\alpha_i^{(n)} - \alpha_i\big)\big] \cdots \big(\alpha_i^{(n)} - \alpha_s\big)},$$

with the entry in brackets $[\ldots]$ excluded.

By assumption $\alpha_i^{(n)}$ converges to α_i, and hence also R_n converges to some nonzero constant R. Given $\delta > 0$, let $N_1 \geq N_0$ be such that

$$|\mathrm{Arg}(R) - \mathrm{Arg}(R_n)| < \frac{\delta}{2} \tag{6}$$

and

$$\left|\mathrm{Arg}(\alpha_i) - \mathrm{Arg}\big(\alpha_i^{(n)}\big)\right| < \frac{\delta}{2n}, \tag{7}$$

for all $n \geq N_1$. Then, we have

$$\begin{aligned} \mathrm{Arg}\big(\alpha_i^{(n)} - \alpha_i\big) &= \mathrm{Arg}(R_n) - n\,\mathrm{Arg}\big(\alpha_i^{(n)}\big) \\ &= \mathrm{Arg}(R) - n\,\mathrm{Arg}(\alpha_i) + \delta_n \end{aligned}$$

where δ_n is the sum of the left sides of (6) and (7). This proves the claim, with $c = \mathrm{Arg}(R)$. □

EXAMPLE. Let

$$P(x) = x^3 + x^2 - 1.$$

Then $P(x)$ is irreducible and has exactly two roots α and $\overline{\alpha}$ outside C. We claim that $\mathrm{Arg}(\alpha)$ is irrational. Consider the ratio

$$\omega = \frac{\alpha}{\overline{\alpha}}.$$

Then, since the Galois group of $P(x)$ over the rationals is S_3, ω must have an algebraic conjugate not on the unit circle, for example,

$$\frac{\beta}{\overline{\alpha}},$$

where β is the real root of $P(x)$. Thus, ω is not a root of unity. Since $\mathrm{Arg}(\omega) = 2\,\mathrm{Arg}(\alpha)$, it follows that $\mathrm{Arg}(\alpha)$ is irrational. Thus, by Theorem 13, the relative angle of $\alpha_i^{(n)}$ to α_i is uniformly distributed as a sequence in n.

Let $\mathrm{Re}(z)$ denote the real part of z. The dot product between two vectors $\overrightarrow{0z}$ and $\overrightarrow{0w}$ is $\mathrm{Re}(z\overline{w})$. It follows from the above that there is no arithmetic progression $kn+l$, so that the sign of

$$\mathrm{Re}\left[\left(\alpha_i^{(kn+l)} - \alpha_i\right)\overline{\alpha_i}\right]$$

is constant as a sequence in n. Therefore, $M(Q_n) = \lambda(Q_n)^2$ cannot be monotone for any arithmetic progression in n.

3.2. Perron polynomials. We will show that for the Salem-Boyd sequence $Q_n(t)$ associated to a Perron polynomial, $\lambda(Q_n)$ is eventually monotone, and prove Theorem 4.

Proof of Theorem 4. Let P be a Perron polynomial, and let $Q_n(t)$ be an associated Salem-Boyd sequence. Let $\mu_1, \dots, \mu_s$ be the roots (counted with multiplicity) of P outside C, with $|\mu_1| > |\mu_i|$ for all $i = 2, \dots, s$. By multiplying P by a large enough power of t (this doesn't change P_*), we can assume that Q_n has roots $\lambda_1^{(n)}, \dots, \lambda_s^{(n)}$ outside C, and $\left|\lambda_i^{(n)} - \mu_i\right| < \left|\lambda_i^{(n)} - \mu_j\right|$ for $\mu_i \neq \mu_j$, and that Q_n is Perron for all $n \geq 1$. Let $\lambda_1^{(n)}$ be the largest root of Q_n. Then for all n, the root of P closest to $\lambda_1^{(n)}$ is μ_1, and the root of Q_n closest to μ_1 is $\lambda_1^{(n)}$. This also implies that $\lambda_1^{(n)}$ is a simple root of Q_n. Fixing n, we will show that $\lambda_1^{(n+1)}$ lies strictly between $\lambda_1^{(n)}$ and μ_1.

Consider the equations

$$0 = Q_n\left(\lambda_1^{(n)}\right) = \left(\lambda_1^{(n)}\right)^n P\left(\lambda_1^{(n)}\right) \pm P_*\left(\lambda_1^{(n)}\right), \tag{8}$$

and

$$Q_{n+1}(\mu_1) = \pm P_*(\mu_1) = Q_n(\mu_1).$$

Since each of the Q_n are increasing for $t > \lambda_1^{(n)}$, and Q_n does not have any roots strictly between μ_1 and $\lambda_1^{(n)}$, it follows that the sign of $\mu_1 - \lambda_1^{(n)}$ equals the sign of $\pm P_*(\mu_1)$ and does not depend on n.

Suppose $\lambda_1^{(n)} < \mu_1$. Then, using (8) in the second line below, we have

$$\begin{aligned}
Q_{n+1}\left(\lambda_1^{(n)}\right) &= \left(\lambda_1^{(n)}\right)^{n+1} P\left(\lambda_1^{(n)}\right) \pm P_*\left(\lambda_1^{(n)}\right) \\
&= \lambda_1^{(n)}\left(\mp P_*\left(\lambda_1^{(n)}\right)\right) \pm P_*\left(\lambda_1^{(n)}\right) \\
&= \pm P_*\left(\lambda_1^{(n)}\right)\left(1 - \lambda_1^{(n)}\right).
\end{aligned}$$

By assumption $\lambda_1^{(n+1)} > 1$. Also, $P\left(\lambda_1^{(n)}\right) < 0$, since otherwise P would have a real root between $\lambda_1^{(n)}$ and μ_1, contradicting the assumption that $\lambda_1^{(n)}$ is closer to μ_1 than any other root of P. This implies that $\pm P_*\left(\lambda_1^{(n)}\right) > 0$, and hence $Q_{n+1}\left(\lambda_1^{(n)}\right) < 0$, and $\lambda_1^{(n)} < \lambda_1^{(n+1)}$.

If $\lambda_1^{(n)} > \mu_1$, then $P\left(\lambda_1^{(n)}\right) > 0$, and hence $\pm P_*\left(\lambda_1^{(n)}\right) < 0$. We thus have

$$Q_n\left(\lambda_1^{(n+1)}\right) = \pm P_*\left(\lambda_1^{(n)}\right)\left(1 - \frac{1}{\lambda_1^{(n+1)}}\right) < 0,$$

and $\lambda_1^{(n)} > \lambda_1^{(n+1)}$. □

The monotonicity property of Salem-Boyd sequences Q_n associated to a Perron polynomial P allows us to give a lower bound greater than one for the sequences $\lambda(Q_n)$.

Proposition 14. *If $Q_n(t)$ is defined by*

$$Q_n(t) = t^n P(t) \pm P_*(t),$$

where P is a Perron polynomial, and n_0 is such that $\lambda(Q_n)$ is monotone for $n \geq n_0$, then

$$\lambda(Q_n) \geq \min\left\{\lambda\left(Q_{n_0^{\pm}}\right), \lambda(P)\right\}$$

for all $n \geq n_0$.

3.3. P-V and Salem polynomials. We now consider the case when $P = P^{\pm}_{\Sigma,\tau}$ belongs to a special class of Perron polynomials, namely those satisfying $N\left(P^{\pm}_{\Sigma,\tau}\right) = 1$.

A *P-V number* is a real algebraic integer $\alpha > 1$ such that all other algebraic conjugates lie strictly within C. A *Salem number* is a real algebraic integer $\alpha > 1$ such that all other algebraic conjugates lie on or within C with at least one on C. If f is an irreducible monic integer polynomial with $N(f) = 1$, then the root of f outside C has absolute value equal to either a Salem number, if f has degree greater than 2 and is reciprocal, or a P-V number otherwise. If f is reciprocal and $N(f) = 1$, then $\lambda(f)$ is either a Salem number or a quadratic P-V number.

The polynomials $Q_n^{\pm}(t)$ were originally studied by Salem [12] in the case when $P(t)$ is a P-V polynomial to show that every P-V number is the upper and lower limit of Salem numbers. Boyd [1] showed that any Salem number occurs as $M(Q_n^{\pm})$ for some P-V polynomial $P(t)$.

Assume that $P(t)$ has no reciprocal factors and $P(1) \neq 0$. Let

$$n_0^-(P) = d - 2\frac{P'(1)}{P(1)} + 1$$

where d is the degree of P, and let

$$n_0^+(P) = 1$$

for all P. For any polynomial (or Laurent polynomial) P, let $l(P)$ be the sign of the lowest degree coefficient of P. The following Proposition is proved in Boyd's discussion in ([1] p.320–321), and implies Theorem 5.

Proposition 15. *If P is a P-V polynomial for the P-V number θ, then the polynomial $Q_n^{\pm}(t)$ has a real root greater than one if and only if $n \geq n_0^{\pm}(P)$. Furthermore, the sequences of resulting Salem numbers $\alpha_n^{\pm}$ is monotone increasing* (*decreasing*) *if and only if* $\pm l(P) > 0$ (< 0).

Proof. The proof follows from looking at the real graphs of $Q_n^{\pm}(t)$ and of P. Since $P(1) = P_*(1) < 0$, $Q_n^+(1)$ must be strictly negative. Thus, Q_n^+ must have a root larger than 1 for all n, and we can set $n_0^+ = 1$. The graph of $y = Q_n^-(t)$ passes through the real axis at $t = 1$. Thus, $Q_n^-(t)$ has a positive real root if and only if the derivative of Q_n^- is negative. Note that $Q_n^-(t)$ cannot have a negative real root by the argument in the proof of Lemma 11. This proves the first part of the Proposition.

For the second part, note that since P has only one root θ outside C, $P_*(\theta)$ and $\pm l(P)$ must have the same sign. Suppose, for example, that $\pm l(P) > 0$. Put $\alpha_n^{\pm} = \lambda(Q_n^{\pm})$. Then $Q_n^{\pm}(\theta) > 0$, and hence $\theta > \alpha_n^{\pm}$ for all n. This implies that $P(\alpha_{n+1}^{\pm}) < 0$. Now consider the equations:

$$\begin{aligned} Q_n^{\pm}\left(\alpha_{n+1}^{\pm}\right) &= Q_n^{\pm}\left(\alpha_{n+1}^{\pm}\right) - Q_{n+1}^{\pm}\left(\alpha_{n+1}^{\pm}\right) \\ &= \left(\left(\alpha_{n+1}^{\pm}\right)^n - \left(\alpha_{n+1}^{\pm}\right)^{n+1}\right) P\left(\alpha_{n+1}^{\pm}\right). \end{aligned}$$

The bottom formula is a product of negative numbers. Hence, $Q_n^{\pm}(\alpha_n^{\pm}) > 0$, and $\alpha_{n+1}^{\pm} > \alpha_n^{\pm}$. The case $\pm l(P) < 0$ is proved in an analogous way. □

4. Poset structure on fibered links

We now apply results of the previous sections to sequences of fibered links obtained by iterated trefoil plumbings. Let (K, Σ) be a fibered link, and let P be the polynomial produced by a given locus of plumbing τ. Let $\Delta_n = \Delta_{(K_n, \Sigma_n)}$ be the Alexander polynomials of the iterated trefoil plumbings. If P is a Perron polynomial, then Proposition 14 implies that one can find lower bounds for $\lambda(\Delta_n)$, and hence for $M(\Delta_n)$ at least for large n. The situation is even better when P is a P-V polynomial. In this case, we can explicitly find the minimal $\lambda(\Delta_n)$ and hence $M(\Delta_n)$ in the sequence by comparing $\lambda(\Delta_{n_0})$ and $\lambda(P)$, where n_0 is as in Proposition 15. Furthermore, any P-V polynomial satisfies the inequality (see [13])

$$\lambda(P) \geq \lambda(x^3 - x - 1) \approx 1.32472.$$

It is not known in general if there is a lower bound greater than one for Salem numbers.

A fibered link (K, Σ) will be called a *Salem fibered link,* if the following equivalent statements hold:

(1) $N(\Delta_{(K,\Sigma)}) = 1$;
(2) $\lambda(\Delta_{(K,\Sigma)}) = M(\Delta_{(K,\Sigma)})$; and
(3) $M(\Delta_{(K,\Sigma)})$ is a Salem number or a quadratic P-V number.

Let $\mathcal{S}$ be the set of Salem fibered links, and write

$$(K_1, \Sigma_1) \prec_S (K_2, \Sigma_2)$$

if (K_2, Σ_2) can be obtained from (K_1, Σ_1) be a sequence of trefoil plumbings, where the polynomial $P^{\pm}_{\Sigma,\tau}$ corresponding to the plumbing locus at each stage is a P-V polynomial. If $(K_1, \Sigma_1) \prec_S (K_2, \Sigma_2)$, then the topological Euler characteristic of Σ_1 is strictly less than that of Σ_2. Thus, $\prec_S$ defines an (anti-symmetric) partial order on Salem fibered links. Proposition 14 implies the following.

Proposition 16. *If* $(K_1, \Sigma_1) \prec_S (K_2, \Sigma_2)$, *then*

$$M(\Delta_{(K_2,\Sigma_2)}) \geq \min\{M(\Delta_{(K_1,\Sigma_1)}), \theta_0\}$$

where $\theta_0 \approx 1.32472$ *is the smallest P-V number*.

Consider the graph structure of $\mathcal{S}$ with respect $\prec_S$. By Proposition 16, for any connected subgraph of $\mathcal{S}$, the minimal Salem number can be determined by comparing the minimal elements with respect to $\prec_S$.

QUESTION 17. Is $\mathcal{S} \cap \mathcal{K}$ connected with respect to $\prec_S$?

It is not difficult to produce examples of Salem fibered links (K, Σ) and a locus for plumbing τ such that $P_{\Sigma,\tau}$ is not a P-V polynomial (see Section 5). We will say a Salem fibered link $(K, \Sigma) \in \mathcal{S} \cap \mathcal{K}$ is *isolated* if for all loci of plumbing τ on Σ, the corresponding polynomial P is not a P-V polynomial.

QUESTION 18. Are there isolated Salem links?

Although we do not know of any isolated Salem links, Salem fibered links do appear sporadically in Salem-Boyd sequences not associated to P-V polynomials as seen in the table at the end of Section 5.

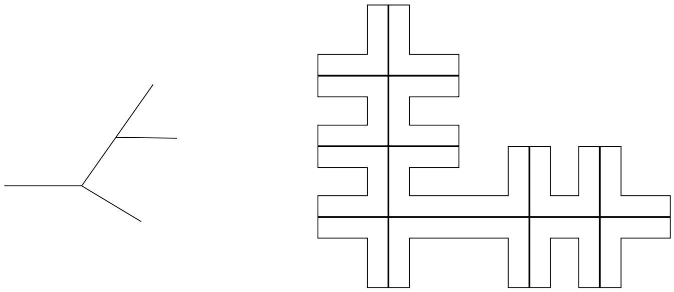

Fig. 4. Construction of fibering surface for arborescent link

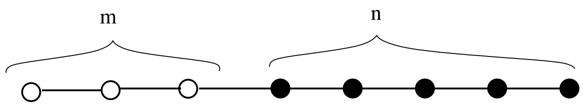

Fig. 5. Plumbing graph with positive (negative) vertices filled black (white)

5. A family of fibered two bridge links

The simplest examples to consider are those coming from arborescent links. Let Γ be a tree, with vertices ν with labels $m(\nu) = \pm 1$. Let $\mathcal{L}$ be a union of line segments in the plane, intersecting transversally, whose dual graph is Γ, and let $U(\mathcal{L})$ be the surface obtained by thickening $\mathcal{L}$. This is illustrated in Fig. 4.

Consider the surface in Fig. 4 as a subspace of S^3 and glue together opposite sides in the diagram that are connected by a vertical or horizontal path with a positive or negative full-twist according to the labeling on the graph. The resulting surface Σ is a fibering surface for $K = \partial\Sigma$ by [15], since it can be obtained by a sequence of Hopf plumbings on the unknot. The line segments of $\mathcal{L}$ close up to form a free basis for $\mathrm{H}_1(\Sigma;\mathbb{R})$. Thus, the vertices of Γ can be thought of as basis elements of $\mathrm{H}_1(\Sigma;\mathbb{R})$. Let S_Γ be the matrix where the rows and columns correspond to vertices $\nu_1, \dots, \nu_k$ of Γ, and the entries $a_{i,j}$ are given by

$$a_{i,j} = \begin{cases} -1 & \text{if} \quad i < j, \quad \text{and} \quad \nu_i \quad \text{and} \quad \nu_j \quad \text{are connected by an edge} \\ m(\nu_i) & \text{if} \quad i = j, \quad \text{and} \\ 0 & \text{otherwise.} \end{cases}$$

Then S is a Seifert matrix for (K, Σ). It follows that although there may be several fibered links (K, Σ) associated to a given labeled graph Γ, the Seifert matrix, and hence the Alexander polynomial, is determined by Γ.

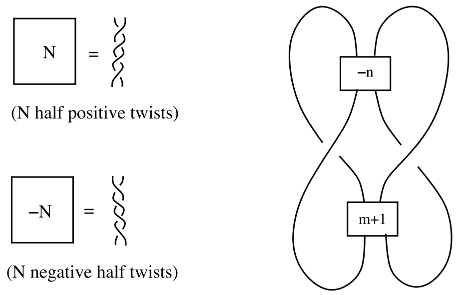

Fig. 6. Two bridge link associated to $\Gamma_{m,n}$

Consider the family of examples $\Gamma_{m,n}$ in Fig. 5. The associated fibered links $(K_{m,n}, \Sigma_{m,n})$ (determined uniquely by $\Gamma_{m,n}$) are the two-bridge link drawn in Fig. 6.

Fixing m, and letting n vary gives a sequence of fibered links $(K_{m,n}, \Sigma_{m,n})$ that are obtained by iterated plumbing on $(K_{m,1}, \Sigma_{m,1})$. Thus, the Alexander polynomials $\Delta_{m,n} = \Delta_{K_{m,n},\Sigma_{m,n}}$ are Salem-Boyd sequences associated to some polynomials P_m. We will compute the P_m, and their numerical invariants.

Considering the vertices of $\Gamma_{m,1}$ as basis elements in $\mathrm{H}_1(\Sigma_{m,1}, \mathbb{R})$, the path τ is dual to the right-most vertex. We start with $\Gamma_{1,1}$. The link $K_{1,1}$ is the figure-eight knot, or 4_1 in Rolfsen's table [11]. We will use Equation (2) to find P_1. Thus, P_1 is given by

$$\begin{aligned} P_1(t) &= \mathrm{s}(S)\left| t\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}\right| \\ &= t(t-2) \end{aligned}$$

Since P_1 has only one root outside C, we have the following Proposition.

Proposition 19. *The links $(K_{1,n}, \Sigma_{1,n})$ are Salem fibered links.*

The Salem numbers $\lambda(\Delta_{1,n})$ converge to $\lambda(P) = 2$, from above for n odd, and from below for n even. The smallest Salem number in this sequence occurs for $(K_{1,4}, \Sigma_{1,4})$, and is approximately 1.8832.

From P_1 it is possible to compute all the P_m using Equation 3. We first recall that $(K_{m,0}, \Sigma_{m,0})$ is the $(2, m+1)$ torus link, $T_{(2,m+1)}$. The Alexander polynomial is given by

$$\Delta_{m,0}(t) = \frac{t^{n+1} + (-1)^n}{t+1}.$$

Since $P_1(t) = t(t-2)$, and $K_{1,1}$ has one component, we also have

$$\Delta_{1,n}(t) = \frac{t^n P_1(t) + (-1)^{n+1}(P_1)_*(t)}{t+1}$$

$$= \frac{t^{n+1}(t-2) + (-1)^{n+1}(-2t+1)}{t+1}$$
$$= \frac{t^{n+1}(t-2) + (-1)^{n}2t + (-1)^{n+1}}{t+1}$$

Furthermore, $\Gamma_{m,0}$ can be thought of as a subgraph of $\Gamma_{m,1}$, and if $S_{m,0}$ and $S_{m,1}$ are their associated Seifert surfaces, we have

$$\mathrm{s}(S_{m,0}) = \mathrm{s}(S_{m,1}).$$

By Equation 3, we have

$$\begin{aligned} P_m(t) &= \Delta_{m,1}(t) + \Delta_{m,0}(t) \\ &= \frac{t^{m+1}(t-2) + (-1)^m 2t + (-1)^{m+1} + t^{m+1} + (-1)^m}{t+1} \\ &= \frac{t^{m+2} - t^{m+1} + (-1)^m 2t}{t+1} \\ &= \frac{t(t^m(t-1) + (-1)^m 2)}{t+1} \end{aligned}$$

Since we are only concerned with $\Delta_{m,n}$ and hence P_m up to products of cyclotomic polynomials, it is convenient to rewrite P_m as

$$P_m(t) = t(t^m(t-1) + (-1)^m 2).$$

Proposition 20. *All roots of $P_m(t)$ other than 0 and -1 lie outside C, hence*

$$M(P_m) = 2 \quad \textit{and} \quad N(P_m) = m.$$

Proof. Suppose $|t| \leq 1$, then $|t^m(t-1)| \leq 2$ with equality if and only if $t = -1$. ☐

Proposition 21.

$$\lim_{m\to\infty} \lambda(P_m) = 1.$$

Proof. Take any $\epsilon > 0$. Let $D_\epsilon = \{z \in \mathbb{C}\colon |z| > 1 + \epsilon\}$. Let $\overline{D_\epsilon}$ be the closure of $\mathbb{C}$ in the Riemann sphere. Then for large m

$$\frac{2}{|t^m|} < \frac{|t-1|}{|t|}$$

for all t on the boundary of $\overline{D_\epsilon}$ and both sides are analytic on $\overline{D_\epsilon}$. Therefore, by Rouché's theorem P_m has no roots on $\overline{D_\epsilon}$ for large m. ☐

Corollary 22. *The homological dilatations of* $(K_{m,n}, \Sigma_{m,n})$ *can be made arbitrarily small by taking* m *and* n *large enough.*

Salem fibered links appear sporadically as homological dilatations of $(K_{m,n}, \Sigma_{m,n})$ for $m, n > 1$. A list for $1 < m, n < 60$ found by computer search is given in the table below. The minimal polynomials, which are reciprocal, are denoted by a list of the first half of the coefficients.

(m, n)	Salem number	Minimal polynomial
(3, 5)	1.63557	$1\ -2\ 2\ -3$
(3, 8)	1.50614	$1\ -1\ 0\ -1$
(5, 9)	1.42501	$1\ -1\ 0\ -1\ 1$

QUESTION 23. Are the Salem fibered links in the table above isolated in the sense of Section 4?

Salem numbers also appear as roots of irreducible factors of the Alexander polynomial. For example, the Alexander polynomial for $K_{11,21}$ has largest root equal to the 7th smallest known Salem number [10]. Its minimal polynomial is given by

$$\Delta_{K_{11,21}}(x) = x^{10} - x^7 - x^5 - x^3 + 1.$$

The monodromy $h_{m,n}$ of the fibered links $(K_{m,n}, \Sigma_{m,n})$ were also studied by Brinkmann [3], who showed that $h_{m,n}$ is pseudo-Anosov for all m, n, and that the dilatations converge to 1 as m, n approach infinity.

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