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LOCALLY NILPOTENT DERIVATIONS ON AFFINE SURFACES WITH A $\mathbb{C}^*$-ACTION

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Abstract
We give a classification of normal affine surfaces admitting an algebraic group action with an open orbit. In particular an explicit algebraic description of the affine coordinate rings and the defining equations of such varieties is given. By our methods we recover many known results, e.g. the classification of normal affine surfaces with a 'big' open orbit of Gizatullin [19, 20] and Popov [31] or some of the classification results of Danilov-Gizatullin [12], Bertin [6, 7] and others.

Introduction

Let G be an algebraic group acting on a normal affine algebraic surface V. By classical results of Gizatullin [19] and Popov [31], if V is smooth and G has a big open orbit O (that is, $V\backslash O$ is finite), then V is one of the surfaces

$$\mathbb{C}^{*2}, \quad \mathbb{A}^2_{\mathbb{C}}, \quad \mathbb{C}^* \times \mathbb{A}^1_{\mathbb{C}}, \quad \mathbb{P}^1 \times \mathbb{P}^1 \backslash \Delta, \quad \mathbb{P}^2 \backslash \bar{\Delta},$$

where $\Delta \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is the diagonal and $\bar{\Delta} \subseteq \mathbb{P}^2$ is a nondegenerate quadric. Furthermore, if V is singular then $V \cong V_d$ is the Veronese cone $\mathbb{A}^2_{\mathbb{C}}/\mathbb{Z}_d$, where $\mathbb{Z}_d$ acts on $\mathbb{A}^2_{\mathbb{C}}$ via multiplication with the group of d-th roots of unity (see Example 5.2).

The aim of this paper is to give more generally *a description of all normal affine surfaces* $V = \operatorname{Spec} A$ (*over the ground field* $\mathbb{C}$) *that admit an action of an algebraic group with an open orbit*. As was suggested by Popov [31] and confirmed in the smooth case by Bertin [7], either such a surface V is isomorphic to $\mathbb{C}^{*2}$, or a semidirect product of $\mathbb{C}^*$ and $\mathbb{C}_+$ acts on V with an open orbit (Proposition 2.10). We provide a classification of all such surfaces in Section 3. This leads to a new proof of the Gizatullin-Popov Theorem above (see Section 4.4) which uses only elementary facts from Lie theory. For generalizations of this result see also [2, 21].

Our interest in such actions is inspired by the role that they play in certain classification problems, e.g. in the proof of linearization of regular $\mathbb{C}^*$-actions on $\mathbb{A}^3_{\mathbb{C}}$ [23]. Usually in applications, to an affine variety V with a $\mathbb{C}_+$-action one associates (non canonically) another one, say, V' with a $\mathbb{C}^*$- and $\mathbb{C}_+$-action (see e.g., [26, 36] and Remark 3.13.3 below). Therefore it is of particular importance to classify such varieties.

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$\mathbb{C}^*$-actions on algebraic surfaces were extensively studied in the literature, see [17] and the references given therein, and also [3] for a generalization to higher dimensions. A $\mathbb{C}^*$-action on V gives rise to a grading $A = \bigoplus_{i\in\mathbb{Z}} A_i$. We will rely here on our previous paper [17] to describe the graded components A_i in terms of the Dolgachev-Pinkham-Demazure construction (the *DPD construction*, for short).

Classification results for $\mathbb{C}_+$-actions on affine surfaces can be found in [11, 16, 27, 28, 35], [4]–[5], [9, 10], and [13]–[15]. It is well known [32] that a $\mathbb{C}_+$-action gives rise to a locally nilpotent derivation ∂ of A (see Proposition 1.1). The condition that a semidirect product of $\mathbb{C}^*$ and $\mathbb{C}_+$ acts on V is equivalent to the condition that ∂ is a homogeneous derivation (cf. Lemma 2.2). Thus we are led to pairs

$$(A, \partial), \quad e = \deg \partial,$$

where ∂ is a homogeneous locally nilpotent derivation on A of a certain degree e. Our classification of such pairs is as follows.

Elliptic case: In this case $A_0 \cong \mathbb{C}$, and A is positively graded so that the associated $\mathbb{C}^*$-surface $V = \operatorname{Spec} A$ has a unique fixed point given by the maximal ideal $A_+ := \bigoplus_{i>0} A_i$. If V also admits a nontrivial $\mathbb{C}_+$-action then by [18, Lemmas 2.6 and 2.16], $V \cong \mathbb{A}^2_{\mathbb{C}}/\mathbb{Z}_d$ is a quotient of $\mathbb{A}^2_{\mathbb{C}}$ by a small cyclic subgroup of $\mathbf{GL}_2(\mathbb{C})$. More precisely, we show in Theorem 3.3 that $A \cong \mathbb{C}[X,Y]^{\mathbb{Z}_d}$, *where the cyclic group* $\mathbb{Z}_d := \mathbb{Z}/d\mathbb{Z} = \langle\zeta\rangle$ *generated by a primitive d-th root of unity ζ acts on* $\mathbb{C}[X,Y]$ *via* $\zeta.X = \zeta X$ *and* $\zeta.Y = \zeta^e Y$ *with* $e \geq 0$, $\gcd(e,d) = 1$, *and* $\partial = X^e\partial/\partial Y$. In particular, $V \cong V_{d,e}$ is an affine toric surface (see Example 2.8).

Parabolic case: Here again A is positively graded, but $A_0 \not\cong \mathbb{C}$. Thus $C = \operatorname{Spec} A_0$ is a smooth affine curve, and V is fibered over C with general fiber $\mathbb{A}^1_{\mathbb{C}}$. Using the DPD construction it follows that $A = A_0[D]$ for some $\mathbb{Q}$-divisor D on C (see [17, Theorem 3.2]). More precisely, if K_0 denotes the field of fractions $\operatorname{Frac}(A_0)$ then $A = A_0[D] \subseteq K_0[u]$ is the graded subring with

$$A_n = \left\{ fu^n \in K_0 \cdot u^n \;\middle|\; \operatorname{div} f + nD \geq 0 \right\}.$$

If such a surface admits also a $\mathbb{C}_+$-action given by a homogeneous locally nilpotent derivation ∂ then either $\mathbb{C}_+$ acts *vertically* (that is fiberwise), so that the orbits are contained in the fibers of the projection $V \to C$, or the orbits map onto the base curve C (*horizontal case*). In both cases we classify all possible actions (see Theorems 3.12 and 3.16). For instance, in the horizontal case $V \cong V_{d,e} \cong \mathbb{A}^2_{\mathbb{C}}/\mathbb{Z}_d$ is again an affine toric surface and the derivation ∂ is as described in the elliptic case. These are the only normal affine surfaces with an elliptic or parabolic $\mathbb{C}^*$-action and with a trivial Makar-Limanov invariant that is, admitting two non-trivial $\mathbb{C}_+$-actions with different orbit maps (see Definition 4.2 and Theorem 4.3).

Hyperbolic case: In this case $A_i \neq 0$ for all $i \in \mathbb{Z}$, and the surface $V = \operatorname{Spec} A$ is fibered over the base curve $C = \operatorname{Spec} A_0$ with general fiber $\mathbb{C}^*$. By [17, Theorem 4.3]

$A = A_0[D_+, D_-] \subseteq \mathrm{Frac}(A_0)[u, u^{-1}]$ with a pair of $\mathbb{Q}$-divisors $D_\pm$ on C satisfying $D_+ + D_- \leq 0$. This means that $A_{\geq 0} = A_0[D_+] \subseteq K_0[u]$ and $A_{\leq 0} = A_0[D_-] \subseteq K_0[v]$ are as above with $v = u^{-1}$. Furthermore, the pair (D_+, D_-) is determined uniquely up to an arbitrary shift $(D_+, D_-) \rightsquigarrow (D_+ + \operatorname{div}\varphi, D_- - \operatorname{div}\varphi)$ with $\varphi \in \operatorname{Frac} A_0$. In Corollary 3.23 we show that *A admits a homogeneous locally nilpotent derivation ∂ of positive degree e if and only if $C \cong \mathbb{A}^1_{\mathbb{C}}$ i.e., $A_0 \cong \mathbb{C}[t]$, and $A \cong A_0[D_+, D_-]$, where $D_+ = -(e'/d)[p]$ is supported at one point, $0 \leq e' < d$ and $ee' \equiv 1 \mod d$. Moreover, ∂ is uniquely determined up to a constant by its degree*. Alternatively, such surfaces can be described as cyclic quotients of the normalizations of hypersurfaces $\{u^d v - p(t) = 0\}$ in $\mathbb{A}^3_{\mathbb{C}}$, where $p \in \mathbb{C}[t]$ (see [17, Proposition 4.14] and Corollary 3.30 below).

$\mathbb{C}_+$-actions on a normal affine surface V are related to affine rulings $V \to \Gamma$ (that is, rulings into affine lines) with Γ being a smooth affine curve (see Lemma 1.6). If $V = \operatorname{Spec} A$ with $A = A_0[D_+, D_-]$ as above, where $A_0 = \mathbb{C}[t]$ and $D_+ + D_- \not\equiv 0$, then *there exists an affine ruling $V \to \mathbb{A}^1_{\mathbb{C}}$ if and only if the fractional part $\{D_\pm\}$ of at least one of the $\mathbb{Q}$-divisors $D_\pm$ is supported at one point or is zero. Such an affine ruling is unique unless both $\{D_+\}$ and $\{D_-\}$ are either zero or supported at points $\{p_\pm\}$, and if and only if, for a homogeneous element $v \in A\backslash\mathbb{C}$, $\ker\partial \supseteq \mathbb{C}[v]$ for every locally nilpotent derivation $\partial \in \operatorname{Der} A$* (Corollary 3.23 and Theorem 4.5). Otherwise V allows continuous families of affine rulings, of $\mathbb{C}^*$-actions and of $\mathbb{C}_+$-actions with generically different orbit maps (Corollary 4.11). The same is also true in the elliptic and the parabolic cases.

In the first two sections we summarize some facts on $\mathbb{C}_+$-actions and on algebraic group actions on normal affine surfaces. Section 3 contains the principal classification results. In Section 4 we classify all $\mathbb{C}^*$-surfaces which have a trivial Makar-Limanov invariant (Corollary 4.4 and Theorem 4.5). Finally, in Section 5 we discuss concrete examples and compare different approaches.

Throughout the paper we use the notation $\mathbf{GL}_2 = \mathbf{GL}(2, \mathbb{C})$, $\mathbf{SL}_2 = \mathbf{SL}(2, \mathbb{C})$, etc.

1. $\mathbb{C}_+$-actions and locally nilpotent derivations

We frequently use the following well known facts.

Proposition 1.1 (see e.g., [26, 32, 36]). *Let $V = \operatorname{Spec} A$ be an affine algebraic $\mathbb{C}$-scheme. Then the following hold*:

(a) *If $\mathbb{C}_+$ acts on V then the associated derivation ∂ on A is locally nilpotent, i.e. for every $f \in A$ we can find $n \in \mathbb{N}$ such that $\partial^n(f) = 0$. Conversely, given a locally nilpotent $\mathbb{C}$-linear derivation $\partial\colon A \to A$ the map $\varphi\colon \mathbb{C}_+ \times A \to A$ with $\varphi(t, f) := e^{t\partial} f$ defines an action of $\mathbb{C}_+$ on V.*

(b) *Assume that A is a domain and let $\partial \in \operatorname{Der}_{\mathbb{C}} A$ be a locally nilpotent derivation of A. Then the subalgebra $\ker\partial = A^{\mathbb{C}_+} \subseteq A$ is algebraically and factorially closed (or*

inert)[1] *in* A, *and for* $\partial \neq 0$ *the field extension* $\operatorname{Frac}(\ker\partial) \subseteq \operatorname{Frac} A$ *has transcendence degree* 1. *Moreover, for any* $u \in \operatorname{Frac} A$ *with* $u\partial(A) \subseteq A$, *the derivation* $u\partial \in \operatorname{Der}_{\mathbb{C}} A$ *is locally nilpotent if and only if* $u \in \operatorname{Frac}(\ker\partial)$.
(c) *If* $\mathbb{C}_+$ *acts non-trivially on an irreducible reduced affine curve* C *then* $C \cong \mathbb{A}^1_{\mathbb{C}}$.

Corollary 1.2. *For an algebraic* $\mathbb{C}$*-scheme* A *and a locally nilpotent derivation* $\partial \neq 0$ *on* A, *the following hold.*
(a) *The algebra of invariants* $\ker\partial = A^{\mathbb{C}_+}$ *is integrally closed in* A. *Consequently, if* A *is normal and the ring of invariants* $A^{\mathbb{C}_+}$ *is finitely generated then the orbit space* $\operatorname{Spec} A^{\mathbb{C}_+}$ *of the associate* $\mathbb{C}_+$*-action on* V *is also normal.*
(b) *For an element* $v \in A$, *the principal ideal* $(v) = vA$ *is* ∂*-invariant if and only if* $v \in \ker\partial$.
(c) *If* $\dim A \geq 2$ *then the automorphism group* $\operatorname{Aut} A$ *is of infinite dimension.*

Proof. (a) immediately follows from Proposition 1.1 (b). To show (b) we fix $n \geq 1$ such that $u := \partial^{n-1}(v) \neq 0$ and $\partial u = 0$. If the ideal (v) is ∂-invariant then $u \in \ker\partial \cap (v)$ can be written as $u = fv$ with $f \in A$. As $\ker\partial$ is inert (see Proposition 1.1 (b)) and $u = fv \in \ker\partial$ we have $v \in \ker\partial$, as required. The proof of the converse is trivial. As $e^{a\partial} \in \operatorname{Aut} A$ $\forall a \in \ker\partial$ and $\dim\ker\partial \geq 1$, (c) also follows from Proposition 1.1 (b). $\square$

1.3. Let us recall some well known facts on the surface geometry in presence of a $\mathbb{C}_+$-action; see e.g., [4, 28, 29]. For a normal affine surface V we denote $V_{\text{reg}} = V \setminus \operatorname{Sing} V$. A *cylinder* in V is a Zariski open subset $U \cong \Gamma_0 \times \mathbb{A}^1_{\mathbb{C}}$, where Γ_0 is a smooth curve. An *affine ruling* on V is a morphism $V \to \Gamma$ onto a smooth curve Γ with general fibers isomorphic to $\mathbb{A}^1_{\mathbb{C}}$. Two affine rulings coincide if they have the same fibers.

Lemma 1.4 ([29, Ch. 3, Lemma 1.3.1, Theorem 1.3.2 and Lemma 1.4.4 (1)]). *For a normal affine surface* V *the following conditions are equivalent*:
(i) V *is affine ruled.*
(ii) V *contains a cylinder.*
(iii) *There exists an affine Zariski open subset* $W \subseteq V_{\text{reg}}$ *with* $\bar{k}(W) = -\infty$.[2]
Moreover, under these conditions V *has at most cyclic quotient singularities.*

REMARK 1.5. If V is smooth then any degenerate fiber of an affine ruling on V consists of disjoint components isomorphic to $\mathbb{A}^1_{\mathbb{C}}$ (see [6, 16]). If V is only normal then any such component has a normalization isomorphic to $\mathbb{A}^1_{\mathbb{C}}$, contains at most one singular point of V and is smooth off this point ([29, Ch. 3, Lemmas 1.4.2 and 1.4.4]).

[1] The latter means that $ab \in \ker\partial \Rightarrow a, b \in \ker\partial$.
[2] As usual, $\bar{k}$ stands for the logarithmic Kodaira dimension.

Suppose that a normal surface $V = \operatorname{Spec} A$ admits a non-trivial $\mathbb{C}_+$-action. The orbit morphism $\pi_+ : V \to \Gamma := V /\!/ \mathbb{C}_+$ then yields an affine ruling on V over a smooth affine curve $\Gamma \cong \operatorname{Spec} A^{\mathbb{C}_+}$. Therefore [4, Remark 1], an affine ruling on V over a projective base cannot be produced in this way. For instance, the latter concerns the projection $\mathrm{pr}_1 : (\mathbb{P}^1 \times \mathbb{P}^1)\backslash\Delta \to \mathbb{P}^1$, where $\Delta \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is the diagonal. The following simple lemma clarifies the situation (cf. [4, Prop. 2]).

Lemma 1.6. *For a normal affine surface V the following are equivalent*:
(i′) *V admits an affine ruling $V \to \Gamma$ over an affine base Γ.*
(ii′) *V contains a cylinder $U \cong \Gamma_0 \times \mathbb{A}^1_{\mathbb{C}}$ which is a principal Zariski open subset.*
(iii′) *There exists a non-trivial regular $\mathbb{C}_+$-action on V.*

Proof. The implication (iii′) $\Rightarrow$ (i′) has been noted above. The proof of (i′) $\Rightarrow$ (ii′) follows that of (i) $\Rightarrow$ (ii) in Lemma 1.4; it suffices to note that, because $\Gamma_0 \subseteq \Gamma$ can be taken principal, so does the cylinder $U \subseteq \pi^{-1}(\Gamma_0)$.

To show (ii′) $\Rightarrow$ (iii′) we let $U \cong \Gamma_0 \times \mathbb{A}^1_{\mathbb{C}}$ be a principal cylinder in $V = \operatorname{Spec} A$ given via $A[1/f_0] \cong B[t]$ with $f_0 \in A$, where $\Gamma_0 = \operatorname{Spec} B$. We consider the derivation $\partial = \partial/\partial t \in \operatorname{Der} B[t]$. Given a system of generators $g_1, \ldots, g_n$ of the algebra A we can write $\partial g_i = h_i / f_0^{k_i}$, where $h_i \in A$ and $k_i \geq 0$ $(i = 1, \ldots, n)$. Since $f_0^e \partial(g_i) \in A$ $\forall i$, where $e := \max_{1 \leq i \leq n} k_i$, we have $\partial_e := f_0^e \partial \in \operatorname{Der} A$. Moreover, $\partial_e f_0 = 0$ as $f_0 \in B[t]$ is a unit. Hence ∂_e is locally nilpotent and so defines a non-trivial $\mathbb{C}_+$-action on $V = \operatorname{Spec} A$, as required. □

1.7. If a ramified covering of normal varieties $Y \to X$ is unramified in codimension 1 then any $\mathbb{C}_+$-action on X lifts to Y [18, proofs of Lemmas 2.15 and 2.16]. In the following lemma we show that, under certain circumstances, it still lifts to a cyclic covering ramified in codimension 1, provided the latter is defined by an invariant.

Lemma 1.8. *Let A be a normal domain of finite type over $\mathbb{C}$ and let $\partial \in \operatorname{Der} A$ be a non-zero locally nilpotent derivation. For a non-zero element $v \in \ker \partial$ and for $n \in \mathbb{N}$ denote A' the normalization of the cyclic ring extension $A[u'] \supseteq A$ with $(u')^n = v$. Then the following hold*:
(a) *A' is a normal affine $\mathbb{C}$-algebra of finite type, and the elements of A are not zero divisors on A'.*
(b) *∂ extends uniquely to a locally nilpotent derivation $\partial' \in \operatorname{Der} A'$ with $\partial'(u') = 0$.*
(c) *If, moreover, A is a graded domain and v and ∂ are homogeneous with $\deg v = n$ then A' is graded as well, and u' and ∂' are homogeneous with $\deg u' = 1$ and $\deg \partial' = \deg \partial$.*
(d) *Furthermore, if the polynomial $x^n - v \in A[x]$ is irreducible over A then the cyclic group $\mathbb{Z}_n = \langle \zeta \rangle$, where ζ is a primitive n-th root of unity, acts on A' with $\zeta | A = \mathrm{id}$, $\zeta . u' = \zeta u'$, and $A = (A')^{\mathbb{Z}_n}$ is the ring of invariants of this action.*

Proof. The proofs of (a), (c) and (d) are easy and we omit them. To show (b) note that the derivation $\partial' \in \operatorname{Der} A[u']$ defined by $\partial'|A = \partial$ and $\partial'(u') = 0$ is locally nilpotent. By [33] its extension to $\operatorname{Frac}(A[u'])$ stabilizes the integral closure A' of $A[u']$. By [34] (see also [18, Lemma 2.15 (a)]) this extension ∂' of ∂ to A' is again locally nilpotent, as stated. □

2. Algebraic group actions on affine surfaces

2.1. $\mathbb{C}_+$-actions on graded rings. We let $V = \operatorname{Spec} A$ be an affine variety over $\mathbb{C}$ with an effective $\mathbb{C}^*$-action, which corresponds to a grading $A = \bigoplus_{i\in\mathbb{Z}} A_i$.

Lemma 2.1. [32] *If ∂ is a locally nilpotent derivation on A and $\partial = \sum_{i=k}^{l} \partial_i$ is the decomposition of ∂ into graded components then δ_k and δ_l are again locally nilpotent.*

Homogeneous locally nilpotent derivations on $A = \bigoplus_{i\in\mathbb{Z}} A_i$ correspond to actions of certain semidirect products of $\mathbb{C}^*$ and $\mathbb{C}_+$ on A. Indeed, we have the following lemma (cf. [31], [7, (2.5)]).

Lemma 2.2. (a) *Let $\partial\colon A \to A$ be a homogeneous locally nilpotent derivation of degree e and consider the action of $\mathbb{C}^*$ on $\mathbb{C}_+$ given by $\tau_e(t,\alpha) := t.\alpha = t^e\alpha$, where $t \in \mathbb{C}^*$, $\alpha \in \mathbb{C}_+$. Then the semidirect product*

$$G_e := \mathbb{C}^* \ltimes_{\tau_e} \mathbb{C}_+$$

(*with $\mathbb{C}_+$ as a normal subgroup*) *acts on A, and hence on V, via*

$$(s,\alpha).f := s.e^{\alpha\partial}(f), \quad \textit{where} \quad (s,\alpha) \in G_e \quad \textit{and} \quad f \in A.$$

This action restricts to the given actions on the subgroups $\mathbb{C}_+$ and $\mathbb{C}^*$ of G_e.
(b) *Conversely, if there is an action of G_e on $V = \operatorname{Spec} A$ restricting to the given action of $\mathbb{C}^* \subseteq G_e$ on A, then $\mathbb{C}_+ \subseteq G_e$ acts on V and the associated derivation ∂ on A is homogeneous of degree e.*

Proof. (a) The multiplication on G_e is given by

$$(s,\alpha)(t,\beta) = (st, t^{-e}\alpha + \beta) \quad \text{with} \quad s,t \in \mathbb{C}^*,\ \alpha,\beta \in \mathbb{C}_+.$$

Since ∂ is homogeneous of degree e it follows that $s.(\partial(f)) = s^e\partial(s.f)$, and so

$$s.e^{\alpha\partial}(f) = \sum_{\nu=0}^{\infty} s.\frac{\alpha^\nu\partial^\nu(f)}{\nu!} = \sum_{\nu=0}^{\infty} \frac{\alpha^\nu s^{e\nu}\partial^\nu(s.f)}{\nu!} = e^{s^e\alpha\partial}(s.f),$$

hence

$$((s,\alpha)(t,\beta)).f = (st).e^{(t^{-e}\alpha+\beta)\partial}(f) = s.(t.e^{t^{-e}\alpha\partial}e^{\beta\partial}(f)) = s.e^{\alpha\partial}t.e^{\beta\partial}(f) = (s,\alpha).((t,\beta).f).$$

This shows that G_e acts indeed on A and hence on V.

(b) Conversely, suppose that G_e acts on A restricting to the given action of $\mathbb{C}^*$ on A. Then for $\mathbb{C}_+ \ni a = (1,a) \in G_e$ we have $a.f = e^{a\partial}(f)$, and so

$$s.e^{a\partial}(f) = (s,a).f = (1,s^e a)((s,0).f) = e^{s^e a\partial}(s.f).$$

Differentiating this equation with respect to a and taking $a = 0$ one gets

$$s.\partial(f) = s^e\partial(s.f).$$

It follows that ∂ is homogeneous of degree e. □

REMARKS 2.3. 1. For any non-zero homogeneous element $u \in \ker\partial$ of degree n, the derivation $\partial' := u^m\partial \in \operatorname{Der} A$ is again locally nilpotent (see Proposition 1.1 (b)) of degree $e+mn$. Thus for every $m \geq 0$ the group G_{e+mn} also acts on A restricting to the given $\mathbb{C}^*$-action on A. The inversion $\lambda \mapsto \lambda^{-1}$ provides an isomorphism $G_e \cong G_{-e}$, and so $G_{e'}$ acts on V for any $e' \equiv \pm e \mod n$.
2. For instance, a Borel subgroup $B \subseteq \mathbf{SL}_2$ is isomorphic to G_2 and acts effectively on $V = \mathbb{A}^2_C$ with an open orbit. Similarly, the Borel subgroup $B' := B/\mathbb{Z}_2$ in $\mathbf{PGL}_2 = \mathbf{SL}_2/\mathbb{Z}_2$, where $\mathbb{Z}_2 = \{\pm I_2\}$ is the center of $\mathbf{SL}_2$ (and of B), is isomorphic to G_1 and acts effectively on the Veronese cone $V_{2,1} := \mathbb{A}^2_{\mathbb{C}}/\mathbb{Z}_2 \cong \operatorname{Spec}\mathbb{C}[t,u,v]/(uv-t^2) \subseteq \mathbb{A}^3_{\mathbb{C}}$ with an open orbit (cf. Example 5.2).
3. For $e > 0$, G_e is a metabelian solvable Lie group with a cyclic center $Z(G_e) = \mathbb{Z}_e \ltimes \{0\} \subseteq \mathbb{C}^* \ltimes \mathbb{C}_+$, and so is an étale covering group of G_1 via $G_e \stackrel{e:1}{\to} G_1 \cong G_e/Z(G_e)$. The Lie algebra $\mathfrak{g} = \operatorname{Lie} G_e$ is isomorphic to $\mathbb{A}^2_{\mathbb{C}}$ with Lie bracket $[\vec{v}_1,\vec{v}_2] = (0, \vec{v}_1 \wedge \vec{v}_2)$.

Actually, an effective G_e-action on A with $e \neq 0$ permits to produce a continuous family of gradings on A.

Proposition 2.4. *Let $A = \bigoplus_{i\in\mathbb{Z}} A_i$ be a graded $\mathbb{C}$-algebra of finite type and $\partial \in \operatorname{Der} A$ be a homogeneous locally nilpotent derivation on A of degree $e \neq 0$. If the orbit closures of the associated $\mathbb{C}^*$- and $\mathbb{C}_+$-actions on $V := \operatorname{Spec} A$ are generically different then A admits a continuous family of generically distinct gradings.*

Proof. For $\alpha \in \mathbb{C}_+$, $\alpha \neq 0$, we consider a new G_e-action on $V_\alpha := V$ induced by the isomorphism $\alpha\colon V \to V_\alpha$ that is, conjugated with the original G_e-action on V by means of α. More precisely, we have a commutative diagram

$$\begin{tikzcd} G_e \arrow[r, "\xi_\alpha"] \arrow[d, "-.\alpha^{-1}(x)"'] & G_e \arrow[d, "-.x"] \\ V \arrow[r, "\alpha.-"] & V_\alpha \end{tikzcd}$$

where the vertical arrow on the right is the new G_e-action on V_α and

$$\xi_\alpha(g) = \alpha g \alpha^{-1} = (t, \beta + t^{-e}\alpha - \alpha) \quad \text{for} \quad g = (t, \beta) \in G_e.$$

The $\mathbb{C}^*$-orbit of $(1, \beta) \in G_e = \mathbb{C}^* \ltimes_{\tau_e} \mathbb{C}_+$ is equal to $\mathbb{C}^* \times \{\beta\}$ and is mapped under ξ_α onto the set

$$\left\{ \left(t, \beta + t^{-e}\alpha - \alpha\right) \;\middle|\; t \in \mathbb{C}^* \right\},$$

which is not an orbit of the $\mathbb{C}^*$-action on G_e. Since by our assumption for a general $x \in V$ the orbit $G_e.x$ has dimension 2, the generic $\mathbb{C}^*$-orbit in V is not mapped onto a $\mathbb{C}^*$-orbit of V_α. $\square$

In the surface case we have the following elementary lemma.

Lemma 2.5. *For a G_e-action on an affine surface $V = \operatorname{Spec} A$ the following conditions are equivalent.*

(i) *It has an open orbit.*

(ii) $A^{\mathbb{C}_+} \neq A^{\mathbb{C}^*}$ ($\Leftrightarrow \ker \partial \neq A_0$)[3].

(iii) $\ker \partial = \mathbb{C}[v]$ *or* $\ker \partial = \mathbb{C}[v, v^{-1}]$, *where* $v \in A_d$ *with* $d \neq 0$.

Under these equivalent conditions the surface V is rational, and the affine ruling $v \colon V \to \Gamma := \operatorname{Spec} A^{\mathbb{C}_+}$ has at most one degenerate fiber $v = 0$ consisting of $\mathbb{C}^*$-orbit closures[4].

Proof. Since $\partial \in \operatorname{Der} A$ is homogeneous, its ring of invariants $\ker \partial = A^{\mathbb{C}_+}$ is a graded subring of A. Thus the normal (hence smooth) affine curve $\Gamma = \operatorname{Spec} A^{\mathbb{C}_+}$ also carries a $\mathbb{C}^*$-action, and the quotient morphism $V \to \Gamma = V/\!/\mathbb{C}_+$ (which provides an affine ruling on $V = \operatorname{Spec} A$) is $\mathbb{C}^*$-equivariant. In case $A^{\mathbb{C}_+} \neq A^{\mathbb{C}^*}$ (that is, $\ker \partial \neq A_0$) the induced $\mathbb{C}^*$-action on Γ is non-trivial, hence $\Gamma \cong \mathbb{A}^1_{\mathbb{C}}$ or $\mathbb{C}^*$. In this case $\ker \partial = \mathbb{C}[v]$ and $\mathbb{C}[v, v^{-1}]$, respectively, where $v \in A_d \cap \ker \partial$ is homogeneous and $d \neq 0$.

The rationality of V follows from Lüroth's Theorem. The rest of the proof is easy and can be omitted. $\square$

[3] I.e., the $\mathbb{C}_+$-action is horizontal w.r.t. the given $\mathbb{C}^*$-action.

[4] Cf. Remarks 1.5, 3.13 (iii) and Lemma 3.24 below.

2.2. Actions with an open orbit. The next simple observations will be used in the proofs below (cf. Remark 1 in [25, II.4.3.B]).

Lemma 2.6. (a) *If a connected Lie group L and a finite group G act on an algebraic $\mathbb{C}$-scheme $V = \operatorname{Spec} A$ then the action of L descends to V/G if and only if the actions of G and L on V commute.*
(b) *Conversely, suppose that a connected and simply connected Lie group L acts on the quotient V/G of V by a free action of a finite group G. Then the action of L lifts to V commuting with the action of G.*

Proof. (a) Suppose first that the action of L on V descends to V/G. We may also assume that G acts faithfully on V. It follows that L preserves the G-orbits, and so, if $w = g.z$ for some $z, w \in V$ and some $g \in G$ then for any $\lambda \in L$ there is an element $g' = g'(\lambda) \in G$ such that $\lambda.w = g'.\lambda.z$. This implies the equality $\lambda g.z = g'\lambda.z$. Since $g'(\lambda)$ is a continuous function on the connected Lie group L with values in G it must be constant, i.e., $g' = g$, and so $g\lambda = \lambda g$ for all $g \in G$ and $\lambda \in L$. Thus the actions of L and of G commute, as stated in (a). The proof of the remaining assertions is easy and will be omitted. □

Lemma 2.7. (a) *If a complex unipotent Lie group U acts on an affine variety V with an open orbit then $V \cong \mathbb{A}^{\dim V}_{\mathbb{C}}$.*
(b) *If a complex reductive Lie group G acts effectively on a connected algebraic variety V with a fixed point $p \in V$ then the induced representation $\tau_p \colon G \to \mathbf{GL}(T_p V)$ on the Zariski tangent space of V at p is faithful.*
(c) *Any affine toric surface V non-isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ admits a G_l-action with an open orbit for every $l \in \mathbb{Z}$.*

Proof. (a) Since any orbit of U is closed in V [22, Exercise 8 in Section 17], [25, III.2.5.3], the open U-orbit is the whole V. Thus $V \cong U/H \cong \mathbb{A}^{\dim V}_{\mathbb{C}}$, where $H \subseteq U$ is a closed subgroup (see [31, Corollary of Theorem 2]). This shows (a).

(b) is well known and follows for instance from Luna's étale slice theorem or from the identity theorem [1, Sect. 2.1]. Alternatively, this can be seen by the following elementary argument: for $n \gg 0$ the induced action of G on $A_n := \mathcal{O}_{V,p}/\mathfrak{m}^{n+1}$ is easily seen to be faithful, i.e. the map $\rho_n \colon G \to \operatorname{Aut}(A_n)$ is injective, where $\operatorname{Aut}(A_n)$ denotes the Lie group of $\mathbb{C}$-algebra automorphisms of A_n. The subgroup N_n of $\operatorname{Aut}(A_n)$ consisting of automorphisms f with $f \equiv \mathrm{id} \mod \bar{\mathfrak{m}}^2$ is a normal unipotent subgroup, so $\rho_n^{-1}(N_n)$ is also normal and unipotent and thus trivial. It follows that already the map $G \to \operatorname{Aut}(A_1) \cong \operatorname{Aut}(A_n)/N_n$ is injective, which implies that G acts effectively on $T_p V$.

(c) As $G_l \cong \mathbb{C}^* \times \mathbb{A}^1_{\mathbb{C}}$ this is evident in case that $V \cong \mathbb{C}^* \times \mathbb{A}^1_{\mathbb{C}}$. Otherwise (c) is shown in Example 2.8 below. □

EXAMPLE 2.8. *Affine toric surfaces.* Given two natural numbers d, e' with $0 \leq e' < d$, $\gcd(e', d) = 1$, we consider the affine toric surface $V_{d,e'} = \operatorname{Spec} A_{d,e'}$, where

$$(1)\quad A_{d,e'} = \mathbb{C}[X,Y]^{\mathbb{Z}_d} \cong \bigoplus_{b\geq 0,\, ad-be'\geq 0} \mathbb{C}\cdot x^a y^b \subseteq \mathbb{C}[x,y] \quad \text{with} \quad x = X^d,\ \ y = Y/X^{e'}$$

is the semigroup algebra of the cone $\sigma^\vee = C(\vec{e}_1, e'\vec{e}_1 + d\vec{e}_2)$ in $\mathbb{R}^2$, and where $\mathbb{Z}_d = \langle \zeta \rangle$ acts on $\mathbb{C}[X,Y]$ via

$$(2)\qquad \zeta.X = \zeta X, \quad \zeta.Y = \zeta^{e'} Y$$

(cf. [17, Example 2.3]). This $\mathbb{Z}_d$-action commutes with any $\mathbb{C}^*$-action on $\mathbb{C}[X,Y]$ of the form

$$\lambda.X = \lambda^{d_X} X, \quad \lambda.Y = \lambda^{d_Y} Y,$$

where $(d_X, d_Y) \in \mathbb{Z}^2$. It also commutes with the locally nilpotent derivations

$$(3)\qquad \partial_{X,e''} = X^{e''}\frac{\partial}{\partial Y} \text{ and } \partial_{Y,e} = Y^e \frac{\partial}{\partial X} \in \operatorname{Der}\ \mathbb{C}[X,Y],$$

where $e'', e \geq 0$ are such that $e'' \equiv e' \mod d$ and $e \cdot e' \equiv 1 \mod d$ if $e' \neq 0$, $e = 0$ if $e' = 0$. Therefore by Lemma 2.6 the $\mathbb{C}_+$-actions on $\mathbb{C}[X,Y]$ induced by $\partial_{X,e''}$ and $\partial_{Y,e}$ stabilize the ring of $\mathbb{Z}_d$-invariants $A_{d,e'} = \mathbb{C}[X,Y]^{\mathbb{Z}_d}$, hence descend from $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[X,Y]$ to the quotient surface $V_{d,e'} = \operatorname{Spec} A_{d,e'} = \mathbb{A}^2_{\mathbb{C}}/\mathbb{Z}_d$. Note that any affine toric surface non-isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ or $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{C}^*$, is isomorphic to $V_{d,e'}$ for some d, e' as above. Consequently, any such surface admits two $\mathbb{C}_+$-actions with different general orbits (cf. Corollary 4.4 below).

Letting above e.g., $d_X = 0$, $d_Y = -l$ we obtain that $\deg \partial_{X,e'} = l$, and so by Lemma 2.2 (b) the group G_l acts effectively on the ring $A = A_{d,e'}$.

Lemma 2.9. *Let G be a connected complex algebraic Lie group acting effectively on a normal affine surface $V = \operatorname{Spec} A$.*
(a) *If G is unipotent and $V \not\cong \mathbb{A}^2_{\mathbb{C}}$, then G is commutative and the orbits of G are 1-dimensional.*
(b) *If G is solvable and acts on V with an open orbit O, then O is isomorphic to one of the surfaces $\mathbb{C}^* \times \mathbb{C}^*$, $\mathbb{C}^* \times \mathbb{A}^1_{\mathbb{C}}$ or $\mathbb{A}^2_{\mathbb{C}}$. Moreover, if O is big that is, $V \backslash O$ is finite, then $O = V$.*
(c) *G is solvable if and only if it does not contain a subgroup isomorphic to $\mathbf{SL}_2$ or to $\mathbf{PSL}_2$.*

Proof. (a) The orbits of G are closed in V and generically one-dimensional, since otherwise $V \cong \mathbb{A}^2_{\mathbb{C}}$ by Lemma 2.7 (a). We let $\pi : V \to \Gamma := \operatorname{Spec} A^G$ be the quotient map. The Lie algebra $\mathfrak{g} = \operatorname{Lie} G$ consists of vector fields tangent along the fibers of π. Any such vector field $\partial \in \mathfrak{g}$ is an infinitesimal generator of a one-parameter subgroup of G isomorphic to $\mathbb{C}_+$ and so is a locally nilpotent derivation on A. Being proportional, every two such nonzero derivations ∂_1, ∂_2 are equivalent i.e., $b_1\partial_1 = b_2\partial_2$ for some $b_1, b_2 \in A^G$. Thus $\partial_1 = b\partial_2$ with $b := b_2/b_1 \in \operatorname{Frac} A^G$ and so $0 = [\partial_1, b\partial_2] = b[\partial_1, \partial_2]$. This shows that ∂_1 and ∂_2 commute, proving (a).

(b) We may suppose that $V \not\cong \mathbb{A}^2_{\mathbb{C}}$. In the decomposition $G = \mathbb{T} \ltimes N$ [22, Theorem 19.3 (b)], where $\mathbb{T}$ is a maximal torus and N is the unipotent radical of G, we have $N \cong \mathbb{C}_+^r$ by (a). If $r = 0$ then clearly $O \cong \mathbb{C}^* \times \mathbb{C}^*$. In case $r > 0$ let $\partial_0 \in \operatorname{Lie} N$ be a common eigenvector of the adjoint representation of $\mathbb{T}$ on $\operatorname{Lie} N$ and denote $N_0 \subseteq N$ the corresponding one-parameter subgroup. By (a) the orbits of G and of $G_0 := \mathbb{T} \ltimes N_0$ are the same. Thus we may suppose that $N = N_0$ has dimension 1. As G acts effectively on V with an open orbit the torus $\mathbb{T}$ must be of dimension 1 or 2, so $G \cong \mathbb{C}^* \ltimes \mathbb{C}_+$ or $G \cong \mathbb{C}^{*2} \ltimes \mathbb{C}_+$. In the first case the open orbit O of G is isomorphic to G. In case $G \cong \mathbb{C}^{*2} \ltimes \mathbb{C}_+$ the stabilizer $H = \operatorname{Stab}_x \subseteq G$ of a point $x \in O$ has dimension 1 and so $H = N$ or $H \cong \mathbb{C}^*$. If $H = N$ then $O \cong G/H \cong \mathbb{C}^{*2}$. If $H \cong \mathbb{C}^*$ then we may suppose that $H \subseteq \mathbb{T}$. Indeed, any subtorus in G is contained in a maximal torus, which is unique up to a conjugation. But then $O \cong G/H \cong \mathbb{C}^* \times \mathbb{A}^1_{\mathbb{C}}$.

In all cases the open orbit O is affine, hence $V \backslash O$ is either empty or a divisor. Thus, if O is big then $O = V$, proving (b).

(c) is well known and follows from the structure theory of algebraic groups, see [8, 22]. □

To describe all normal affine surfaces V admitting an action of an algebraic group G with an open (not necessarily big) orbit, we follow a suggestion in [31, The concluding remark]. In the particular case of smooth rational surfaces it was confirmed in [7, Proposition 2.5].

Proposition 2.10. *Let $V = \operatorname{Spec} A$ be a normal affine surface non-isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$. If an algebraic group G acts on V with an open orbit then, for some $e \in \mathbb{Z}$, the group $G_e = \mathbb{C}^* \ltimes_{\tau_e} \mathbb{C}_+$ also acts on V with an open orbit.*

Proof. If V is a toric surface then by Lemma 2.7 (c) it admits a G_e-action with an open orbit. So we may suppose in the sequel that V is not toric, in particular $V \not\cong \mathbb{A}^2_{\mathbb{C}}$.

In case $G \cong \mathbf{SL}_2$ we let $B_\pm$ be the Borel subgroups of upper/lower triangular matrices. Their intersection is the torus $\mathbb{T} \cong \mathbb{C}^*$ of diagonal matrices. If both $B_\pm$ act with 1-dimensional orbits on V then their orbits would be equal to the orbit closures of the torus action. Hence also G would act with 1-dimensional orbits contradicting our assumption. Thus at least one of the groups $B_\pm$ has an open orbit in V. Since $B_\pm \cong G_2$ the result follows in this case.

Clearly, the case $G \cong \mathbf{PGL}_2 \cong \mathbf{SL}_2/\{\pm I\}$ reduces to the previous one.

For the remaining cases we may suppose that G acts effectively on V, is connected and does not contain a subgroup isomorphic to $\mathbf{SL}_2$ or $\mathbf{PGL}_2$. By Lemma 2.9 (a), (c) G is solvable and not unipotent. Since V is not toric, the maximal torus $\mathbb{T}$ of G has dimension 1. As in the proof of Lemma 2.9 (b) we can restrict the action of G to a subgroup $H = \mathbb{T} \ltimes \mathbb{C}_+$ of G which still has an open orbit. As $H \cong G_e$ for some e, the result follows. □

3. Classification of affine surfaces with a $\mathbb{C}^*$- and $\mathbb{C}_+$-action

In this section we study normal affine surfaces $V = \operatorname{Spec} A$ endowed with an effective $\mathbb{C}^*$- and a $\mathbb{C}_+$-action. The $\mathbb{C}^*$-action provides a grading $A = \bigoplus_{i\in\mathbb{Z}} A_i$ and the $\mathbb{C}_+$-action a locally nilpotent derivation ∂ of A. Due to Lemma 2.1 we can find a *homogeneous* locally nilpotent derivation on A. Thus in the sequel we consider pairs (A, ∂), where A is the graded coordinate ring of $V = \operatorname{Spec} A$ as above and $\partial \in \operatorname{Der} A$ is a nonzero homogeneous locally nilpotent derivation.

DEFINITION 3.1. We call such a pair (A, ∂) *elliptic* if the $\mathbb{C}^*$-action on V is elliptic i.e., if A is positively graded with $\dim A_0 = 0$, *parabolic* if A is parabolic i.e., positively graded with $\dim A_0 = 1$, and *hyperbolic* if A is hyperbolic, i.e. $A_\pm \neq 0$.

Two such pairs (A, ∂) and (A', ∂') are called *isomorphic* if there is an isomorphism of graded $\mathbb{C}$-algebras $\varphi : A \to A'$ with $\varphi\partial = \partial'\varphi$.

For hyperbolic pairs we will suppose in the sequel that $e := \deg\partial \geq 0$ (indeed, otherwise we can reverse the grading of A).

We can reformulate 2.2 in this setup as follows.

Proposition 3.2. *Let $e \in \mathbb{Z}$ be fixed. There is a* 1-1 *correspondence between isomorphism classes of pairs (A, ∂) with $\deg\partial = e$ as above and normal algebraic affine surfaces V equipped with an effective G_e-action up to equivariant isomorphism.*

Thus to describe normal affine surfaces with a G_e-action up to equivariant isomorphism we classify in this section all elliptic, parabolic and hyperbolic pairs (A, ∂) with $e = \deg\partial$. Our main results are the structure theorems 3.3, 3.12, 3.16, 3.22 and Corollary 3.30. It also turns out that in many cases the isomorphism class of a pair (A, ∂) depends only on the isomorphism class of the graded algebra A, see Proposition 3.7.

3.1. Elliptic case. Let (A, ∂) be an elliptic pair. It is shown in [18, Lemmas 2.6 and 2.16] that $A \cong \mathbb{C}[X, Y]^{\mathbb{Z}_d}$, where $\mathbb{C}[X, Y]$ is graded via $\deg X = d_X > 0$, $\deg Y = d_Y > 0$, and where $G \cong \mathbb{Z}_d$ is a small subgroup of $\mathbf{GL}_2$. In particular $V = \operatorname{Spec} A$ is a toric surface. Moreover, ∂ extends to a homogeneous locally nilpotent derivation also denoted by $\partial : \mathbb{C}[X, Y] \to \mathbb{C}[X, Y]$, and the actions of ∂ and G on

$\mathbb{C}[X, Y]$ commute (see Lemma 2.6(a)).

Theorem 3.3. *If (A, ∂) is an elliptic pair then, after an appropriate change of coordinates, we have $A = \mathbb{C}[X, Y]^G$ with $G = \mathbb{Z}_d = \langle\zeta\rangle$, where ζ is a primitive d-th root of unity generating G, acting on $\mathbb{C}[X, Y]$ via*

$$\zeta.X = \zeta X, \quad \zeta.Y = \zeta^e Y,$$

and ∂ extends to $\mathbb{C}[X, Y]$ via

$$\partial(X) = 0, \quad \partial(Y) = X^e \quad \textit{i.e.}, \quad \partial = X^e \frac{\partial}{\partial Y},$$

where $e \geq 0$, $\gcd(e, d) = 1$.

Proof. Since ∂ is homegeneous locally nilpotent on $\mathbb{C}[X, Y]$ we have $\partial(P) = 0$ for an irreducible quasihomogeneous polynomial $P \in \mathbb{C}[X, Y]$ with $\deg P > 0$ (see Proposition 1.1 (b)). We can write $\partial = P^s\tilde{\partial}$, where $\tilde{\partial}$ is again a locally nilpotent derivation and s is chosen to be maximal. The derivation, say, $\overline{\partial}$ of $\mathbb{C}[X, Y]/(P)$ induced by $\tilde{\partial}$ is then nontrivial, so by Proposition 1.1 (c) above $\mathbb{C}[X, Y]/(P)$ is a polynomial ring in one variable. Since P is quasihomogeneous, it must be linear in X or in Y. After a suitable quasihomogeneous change of variables we may assume that $P = X$ so that $\partial(X) = 0$ and $\ker\partial = \mathbb{C}[X]$. Since ∂ is homogeneous locally nilpotent, $\partial(Y)$ is a homogeneous polynomial in X, i.e., $\partial(Y) = aX^e$ with $a \in \mathbb{C}^*$ and $e \geq 0$ (cf. the proof of Lemma 2.16 in [18]). Replacing Y by Y/a we may suppose that $a = 1$.

Since ∂ commutes with the action of G, for any $g \in G$ we have $\partial(g.X) = g.\partial(X) = 0$, and so $g.X = \alpha(g)X$ for some character $\alpha\colon G \to S^1$. It was shown in the proof of [18, Lemma 2.16] that α is necessarily injective. Thus we can identify G with the cyclic group $\alpha(G) = \langle\zeta\rangle \cong \mathbb{Z}_d$ for a certain primitive d-th root of unity ζ, where $\zeta.X = \zeta X$. We write now $\zeta.Y = \alpha Y + \beta X^\sigma$, where $d_X = \sigma d_Y$ in the case that $\beta \neq 0$. Since $\partial(\zeta.Y) = \zeta.\partial(Y)$ we obtain

$$\alpha X^e = \zeta.X^e = \zeta^e X^e,$$

and therefore $\alpha = \zeta^e$. If $\gcd(d, e) \neq 1$ then $d'e \equiv 0 \mod d$ for some $d' < d$, and so $\zeta^{d'} \neq 1$ acts as a pseudo-reflection on $\mathbb{C}[X, Y]$, which is excluded by our assumption that G is small. Hence $\gcd(d, e) = 1$.

Finally, if $\zeta^e = \zeta^\sigma$ then ζ when considered as an operator on $\mathbb{C}Y + \mathbb{C}X^\sigma$ has infinite order, which is impossible. Hence $\beta = 0$ in this case. If $\zeta^e \neq \zeta^\sigma$ then replacing Y by $Y' := Y + (\beta/(\zeta^e - \zeta^\sigma))X^\sigma$ we can achieve that $\zeta.Y' = \zeta^e Y'$, proving the theorem. □

3.2. Technical lemmas.

NOTATION 3.4. Until the end of this section we let (A, ∂) be a parabolic or hyperbolic pair as in Definition 3.1. Thus ∂ is a homogeneous locally nilpotent derivation on $A = A_+ \oplus A_0 \oplus A_-$ corresponding to a $\mathbb{C}_+$-action, $C := \operatorname{Spec} A_0$ is a smooth curve and $A_+ := \bigoplus_{i>0} A_i \neq 0$. We assume as before that the $\mathbb{C}^*$-action is effective so that $A_1 \neq 0$, and also $A_{-1} \neq 0$ as soon as $A_- := \bigoplus_{i<0} A_i \neq 0$. We let $d = d(A_{\geq 0})$ be the minimal positive integer such that $A_{d+n} = A_d A_n$ for every $n \geq 0$ (see [17, 3.6 and Lemma 3.5]).

Lemma 3.5. *If $\partial|A_0 \neq 0$ then $A_0 = \mathbb{C}[t]$ for a certain $t \in A_0$. Consequently for every m with $A_m \neq 0$ the A_0-module A_m is free of rank* 1.

Proof. The morphism $\pi\colon \operatorname{Spec} A \to C = \operatorname{Spec} A_0$ induced by the inclusion $A_0 \hookrightarrow A$ coincides with the orbit map onto the algebraic quotient $V/\!/\mathbb{C}^*$, hence its general fiber is an orbit closure of the $\mathbb{C}^*$-action on $V = \operatorname{Spec} A$ associated to the given grading. Since $\partial|A_0 \neq 0$ the general orbits of the $\mathbb{C}_+$-action φ_∂ on V belonging to ∂ are not contained in the fibers of π, and so map dominantly onto $\operatorname{Spec} A_0$. These orbits being isomorphic to $\mathbb{A}^1_{\mathbb{C}}$, A_0 is a subring of a polynomial ring $\mathbb{C}[T]$. It is easily seen that A_0 is a normal ring, hence $A_0 = \mathbb{C}[t]$ for some $t \in A_0$, as stated. Now the second statement follows from [17, Lemma 1.3 (b)]. $\square$

For later use we consider in the next lemma more generally a non-homogeneous derivation, but with homogeneous components of only nonnegative degrees.

Lemma 3.6.[5] *Let $\partial = \sum_{i=k}^{l} \partial_i$ be a nonzero locally nilpotent derivation on A decomposed into homogeneous components with $l \geq k \geq 0$. If $d := d(A_{\geq 0})$ and $v \in A_d$ generates A_d as an A_0-module, then* $\ker \partial = \mathbb{C}[v, v^{-1}] \cap A$. *In particular,* $\partial|A_0 \neq 0$.

Proof. Note first that ∂ stabilizes the subring $A_{\geq 0}$. Since by definition of d we have $A_{n+d} = A_n A_d = vA_n$, it stabilizes as well the principal ideal $vA_{\geq 0}$ of $A_{\geq 0}$. Thus by Corollary 1.2 (b) $\partial(v) = 0$ and so $\mathbb{C}[v, v^{-1}] \cap A \subseteq \ker \partial$. To deduce the other inclusion it is sufficient to show that $\mathbb{C}[v, v^{-1}] \cap A$ is integrally closed in A (see Proposition 1.1 (b)). The normalization of $\mathbb{C}[v, v^{-1}] \cap A$ in A is again graded and normal and so is equal to $\mathbb{C}[w, w^{-1}] \cap A$ for some homogeneous element $w \in A$ of positive degree d'. Thus $v = cw^k$ for some $k \geq 0$ and $c \in \mathbb{C}$, and so $d = d'k$. It follows that $A_{n+d} = vA_n = wA_{n+(k-1)d'}$ for all $n \geq 0$. By definition of d, this is only possible in the case $d = d'$, which proves that $\mathbb{C}[v, v^{-1}] \cap A = \ker \partial$. $\square$

This lemma has the following important consequence. Although it also follows from the classification theorems 3.16 and 3.22 we give here an independent proof.

[5] Cf. Lemma 2.5.

Proposition 3.7. *Let A be a parabolic or hyperbolic algebra with $A_0 \cong \mathbb{C}[t]$ as above and let ∂, ∂' be nonzero homogeneous locally nilpotent derivations on A of the same degree e. In the parabolic case assume further that $e \geq 0$. Then ∂ and ∂' are proportional, i.e. $\partial' = c\partial$ for some $c \in \mathbb{C}^*$. In particular, the pairs (A, ∂) and (A, ∂') are isomorphic.*

Proof. If A is hyperbolic we may reverse the grading, so in both cases we may suppose that $e \geq 0$. By Lemmas 3.5 and 3.6 there exists $v \in A_d$ such that $\ker\partial = \ker\partial' = \mathbb{C}[v, v^{-1}] \cap A$. Thus $v\colon V := \operatorname{Spec} A \to \Gamma := \operatorname{Spec}(\mathbb{C}[v, v^{-1}] \cap A)$ is an affine ruling (see also Lemma 2.5), and the vector fields ∂ and ∂' are both tangent to the fibers of v. Hence $\partial' = c\partial$ for some $c \in \operatorname{Frac}(A)$ of degree 0, and because of Proposition 1.1 (b) we have $c \in \ker\partial$. By Lemma 3.6 this implies that $c \in \mathbb{C}$, proving the first assertion.

To deduce the second one, we write $c = \lambda^e$ with $\lambda \in \mathbb{C}^*$. The $\mathbb{C}^*$-action on A induces a $\mathbb{C}^*$-action on $\operatorname{Der}_{\mathbb{C}}(A, A)$ via $(\lambda.\delta)(a) = \lambda.(\delta(\lambda^{-1}.a))$ for $\delta \in \operatorname{Der}_{\mathbb{C}}(A, A)$ and $a \in A$. As ∂ is homogeneous of degree e we have $\lambda.\partial = c\partial = \partial'$, as required. □

Lemma 3.8. *If* $\deg\partial \geq 0$ *and* $\partial(u) = 0$ *for some nonzero element* $u \in A_1$, *then* $A_{\geq 0} \cong \mathbb{C}[t, u]$ *with* $\deg t = 0$, *and* $\partial|A_{\geq 0} = x\partial/\partial t$ *for some homogeneous* $x \in A_{\deg\partial}$.

Proof. First we note that $\partial(x) \neq 0$ for all $x \in A_0 \backslash \mathbb{C}$ by Lemma 3.6. Applying Lemma 3.5 we see that $A_0 = \mathbb{C}[t]$ for some $t \in A_0$ and, moreover, for every $k > 0$ the A_0-module A_k is freely generated by some element $e_k \in A_k$. Therefore $u^k = p_k e_k$ for a certain $p_k \in A_0$. Since $u^k \in \ker\partial$ and $\ker\partial$ is factorially closed, $\partial(p_k) = \partial(e_k) = 0$. Hence $p_k \in \mathbb{C}$ for all $k > 0$, and so $A_{\geq 0} = \mathbb{C}[t, u]$. Since $\partial(u) = 0$ we have $\partial|A_{\geq 0} = x\partial/\partial t$, where $x = \partial(t) \in A_{\deg\partial}$, as required. □

Lemma 3.9. *If* $A_0 \cong \mathbb{C}[t]$ *and* $\deg\partial =: e \geq 0$ *then there is an isomorphism of graded* $\mathbb{C}$*-algebras* $A_{\geq 0} \cong \mathbb{C}[s, u']^{\mathbb{Z}_d}$ *with* $s^d = t$ *and* $u'^d = v$, *where the polynomial ring* $B := \mathbb{C}[s, u']$ *is graded via* $\deg s = 0$, $\deg u' = 1$ *and the cyclic group* $\mathbb{Z}_d = \langle\xi\rangle$ *acts on* B *via*

$$\xi.s = \xi^e s, \quad \xi.u' = \xi u'.$$

Moreover $\gcd(e, d) = 1$, *and* ∂ *is the restriction to* $A_{\geq 0}$ *of the derivation*

$$\partial = u'^e \frac{\partial}{\partial s}.$$

Proof. We may suppose that $A = A_{\geq 0}$, and we let B be the normalization of A in the field of fractions of $A[u']$, where $u' := \sqrt[d]{v}$. In view of the minimality of d the assumptions of Lemma 1.8 are fulfilled. Hence the group $\mathbb{Z}_d \cong \langle\xi\rangle$ acts on B via $\xi|A = \mathrm{id}$, $\xi.u' = \xi u'$, so that $A = (B)^{\mathbb{Z}_d}$, and ∂ extends to a locally nilpotent derivation (also denoted ∂) on B of degree e. As $\partial(u') = 0$ and $\deg u' = 1$ we can apply Lemma 3.8 to obtain that $B \cong \mathbb{C}[s, u']$ for some $s \in B_0$, and $\partial = x\partial/\partial s$ for a certain homogeneous element $x = p(s)u'^e \in (\mathbb{C}[s, u'])_e$. Since $\partial = p(s)u'^e\partial/\partial s$ is locally nilpotent we have $p \in \mathbb{C}^*$. Hence we may assume that $x = u'^e$.

The action of $\mathbb{Z}_d$ on $\operatorname{Spec} B_0 = \operatorname{Spec} \mathbb{C}[s]$ has a fixed point which we may suppose to be given by $s = 0$. Thus $\xi.s = \xi^k s$ for some $k \in \mathbb{Z}$. Since ∂ commutes with the action of $\mathbb{Z}_d$ (see Lemma 2.6) we have

$$\xi^e u'^e = \xi.\partial(s) = \partial(\xi.s) = \xi^k u'^e,$$

i.e., we may assume that $k = e$.

Since $A_1 = (B_1)^{\mathbb{Z}_d} \neq 0$ there exists a non-zero element $f = q(s)u' \in A_1$, where $q(s) = \sum_{m=0}^n q_m s^m \in \mathbb{C}[s]$. The element f being invariant under ξ we obtain

$$\xi.f = q(\xi^e s)\xi u' = q(s)u' = f,$$

i.e., $\xi^{me+1} = 1$ as soon as $q_m \neq 0$. Thus $me + 1 \equiv 0 \mod d$ and so $\gcd(e, d) = 1$. Finally, by Lemma 3.5, $s^d \in \mathbb{C}[s]^{\mathbb{Z}_d} = \mathbb{C}[t] = A_0$ generates A_0. After rescaling we may suppose that $s^d = t$ as claimed. □

REMARK 3.10. In the situation of Lemma 3.9 $\operatorname{Frac}\big(A\big[\sqrt[d]{t}\big]\big) = \operatorname{Frac}\big(A\big[\sqrt[d]{v}\big]\big) = \mathbb{C}(s, u')$.

3.3. Parabolic case.

We are now in position to exhibit the structure of (A, ∂) in the case of a positive grading with $\dim A_0 = 1$. We distinguish the following cases.

DEFINITION 3.11. A parabolic pair (A, ∂) as in Definition 3.1 will be called *vertical* or of *fiber type* if $\partial|A_0 = 0$, and of *horizontal type* if $\partial|A_0 \neq 0$.

Two isomorphic pairs (A, ∂) and (A', ∂') have the same *numerical invariants* (d, e), where $e := \deg \partial$ and $d := d(A)$ is as in 3.4 (see also [17, 3.6]). In Theorem 3.16 below we show the converse, namely, that two parabolic pairs of horizontal type with the same numerical invariants are isomorphic.

A parabolic pair is of fiber type if and only if the general orbits of the corresponding $\mathbb{C}_+$-action on $V = \operatorname{Spec} A$ coincide with the general fibers of the morphism $\pi : V \to C := \operatorname{Spec} A_0$ or, equivalently, if the vector field ∂ on V is tangent to the fibers of π. In contrast, if the pair is of horizontal type then the fibers of the $\mathbb{C}_+$-action map surjectively onto the base curve C and so, $C \cong \mathbb{A}^1_{\mathbb{C}}$ or, equivalently, $A_0 \cong \mathbb{C}[t]$ (see Lemma 3.5).

We start with the case of parabolic pairs of fiber type.

Theorem 3.12. *If (A, ∂) is a parabolic pair of fiber type, then ∂ has degree -1. Furthermore, if we represent A via the DPD construction as*

$$A \cong A_0[D] = \bigoplus_{n\geq 0} H^0(C, \mathcal{O}_C(\lfloor nD\rfloor)) \cdot u^n \subseteq \operatorname{Frac}(A_0)[u]$$

with a $\mathbb{Q}$-divisor D on $C = \operatorname{Spec} A_0$ then ∂ extends to $\operatorname{Frac}(A_0)[u]$ as $\partial = \varphi \cdot \partial/\partial u$, where $\varphi = \partial u$ belongs to $H^0(C, \mathcal{O}_C(\lfloor -D\rfloor))$. Vice versa, any $\varphi \in H^0(C, \mathcal{O}_C(\lfloor -D\rfloor))$ gives rise to a homogeneous locally nilpotent derivation $\partial = \varphi \cdot \partial/\partial u$ on A of degree -1.

Proof. The case $\deg\partial \geq 0$ is impossible by Lemma 3.6. If $\deg\partial < 0$ then $A_0 \subseteq \ker\partial$, and since A_0 is integrally closed in A we have even equality (see Proposition 1.1 (b)). If $\deg\partial < -1$ then any element in A_1 would be in $\ker\partial$, which is a contradiction. It follows that $\deg\partial = -1$.

If φ is a section in $H^0(C, \mathcal{O}_C(\lfloor -D\rfloor))$ then the derivation $\partial = \varphi \cdot \partial/\partial u$ of $\operatorname{Frac}(A_0)[u]$ stabilizes A. Indeed, for $f \in H^0(C, \mathcal{O}_C(\lfloor nD\rfloor))$ we have $\varphi f \in H^0(C, \mathcal{O}_C(\lfloor (n-1)D\rfloor))$ and so $\partial(fu^n) = n\varphi f u^{n-1} \in A_{n-1}$. Conversely, if ∂ is a A_0-linear derivation of A then it extends to $\operatorname{Frac}(A_0)[u]$, and so is of type $\partial = \varphi\cdot\partial/\partial u$ for some $\varphi \in \operatorname{Frac}(A_0)$. If $d \in \mathbb{N}$ is such that dD is integral then multiplication by φ gives a map

$$H^0(C, \mathcal{O}_C(\lfloor dD\rfloor)) \to H^0(C, \mathcal{O}_C(\lfloor (d-1)D\rfloor)),$$

and hence amounts to a section in $H^0(C, \mathcal{O}_C(\lfloor -D\rfloor))$. □

REMARKS 3.13. 1. Our proof shows that

(i) $A \cong A_0[D]$ *always admits a non-zero locally nilpotent derivation of fiber type*, and

(ii) *every homogeneous locally nilpotent derivation on $A \cong A_0[D]$ of negative degree has degree -1 and is of fiber type*.

(i) also follows from Lemma 1.6, since for a parabolic $\mathbb{C}^*$-surface $V = \operatorname{Spec} A_0[D]$ the canonical projection $\pi: V \to C = \operatorname{Spec} A_0$ is an affine ruling.

We claim as well that

(iii) *The reduced fibers of the affine ruling $\pi: V \to C$ are all irreducible and isomorphic to $\mathbb{A}^1_{\mathbb{C}}$.*

To show (iii), with the same argument as in the proof of Proposition 3.8 (b) in [17] we can reduce to the case that $A_0 = \mathbb{C}[t]$ (i.e., $C = \mathbb{A}^1_{\mathbb{C}}$) and $D = -(e'/d)[0]$, where $0 \leq e' < d$ and $\gcd(e', d) = 1$ (see [17, Theorem 3.2 (b)]). In this case the reduced fiber of $\pi: V \to \mathbb{A}^1_{\mathbb{C}}$ over $0 \in \mathbb{A}^1_{\mathbb{C}}$ is isomorphic to $\operatorname{Spec}\mathbb{C}[v]$ with $v := t^{e'}u^d$. In fact, using the presentation of A as in (1) it is readily seen that the radical of $\sqrt{tA}$

is given by

$$\sqrt{tA} \cong \bigoplus_{b\geq 0, ad-be'>0} \mathbb{C}t^a u^b, \quad \text{and so} \quad A/\sqrt{tA} \cong \bigoplus_{b\geq 0, ad-be'=0} \mathbb{C}t^a u^b \cong \mathbb{C}[v].$$

2. The multiple fibers of $\pi: V \to C$ correspond to the points in $|\{D\}|$. More precisely, if $\{D\} = \sum_i (e_i/m_i)a_i$ with $a_i \in C$ and $\gcd(e_i, m_i) = 1$ then $\pi^*(a_i) = m_i\pi^{-1}(a_i)$ (see [17, Theorem 4.18]).
3. Let $W = \operatorname{Spec} B$ be any affine surface with a non-trivial $\mathbb{C}_+$-action. The coordinate ring B is filtered by the kernels $B_n := \ker \partial^n$, where $\partial \in \operatorname{Der} B$ is the corresponding locally nilpotent derivation. Consider the associated graded ring $A := \bigoplus_{i\geq 0} A_i$ with $A_i := B_{i+1}/B_i$ and the associated homogeneous locally nilpotent derivation $\partial' \in \operatorname{Der} A$ of degree -1. Then $\partial'|A_0 = 0$, and so the normalization of A is as in Theorem 3.12.

In the following example we exhibit a particular family of parabolic pairs of horizontal type, and then we show in Theorem 3.16 below that this family is actually exhaustive.

EXAMPLE 3.14. Given coprime integers $e \geq 0$ and $d > 0$ let e' be the unique integer with $0 \leq e' < d$ and $ee' \equiv 1 \mod d$; we note that by this condition $e' = 0$ and $d = 1$ if $e = 0$. Letting $A_0 = \mathbb{C}[t]$, we consider the A_0-algebra A given by the DPD construction as follows:

$$A := A_0[D] \subseteq \operatorname{Frac}(A_0)[u] \quad \text{with} \quad D = -\frac{e'}{d}[0] \in \operatorname{Div}\left(\mathbb{A}^1_{\mathbb{C}}\right).$$

Clearly $d = d(A)$ (see Lemma 3.5 in [17]). According to [17, Proposition 3.8] and Example 2.8 above we can represent A as the ring of invariants

$$A = A'^{\mathbb{Z}_d} \quad \text{with} \quad A' := \mathbb{C}[s, u'], \quad \deg s = 0, \quad \deg u' = 1,$$

where $s^d = t$, $u' = us^{e'}$, and where $\mathbb{Z}_d = \langle \zeta \rangle$ acts on A' via

$$\zeta.s = \zeta s, \quad \zeta.u' = \zeta^{e'} u'.$$

Thus as in Example 2.8 $V = \operatorname{Spec} A \cong V_{d,e'}$ is an affine toric surface, and because of $ee' \equiv 1 \mod d$ the derivation

$$\partial' := u'^e \frac{\partial}{\partial s} \in \operatorname{Der} A' \tag{4}$$

of degree e is locally nilpotent and commutes with the $\mathbb{Z}_d$-action. By Lemma 2.6 it restricts to a locally nilpotent derivation ∂ of A.

DEFINITION 3.15. We call the pair $P_{d,e} := (A, \partial)$ as above *the parabolic (d, e)-pair*.

Note that $P_{d,e}$ is of horizontal type. Moreover, two parabolic pairs $P_{d,e}$ and $P_{\tilde{d},\tilde{e}}$ are isomorphic if and only if $d = \tilde{d}$ and $e = \tilde{e}$ (cf. [17, Corollary 3.4]). In the next result we classify all parabolic pairs of horizontal type.

Theorem 3.16. *Every parabolic pair (A, ∂) of horizontal type is isomorphic to the parabolic (d, e)-pair $P_{d,e}$ with $e := \deg \partial$ and $d := d(A)$.*

Proof. We recall (see [17, Remark 2.5]) that for $e, e' > 0$ and $ee' \equiv 1 \mod d$, the $\mathbb{Z}_d$-actions $G'_{d,e}$ and $G_{d,e'}$ on $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[s, u']$ with

$$G'_{d,e}:\ \xi.(s, u') = (\xi^e s, \xi u') \quad \text{and} \quad G_{d,e'}:\ \zeta.(s, u') = (\zeta s, \zeta^{e'} u'),$$

where ξ, ζ are primitive d-th roots of unity with $\xi = \zeta^{e'}$, have the same orbits, hence also the same rings of invariants. Now Lemma 3.9 shows that (A, ∂) is isomorphic to $P_{d,e}$. This proves the result. □

EXAMPLE 3.17. If A is parabolic and admits a nonzero homogeneous locally nilpotent derivation ∂ of degree 0 then $A \cong \mathbb{C}[t, u]$ and $\partial = \partial/\partial t$. In fact, by the classification above (A, ∂) is the pair $P_{1,0}$ i.e., $e' = 0$, $d = 1$, $s = t$ and $u' = u$ in Example 3.14.

REMARKS 3.18. 1. We note that the derivation ∂ in Example 3.14 naturally extends to $\operatorname{Frac}(A_0)[u, u^{-1}]$ giving the derivation

$$\partial = d \cdot t^{k+1} u^e \frac{\partial}{\partial t} - e' \cdot t^k u^{e+1} \frac{\partial}{\partial u} = t^k u^e \left(d \cdot t \frac{\partial}{\partial t} - e' \cdot u \frac{\partial}{\partial u} \right), \tag{5}$$

where $ee' - 1 = kd$. Indeed, from $t = s^d$ and $u = u' s^{-e'}$ we obtain

$$\partial(t) = d \cdot s^{d-1} u'^e = d \cdot t^{k+1} u^e \quad \text{and} \quad \partial(u) = -e' \cdot s^{-e'-1} u'^{e+1} = -e' \cdot t^k u^{e+1}.$$

2. By virtue of Lemma 3.6, $\ker \partial = \mathbb{C}[v]$. Hence $v\colon V \to \mathbb{A}^1_{\mathbb{C}}$ is the orbit map of the $\mathbb{C}_+$-action $e^{t\partial}$ on V generated by ∂. As v is homogeneous of degree $d = d(A) > 0$, the $\mathbb{C}^*$-action on V acts non-trivially on this affine ruling and on its base. Therefore v can have at most one degenerate fiber $v^{-1}(0)$, which is the fixed point curve $C_+ \cong \mathbb{A}^1_{\mathbb{C}}$ of the $\mathbb{C}^*$-action. Moreover, $\operatorname{div}(v) = dC_+$ (see [17, Remark 3.7]).

Corollary 3.19. *A normal affine surface $V = \operatorname{Spec} A$, where $A = A_0[D]$, admits a horizontal $\mathbb{C}_+$-action if and only if $A_0 \cong \mathbb{C}[t]$ and the fractional part $\{D\}$ of the $\mathbb{Q}$-divisor D on $C \cong \mathbb{A}^1_{\mathbb{C}}$ is supported at one point or is zero.*

Proof. This follows immediately from Theorem 3.16; note that in the case $A_0 \cong \mathbb{C}[t]$ we have $A = A_0[D] \cong A_0[\{D\}]$, see [17, Corollary 3.4 and Proposition 3.8].

Let us provide for the ‘only if’-part an independent geometric argument. For this consider more generally a morphism $\pi\colon V \to C$ of a normal affine surface V onto a smooth affine curve C with only irreducible fibers. We claim that *if there exists an affine ruling $\upsilon\colon V \to \Gamma$ different from π then $C \cong \mathbb{A}^1_{\mathbb{C}}$ and π has at most one multiple fiber*. Clearly, this claim implies our assertion (see Remark 3.13.2). To show the claim, we let $G \cong \mathbb{A}^1_{\mathbb{C}}$ be a general fiber of υ, and we assume on the contrary that π has at least two fibers F_i of multiplicity $m_i \geq 2$, $i = 0, 1$. As $\pi|G\colon G \to C$ is dominant it follows that $C \cong \mathbb{A}^1_{\mathbb{C}}$, and so $\pi|G : G \to C$ can be viewed as a non-constant polynomial $\nu \in \mathbb{C}[t]$. We also may assume that $F_0 = \pi^{-1}(0)$ and $F_1 = \pi^{-1}(1)$. As G is a general fiber of υ it meets F_i at smooth points of V only, with the intersection multiplicities in $G \cdot \pi^*(i)$ being a multiple of m_i $(i = 0, 1)$. Thus m_0, m_1, divides the multiplicity of any root of the polynomial ν, $\nu - 1$, respectively. Hence $\nu = \lambda^{m_0} = \mu^{m_1} + 1$ for some non-constant polynomials $\lambda, \mu \in \mathbb{C}[t]$. The pair (λ, μ) defines a dominant map $\mathbb{A}^1_{\mathbb{C}} \to \Gamma_{m_0,m_1}$, where Γ_{m_0,m_1} is the smooth plane affine curve $x^{m_0} - y^{m_1} = 1$. But the existence of such a map contradicts the Riemann-Hurwitz formula, which proves our claim. □

3.4. Hyperbolic case. In this subsection we assume that A is hyperbolic, so that $A_\pm \neq 0$. If ∂ is a homogeneous locally nilpotent derivation on A of degree e with $e < 0$ then by reversing the grading of A we obtain a derivation of positive degree. Thus it is sufficient to classify the hyperbolic pairs (A, ∂) as in Definition 3.1.

Lemma 3.20. *If (A, ∂) is a hyperbolic pair then ∂ stabilizes $A_{\geq 0} \subseteq A$, and $(A_{\geq 0}, \partial)$ is a parabolic pair of horizontal type.*

Proof. It follows immediately from the definitions that $(A_{\geq 0}, \partial)$ is a parabolic pair. If it were of fiber type then the orbits of the corresponding $\mathbb{C}_+$-action on $V = \operatorname{Spec} A$ would be the fibers of $\pi\colon V \to C = \operatorname{Spec} A_0$. As the general fiber of π is $\mathbb{C}^*$, this leads to a contradiction. □

Thus by Theorem 3.16 $(A_{\geq 0}, \partial)$ is isomorphic to the (d, e)-pair $P_{d,e}$, where $e = \deg \partial$ and $d = d(A_{\geq 0}) = d(\partial)$ (see 3.4 and Lemma 3.9). In particular, $A_0 = \mathbb{C}[t]$ and $A_{\geq 0} = A_0[-(e'/d)[0]] \subseteq \operatorname{Frac}(A_0)[u]$, where 0 is the origin in $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} A_0$ (see Example 3.14). Moreover ∂ is given as in (4) or, alternatively, as in (5). The following lemma is crucial in our classification.

Lemma 3.21. *Let D_+, D_- be $\mathbb{Q}$-divisors on $C := \operatorname{Spec} A_0$ with $A_0 = \mathbb{C}[t]$ satisfying $D_+ + D_- \leq 0$, where $D_+ = -(e'/d)[0]$ with $0 \leq e' < d$ and $\gcd(e', d) = 1$. The derivation $\partial\colon A_0[D_+] \to A_0[D_+]$ of degree $e \geq 0$ as in* (5) *extends to*

$$A = A_0[D_+, D_-] \subseteq \operatorname{Frac}(A_0)\left[u, u^{-1}\right]$$

if and only if the following two conditions are satisfied.
(i) *If* $D_-(0) \neq e'/d$ *then* $(ee'-1)/d - eD_-(0) \geq 0$ *i.e.,* $-e(D_+(0)+D_-(0)) \geq 1/d$.
(ii) *If* $a \in \mathbb{A}^1_{\mathbb{C}}$ *with* $a \neq 0$ *and* $D_-(a) \neq 0$ *then* $-1 - eD_-(a) \geq 0$.

Proof. Note that ∂ extends in a unique way to a derivation of $\mathrm{Frac}(A_0)[u, u^{-1}]$ also denoted ∂. We must show that ∂ stabilizes A if and only if (i) and (ii) are satisfied.

Let us first treat the case $d = 1$ so that $e' = 0$ and $D_+ = 0$. Then (i) and (ii) can be reduced to the condition

$$-1 - eD_-(a) \geq 0 \quad \forall a \in \mathbb{A}^1_{\mathbb{C}}. \tag{6}$$

Moreover, $k = -1$ and so according to (5) $\partial = u^e \cdot \partial/\partial t$ acts on a homogeneous element $f(t)u^{-m} \in \mathbb{C}(t)u^{-m}$ by

$$\partial(f(t)u^{-m}) = f'(t)u^{e-m}. \tag{7}$$

Thus ∂ stabilizes A if and only if $f(t)u^{-m} \in A_{-m}$ $(m \geq 0)$ implies $f'(t)u^{e-m} \in A_{e-m}$ or, equivalently,

$$\operatorname{div} f + mD_- \geq 0 \Rightarrow \begin{cases} \operatorname{div} f' + (m-e)D_- \geq 0 & \text{if } m-e \geq 0 \\ \operatorname{div} f' + (e-m)D_+ \geq 0 & \text{if } m-e \leq 0. \end{cases} \tag{8}$$

If (6) is satisfied then for any $a \in \mathbb{A}^1_{\mathbb{C}}$

$$\operatorname{div}_a f' + (m-e)D_-(a) \geq \operatorname{div}_a f + mD_-(a) - 1 - eD_-(a) \geq \operatorname{div}_a f + mD_-(a),$$

where $\operatorname{div}_a(\,\cdot\,)$ denotes the order at a. Thus (8) is satisfied if $m - e \geq 0$, and since $D_+(a) = 0$ and $-D_-(a) \geq 0$, it also follows for $m - e \leq 0$.

Conversely, assume that ∂ stabilizes A. Consider $m > e$ such that the divisor mD_- is integral. For $a \in \mathbb{A}^1_{\mathbb{C}}$ with $D_-(a) \neq 0$ we let $s := -mD_-(a)$; thus $s > 0$. Consider a polynomial Q without zero at a such that

$$(t-a)^s Q u^{-m} \in A_{-m}.$$

By assumption $\partial((t-a)^s Q u^{-m}) = (s(t-a)^{s-1}Q + (t-a)^s Q')u^{e-m} \in A_{-m+e}$ and so

$$\operatorname{div}_a(s(t-a)^{s-1}Q + (t-a)^s Q') + (m-e)D_-(a) \geq 0.$$

The term on the left is equal to $s - 1$, hence we obtain

$$s - 1 + (m-e)D_-(a) = -1 - eD_-(a) \geq 0,$$

as required in (6).

In case $d \geq 2$ we consider the normalization A' of A in $\mathrm{Frac}(A)\big[\sqrt[d]{v}\big]$ as in Lemma 3.9, and we let $p\colon \mathbb{A}^1_{\mathbb{C}} \cong \operatorname{Spec} A'_0 \to \mathbb{A}^1_{\mathbb{C}} \cong \operatorname{Spec} A_0$, $s \longmapsto s^d$, be the covering induced by the inclusion $A_0 \subseteq A'_0$. By *loc. cit.* $A'_{\geq 0} \cong \mathbb{C}[s, u']$ with $s^d = t$ and $u'^d = v \in \ker \partial$, $\deg u' = 1$, and ∂ extends to the derivation $\partial' = u'^e \cdot \partial/\partial s$ on $A'_{\geq 0}$ and as well on $\mathrm{Frac}(A')$. If ∂ stabilizes A then ∂' stabilizes A' (see Lemma 1.8). Moreover, v can be written as $v = t^{e'} u^d$ (see the proof of Theorem 4.15 in [17]). So, by [17, Proposition 4.12],

$$A' \cong A'_0\left[D'_+, D'_-\right] \subseteq \mathrm{Frac}(A'_0)\left[u', u'^{-1}\right],$$

where $D'_+ = 0$ and $D'_- = p^*(D_+ + D_-)$. Using the first part of the proof we get that $D'_-(a') < 0$ implies $-1 - eD'_-(a') \geq 0$. If $a = p(a') \neq 0$ then $D_-(a) = D_-(a')$, hence (ii) follows. Similarly, if $p(a') = 0$ then $D'_-(a') = -e' + dD_-(0)$ and (i) follows.

Conversely, assume that (i) and (ii) are satisfied. Reversing the reasoning above we obtain that $-1 - D'_-(a') \geq 0$ if $D'_-(a') \neq 0$. Thus by the first part ∂' stabilizes A'. Taking invariants ∂ stabilizes $A = (A')^{\mathbb{Z}_d}$, as desired. ☐

Summarizing we state now our main classification result for hyperbolic pairs.

Theorem 3.22. *If (A, ∂) is a hyperbolic pair with $d := d(A_{\geq 0})$ and $e := \deg \partial \geq 0$, then $A_0 \cong \mathbb{C}[t]$ and $A \cong A_0[D_+, D_-]$ for two $\mathbb{Q}$-divisors D_+, D_- on $\mathbb{A}^1_{\mathbb{C}}$ with $D_+ + D_- \leq 0$, where the following conditions are satisfied*:

(i) $D_+ = -(e'/d)[0]$ *with* $0 \leq e' < d$ *and* $ee' \equiv 1 \mod d$.

(ii) *If* $D_+(a) + D_-(a) \neq 0$ *then* $-(D_+(a) + D_-(a))^{-1} \leq \begin{cases} de, & a = 0 \\ e, & a \neq 0 \end{cases}.$

(iii) ∂ *is defined by* (5) *in* Remark 3.18.

Conversely, given two $\mathbb{Q}$-divisors D_+ and D_- on $\mathbb{A}^1_{\mathbb{C}}$ with $D_+ + D_- \leq 0$ satisfying (i) *and* (ii) *there exists a unique, up to a constant, locally nilpotent derivation ∂ of degree e on $A = A_0[D_+, D_-]$, and this derivation is as in* (iii). *In particular, isomorphism classes of hyperbolic pairs are in* 1-1 *correspondence to pairs $(P_{d,e}, D_-)$, where D_- is a $\mathbb{Q}$-divisor on $\mathbb{A}^1_{\mathbb{C}}$ satisfying* (ii).

Proof. By Theorem 3.16, $(A_{\geq 0}, \partial)$ is isomorphic to the parabolic pair $P_{d,e}$. In particular, (i) and (iii) are satisfied. By Lemma 3.21 also (ii) holds, proving the theorem. ☐

Corollary 3.23. *A two-dimensional normal graded $\mathbb{C}$-algebra $A = \bigoplus_{m \in \mathbb{Z}} A_m$ with $A_\pm \neq 0$ admits a homogeneous locally nilpotent derivation ∂ of positive degree if and only if $A_0 \cong \mathbb{C}[t]$ and $A \cong A_0[D_+, D_-]$, where the fractional part $\{D_+\}$ is supported at one point or is zero.*

In order to study more closely the structure of the affine ruling which corresponds to the $\mathbb{C}_+$-action with generator ∂ as above, we need a simple lemma. We let $A = A_0[D_+, D_-] \subseteq \operatorname{Frac}(A_0)[u, u^{-1}]$ be a normal graded $\mathbb{C}$-algebra, and we consider the associated $\mathbb{C}^*$-fibration $\pi\colon V = \operatorname{Spec} A \to C := \operatorname{Spec} A_0$ over the curve C. It was shown in [17, Theorem 4.18] that the fiber over a point $a \in C$ with $D_+(a) + D_-(a) < 0$ consists of two $\mathbb{C}^*$-orbit closures $\bar{O}_a^{\pm}$. Moreover, if $D_+(a) = -(e_+/m_+)$, $D_-(a) = e_-/m_-$, where $m_+ > 0$, $m_- < 0$ and $\gcd(e_\pm, m_\pm) = 1$, then

$$(9)\qquad \pi^*(a) = m_+[\bar{O}_a^+] - m_-[\bar{O}_a^-] \quad \text{and} \quad \operatorname{div} u = -e_+[\bar{O}_a^+] + e_-[\bar{O}_a^-] + \cdots,$$

where the terms in dots correspond to points in $|D_+| \cup |D_-|$ different from a. Letting $v_\pm \in A_{m_\pm}$ be an element with $A_{m_\pm} = v_\pm A_0$ near a, we have the following observation.

Lemma 3.24. (a) *The orbit closures* $\bar{O}_a^{\pm} \cong \operatorname{Spec} \mathbb{C}[v_\pm]$ *are smooth affine lines.*
(b) $\operatorname{div}(v_+) = \Delta(a)[\bar{O}_a^-]$ *and* $\operatorname{div}(v_-) = \Delta(a)[\bar{O}_a^+]$, *where* $\Delta(a) := m_+e_- - m_-e_+$.

Proof. With the same argument as in the proof of Proposition 3.8 (b) in [17] we can reduce to the case where $A_0 = \mathbb{C}[t]$ and $|D_+| \cup |D_-|$ is the point $a = 0 \in C = \mathbb{A}^1_{\mathbb{C}}$. We may also suppose that $D_+(0) + D_-(0) < 0$. Recall (see the proof of Theorem 4.15 in [17]) that $v_\pm = t^{e_\pm} u^{m_\pm}$ up to a constant in $\mathbb{C}^*$.

(a) The ideal of $\bar{O}_a^+$ coincides with the radical $\sqrt{v_- A}$, see the proof of Theorem 4.18 in [17]. Thus it suffices to show that

$$\sqrt{v_- A} = A_- \oplus tA \oplus \bigoplus_{m_+ \nmid \beta} A_\beta.$$

As $v_+ \notin \sqrt{v_- A}$ we have the inclusion '$\subseteq$'. To deduce '$\supseteq$' we note first that $A_- \oplus tA \subseteq \sqrt{v_- A}$. Suppose that $t^\alpha u^\beta \in A$, where $\beta > 0$ and $m_+ \nmid \beta$, and let us show that $t^\alpha u^\beta \in \sqrt{v_- A}$. For this we need to prove that $v_- = t^{e_-} u^{m_-}$ divides $t^{n\alpha} u^{n\beta}$ in A for $n \gg 0$ or, equivalently, that $t^{n\alpha - e_-} u^{n\beta - m_-} \in A_{n\beta - m_-}$. This amounts to

$$(10)\qquad (n\alpha - e_-) + (n\beta - m_-)D_+(0) \geq 0 \iff n(\alpha + \beta D_+(0)) \geq e_- + m_- D_+(0).$$

Because of our assumptions $t^\alpha u^\beta \in A$ and $m_+ \nmid \beta$ we have $\alpha + \beta D_+(0) \geq 0$ and $\beta D_+(0) \notin \mathbb{Z}$, so $\alpha + \beta D_+(0) > 0$. Hence (10) is satisfied for $n \gg 0$, as required.

(b) follows from (9) by virtue of the equalities

$$\operatorname{div} v_+ = e_+ \operatorname{div} t + m_+ \operatorname{div} u \quad \text{and} \quad \operatorname{div} v_- = e_- \operatorname{div} t + m_- \operatorname{div} u. \qquad \square$$

We consider below a hyperbolic pair (A, ∂) as in Theorem 3.22, and we let $v \in A_d$ be a generator of A_d over $A_0 = \mathbb{C}[t]$ (cf. Lemma 3.6). Then $v\colon V = \operatorname{Spec} A \to \Gamma = \mathbb{A}^1_{\mathbb{C}}$ provides an affine ruling which is the quotient map of the $\mathbb{C}_+$-action on V induced by ∂. In the next proposition we describe the multiplicities which occur in the degenerate fibers of this affine ruling (cf. Remark 3.18.2).

Proposition 3.25. *The fiber of the affine ruling $v\colon V \to \Gamma = \mathbb{A}^1_{\mathbb{C}}$ over a point $p \neq 0$ is smooth, reduced and consists of just one $\mathbb{C}_+$-orbit, whereas the fiber over $p = 0$ is a disjoint union of $\mathbb{C}^*$-orbit closures isomorphic to affine lines, one for each point $a \in C$ with $D_+(a) + D_-(a) < 0$. Moreover*

$$\operatorname{div}(v) = d_+ \sum_{a\in\mathbb{A}^1_{\mathbb{C}}} m_-(a)\,(D_+(a) + D_-(a))\,\left[\bar{O}^-_a\right], \tag{11}$$

where the integer $m_-(a) < 0$ is defined by $D_-(a) = e_-(a)/m_-(a)$ with $\gcd(e_-(a), m_-(a)) = 1$.

Proof. As v is homogeneous of degree $d = d_+ := d(A_{\geq 0})$ the affine ruling $v\colon V \to \mathbb{A}^1_{\mathbb{C}}$ is equivariant if we equip $\mathbb{A}^1_{\mathbb{C}}$ with the $\mathbb{C}^*$-action $\lambda.t = \lambda^d t$. This implies that for every point $p \neq 0$, the fiber of v over p is smooth, reduced and consists of just one $\mathbb{C}_+$-orbit. By the previous lemma, $\operatorname{div}(v)$ is a linear combination of the divisors $\bar{O}^-_a$, where a runs through all points of $C = \mathbb{A}^1_{\mathbb{C}}$ with $D_+(a) + D_-(a) < 0$. We compute the multiplicities separately in the cases where $a = 0$ and $a \neq 0$.

If $a = 0$ then $D_+(0) = -(e_+/d_+)$ with $e_+ = e'$ and $D_-(0) = e_-/m_-$ with $e_- = e_-(0)$, $m_- = m_-(0)$, so by Lemma 3.24 the coefficient of $\bar{O}^-_0$ in $\operatorname{div}(v)$ is $\Delta(0) = -e_+m_- + e_-m_+ = d_+m_-(D_+(0) + D_-(0))$, which agrees with (11).

If $a \neq 0$ then $m_+(a) = 1$ and so $\Delta(a) = m_-(a)(D_+(a) + D_-(a))$. Letting $v_* \in A_1$ be an element generating A_1 over A_0 near a, we can write $v = \varepsilon v_*^{d_+}$, where $\varepsilon \in A_0$ is a unit near a i.e., $\varepsilon(a) \neq 0$. By Lemma 3.24 $\bar{O}^-_a$ occurs with multiplicity $\Delta(a)$ in $\operatorname{div}(v_*)$, and so it occurs with multiplicity $d_+\Delta(a) = d_+m_-(a)(D_+(a)+D_-(a))$ in $\operatorname{div}(v)$, as required in (11). □

REMARK 3.26. We note that $\operatorname{div}(v)$ is the exceptional divisor of the birational morphism $\sigma_+\colon V \to V_+ = \operatorname{Spec} A_{\geq 0}$ induced by the inclusion $A_{\geq 0} \hookrightarrow A$. Indeed, the divisor $\operatorname{div}(v) = d_+C_+$ on V_+ is supported by the fixed point curve $C_+ \cong \mathbb{A}^1_{\mathbb{C}}$ of the $\mathbb{C}^*$-action on V_+ (see Remark 3.18). For every point $a \in C = \mathbb{A}^1_{\mathbb{C}}$ with $D_+(a)+D_-(a) < 0$ there is a unique point a' over a on C_+, and σ_+ is the affine modification consisting in an equivariant blowing up of V_+ with center supported at all those points $a' \in C_+$ and deleting the proper transform of the divisor C_+ (see [17, Remark 4.20]).

If v is a unit in A then $D_+ + D_- = 0$, $v\colon V \to \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ is the quotient map, and all fibers of v are smooth affine lines. More precisely the following result holds.

Corollary 3.27. *Let (A, ∂) be a hyperbolic pair and $d := d(A_{\geq 0})$. If one of the following two conditions is satisfied*:

(i) $e := \deg \partial = 0$, *or*

(ii) *A contains a unit of non-zero degree,* *then*

$$A \cong \mathbb{C}\left[z, v, v^{-1}\right] \quad (\Rightarrow V \cong \mathbb{A}^1_{\mathbb{C}} \times \mathbb{C}^*) \quad \textit{and} \quad \partial = \frac{\partial}{\partial z},$$

where $\deg z = -e$ *and* $\deg v = d$.

Proof. In case (i) Theorem 3.22 (i) shows that $d = 1$ and $e' = 0$, so $D_+ = 0$, and moreover by 3.22 (ii) $D_+(a)+D_-(a) = 0$ for all closed points $a \in \mathbb{A}^1_{\mathbb{C}}$. Thus $D_+ = D_- = 0$ and $A = A_0[u, u^{-1}]$ for some element $u \in A_1$. By 3.22 (iii) and Remark 3.18.1 ∂ is the derivation $\partial = \partial/\partial t$, which proves the result.

In case (ii), by [17, Remark 4.5], $D_+ = -D_-$, and by Theorem 3.22, $D_+ = -(e'/d)[0]$. Therefore, A is the semigroup algebra generated over $\mathbb{C}$ by all monomials $t^a u^b$ with $ad - be' \geq 0$, $a, b \in \mathbb{Z}$ (cf. the proof of Theorem 4.15 in [17]). Choose $q \in \mathbb{Z}$ with $\left|\begin{smallmatrix} e & q \\ d & e' \end{smallmatrix}\right| = 1$ and consider the elements

$$v := t^{e'} u^d, \quad v^{-1} \quad \text{and} \quad z := t^{-q} u^{-e} \in A \quad \text{with} \quad \deg v = d, \ \deg z = -e,$$

so that $u = v^{-q} z^{-e'}$ and $t = v^e z^d$. As we have noticed above, a monomial $t^a u^b = v^{ae-bq} z^{ad-be'}$ belongs to A if and only if $ad - be' \geq 0$. Thus $A = \mathbb{C}[v, v^{-1}, z]$. The orbits of the $\mathbb{C}_+$-action on $\operatorname{Spec} \mathbb{C}[v, v^{-1}, z] = \mathbb{A}^1_{\mathbb{C}} \times \mathbb{C}^*$ given by ∂ are necessarily contained in the fibers of the projection to $\mathbb{C}^*$, and $\ker \partial = \mathbb{C}[v, v^{-1}]$ (cf. Lemma 3.6). Since ∂ is homogeneous of degree e, we get $\partial = c v^a z^b \cdot \partial/\partial z$ for suitable $c \in \mathbb{C}^*$, $a \in \mathbb{Z}$, $b \in \mathbb{N}$ with $ad - be = 0$. As ∂ is also locally nilpotent this forces $a = b = 0$ and so $\partial = c \cdot \partial/\partial z$. Replacing z by z/c, the result follows. □

Next we describe explicit equations for hyperbolic pairs in the case that $A = A_0[D_+, D_-]$ with $D_+ = 0$. Similarly as in 3.4 we let $k = d(A_{\leq 0})$ be the minimal positive integer such that $A_{-k-n} = A_{-k} A_{-n}$ for every $n \geq 0$.

Corollary 3.28. *Let* (A, ∂) *be a hyperbolic pair, and suppose that* $A = A_0[D_+, D_-]$ *with* $D_+ = 0$, *so that* $A_{\geq 0} \cong \mathbb{C}[t, u]$ *with* $\deg u = 1$ *and* $\deg t = 0$. *If* $k := d(A_{\leq 0})$ *and* $e := \deg \partial \geq 0$ *then* A *is the normalization of the graded domain*

$$B = B_{k,P} := \mathbb{C}[t, u, v] \big/ \left(u^k v - P(t)\right) \quad \textit{with} \quad \deg v = -k,$$

where

$$P(t) = \prod_{i=1}^{s} (t - a_i)^{r_i} \in \mathbb{C}[t] \quad (r_i \geq 1 \quad \textit{and} \quad a_i \neq a_j \quad \textit{for} \quad i \neq j) \tag{12}$$

is a unitary polynomial uniquely determined by $D_- = -\operatorname{div} P/k$ *and satisfying*

$$\gcd(k, r_1, \ldots, r_s) = 1 \quad \textit{and} \quad e \geq \frac{k}{r_i} \tag{13}$$

for $i = 1, \dots, s$. The derivation ∂ is given (and uniquely determined) by the conditions

$$\partial(u) = 0, \quad \partial(t) = u^e \quad (\Rightarrow \partial(v) = P'(t)u^{e-k}). \tag{14}$$

Conversely, given a polynomial P as in (12) *and* (13) *there is up to a constant a unique locally nilpotent derivation ∂ of degree e of the normalization A of $B_{k,P}$ satisfying* (14).

Proof. As was shown in [17, Example 4.10 and Proposition 4.11], A is the normalization of the algebra $B_{k,P}$, where P is a unitary polynomial uniquely determined by $D_- = -\operatorname{div}(P)/k$. Since k is minimal with kD_- integral, we have $\gcd(k, r_1, \dots, r_n) = 1$. By Theorem 3.22 (ii), (iii) it follows that $e \geq k/r_i$ and that ∂ has the stated form (14). Conversely, given P the normalization A of $B_{k,P}$ is isomorphic to $A_0[D_+, D_-]$ with $D_+ = 0$ and $D_- = -\operatorname{div}(P)/k$. If $e \geq k/r_i$ for all i then the conditions (i), (ii) in Theorem 3.22 are fulfilled for A, so there is a locally nilpotent derivation ∂ of A satisfying (14), and ∂ is uniquely determined up to a constant factor. □

REMARKS 3.29. 1. Over each of the points $t = a_i \in \mathbb{A}^1_{\mathbb{C}}$, the surface $V = \operatorname{Spec} A$ considered in Corollary 3.28 has a unique fixed point a_i' of the $\mathbb{C}^*$-action. This point $a_i' \in V$ is a quotient singularity of type (d_i, e_i), where $r_i/k = d_i/e_i'$ with d_i, e_i' coprime and $0 \leq e_i < d_i$, $e_i \equiv e_i' \mod d_i$. This follows from Theorem 4.15 in [17], since $D_+(a_i) = 0$ and $D_-(a_i) = r_i/k$. In particular, the surface V is smooth if and only if $r_i|k$ for all i (cf. Corollary 4.16 in [17]).
2. A description of the automorphism group $\operatorname{Aut} V_{k,P}$ for a smooth surface $V_{k,P} := \operatorname{Spec} B_{k,P}$, where $B_{k,P}$ is as in Corollary 3.28, can be found in [6, (2.3)–(2.4)] and [27, Theorem 1].
3. For any $e \geq k$ the derivation ∂ described in Corollary 3.28 stabilizes the ring B and induces a $\mathbb{C}_+$-action (actually, a G_e-action, see Lemma 2.2) on $\mathbb{A}^3_{\mathbb{C}}$ which leaves the surface $V_{k,P} = \operatorname{Spec} B \subseteq \mathbb{A}^3_{\mathbb{C}}$ invariant. In case $e < k$, however, ∂ does not induce a derivation on B. The simplest example of such a surface $V_{k,P}$ is with $P = t^3$ and $k = 2$, $e = 1$. Here the element $\partial(v) = 3t^2u^{-1}$ is not in B but is integral over B as its square is equal to $9tv \in B$.
4. The $\mathbb{C}_+$-action associated to the derivation ∂ in Corollary 3.28 is

$$\alpha.(t, u, v) = (t + \alpha u^e, u, u^{-k}P(t + \alpha u^e)), \quad \alpha \in \mathbb{C}_+, \tag{15}$$

with fixed point set $\{u = 0\}$. Again, for $e \geq k$ this $\mathbb{C}_+$-action extends to $\mathbb{A}^3_{\mathbb{C}}$.

In the case $D_+ = -(e'/d)[0] \neq 0$ a suitable cyclic covering of $V = \operatorname{Spec} A$ can be described as in Corollary 3.28. This leads to the following alternative description of arbitrary hyperbolic pairs (A, ∂).

Corollary 3.30. *We let (A,∂) be a hyperbolic pair with invariants $d := d(A_{\geq 0})$, $k := d(A_{\leq 0})$, $e := \deg\partial > 0$. If $A \cong A_0[D_+, D_-]$, where $D_+ = -(e'/d)[0]$ and $D_-(0) = -(l/k)$, then there exists a unitary polynomial $Q \in \mathbb{C}[t]$ with $Q(0) \neq 0$ and $\operatorname{div}(Qt^l) = -kD_-$. Moreover if $A' = A_{k,P}$ is the normalization of*

$$B_{k,P} = \mathbb{C}[s,u,v]\big/\big(u^k v - P(s)\big), \quad \text{where} \quad P(s) := Q(s^d)s^{ke'+dl}, \tag{16}$$

then the group $\mathbb{Z}_d = \langle\zeta\rangle$ acts on $B_{k,P}$ and also on A' via

$$\zeta.s = \zeta s, \quad \zeta.u = \zeta^{e'}u \quad \text{and} \quad \zeta.v = v \tag{17}$$

so that $A \cong A'^{\mathbb{Z}_d}$. Furthermore, $ee' \equiv 1 \mod d$ and, up to a constant factor, ∂ is the restriction of the derivation $u^e \cdot \partial/\partial s$ to A.

Proof. The inequality $D_+(0)+D_-(0) \leq 0$ is equivalent to $ke'+dl \geq 0$. This implies that there are unitary polynomials $Q(t) \in \mathbb{C}[t]$ and $P(s) \in \mathbb{C}[s]$ such that $\operatorname{div}(Qt^l) = -kD_-$ and $P(s) = Q(s^d)s^{ke'+dl}$.

The isomorphism $A \cong A'^{\mathbb{Z}_d}$ was established in Example 4.13 and Proposition 4.14 of [17]. The derivation $u^e \cdot \partial/\partial s$ commutes with the $\mathbb{Z}_d$-action (17), and so restricts to a homogeneous locally nilpotent derivation ∂' of degree e on A, iff $ee' \equiv 1 \mod d$ (see Theorem 3.22 (i)). Thus by Corollary 3.28 it is equal to ∂ up to a constant. The rest of the proof can be left to the reader. □

4. Applications

4.1. Preliminaries. Sometimes the surfaces $V = \operatorname{Spec} A$ as above admit two $\mathbb{C}_+$-actions with different orbit maps; see e.g. Example 2.8. The following example is also well known.

EXAMPLE 4.1. We let A be the normalization of the ring $B_{1,P} = \mathbb{C}[t,u_+,u_-]/(u_+u_- - P(t))$, where $P \in \mathbb{C}[t]$ is a unitary polynomial and the grading is given by $\deg t = 0$, $\deg u_\pm = \pm 1$. By Corollary 3.28, for every $e \geq 1$ there are homogeneous locally nilpotent derivations of degree e as well as of degree $-e$ on A. More explicitly these are given (up to a constant factor) by

$$\partial_+ = u_+^e\frac{\partial}{\partial t} + P'(t)u_+^{e-1}\frac{\partial}{\partial u_-} \quad \text{and} \quad \partial_- = u_-^e\frac{\partial}{\partial t} + P'(t)u_-^{e-1}\frac{\partial}{\partial u_+}; \tag{18}$$

cf. (14). Note that $\ker(\partial_\pm) = \mathbb{C}[u_\pm]$, hence the corresponding $\mathbb{C}_+$-actions φ_+ and φ_- preserve the affine rulings $u_\pm \colon V \to \mathbb{C}$ of $V = \operatorname{Spec} A$, respectively. These rulings are different provided that P is a non-constant polynomial.

In view of (15) φ_+ is given by

$$\alpha.(t,u_+,u_-) = \left(t+\alpha u_+^e, u_+, u_+^{-1}P\left(t+\alpha u_+^e\right)\right), \quad \alpha \in \mathbb{C}_+.$$

As $\ker(\partial_-) = \mathbb{C}[u_-]$ the conjugated locally nilpotent derivation

$$\partial_\alpha := \alpha.\partial_-.\alpha^{-1} \in \operatorname{Der} A$$

has kernel $\ker(\partial_\alpha) = \mathbb{C}[u_\alpha]$, where

$$u_\alpha := \alpha(u_-) = u_+^{-1} P(t + \alpha u_+^e) = u_- + \sum_{j=1}^{\deg P} P^{(j)}(t) \frac{\alpha^j}{j!} u_+^{je-1}.$$

As $\alpha \in \mathbb{C}_+$ varies, the affine rulings $u_\alpha \colon V \to \mathbb{A}^1_{\mathbb{C}}$ also vary in a continuous family.

DEFINITION 4.2. One says that two $\mathbb{C}_+$-actions on an affine variety $V = \operatorname{Spec} A$ are *equivalent* if their general orbits are the same, or in other words, if they define the same affine ruling on V.

If ∂ and $\partial' \in \operatorname{Der} A$ are the associated locally nilpotent derivations then the $\mathbb{C}_+$-actions are equivalent if and only if $\ker \partial = \ker \partial'$, and if and only if $a\partial = a'\partial'$ for some elements $a, a' \in \ker \partial$ (see [24, Lemma 2.1] or Proposition 1.1 (b)). Consequently, any two equivalent locally nilpotent derivations ∂ and ∂' commute: $[\partial, \partial'] = 0$.

We recall [24, 36] that the *Makar-Limanov invariant* of an affine variety $V = \operatorname{Spec} A$ is $\operatorname{ML}(V) = \operatorname{ML}(A) = \bigcap \ker \partial$, where ∂ runs over the set of all locally nilpotent derivations of A.

Certainly, a surface V has a trivial Makar-Limanov invariant $\operatorname{ML}(V) = \mathbb{C}$ if and only if V admits two non-equivalent $\mathbb{C}_+$-actions, or two different affine rulings over affine bases, or else two non-equivalent nonzero locally nilpotent derivations of A.

A useful characterization of surfaces with a trivial Makar-Limanov invariant is the following result due to Gizatullin [20, Theorems 2 and 3], Bertin [7, Theorem 1.8], Bandman and Makar-Limanov [5] in the smooth case, and to Dubouloz [13] in the normal case.

Theorem 4.3. *For a normal affine surface V non-isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ and $\mathbb{C}^* \times \mathbb{A}^1_C$, the following conditions are equivalent.*

(i) *The Makar-Limanov invariant of V is trivial.*

(ii) *The automorphism group* $\operatorname{Aut} V$[6] *acts on V with an open orbit O such that the complement $V \backslash O$ is finite.*

(iii) *V admits a compactification by a* zigzag *that is, by a linear chain of smooth rational curves.*

Thus an affine ruling $V \to \mathbb{A}^1_{\mathbb{C}}$ on a normal affine surface V is unique (in other words, any two $\mathbb{C}_+$-actions on V are equivalent) unless V admits a smooth compact-

[6]Which is not necessarily an algebraic group, see Example 5.3 below.

ification by a zigzag. In the latter case there are, indeed, at least two different affine rulings $V \to \mathbb{A}^1_{\mathbb{C}}$, hence also two non-equivalent $\mathbb{C}_+$-actions on V.

Note that all surfaces as in Theorem 4.3 are rational and allow a constructive description, see [20, Proposition 3] or [13]. The automorphism group $\operatorname{Aut} V$ of such a surface V is infinite dimensional and admits an amalgamated free product structure [12].

4.2. $\mathbb{C}^*$-surfaces with trivial Makar-Limanov invariant. Some interesting classes of normal affine surfaces with a trivial Makar-Limanov invariant were discussed e.g., in [4, 5, 9, 10, 14, 15] and [28]. *If, for instance, such a surface V is smooth and its canonical bundle K_V is trivial* (*e.g., if V is a smooth complete intersection*) *then* $V \cong \operatorname{Spec} \mathbb{C}[t,u,v]/(uv - P(t))$ *for a polynomial* $P \in \mathbb{C}[t]$ *with simple roots* [5] (cf. Example 4.1). Here we concentrate on such surfaces which also admit a $\mathbb{C}^*$-action. From Theorems 3.3 and 3.16 we deduce:

Corollary 4.4. *A normal affine surface V with an elliptic or a parabolic $\mathbb{C}^*$-action has a trivial Makar-Limanov invariant if and only if $V \cong V_{d,e} \cong \mathbb{A}^2_{\mathbb{C}}/\mathbb{Z}_d$ is an affine toric surface as in* Example 2.8.

Actually V as in the corollary admits a parabolic $\mathbb{C}^*$-action, and so by Remark 3.13.1 (i) it has a $\mathbb{C}_+$-action of fiber type and also a $\mathbb{C}_+$-action of horizontal type (see Examples 2.8 and 3.14).

The following theorem together with Corollary 4.4 describes all normal affine $\mathbb{C}^*$-surfaces with a trivial Makar-Limanov invariant.

Theorem 4.5. *We let $A = A_0[D_+, D_-]$, where $A_0 = \mathbb{C}[t]$ and D_+, D_- are $\mathbb{Q}$-divisors on $\mathbb{A}^1_{\mathbb{C}}$ with $D_+ + D_- \le 0$. The following conditions are equivalent.*

(i) *The Makar-Limanov invariant of V is trivial.*

(ii) *A admits two homogeneous locally nilpotent derivations ∂_+, ∂_- of positive and negative degree, respectively, such that the orbits of the corresponding $\mathbb{C}_+$-actions are generically different.*

(iii) *There are* (*not necessarily distinct*) *points $p_+, p_- \in \mathbb{A}^1_{\mathbb{C}}$ such that the fractional part $\{D_\pm\}$ of $D_\pm$ is zero or is supported in $p_\pm$, and $D_+ + D_- \ne 0$.*

Proof. The implication (ii) $\Rightarrow$ (i) is evident. For the proof of the converse, assuming (i) there exist two non-equivalent locally nilpotent derivations on A, which means that they have different kernels. By Lemma 3.6 not both of them can be linear combinations of derivations of positive degrees, and similarly not both of them can have homogeneous components of only negative degree. Thus there are also homogeneous locally nilpotent derivations on A of positive and of negative degree. To show that the corresponding $\mathbb{C}_+$-actions are not equivalent, we let v_+ and v_- be generators of the A_0-modules A_{d_+} and A_{-d_-}, respectively, where $d_+ := d(A_{\geq 0})$ and $d_- := d(A_{\leq 0})$. By Lemma 3.6 $\ker \partial_\pm = \mathbb{C}[v_\pm, v_\pm^{-1}] \cap A$. Thus, if ∂_+ and ∂_- were equivalent then $v_\pm$ would be units and so by Corollary 3.27 we would have $A \cong \mathbb{C}[z, v_+, v_+^{-1}]$. As the latter ring does not admit two non-equivalent $\mathbb{C}_+$-actions, (ii) follows.

(iii) $\Rightarrow$ (ii). Assuming (iii) Corollary 3.23 shows that there are homogeneous derivation ∂_+ and ∂_- of positive and negative degree, respectively. By our assumption $D_+ + D_- \neq 0$, hence $A^\times = \mathbb{C}$ and so, the elements v_+ and v_- are not units (see [17, Remark 4.5]). Thus with the same arguments as above the derivations ∂_+ and ∂_- are not equivalent.

(ii) $\Rightarrow$ (iii). Conversely, if (ii) holds then by Corollary 3.23 the first two conditions in (iii) are satisfied. With the same arguments as above A cannot contain a non-constant unit, hence again by [17, Remark 4.5] we have $D_+ + D_- \neq 0$. □

REMARK 4.6. For explicit equations of $\mathbb{C}^*$-surfaces with a trivial Makar-Limanov invariant we refer the reader to Proposition 4.8 in [17], where for $\{-D_\pm\} = e'_\pm/d_\pm[p_\pm]$ one must let $P_\pm := (t - p_\pm)^{d'_\pm - e'_\pm}$ with $d'_\pm := d_\pm/k$ and $k := \gcd(d_+, d_-)$.

We note that the two locally nilpotent derivations as in Theorem 4.5 (ii) do not commute except in the case $V \cong \mathbb{A}^2_{\mathbb{C}}$. This is a consequence of the next result. Although it follows immediately from Lemma 2.7 (a), we provide a direct argument.

Corollary 4.7. *If a normal affine variety $V = \operatorname{Spec} A$ of dimension n admits an effective $\mathbb{C}_+^n$-action, then $V \cong \mathbb{A}^n_{\mathbb{C}}$.*

Proof. Let $\mathbb{A}^n_{\mathbb{C}} \cong \mathbb{C}^n_+.p \hookrightarrow V$ be an open orbit and consider the associated inclusion of $\mathbb{C}$-algebras $A \hookrightarrow B := \mathbb{C}[X_1, \ldots, X_n]$. The derivations $\partial_i := \partial/\partial X_i$ on B stabilize A and the restrictions $\partial_i|A$ are the infinitesimal generators of the actions of the factors of $\mathbb{C}^n_+$ on A. By Proposition 1.1 (b), for every $1 \leq i \leq n$ the intersection

$$K_i := A \cap \bigcap_{j \neq i} \ker \partial_j = A \cap \mathbb{C}[X_i]$$

has transcendence degree 1, hence $K_i \neq \mathbb{C}$. As ∂_i acts on K_i and decreases the degree of polynomials in K_i by 1, $K_i \subseteq A$ must contain a linear polynomial $a_i X_i + b_i$ and hence also X_i. It follows that $A = B$, as required. □

For a normal affine surface $V = \operatorname{Spec} A$ with two different affine rulings $v_+, v_- \colon V \to \mathbb{A}^1_{\mathbb{C}}$, Miyanishi and Masuda [28] introduced a useful invariant $\iota(v_+, v_-) \in \mathbb{N}$, called the *intertwining number* of v_+ and v_-, which is the intersection number of two general fibers of v_+ and v_-, respectively. Actually $\iota(v_+, v_-) = \operatorname{trdeg}(\operatorname{Frac} A : \mathbb{C}(v_+, v_-))$.

DEFINITION 4.8. Let us call the *Miyanishi-Masuda invariant* of V the integer

$$\mathrm{MM}(V) := \min_{(v_+, v_-)} \iota(v_+, v_-),$$

where the minimum is taken over all possible choices of pairs (v_+, v_-) as above. In case that V is endowed with an effective $\mathbb{C}^*$-action, we also consider the homogeneous version

$$\mathrm{MM}_h(V) := \min_{(v_+, v_-)} \iota(v_+, v_-),$$

under the additional assumption that v_+ and $v_- \in A$ as above are homogeneous.[7]

We let as before $d_+ := d(A_{\geq 0})$ and $d_- := d(A_{\leq 0})$. We recall [17, Lemma 3.5] that $d(A_0[D])$ is equal to the minimal integer $d \geq 1$ such that the divisor dD is integral.

Lemma 4.9. *For a normal affine $\mathbb{C}^*$-surface $V = \operatorname{Spec} A$ with a trivial Makar-Limanov invariant the following hold.*
(a) *If $A = A_0[D]$ then $\mathrm{MM}_h(V) = d(A)$.*
(b) *If $A = A_0[D_+, D_-]$ then $\mathrm{MM}_h(V) = -d_+d_- \deg(D_+ + D_-)$.*
(c) *If $\mathrm{MM}_h(V) = 1$ then $V \cong \mathbb{A}^2_{\mathbb{C}}$.*

Proof. (a) In this case the grading on A is parabolic, so V is a toric surface $V_{d,e}$, where $d = d(A)$, and the two $\mathbb{C}^*$-equivariant affine rulings on V are provided by elements $t \in A_0 = \mathbb{C}[t]$ and $v \in A_d = vA_0$ (see Corollary 4.4). Since the restriction of v onto a general fiber of t has degree d, the result follows.

(b) In this case the grading on A is hyperbolic, and so the two $\mathbb{C}^*$-equivariant affine rulings on V are provided by elements $v_\pm \in A_{\pm d_\pm}$ with $A_{\pm d_\pm} = v_\pm A_0$ (see the Proof of Theorem 4.5). By Proposition 4.8 in [17], V is a cyclic branch covering of degree $k := \gcd(d_+, d_-)$ of the normalization of the hypersurface $\left\{v_+^{d'_-} v_-^{d'_+} - P(t) = 0\right\}$ in $\mathbb{A}^3_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[t, v_+, v_-]$, where $d'_\pm := d_\pm/k$. Hence $\mathrm{MM}_h(V) = k \deg P(t)$. By Lemma 4.7 in *loc. cit.* we have

$$D_+ = D_0 + \{D_+\} \quad \text{and} \quad D_- = \{D_-\} - D_0 - \operatorname{div} Q,$$

where $Q \in \mathbb{C}[t]$. From (8) and (10) in *loc. cit.* we obtain

$$\operatorname{div} P = kd'_+d'_- \operatorname{div} Q - d'_- \operatorname{div} \ P_+ - d'_+ \operatorname{div} P_-,$$

where $\operatorname{div} \ P_\pm = d_\pm \operatorname{div}\{D_\pm\}$. Therefore

$$\operatorname{div} \ P = kd'_+d'_- \operatorname{div}(Q - \{D_+\} - \{D_-\}) = -kd'_+d'_-(D_+ + D_-). \tag{19}$$

[7] Clearly $\mathrm{MM}_h \geq \mathrm{MM}(V)$, where presumably the equality holds.

Now

$$\mathrm{MM}_h(V) = \iota(v_+, v_-) = k \deg P(t) = -k^2 d'_+ d'_- \deg(D_+ + D_-) = -d_+ d_- \deg(D_+ + D_-),$$

as stated.

(c) The equalities $\mathrm{MM}_h(V) = k \deg P(t) = 1$ imply that $k = 1$ and $\deg P(t) = 1$. Now the assertion easily follows. □

4.3. Families of $\mathbb{C}_+$-actions on a $\mathbb{C}^*$-surface. We show in Corollary 4.11 below that any $\mathbb{C}^*$-surface with a trivial Makar-Limanov invariant admits a continuous family of generically non-equivalent locally nilpotent derivations (cf. Proposition 2.4). This is based on the following general observation.

Proposition 4.10. *If a domain A of finite type admits two non-commuting locally nilpotent derivations $\partial, \delta \in \operatorname{Der} A$, then A also admits a continuous family of generically non-equivalent locally nilpotent derivations $\{\partial_t\}_{t\in\mathbb{C}^*} \subseteq \operatorname{Der} A$.*

Proof. Letting $\varphi_t = \exp(t\partial)$, $\psi_t = \exp(t\delta)$ be the associated $\mathbb{C}_+$-actions on A, we consider the following two families of conjugated locally nilpotent derivations on A:

$$\partial_t := \psi_t \circ \partial \circ \psi_t^{-1} \quad \text{and} \quad \delta_t := \varphi_t \circ \delta \circ \varphi_t^{-1}.$$

Suppose in contrary that none of these has the desired property that is, the derivations in each family $\{\partial_t\}_{t\in\mathbb{A}^1_{\mathbb{C}}}$ and $\{\delta_t\}_{t\in\mathbb{A}^1_{\mathbb{C}}}$ are mutually equivalent. It follows that

$$\partial_t = f(t)\partial_0 = f(t)\partial \quad \text{and} \quad \delta_t = g(t)\delta \quad \forall t \in \mathbb{A}^1_{\mathbb{C}}, \tag{20}$$

where $f(t) \in \ker\partial$, $g(t) \in \ker\delta$ $\forall t \in \mathbb{A}^1_{\mathbb{C}}$ (see Definition 4.2 and Proposition 1.1 (b)) and $f(0) = g(0) = 1$. Moreover $f \in (\ker\partial)[t]$, since f is an everywhere defined ratio of two proportional regular vector fields ∂_t and ∂_0 on the affine scheme $(\operatorname{Spec} A)\times\mathbb{A}^1_{\mathbb{C}}$. Similarly, $g \in (\ker\delta)[t]$. In particular $\partial((f(t)-1)/t) = 0$, so taking the limit as $t \to 0$ gives $f'(0) \in \ker\partial$ and, similarly, $g'(0) \in \ker\delta$. From (20) we get:

$$\psi_t \circ \partial = f(t)\partial \circ \psi_t \Rightarrow \frac{\psi_t \circ \partial - \partial}{t} = \frac{f(t)\partial \circ \psi_t - \partial}{t}$$
$$\Rightarrow \frac{\psi_t - \mathrm{id}}{t} \circ \partial = \partial \circ \frac{f(t)\psi_t - \mathrm{id}}{t} = \partial \circ \left(f(t)\frac{\psi_t - \mathrm{id}}{t} + \frac{f(t)-1}{t}\,\mathrm{id}\right).$$

Taking the limit as $t \to 0$ we obtain

$$\delta \circ \partial = \partial \circ \delta + f'(0)\partial$$

and, similarly,

$$\partial \circ \delta = \delta \circ \partial + g'(0)\delta,$$

whence

$$[\partial, \delta] = g'(0)\delta = -f'(0)\partial.$$

As observed above, $g'(0) \in \ker \delta$ and $f'(0) \in \ker \partial$, thus ∂ and δ are equivalent and so commute, contradicting our assumption. $\square$

Corollary 4.11. *Any normal affine surface $V = \operatorname{Spec} A$ with a $\mathbb{C}^*$-action and a trivial Makar-Limanov invariant admits continuous families of $\mathbb{C}_+$-actions and of generically distinct affine rulings $V \to \mathbb{A}^1_{\mathbb{C}}$.*

4.4. Actions with a big orbit. As an application of our results we give below a new proof for the classification due to Gizatullin [19] and Popov [31], mentioned in the introduction. Let us recall it again.

Theorem 4.12. *Let a normal affine surface V admits an action of an algebraic group G with an open orbit O such that $V \backslash O$ is finite. If V is smooth then V is isomorphic to one of the following 5 surfaces:*

$$\mathbb{A}^2_{\mathbb{C}}, \quad \mathbb{A}^1_{\mathbb{C}} \times \mathbb{C}^*, \quad \mathbb{C}^* \times \mathbb{C}^*, \quad (\mathbb{P}^1 \times \mathbb{P}^1)\backslash\Delta, \quad \mathbb{P}^2\backslash\bar{\Delta}, \tag{21}$$

where $\Delta \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is the diagonal and $\bar{\Delta} \subseteq \mathbb{P}^2$ is a smooth conic. If V is singular then V is isomorphic to a Veronese cone $V_{d,1}$ for some $d \geq 2$ (see Example 5.2).

REMARK 4.13. Popov [31] listed as well all affine surfaces with a big open orbit without the assumption of normality.

Proof of Theorem 4.12. We note first that all surfaces listed in 4.12 admit an action of an algebraic group with a big open orbit (see Examples 5.1 and 5.2). Conversely, suppose that V admits an effective G-action with a big open orbit. If G is solvable then by Lemma 2.9 (b) V is isomorphic to $\mathbb{A}^2_{\mathbb{C}}$, $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{C}^*$ or $\mathbb{C}^{*2}$. Otherwise by Lemma 2.9 (c) G contains a subgroup isomorphic to $\mathbf{SL}_2$ or $\mathbf{PGL}_2$. Now the conclusion follows from the next result. $\square$

Proposition 4.14. *If $\mathbf{SL}_2$ acts nontrivially on a normal affine surface $V = \operatorname{Spec} A$ then V is isomorphic either to one of the surfaces $\mathbb{P}^1 \times \mathbb{P}^1\backslash\Delta$, $\mathbb{P}^2\backslash\bar{\Delta}$ or to a Veronese cone $V_{d,1}$. Moreover, any two such $\mathbf{SL}_2$-actions on V are conjugated in* $\operatorname{Aut}(V)$.

The proof is preceded by the following observations and by Lemma 4.17 below.

4.15. With the assumptions of 4.14, the kernel of $\mathbf{SL}_2 \to \operatorname{Aut}(V)$ is either trivial or equal to the center $Z(\mathbf{SL}_2) = \{\pm I_2\}$, so one of the groups $G = \mathbf{SL}_2$ or $G = \mathbf{PGL}_2$ acts effectively on V. We let $e = e(G)$ be the order of the center $Z(G)$ that is, $e = 2$ if

$G = \mathbf{SL}_2$ and $e = 1$ if $G = \mathbf{PGL}_2$. The effective $\mathbb{C}^*$-action on V provided by the maximal torus of diagonal matrices $\mathbb{T}$ of G defines a grading $A = \bigoplus_{i\in\mathbb{Z}} A_i = A_+ \oplus A_0 \oplus A_-$. The Borel subgroups $B_\pm \cong G_e$ (cf. Remark 2.3.2) act effectively on V, and the infinitesimal generators of the unipotent subgroups $U_\pm \cong \mathbb{C}_+$ of upper/lower triangular matrices with 1 on the diagonal induce nonzero homogeneous locally nilpotent derivations $\partial_\pm \in \operatorname{Der} A$ of degree $\pm e$ (see Lemma 2.2). We let $\delta \in \operatorname{Der} A$ be the infinitesimal generator of $\mathbb{T}$ so that $\delta(a) = \deg a \cdot a$ for $a \in A$ homogeneous. If $\partial \in \operatorname{Der} A$ is a homogeneous derivation then $[\delta, \partial] = \deg \partial \cdot \partial$; in particular

$$[\delta, \partial_\pm] = \pm e \partial_\pm \quad \text{and moreover} \quad [\partial_+, \partial_-] = \delta.$$

The adjoint action on $\mathbb{T}$ of the element $\tau = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in G$ of order $2e$ is given by $\operatorname{Ad}\tau \colon \delta \mapsto -\delta$. Hence τ acts on A homogeneously by reversing the grading, i.e. $\tau(A_i) = A_{-i}$, and the action of $\operatorname{Ad}\tau$ on the Lie algebra $\mathfrak{g} \cong \mathfrak{sl}_2 \cong \mathbb{C}\delta \oplus \mathbb{C}\partial_+ \oplus \mathbb{C}\partial_-$ of G is given by $\partial_\pm \mapsto -\partial_\mp$. In particular, the $\mathbb{C}^*$-action on V defined by $\mathbb{T}$ is hyperbolic.

DEFINITION 4.16. We say that two pairs (D_+, D_-) and $(\hat{D}_+, \hat{D}_-)$ of $\mathbb{Q}$-divisors on $\mathbb{A}^1_{\mathbb{C}}$ are equivalent if one can be obtained from the other by applying an affine transformation $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ and a shift $D_\pm \mapsto D_\pm \pm D_0$ with an integral divisor D_0.

Lemma 4.17. *Let the assumptions be as in* Proposition 4.14. *If $A = A_0[D_+, D_-]$ with $A_0 = \mathbb{C}[t]$ and $D_+ + D_- \le 0$ is a DPD representation for A graded via the $\mathbb{T}$-action, then (D_+, D_-) is equivalent to one of the following pairs*:
(1) $(0, -[1] - [-1])$; *here $e = 1$ and $V \cong \mathbb{P}^1 \times \mathbb{P}^1 \backslash \Delta$, see* (23);
(2) $((1/2)[0], -(1/2)[0] - [1])$; *here $e = 1$ and $V \cong \mathbb{P}^2 \backslash \bar{\Delta}$, see* (25);
(3) $(-(1/d)[0], -(1/d)[0])$ *with $d \ge 1$; here $e = 1$ and $V \cong V_{2d,1}$, see* (26);
(4) $(-(e'/d)[0], ((e'-1)/d)[0])$ *with $d = 2e' - 1 \ge 1$; here $e = 2$ and $V \cong V_{d,1}$, see* (27).

Proof. We start with the following

CLAIM. If the divisors $D_\pm$ are integral then, in a suitable coordinate t on $\mathbb{A}^1_{\mathbb{C}}$, one of the following 3 cases occurs:
(α) $e = 2$, $D_+ + D_- = -[0]$.
(β) $e = 1$, $D_+ + D_- = -2[0]$.
(γ) $e = 1$, $D_+ + D_- = -[1] - [-1]$.
In particular, (D_+, D_-) is equivalent to one of the integral pairs in (1)–(4).

To prove the claim, we note first that $D_\pm$ being integral $A \cong A_0[0, D_+ + D_-]$ is the normalization of the ring $B_{1,P} = \mathbb{C}[t, u_+, u_-]/(u_+u_- - P(t))$, where $P \in \mathbb{C}[t]$ is a unitary polynomial with $\operatorname{div}(P) = -(D_+ + D_-)$ (see Corollary 3.28 and Example 4.1).

After multiplying u_+, u_- with suitable constants we have

$$\partial_\pm(u_\pm) = 0, \quad \partial_\pm(t) = u_\pm^e, \quad \text{and} \quad \partial_\pm(u_\mp) = P'(t)P^{e-1}(t)u_\mp^{-e+1}$$

(cf. (18)). Hence

$$[\partial_+, \partial_-](u_+) = \partial_+\left(P'(t)P^{e-1}(t)u_+^{-e+1}\right) = \frac{d}{dt}\left(P'(t)P^{e-1}(t)\right)u_+.$$

On the other hand, $[\partial_+, \partial_-] = \delta$ and $\delta(u_+) = u_+$, therefore

$$\frac{d}{dt}\left(P'(t)P^{e-1}(t)\right) = 1.$$

Thus either $e = 2$ and $\deg P = 1$ or $e = 1$ and $\deg P = 2$. Since $D_+ + D_- = -\operatorname{div}(P)$, the claim follows.

For the rest of the proof we may assume that $D_\pm$ are not both integral. By Theorem 4.5 (ii) $\Rightarrow$ (iii), the fractional parts $\{D_\pm\}$ are concentrated in points $p_\pm \in \mathbb{A}^1_{\mathbb{C}}$. Clearly, τ yields an isomorphism $A_0[D_+, D_-] \cong A_0[\tau_0^*(D_-), \tau_0^*(D_+)]$, where $\tau_0 \colon \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ is the affine transformation of $\operatorname{Spec} A_0 = \mathbb{A}^1_{\mathbb{C}}$ induced by $\tau_0 := \tau|A_0$. By Theorem 4.3 (b) in [17] there is an integral divisor D_0 with

$$D_+ = \tau_0^*(D_-) + D_0, \quad D_- = \tau_0^*(D_+) - D_0 \quad \Rightarrow \quad D_+ + D_- = \tau_0^*(D_+ + D_-). \tag{22}$$

It follows that $\tau_0^*(\{D_\pm\}) = \{D_\mp\} \neq 0$ and so $\tau_0(p_\pm) = p_\mp$. With a suitable choice of t then either (i) $p_+ = p_- = 0$ is a fixed point of τ_0, or (ii) $\tau_0 \colon t \mapsto -t$ and $p_- = -p_+ \neq 0$.

We claim that the case (ii) cannot occur. In fact, in this case we have $\tau_0 \neq \mathrm{id}$, and because of (22) and Theorem 3.22 we may suppose that

$$D_+(p_+) = \frac{-e'}{d}, \quad D_-(p_+) = -a, \quad \text{and} \quad D_-(p_-) = \frac{-e'}{d}, \quad D_+(p_-) = -a,$$

where $d \geq 2$, $0 < e' < d$ and $ee' \equiv 1 \mod d$. In particular, $(D_+ + D_-)(p_\pm) = -(e'/d) - a < 0$ and so the $\mathbb{T} \cong \mathbb{C}^*$-action on V has a unique fixed point $p'_\pm \in \pi^{-1}(p_\pm)$ over $p_\pm$ (see Theorem 4.15 in [17]). If $p'_\pm$ were singular points of V, then they would be fixed under the action of the connected group G contradicting $\tau_0(p_-) = p_+ \neq p_-$. Thus V is smooth in $p'_\pm$ and hence by [17, Theorem 4.15] $e' + ad = 1$, forcing $e' = 1$ and $a = 0$. The condition $ee' \equiv 1 \mod d$ then implies $e = 1$. By Theorem 3.22 (ii) we also have $-e(D_+(p_+) + D_-(p_+)) \geq 1$, which gives $e(e'/d + a) \geq 1$. This is a contradiction.

Thus in fact (ii) is impossible and so $p_+ = p_- = 0$. We can write $D_+ = -(e'/d)[0]$ and $D_- = -(e'/d)[0] + E_0$ on $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} A_0$ with $d \geq 2$, where E_0 is integral and $D_+ + D_- \leq 0$. Let $v_\pm \in A_{\pm d}$ be a generator of $A_{\pm d}$ over A_0. Due to Lemmas 1.8, 3.9 and Remark 3.10 the fraction fields of $A\left[\sqrt[d]{v_\pm}\right]$ and $A\left[\sqrt[d]{t}\right]$ are equal, the normalization A' of A in this field is again graded, and $\partial_\pm$ extend to locally nilpotent derivations on A'. Thus A' admits again an $\mathfrak{sl}_2$-action. Applying Proposition 4.12 in [17] $A' \cong A'_0[D'_+, D'_-]$, where $A'_0 = \mathbb{C}[s]$, $s^d = t$, and $D'_\pm = \pi^*(D_\pm)$ with $\pi : s \mapsto s^d$. Since the divisors $D'_\pm$ are integral, their sum

$$D'_+ + D'_- = \pi^*(D_+ + D_-) = -2e'[0] + \pi^*(E_0)$$

is as in (α)–(γ) above. In case (α) or (β) clearly $E_0 = -b[0]$ with $b \in \mathbb{Z}$. In case (α) we have $2e' + db = 1$, and since $0 < e' < d$ this implies $b = -1$, so $d = 2e' - 1$ and (D_+, D_-) is as in (4). Similarly, in case (β) we have $2e' + db = 2$, $e = 1$, so $ee' \equiv 1 \mod d$ implies $e' = 1$, $b = 0$, thus $E_0 = 0$ and we are in case (3).

In the remaining case (γ) we have $e = 1$. Letting $E_0 = -b[0] + E'_0$ with $E'_0(0) = 0$, we obtain that $-(2e' + db)[0] + \pi^*(E'_0) = -[p] - [q]$ with $p \neq q$. Therefore either

$$2e' + db = 0 \quad \text{and} \quad \pi^*(E'_0) = -[p] - [q] \quad \text{with} \quad p, q \neq 0,$$

or, up to interchanging p and q,

$$p = 0, \quad 2e' + db = 1 \quad \text{and} \quad \pi^*(E'_0) = -[q] \neq [0].$$

Actually this latter case cannot occur since $d \geq 2$ divides $\deg \pi^*(E'_0)$. Thus we must have $d = 2$, $e' = 1$, $b = -1$ and $p = -q \neq 0$. Letting e.g., $p = 1$ we obtain that (D_+, D_-) is as in (2). This proves the lemma. □

Proof of Proposition 4.14. Lemma 4.17 implies that a surface with an $\mathbf{SL}_2$-action is isomorphic to one of the surfaces listed in the proposition. It remains to show that this isomorphism can be chosen to be equivariant with respect to the given $\mathbf{SL}_2$-actions. For this we restrict to the case $\mathbb{P}^2 \backslash \bar{\Delta}$, the argument in the other cases being similar.

Let $V = \operatorname{Spec} A$ be an $\mathbf{SL}_2$-surface as in Lemma 4.17 (2) and denote $V' := \mathbb{P}^2 \backslash \bar{\Delta}$ with its standard action as in Example 5.1. Both A and the affine coordinate ring A' of V' are equipped with the grading coming from the maximal torus in $\mathbf{SL}_2$, and by the construction in Lemma 4.17 the isomorphism $A \cong A'$ is compatible with these gradings. Let $(\delta, \partial_+, \partial_-)$ be the triplet of derivations on A as in 4.15, and let $(\delta', \partial'_+, \partial'_-)$ denote the corresponding derivations on A'. Using Lemma 4.17 again $e = \deg \partial_\pm = \pm 1$; as $\mathbf{PGL}_2$ acts on V' (cf. Example 5.1) we also have $\deg \partial'_\pm = \pm 1$.

Now Proposition 3.7 shows that the pairs (A, ∂_+) and (A', ∂'_+) are isomorphic, so there is a graded isomorphism $f\colon A \to A'$ with $f_*(\delta) = \delta'$ and $f_*(\partial_+) = \partial'_+$. Again by Proposition 3.7 $f_*(\partial_-) = c\partial'_-$ for some constant $c \in \mathbb{C}^*$. As $\delta = [\partial_+, \partial_-]$ it follows that $\delta' = f_*(\delta) = f_*([\partial_+, \partial_-]) = c[\partial'_+, \partial'_-] = c\delta'$. Hence $c = 1$ and so $f_*(\partial_\pm) = \partial'_\pm$. By Proposition 3.2 this means that the induced isomorphism $V \cong V'$ is equivariant with respect to the Borel subgroups $B_\pm$ of $\mathbf{SL}_2$ and so it is $\mathbf{SL}_2$-equivariant, as desired. □

REMARK 4.18. Proposition 4.14 shows in particular that any $\mathbf{SL}_2$-action on the plane $\mathbb{A}^2_{\mathbb{C}}$ is conjugated in $\operatorname{Aut}\mathbb{A}^2_{\mathbb{C}}$ to the standard linear representation.

5. Concluding remarks: Examples

Here we illustrate our methods in concrete examples. According to Gizatullin's Theorem cited in 4.12, there are only 5 different homogeneous affine surfaces (21). In the following example we consider more closely the last two of these surfaces $\mathbb{P}^1 \times \mathbb{P}^1 \backslash \Delta$ and $\mathbb{P}^2 \backslash \bar{\Delta}$ (cf. [31, Lemma 2]).

EXAMPLE 5.1. Let $V \cong \mathbb{C}^2$ be a 2-dimensional vector space. The group $\mathbf{PGL}_2 \cong \mathbf{PGL}(V)$ then acts on $\mathbb{P}^1 = \mathbb{P}(V)$ as well as on the projectivized space of binary quadrics $\mathbb{P}^2 = \mathbb{P}(S^2V)$. Since $\mathbf{PGL}_2$ acts doubly transitive on $\mathbb{P}^1$, the diagonal action on $\mathbb{P}^1 \times \mathbb{P}^1$ has an open orbit $\mathbb{P}^1 \times \mathbb{P}^1 \backslash \Delta$, where Δ is the diagonal. Similarly, the action of $\mathbf{PGL}_2$ on $\mathbb{P}^2$ leaves the degenerate quadrics invariant thus providing an action on $\mathbb{P}^2 \backslash \bar{\Delta}$, where $\bar{\Delta}$ is the space of degenerate binary forms.

The symmetric product $V \times V \to S^2V$, $(v, w) \mapsto v \vee w$, induces a natural unramified 2:1 covering

$$p\colon \mathbb{P}^1 \times \mathbb{P}^1 \backslash \Delta \to \mathbb{P}^2 \backslash \bar{\Delta},$$

where the covering involution is the map interchanging the two factors of $\mathbb{P}^1 \times \mathbb{P}^1$.

To make the situation more explicit, let us fix a basis v_0, v_1 of V so that the points of $\mathbb{P}(V)$ can be represented in coordinates $[x_0, x_1] \mathrel{\hat=} [x_0v_0 + x_1v_1]$. With respect to the basis v_0^2, $2v_0v_1$, v_1^2 of S^2V the points of $\mathbb{P}^2 = \mathbb{P}(S^2V)$ have then coordinates $[u'_+, s, u'_-]$. Clearly $\bar{\Delta} = \left\{\left[(x_0v_0 + x_1v_1)^2\right] \in \mathbb{P}(S^2V)\colon x_0, x_1 \in K\right\}$ has equation $Q := s^2 - u'_+u'_- = 0$. The map p factors through

$$\mathbb{P}^1 \times \mathbb{P}^1 \backslash \Delta \xrightarrow{\tilde{p}} H \xrightarrow{\text{can}} \mathbb{P}^2 \backslash \bar{\Delta},$$

where H is the affine quadric $\{Q = 1\} \subseteq \mathbb{A}^3_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[u'_+, s, u'_-]$ and $\tilde{p}$ is the isomorphism given by

$$([x_0, x_1], [y_0, y_1]) \mapsto \frac{1}{x_0y_1 - x_1y_0}(2x_0y_0, x_0y_1 + x_1y_0, 2x_1y_1).$$

This isomorphism identifies the factors interchanging involution of $\mathbb{P}^1 \times \mathbb{P}^1$ with the map $(u_+, s, u_-) \mapsto -(u_+, s, u_-)$. Thus $\mathbb{P}^1 \times \mathbb{P}^1 \backslash \Delta \cong H \cong \operatorname{Spec} A'$, where according to Example 4.10 in [17]

$$A' := \mathbb{C}\left[u'_+, s, u'_-\right] \big/ \left(u'_+u'_- - s^2 + 1\right) \cong A'_0[D'_+, D'_-] \qquad (23)$$

with $A'_0 := \mathbb{C}[s]$, $D'_+ = 0$ and $D'_- = -[1] - [-1]$. This isomorphism determines a hyperbolic grading with $\deg s = 0$ and $\deg u'_\pm = \pm 1$.

Next we turn to the surface $\mathbb{P}^2\backslash\overline{\Delta}$, which is the spectrum of the invariant ring $A := A'^{\mathbb{Z}_2}$. As noted above the action of $\mathbb{Z}_2$ on A' is given by $(u'_+, s, u'_-) \mapsto -(u'_+, s, u'_-)$. The algebra of invariants is generated by the degree 2 monomials in $s, u'_\pm$

$$u_\pm := su'_\pm, \quad v_\pm := u'^2_\pm \quad \text{and} \quad t := s^2$$

satisfying the relations

$$u_- = t(t-1)u_+^{-1}, \quad v_+ = t^{-1}u_+^2, \quad v_- = t(t-1)^2u_+^{-2}$$

(observe that $u'_+u'_- = s^2 - 1 = t - 1$ in A'). Thus $A = \mathbb{C}[t][v_-, u_-, u_+, v_+]$ can be presented as

$$A = \mathbb{C}[t]\left[t(t-1)^2u_+^{-2}, t(t-1)u_+^{-1}, u_+, t^{-1}u_+^2\right] \subseteq \mathbb{C}(t)[u_+, u_+^{-1}]. \tag{24}$$

By virtue of (24) and Lemma 4.6 in [17],

$$A \cong A_0[D_+, D_-] \quad \text{with} \quad D_+ = \frac{1}{2}[0], \quad D_- = -\frac{1}{2}[0] - [1]. \tag{25}$$

Indeed, according to this lemma

$$D_+ = -\min\left\{0, -\frac{1}{2}[0]\right\} = \frac{1}{2}[0],$$
$$D_- = -\min\left\{\operatorname{div}\ t(t-1), \frac{\operatorname{div}\ t(t-1)^2}{2}\right\} = -\frac{1}{2}[0] - [1]$$

and so $D_+ + D_- = -[1]$.

With this example one can also make some of the previous results quite explicit. For instance, $\pi^*(D_+) = D'_+ + D_0$ with $D_0 := [0]$ and $\pi^*(D_-) = -[0] - [1] - [-1] = D'_- - D_0$ with $\pi\colon \mathbb{A}^1_\mathbb{C} \to \mathbb{A}^1_\mathbb{C}$ being the base change $s \mapsto s^2 = t$, which agrees with Proposition 4.12 in [17] applied to the Galois $\mathbb{Z}_2$-extension $A \hookrightarrow A'$. Further, the fractional parts $\{D_\pm\}$ of $D_\pm$ are supported at one point; compare with Theorem 4.5 above.

For every $\lambda = (\lambda_+, \lambda_0, \lambda_-)$ with $\lambda_0^2 = 4\lambda_+\lambda_-$ the hyperplane in $\mathbb{P}^2$ given by $f_\lambda := \lambda_0 s + \lambda_+ u_+ + \lambda_- u_- = 0$ intersects $\overline{\Delta}$ in one point. It follows that the maps

$$f_\lambda : H \to \mathbb{A}^1_\mathbb{C} \quad \text{and} \quad g_\lambda := f_\lambda^2/Q\colon \mathbb{P}^2\backslash\overline{\Delta} \to \mathbb{A}^1_\mathbb{C}$$

provide explicit families of affine rulings compatible with p (cf. Proposition 4.10). By [7, Proposition 1.11] any affine ruling $\mathbb{P}^2\backslash\overline{\Delta} \to \mathbb{A}^1_\mathbb{C}$ is given by a certain g_λ; they can be visualized via the Segre and Veronese embeddings $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$.

Finally it is easy to see (and left as an exercise to the reader) that the locally nilpotent derivations $\partial_\pm$ defined by the unipotent subgroups $U_\pm \subseteq \mathbf{PGL}_2$ of upper/lower triangular matrices with 1 on the diagonal are of degree 1 and are given by the formulas in Remark 3.18 (1) (compare also with the Proof of Lemma 4.14).

EXAMPLE 5.2. *Veronese cones*. For $d \geq 1$ and $e = 1$, $A_{d,1} = \bigoplus_{\nu\geq 0} \mathbb{C}[X, Y]_{\nu d}$ is the d-th Veronese subring of the polynomial ring $\mathbb{C}[X, Y]$. The standard $\mathbf{SL}_2$-action on $\mathbb{C}[X, Y]$ stabilizes $A_{d,1}$ and so, induces an $\mathbf{SL}_2$-action on the normal affine surface $V_{d,1} := \operatorname{Spec} A_{d,1}$. This $\mathbf{SL}_2$-action has a unique fixed point $\bar{0} \in V_{d,1}$ and is transitive on $V_{d,1}\backslash\{\bar{0}\}$.

The algebra $A_{d,1}$ is generated by the monomials $X^i Y^{d-i} \in (A_{d,1})_1$ ($i = 0, \ldots, d$), and these define an embedding $\rho\colon V_{d,1} \hookrightarrow \mathbb{A}^{d+1}_{\mathbb{C}}$ onto the affine cone over the degree d rational normal curve $\Gamma_d \cong \operatorname{Proj} A_{d,1}$ in $\mathbb{P}^d$. The morphism ρ is equivariant with respect to the standard irreducible representation of $\mathbf{SL}_2$ on the space $\mathbb{A}^{d+1}_{\mathbb{C}}$ of degree d binary forms. The group $\mathbf{SL}_2$ (respectively, $\mathbf{PGL}_2$) acts effectively on $V_{d,1}$ if d is odd (respectively, even). The stabilizer subgroup

$$N_d := \left\{ \begin{pmatrix} \varepsilon & \alpha \\ 0 & \varepsilon^{-1} \end{pmatrix} \,\middle|\, \varepsilon^d = 1, \alpha \in \mathbb{C}_+ \right\}$$

of the binary form $X^d \in A_{d,1}$ is a cyclic extension of the maximal connected unipotent subgroup $N = N_1$ of $\mathbf{SL}_2$. Clearly, $V_{d,1} \cong \operatorname{Spec} \mathcal{O}(\mathbf{SL}_2/N_d)$, as $\mathbf{SL}_2/N_d \cong V_{d,1}\backslash\{\bar{0}\}$ and $V_{d,1}$ is normal [31].

To represent the Veronese cones via the DPD construction, note first that the action of the torus $\mathbb{T} = \left\{ \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) \,\middle|\, \lambda \in \mathbb{C}^* \right\} \subseteq \mathbf{SL}_2$ provides a grading on the ring $A = \mathbb{C}[X, Y]$ with $\deg X = 1, \deg Y = -1$, and so induces a grading on the d-th Veronese subring $A_{d,1} = A^{(d)} = \bigoplus_{i\in\mathbb{Z}} A^{(d)}_i$. We consider separately the case where d is even or odd.

(1) For $d = 2d'$ even, the $\mathbb{T}$-action on $A^{(d)}$ factorizes through an action of $\mathbb{T}/\mathbb{Z}_2$, which corresponds to letting $\deg X = 1/2, \deg Y = -1/2$. With $t := (XY)^{d'} \in A^{(d)}_0 = \mathbb{C}[t]$ and $u := X/Y \in (\operatorname{Frac} A_0)A^{(d)}_1$, we have

$$u_i := X^{d'+i} Y^{d'-i} = tu^i \in A^{(d)}_i, \quad -d' \leq i \leq d'.$$

As $A^{(d)} = \mathbb{C}[u_{-d'}, \ldots, u_{d'}]$ by Lemma 4.6 in [17] $A^{(d)} \cong A^{(d)}_0[D_+, D_-]$, where

$$D_+ = -\min_{1\leq i\leq d'} \left\{ -\frac{1}{i}[0] \right\} = -\frac{1}{d'}[0], \quad D_- = -\min_{-d'\leq i\leq -1} \left\{ -\frac{1}{-i}[0] \right\} = -\frac{1}{d'}[0] \tag{26}$$

and so $D_+ + D_- = -(2/d')[0]$.

(2) For $d = 2e' - 1$ odd, the torus $\mathbb{T}$ acts effectively on $A^{(d)}$. We let $t := (XY)^d \in A^{(d)}_0 = \mathbb{C}[t]$, $u_1 := X^{e'} Y^{e'-1} \in A^{(d)}_1$ and

$$u_{2k-1} := X^{e'+k-1} Y^{e'-k} = t^{-k+1} u_1^{2k-1} \in A^{(d)}_{2k-1}, \quad -e' + 1 \leq k \leq e'.$$

As $A^{(d)} = \mathbb{C}[u_{-d}, \ldots, u_d]$ then by Lemma 4.6 in [17] $A^{(d)} \cong A^{(d)}_0[D_+, D_-]$, where

$$D_+ = -\min_{1\leq k\leq e'} \left\{ -\frac{k-1}{2k-1}[0] \right\} = \frac{e'-1}{d}[0], \quad D_- = -\min_{-e'+1\leq k\leq 0} \left\{ \frac{k-1}{2k-1}[0] \right\} = -\frac{e'}{d}[0]$$

and so, $D_+ + D_- = -(1/d)[0]$. We notice that

$$(D_+, D_-) = \left(\frac{e'-1}{d}[0], -\frac{e'}{d}[0]\right) \sim \left(-\frac{e'}{d}[0], \frac{e'-1}{d}[0]\right) \tag{27}$$

via the shift $(D_+, D_-) \mapsto (D_+ - [0], D_- + [0])$.

Alternatively, the Veronese cone $V_{d,1}$ can be obtained from the Hirzebruch surface $\Sigma_d := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-d))$ by deleting a section C_d with $C_d^2 = d$ and contracting the exceptional section E_d with $E_d^2 = -d$ [12, §11, Example 1]. This leads [12] to a description of the automorphism groups $\operatorname{Aut}(V_{d,1})$.

In the next example we exhibit affine surfaces V such that the automorphism group $\operatorname{Aut} V$ acts on V with a big open orbit O and there are algebraic group actions on V with an open orbit, whereas there is no such action with a big open orbit.

EXAMPLE 5.3. (*Actions on surfaces with a big open orbit.*) Let $D_\pm$ be two $\mathbb{Q}$-divisors on $\mathbb{A}^1_{\mathbb{C}}$ with $D_+ + D_- \le 0$ such that the supports of the fractional parts $\{D_\pm\}$ are contained in (possibly the same) points $\{p_\pm\}$. According to Theorem 4.5 the ring $A := A_0[D_+, D_-]$ with $A_0 := \mathbb{C}[t]$ admits locally nilpotent derivations $\partial_\pm$ of positive and negative degree. The associated $\mathbb{C}_+$-actions φ_+ and φ_- on V are not equivalent provided that $D_+ + D_- \neq 0$ (see Definition 4.2).

Consider the subgroup $G := \langle \varphi_+, \lambda, \varphi_- \rangle \subseteq \operatorname{Aut} V$ generated by $\varphi_\pm$ and the $\mathbb{C}^*$-action λ on V. The fixed points set of G is finite as it is contained in the fixed points set F of the $\mathbb{C}^*$-action on V. Recall that F has exactly one point a' over every point $a \in \mathbb{A}^1_{\mathbb{C}}$ with $D_+(a) + D_-(a) < 0$ [17, Theorem 4.18 (b)]. We claim that G acts transitively on the complement $V \backslash F$. Indeed, the algebraic subgroup $G_{e_\pm} := \langle \varphi_\pm, \lambda \rangle$ of G acts on V with an open orbit which contains $V \backslash v_\pm^{-1}(0)$. Hence for a general point $x \in V$, the orbit $G.x$ contains $V \backslash \{v_+^{-1}(0) \cap v_-^{-1}(0)\} = V \backslash F$ (cf. Proposition 3.25).

Thus G acts on V with a big open orbit. However, such a surface V does not admit an action of an *algebraic* group with a big open orbit unless it is isomorphic to one of the surfaces from Theorem 4.12. For instance, this is the case if V has two or more singular points (cf. [17, Theorem 4.15]), or is an affine toric surface $V_{d,e}$ with $d > e > 1$.

A particular case is provided by the dihedral surfaces $V_{d,d-1} \cong \operatorname{Spec} A_{d,d-1}$, where $A_{d,d-1} \cong \mathbb{C}[t, u_+, u_-]/(u_+u_- - t^d)$ and $d \ge 3$. We have $A_{d,d-1} \cong_{A_0} A_0[D_+, D_-]$ with $D_+ = 0$ and $D_- = -d[0]$ for a grading on $A_{d,d-1}$ with $\deg t = 0, \deg u_\pm = \pm 1$ (see Corollary 3.28). The derivations

$$\partial_\pm = u_\pm \frac{\partial}{\partial t} + d \cdot t^{d-1} \frac{\partial}{\partial u_\mp}$$

with $\deg \partial_\pm = \pm 1$ are locally nilpotent on $A_{d,d-1}$. The associated $\mathbb{C}_+$-actions φ_+ and φ_- on $V_{d,d-1}$ generate a subgroup G of $\operatorname{Aut} V_{d,d-1}$. Using e.g., Remark 3.29.4 it is easily seen that G acts with a big open orbit $V_{d,d-1}\backslash\{\bar{0}\}$, where $\bar{0} \in V_{d,d-1}$ denotes the unique singular point. The dihedral surfaces $V_{d,d-1}$ with $d \geq 3$ are not isomorphic to Veronese cones, since the exceptional set of the minimal resolution of $V_{d,d-1}$ is a chain of $d-1$ rational (-2)-curves, whereas it is just one rational curve for every singular Veronese cone. Hence by Popov's Theorem 4.12 there is no algebraic group action with a big open orbit on $V_{d,d-1}$.

We continue with examples that illustrate Corollaries 3.28 and 3.30.

EXAMPLE 5.4. *Danielewski's surfaces.* These are the smooth surfaces

$$W_d := \{u^d v = t^2 + t\} \subseteq \mathbb{A}^3_{\mathbb{C}} \quad (d \geq 1).$$

Thus $W_d = \operatorname{Spec} B_{d,P}$ with $P(t) := t^2 + t$ is one of the surfaces studied in Corollary 3.28. So it admits a $\mathbb{C}^*$-equivariant $\mathbb{C}_+$-action along the fibers of the affine ruling $u\colon W_d \to \mathbb{A}^1_{\mathbb{C}}$. Note that $W_1 \cong (\mathbb{P}^1 \times \mathbb{P}^1)\backslash\Delta$ has a continuous family of affine rulings over $\mathbb{A}^1_{\mathbb{C}}$ (see Example 5.1), whereas for every $d \geq 2$, such a ruling on W_d is unique and $\mathrm{ML}(W_d) = \mathbb{C}[u]$. The latter follows from Theorem 4.5 as $A = B_{d,P} \cong A_0[D_+, D_-]$ with $A_0 = \mathbb{C}[t]$, $D_+ = 0$ and $D_- = -(1/d)([0] + [-1])$, where the fractional part $\{D_-\}$ of the $\mathbb{Q}$-divisor D_- is supported at two points (see Example 4.10 in [17]).

According to Corollaries 4.24 and 4.25 in [17] we have $\mathrm{Pic}(W_d) \cong \mathbb{Z}$ generated e.g., by $[\bar{O}_0^-]$, whereas $K_{W_d} = 0$.

We recall [11][8] that these surfaces provide examples of non-cancellation, that is $W_d \times \mathbb{A}^1_{\mathbb{C}} \cong W_{d'} \times \mathbb{A}^1_{\mathbb{C}}$ $\forall d, d' \in \mathbb{N}$, whereas $W_d \not\cong W_{d'}$ if $d \neq d'$.

EXAMPLE 5.5. *Bertin's surfaces.* These are the smooth affine surfaces

$$W_{d,n} := \{x^d y = x + z^n\} \subseteq \mathbb{A}^3_{\mathbb{C}}; \tag{28}$$

they admit an algebraic group action with an open orbit [7]. Note that $W_{d,1} \cong \mathbb{A}^2_{\mathbb{C}}$ and $W_{1,n} \cong V_{n,n-1}$ admit continuous families of affine rulings over $\mathbb{A}^1_{\mathbb{C}}$. Thus we will suppose in the sequel that $d, n \geq 2$. The defining equation of $W_{d,n}$ is quasihomogeneous with weights

$$\deg x = n, \quad \deg y = -n(d-1), \quad \deg z = 1.$$

To compute a DPD presentation of the coordinate ring $A = \mathbb{C}[x, y, z]/(x^d y - x - z^n)$, we note that $A_0 = \mathbb{C}[t]$ with $t := x^{d-1}y - 1$. Moreover the equations $xt = z^n$ and $y = (t+1)t^{d-1}z^{-n(d-1)}$ show that $A = A_0[z, t^{-1}z^n, (t+1)t^{d-1}z^{-n(d-1)}]$, and so by [17, Lemma 4.6]

$$A \cong A_0[D_+, D_-] \quad \text{with} \quad D_+ = \frac{1}{n}[0] \quad \text{and} \quad D_- = -\frac{1}{n}[0] - \frac{1}{n(d-1)}[-1].$$

[8] Cf. also [4, 16, 35].

A homogeneous locally nilpotent derivation ∂ on A of degree $nd-1$ can be given by

$$\partial(x)=0,\quad \partial(y)=nz^{n-1},\quad \partial(z)=x^d.$$

According to Corollary 3.30 Bertin's surfaces can be described as cyclic quotients of $V_{d',P}$ for a suitable pair (d',P). To find such a presentation one takes the normalization A' of A in the quotient field of $A[u]$, where $u:=x^{1/n}$. The equation $z^n/u^n=t$ shows that $s:=z/u\in A'$. Thus A' contains $\mathbb{C}[s,u,y]/(u^{n(d-1)}y-1-s^n)$, and since the latter ring is normal, these two rings are equal. The derivation ∂ extends to A' via $\partial(u)=0$ and $\partial(s)=u^{nd-1}$ commuting with the homogeneous $\mathbb{Z}_n$-action on A'

$$\zeta.s=\zeta^{-1}s,\quad \zeta.u=\zeta u\quad\text{and}\quad \zeta.y=y,$$

where ζ is a primitive n-th root of unity. This action on $V'=\operatorname{Spec}A'$ is fixed point free and $A=A'^{\mathbb{Z}_n}$ i.e., $W_{d,n}\cong V'/\mathbb{Z}_n=V_{d',P}/\mathbb{Z}_n$, where $d':=n(d-1)$ and $P:=s^n+1$.

For every $d,n\geq 2$ the fractional part $\{D_-\}$ of the $\mathbb{Q}$-divisor D_- is supported at two points. Hence according to Theorem 4.5, $x\colon W_{d,n}\to\mathbb{A}^1_{\mathbb{C}}$ gives a unique affine ruling on $W_{d,n}$ over an affine base, and $\mathrm{ML}(W_{d,n})=\mathbb{C}[x]$ (cf. [27]). The latter also follows from [7, Theorem 1.8 and Example 4.11 (iii)] (cf. Theorem 4.3) due to the fact that the dual graph of a minimal compactification of $W_{d,n}$ is not linear.

It can be readily seen that $\mathrm{Pic}(W_{d,n})\cong\mathbb{Z}/n\mathbb{Z}$ generated e.g., by $[O_0]$, whereas $K_{W_{d,n}}=0$ (see e.g., Corollaries 4.24 and 4.25 in [17]).

REMARK 5.6. Any affine surface $V\not\cong\mathbb{A}^2_{\mathbb{C}}$ which admits an elliptic $\mathbb{C}^*$-action is singular. If V is equipped with a parabolic $\mathbb{C}^*$-action and a horizontal $\mathbb{C}_+$-action then by Theorem 3.19 it has a quotient singularity. Thus being smooth the surfaces $\mathbb{P}^1\times\mathbb{P}^1\backslash\Delta$, $\mathbb{P}^2\backslash\bar{\Delta}$, W_d and $W_{d,n}$ with $d,n\geq 2$ admit neither elliptic nor parabolic $\mathbb{C}^*$-actions.

Correction to our paper [17]. Due to an error in the printing process the letter ℓ in Example 4.13 was printed as e. Thus the first 4 lines of the second paragraph of 4.13 have to be replaced by:

More concretely, if $k:=d_-(A)$, $l:=kD_-(0)$ and if we choose a unitary polynomial $Q\in\mathbb{C}[t]$ with $D_-=-(\mathrm{div}(Qt^l)/k)$ then $D'_+ + D'_- = -(\mathrm{div}(Q(s^d)s^{ke+dl})/k)$. By Example 4.10 $A'\cong A_{k,P}$ is the normalization of

$$B_{k,P}=\mathbb{C}[s,\tilde{u},v]\big/\big(\tilde{u}^k v-P(s)\big),\quad\text{where}\quad P(s):=Q(s^d)s^{ke+dl}. \tag{12}$$

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