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SOME QUOTIENT ALGEBRAS ARISING FROM THE QUANTUM TOROIDAL ALGEBRA $U_q(sl_2(\mathcal{C}_\gamma))$

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Abstract

Some quotient algebras arising from the quantum toroidal algebra $U_q(sl_2(\mathcal{C}_\gamma))$ are considered. They are related to integrable highest weight representations of the algebra and are shown to be isomorphic to direct sums of tensor products of two algebras of symmetric Laurent polynomials and Macdonald's difference operators.

1. Introduction

The quantum toroidal algebras were introduced in [1] and [2] as q deformations of the universal enveloping algebras of toroidal Lie algebras [3]. Since then, the algebras and representations of them have been studied in [2], [4]–[11]. In particular the connection of representations of the quantum toroidal algebra of type sl_n ($n \geq 3$) with the double affine Hecke algebra was found in [2] and this quantum toroidal algebra was shown to reduce to the universal enveloping algebra of the universal covering of sl_n coordinatized by quantum torus in the limit $q \to 1$ in [4].

In this paper we are interested in the quantum toroidal algebra of type sl_2 and quotient algebras arising from this. Let $\mathcal{C}_\gamma$ be the ring of Laurent polynomials in noncommutative variables x, y satisfying $xy = \gamma^2 yx$ and set $\mathcal{L} = sl_2(\mathcal{C}_\gamma) := [gl_2(\mathcal{C}_\gamma), gl_2(\mathcal{C}_\gamma)]$. Lie algebras of this kind and central extensions of them were considered in the study of extended affine Lie algebras in [12] and representations of these algebras were studied in [13]–[19]. The quantum toroidal algebra which we consider is a q deformation of the universal enveloping algebra of this Lie algebra $\mathcal{L}$. Let us briefly explain what quotient algebras we consider and why we study them. In [19] we studied integrable highest weight representations of the Lie algebra $\mathcal{L}$ and obtained the following result. For an integer m let $U(\mathcal{L})_m = \{u \in U(\mathcal{L}) \mid [h, u] = 2mu\}$ where $h = E_{11} - E_{22}$. Set $I = \sum_{m>0} U(\mathcal{L})_{-m} U(\mathcal{L})_m$ and let I_N be the ideal of $U(\mathcal{L})_0$ generated by I, $\sum_{m>N} U(\mathcal{L})_m U(\mathcal{L})_{-m}$ and $h - N$ for a nonngegative integer N. Any integrable highest weight representation V admits a weight decomposition $V = \bigoplus_{m=0}^{N} V_{(N-2m)/2}$ for some nonnegative integer N where $V_{m/2}$ is the eigenspace of h corresponding to the eigenvalue m. The weight space $V_{N/2}$ becomes a $U(\mathcal{L})_0/I_N$ module and the classification of irreducible integrable highest weight modules is reduced to that of irreducible $U(\mathcal{L})_0/I_N$ modules. The quotient algebras which we study in this paper are q

analogues of the $U(\mathcal{L})_0/I_N$. Our main result is that they are isomorphic to direct sums of tensor products of two algebras of symmetric Laurent polynomials and Macdonald's difference operators [20]. (See Proposition 4.4, Theorem 4.1 and Corollary 4.3 for precise statements.) We expect that this result will be of use for the study of integrable highest weight representations of $U_q(\mathcal{L})$ as in the case $q = 1$. The appearance of Macdonald's difference operators is not so unexpected since the connection of the quantum toroidal algebra with the double affine Hecke algebra is already known [2]. Finally we remark that we are motivated to consider the quotient algebras by [21], [22] and [23], which investigated integrable representations of affine Lie algebras and quantum affine algebras.

This paper is organized as follows. In Section 2, presentations of the universal enveloping algebra of $\mathcal{L}$ and that of its central extension $\hat{\mathcal{L}}$ in terms of generators and relations are given and the results on quotients algebras of $U(\mathcal{L})_0$ of [19] are reviewed. In Section 3 $U_q\big(\hat{\mathcal{L}}\big)$ is defned and several properties of it are derived. In Section 4 we study quotients algebra of $U_q(\mathcal{L})_0$ and in Section 5 we compare our results with those of [2]. In Sections 6, 7 and 8, the proofs of some technical details are given.

2. The Lie algebra $\mathcal{L}$ and quotient algebras of $U(\mathcal{L})_0$

2.1. The Lie algebra $\mathcal{L}$ and its central extension $\hat{\mathcal{L}}$. Let γ be a formal variable and set $p = \gamma^{-2}$. Let $\mathcal{C}_\gamma$ and $\mathcal{C}_{\gamma^{-1}}$ be the $\mathbf{C}(\gamma)$ algebras of Laurent polynomials in noncommutative variables x and y satisfying $xy = \gamma^2 yx$ and $xy = \gamma^{-2} yx$, respectively.

We consider the $\mathbf{C}(\gamma)$ Lie algebra $\mathcal{L} = sl_2(\mathcal{C}_\gamma) := [gl_2(\mathcal{C}_\gamma), gl_2(\mathcal{C}_\gamma)]$. Set $\mathbf{0} = (0,0)$. For $\mathbf{k} = (k,l) \in \mathbf{Z}^2$, $\mathbf{m} = (m,n) \in \mathbf{Z}^2 \setminus \{\mathbf{0}\}$ and $i = 1,2$, define the following elements of $\mathcal{L}$:

$$e(\mathbf{k}) = E_{12}x^k y^l, \quad f(\mathbf{k}) = E_{21}x^k y^l, \quad h = E_{11} - E_{22}, \quad \epsilon_i(\mathbf{m}) = E_{ii}x^m y^n$$

where the E_{ij} are matrix units. Then these elements form a basis of $\mathcal{L}$ and satisfy the relations

$$\begin{aligned}
&[e(\mathbf{k}), f(\mathbf{m})] = \begin{cases} p^{lm}\epsilon_1(\mathbf{k}+\mathbf{m}) - p^{nk}\epsilon_2(\mathbf{k}+\mathbf{m}) & \text{if } \mathbf{k}+\mathbf{m} \neq \mathbf{0}, \\ p^{lm}h & \text{if } \mathbf{k}+\mathbf{m} = \mathbf{0}, \end{cases}\\
&[e(\mathbf{k}), e(\mathbf{m})] = 0 = [f(\mathbf{k}), f(\mathbf{m})],\\
&[h, e(\mathbf{k})] = 2e(\mathbf{k}), \quad [h, f(\mathbf{k})] = -2f(\mathbf{k}), \quad [h, \epsilon_i(\mathbf{m})] = 0,\\
&[\epsilon_1(\mathbf{k}), e(\mathbf{m})] = p^{lm}e(\mathbf{k}+\mathbf{m}), \quad [\epsilon_2(\mathbf{k}), e(\mathbf{m})] = -p^{nk}e(\mathbf{k}+\mathbf{m}),\\
&[\epsilon_1(\mathbf{k}), f(\mathbf{m})] = -p^{nk}f(\mathbf{k}+\mathbf{m}), \quad [\epsilon_2(\mathbf{k}), f(\mathbf{m})] = p^{lm}f(\mathbf{k}+\mathbf{m}),\\
&[\epsilon_i(\mathbf{k}), \epsilon_j(\mathbf{m})] = \begin{cases} \delta_{ij}(p^{lm} - p^{nk})\epsilon_i(\mathbf{k}+\mathbf{m}) & \text{if } \mathbf{k}+\mathbf{m} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{k}+\mathbf{m} = \mathbf{0} \end{cases}
\end{aligned}$$

where $\mathbf{k} = (k,l)$ and $\mathbf{m} = (m,n)$.

Let $Q = \mathbf{Z}\alpha_1 \oplus \mathbf{Z}\alpha_0$ and set $\delta = \alpha_1 + \alpha_0$. By assigning $(\alpha_1 + k\delta, l)$ to $e(k,l)$, $(-\alpha_1 + k\delta, l)$ to $f(k,l)$, $(k\delta, l)$ to $\epsilon_i(k,l)$ and $(0,0)$ to h, $\mathcal{L}$ is endowed with a structure of $Q \oplus \mathbf{Z}$ graded Lie algebras. We denote the homogeneous subspace of degree (α, l) by $\mathcal{L}(\alpha, l)$.

Define the Lie algebra $\hat{\mathcal{L}}$ to be the vector space $\mathcal{L} \oplus \mathbf{C}(\gamma)c_1 \oplus \mathbf{C}(\gamma)c_2$ with multiplication rule

$$\begin{aligned}&\left[a_1x^{k_1}y^{l_1} + r_1c_1 + s_1c_2, a_2x^{k_2}y^{l_2} + r_2c_1 + s_2c_2\right]'\\ &= \left[a_1x^{k_1}y^{l_1}, a_2x^{k_2}y^{l_2}\right] + \operatorname{tr}(a_1a_2)\delta_{k_1+k_2,0}\delta_{l_1+l_2,0}(k_1c_1 + l_1c_2)p^{-k_1l_1}\end{aligned}$$

where a_1 and a_2 are 2×2 matrices and $r_i, s_i \in \mathbf{C}(\gamma)$ $(i = 1, 2)$. This central extension was considered in [12] and was shown to be the universal covering of $\mathcal{L}$ if γ is a generic complex number in [13]. By assigning $(0,0)$ to c_1 and c_2, the structure of $Q \oplus \mathbf{Z}$ graded Lie algebras of $\mathcal{L}$ is extended to that of $\hat{\mathcal{L}}$.

As is easily checked, there exist automorphisms ψ, $\mathcal{Y}$ and $\tilde{\mathcal{Y}}$ of $\hat{\mathcal{L}}$ determined by

$$\begin{aligned}\psi\colon\ & e(k,l) \mapsto p^{-kl}e(l,-k), \quad f(k,l) \mapsto p^{-kl}f(l,-k), \quad \epsilon_i(k,l) \mapsto p^{-kl}\epsilon_i(l,-k),\\ & h \mapsto h, \quad c_1 \mapsto -c_2, \quad c_2 \mapsto c_1,\\ \mathcal{Y}\colon\ & e(k,l) \mapsto -(-\gamma)^k e(k, l-1), \quad f(k,l) \mapsto -(-\gamma)^{-k} f(k, l+1),\\ & \epsilon_i(k,l) \mapsto (-\gamma)^{-\varepsilon_i k}\epsilon_i(k,l), \quad h \mapsto h - c_2, \quad c_i \mapsto c_i,\\ \tilde{\mathcal{Y}}\colon\ & e(k,l) \mapsto -(-\gamma)^l e(k-1, l), \quad f(k,l) \mapsto -(-\gamma)^{-l} f(k+1, l),\\ & \epsilon_i(k,l) \mapsto (-\gamma)^{-\varepsilon_i l}\epsilon_i(k,l), \quad h \mapsto h - c_1, \quad c_i \mapsto c_i\end{aligned}$$

where $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$. These automorphisms satisfy $\tilde{\mathcal{Y}}\psi = \psi\mathcal{Y}$.

Let $\hat{L}$ be the $\mathbf{C}(\gamma)$ Lie algebra defined by generators

$$x^{\pm}_{i,m}, h_{i,r}, c \quad (i = 0, 1,\ m, r \in \mathbf{Z})$$

and relations

$$\left[c, x^{\pm}_{i,m}\right] = 0, \quad [c, h_{i,r}] = 0, \tag{2.1}$$

$$[h_{i,r}, h_{i,s}] = 2r\delta_{r+s,0}c, \tag{2.2}$$

$$[h_{i,r}, h_{1-i,s}] = -r(\gamma^r + \gamma^{-r})\delta_{r+s,0}c, \tag{2.3}$$

$$\left[h_{i,r}, x^{\pm}_{i,m}\right] = \pm 2x^{\pm}_{i,m+r}, \tag{2.4}$$

$$\left[h_{i,r}, x^{\pm}_{1-i,m}\right] = \mp(\gamma^r + \gamma^{-r})x^{\pm}_{1-i,m+r}, \tag{2.5}$$

$$\left[x^{+}_{i,m}, x^{-}_{j,n}\right] = \delta_{ij}(h_{i,m+n} + m\delta_{m+n,0}c), \tag{2.6}$$

$$\left[x^{\pm}_{i,m}, x^{\pm}_{i,n}\right] = 0, \tag{2.7}$$

$$\left[x^{\pm}_{i,m_1}, \left[x^{\pm}_{i,m_2}, \left[x^{\pm}_{i,m_3}, x^{\pm}_{1-i,n}\right]\right]\right] = 0. \tag{2.8}$$

By assigning $(\pm\alpha_i, m)$ to $x^{\pm}_{i,m}$, $(0, r)$ to $h_{i,r}$ and $(0, 0)$ to c, $\hat{L}$ is endowed with a structure of $Q \oplus \mathbf{Z}$ graded Lie algebras. Let $\hat{L}(\alpha, n)$ denote the homogeneous subspace of degree (α, n) of it.

Proposition 2.1. (1) *There exists an isomorphism of $Q \oplus \mathbf{Z}$ graded Lie algebras $\hat{L} \to \hat{\mathcal{L}}$ determined by*

$$\begin{aligned}
&x^+_{1,m} \mapsto e(0,m), \quad x^-_{1,m} \mapsto f(0,m), \quad h_{1,r} \mapsto \epsilon_1(0,r) - \epsilon_2(0,r),\\
&x^+_{0,m} \mapsto \gamma^{-m} f(1,m), \quad x^-_{0,m} \mapsto \gamma^m e(-1,m), \quad h_{0,r} \mapsto \gamma^r \epsilon_2(0,r) - \gamma^{-r}\epsilon_1(0,r),\\
&h_{1,0} \mapsto h, \quad h_{0,0} \mapsto c_1 - h, \quad c \mapsto c_2
\end{aligned}$$

where $r \neq 0$.
(2) *Let L be the quotient algebra of $\hat{L}$ by the ideal generated by the elements c and $h_{1,0} + h_{0,0}$. Then the isomorphism in* (1) *induces an isomorphism $L \to \mathcal{L}$.*

The proof of this proposition will be given in Section 6.2. We shall identify L and $\hat{L}$ with $\mathcal{L}$ and $\hat{\mathcal{L}}$ by the above correspondence, respectively.

2.2. Quotient algebras of $U(\mathcal{L})_0$. Set

$$\mathcal{N}_- = \bigoplus_{\mathbf{k}\in\mathbf{Z}^2} \mathbf{C}(\gamma) f(\mathbf{k}), \quad \mathcal{N}_+ = \bigoplus_{\mathbf{k}\in\mathbf{Z}^2} \mathbf{C}(\gamma) e(\mathbf{k}),$$
$$\mathcal{H} = \bigoplus_{i=0}^{2} \mathcal{H}_i, \quad \mathcal{H}_0 = \mathbf{C}(\gamma) h, \quad \mathcal{H}_i = \bigoplus_{\mathbf{m}\neq\mathbf{0}} \mathbf{C}(\gamma)\epsilon_i(\mathbf{m}) \quad (i = 1, 2).$$

Then these are subalgebras, $[\mathcal{H}_i, \mathcal{H}_j] = 0$ $(i \neq j)$ and $\mathcal{L} = \mathcal{N}_- \oplus \mathcal{H} \oplus \mathcal{N}_+$. Further the multiplication map $U(\mathcal{H}_1) \otimes U(\mathcal{H}_0) \otimes U(\mathcal{H}_2) \to U(\mathcal{H})$ is an isomorphism of algebras and

$$U(\mathcal{L}) \simeq U(\mathcal{N}_-) \otimes U(\mathcal{H}) \otimes U(\mathcal{N}_+) \tag{2.9}$$

as vector spaces.

For a subalgebra K of $\mathcal{L}$ and an integer m, set

$$U(K)_m = \{u \in U(K) \mid [h, u] = 2mu\}.$$

Set $I = \sum_{m>0} U(\mathcal{L})_{-m} U(\mathcal{L})_m$. Then $I = \bigoplus_{m>0} U(\mathcal{N}_-)_{-m} U(\mathcal{H}) U(\mathcal{N}_+)_m$ and $U(\mathcal{L})_0 = U(\mathcal{H}) \oplus I$. Since I is an ideal of $U(\mathcal{L})_0$, we find that

$$U(\mathcal{H}) \simeq U(\mathcal{L})_0 / I$$

as algebras. We denote the projection $U(\mathcal{L})_0 \to U(\mathcal{H})$ with respect to the preceding direct sum by $\|\cdot\|$.

Let $a_1(\mathbf{k}) = \epsilon_1(\mathbf{k})$ and $a_2(\mathbf{k}) = -p^{kl}\epsilon_2(\mathbf{k})$ for $\mathbf{k} = (k,l) \neq \mathbf{0}$. For $i = 1,2$, a nonnegative integer r, $M \geq 1$ and $\mathbf{k}_j = (k_j, l_j) \in \mathbf{Z}^2$ $(1 \leq j \leq M)$, define $A^r_{i,M}(\mathbf{k}_1, \ldots, \mathbf{k}_M) \in U(\mathcal{H}_i)$ by the recurrence relation

$$A^r_{i,1}(\mathbf{k}) = \begin{cases} a_i(\mathbf{k}) & \text{if } \mathbf{k} \neq \mathbf{0}, \\ r & \text{if } \mathbf{k} = \mathbf{0}, \end{cases} \tag{2.10}$$

$$\begin{aligned} A^r_{i,M}(\mathbf{k}_1, \ldots, \mathbf{k}_M) &= A^r_{i,1}(\mathbf{k}_M) A^r_{i,M-1}(\mathbf{k}_1, \ldots, \mathbf{k}_{M-1}) \\ &\quad - \sum_{i=1}^{M-1} p^{\varepsilon_i l_M k_i} A^r_{i,M-1}(\mathbf{k}_1, \ldots, \mathbf{k}_i + \mathbf{k}_M, \ldots, \mathbf{k}_{M-1}) \quad (M \geq 2). \end{aligned} \tag{2.11}$$

In the case $M = 0$ set $A^r_{i,0} = 1$ for $i = 1,2$ and a nonnegative integer r. Note that

$$A^r_{1,M}(\mathbf{k}_1, \ldots, \mathbf{k}_M) = A^r_M(\mathbf{k}_1, \ldots, \mathbf{k}_M), \quad A^r_{2,M}(\mathbf{k}_1, \ldots, \mathbf{k}_M) = p^{\sum_j k_j l_j} B^r_M(\mathbf{k}_1, \ldots, \mathbf{k}_M)$$

in the notation of [19].

For $i = 1,2$ and a nonnegative integer r let I^i_r be the ideal of $U(\mathcal{H}_i)$ generated by the elements $A^r_{i,r+1}(\mathbf{k}_1, \ldots, \mathbf{k}_{r+1})$ $(\mathbf{k}_j \in \mathbf{Z}^2, \forall j)$. For $\mathbf{r} = (r_1, r_2) \in \mathbf{Z}^2_{\geq 0}$ let $I_{\mathbf{r}}$ be the ideal of $U(\mathcal{H})$ generated by $h - (r_1 + r_2)$, $I^1_{r_1}$ and $I^2_{r_2}$.

The following theorem was proved in [19, Propositions 3 and 5].

Theorem 2.1. (1) *As algebras, $U(\mathcal{H})/I_{\mathbf{r}} \simeq U(\mathcal{H}_1)/I^1_{r_1} \otimes U(\mathcal{H}_2)/I^2_{r_2}$ $(\overline{a_1 a_2} \leftrightarrow \overline{a_1} \otimes \overline{a_2})$ where $a_i \in U(\mathcal{H}_i)$ $(i = 1,2)$.*
(2) *There exists an injective homomorphism $\varphi_{i,r_i} \colon U(\mathcal{H}_i)/I^i_{r_i} \to \mathcal{C}^{\otimes r_i}_{\gamma^{\varepsilon_i}}$ determined by*

$$a_i(0,k) \mapsto \sum_{j=1}^{r_i} y_j^k \quad \text{and} \quad a_i(k,0) \mapsto \sum_{j=1}^{r_i} x_j^k$$

where $y_j = 1^{\otimes j-1} \otimes y \otimes 1^{\otimes r-j}$ and x_j is defined similarly.

The purpose of this paper is to obtain a q analogue of this theorem. Later we need another description of the ideal $I_{\mathbf{r}}$. Define $\Lambda_{i,r} \in U(\mathcal{H}_i)$ $(r \in \mathbf{Z}, i = 1,2)$ by the generating series

$$\sum_{r \geq 0} \Lambda_{i,\pm r} z^r = \exp\left(- \sum_{r>0} \frac{a_i(0, \pm r)}{r} z^r \right).$$

These elements were intoduced in [21] in the study of integrable representations of affine Lie algebras and the q analogues of them were considered in [23]. For $\mathbf{r} = (r_1, r_2) \in \mathbf{Z}^2_{\geq 0}$ let $J_{\mathbf{r}}$ be the ideal of $U(\mathcal{H})$ generated by $h - (r_1 + r_2)$ and the elements

$$\begin{aligned} &\Lambda_{i,\pm n} \quad (n > r_i, \ i = 1,2), \quad \Lambda_{i,k}\Lambda_{i,-r_i} - \Lambda_{i,k-r_i} \quad (0 \leq k \leq r_i, \ i = 1,2), \\ &\| e(\mathbf{k}_1) \cdots e(\mathbf{k}_M) f(\mathbf{m}_M) \cdots f(\mathbf{m}_1) \| \quad (M > r_1 + r_2, \ \mathbf{k}_j, \mathbf{m}_j \in \mathbf{Z}^2, \ \forall j). \end{aligned}$$

Then the following proposition holds, the proof of which will be given in Section 8.

Proposition 2.2. $I_{\mathbf{r}} = J_{\mathbf{r}}$.

3. The quantum toroidal algebra $U_q(\hat{\mathcal{L}})$

3.1. $U_q(\hat{\mathcal{L}})$. Let q and γ be formal vairables and set $F = \mathbf{C}(q,\gamma)$. Let $(a_{i,j})_{0\le i,j\le 1}$ be the Cartan matrix for $A_1^{(1)}$. For $n \in \mathbf{Z}$ set $[n] = (q^n - q^{-n})/(q - q^{-1})$. Let $U_q(\hat{\mathcal{L}})$ be the algebra over F defined by generators

$$x_{i,m}^{\pm}, h_{i,r}, k_i^{\pm 1}, C^{\pm 1} \quad (i = 0, 1, \ m \in \mathbf{Z}, \ r \in \mathbf{Z} \setminus \{0\})$$

and relations

$$k_i^{\pm 1} k_i^{\mp 1} = C^{\pm 1} C^{\mp 1} = 1, \tag{3.1}$$

$$C^{\pm 1} \text{ central}, \quad [k_i, k_j] = [k_i, h_{j,r}] = 0, \tag{3.2}$$

$$[h_{i,r}, h_{i,s}] = \delta_{r+s,0} \frac{[2r]}{r} \frac{C^r - C^{-r}}{q - q^{-1}}, \tag{3.3}$$

$$[h_{i,r}, h_{1-i,s}] = -\delta_{r+s,0} \frac{[r](\gamma^r + \gamma^{-r})}{r} \frac{C^r - C^{-r}}{q - q^{-1}}, \tag{3.4}$$

$$k_i x_{j,m}^{\pm} k_i^{-1} = q^{\pm a_{ij}} x_{j,m}^{\pm}, \tag{3.5}$$

$$\left[h_{i,r}, x_{i,m}^{\pm}\right] = \pm \frac{[2r]}{r} C^{(r\mp|r|)/2} x_{i,r+m}^{\pm}, \tag{3.6}$$

$$\left[h_{i,r}, x_{1-i,m}^{\pm}\right] = \mp \frac{[r](\gamma^r + \gamma^{-r})}{r} C^{(r\mp|r|)/2} x_{1-i,r+m}^{\pm}, \tag{3.7}$$

$$\left[x_{i,m}^{+}, x_{j,n}^{-}\right] = \frac{\delta_{ij}}{q - q^{-1}} \left(C^{-n} \Phi_{i,m+n}^{(+)} - C^{-m} \Phi_{i,m+n}^{(-)}\right), \tag{3.8}$$

$$\left[x_{i,m+1}^{\pm}, x_{i,n}^{\pm}\right]_{q^{\pm 2}} + \left[x_{i,n+1}^{\pm}, x_{i,m}^{\pm}\right]_{q^{\pm 2}} = 0, \tag{3.9}$$

$$\mathrm{Sym}_{m_1,m_2,m_3} \left[x_{i,m_1}^{\pm}, \left[x_{i,m_2}^{\pm}, \left[x_{i,m_3}^{\pm}, x_{1-i,n}^{\pm}\right]_{q^{-2}}\right]\right]_{q^2} = 0 \tag{3.10}$$

where $\mathrm{Sym}_{m_1,m_2,m_3}$ means symmetrization in m_1, m_2 and m_3, $[a,b]_r = ab - rba$, $\Phi_{i,\pm r}^{(\pm)} = 0$ $(r < 0)$ and $\Phi_{i,\pm r}^{(\pm)}$ $(r \ge 0)$ is expressed in terms of $k_i^{\pm 1}$ and the $h_{i,s}$ by

$$\sum_{r \ge 0} \Phi_{i,\pm r}^{(\pm)} z^r = k_i^{\pm 1} \exp\left(\pm(q - q^{-1}) \sum_{r > 0} h_{i,\pm r} z^r\right).$$

Note that our $C^{-r/2} h_{i,r}$ corresponds to $h_{i,r}$ in the standard notation. Note also that relations (3.1)–(3.9) and

$$\left[x_{i,0}^{\pm}, \left[x_{i,0}^{\pm}, \left[x_{i,0}^{\pm}, x_{1-i,0}^{\pm}\right]_{q^{-2}}\right]\right]_{q^2} = 0 \tag{3.11}$$

give relation (3.10) as in [10].

REMARK 1. (1) If $q = \gamma^{\pm 1}$, then this algebra is the same as that in [6].
(2) A vertex operator representation of this algebra in terms of two bosons is easily constructed as in [6].

3.2. $U_q\left(\widehat{sl_2}\right)$. Let $U_q\left(\widehat{sl_2}\right)$ [24] be the algebra over $\mathbf{C}(q)$ defined by generators

$$x_m^{\pm}, h_r, k^{\pm 1}, C^{\pm 1} \quad (m \in \mathbf{Z},\ \ r \in \mathbf{Z} \setminus \{0\})$$

and relations

$$\begin{aligned}
&k^{\pm 1}k^{\mp 1} = C^{\pm 1}C^{\mp 1} = 1,\\
&C^{\pm 1} \text{ central}, \quad [k, h_r] = 0,\\
&[h_r, h_s] = \delta_{r+s,0}\frac{[2r]}{r}\frac{C^r - C^{-r}}{q-q^{-1}},\\
&kx_m^{\pm}k^{-1} = q^{\pm 2}x_m^{\pm},\\
&\left[h_r, x_m^{\pm}\right] = \pm\frac{[2r]}{r}C^{(r\mp|r|)/2}x_{r+m}^{\pm},\\
&\left[x_m^{+}, x_n^{-}\right] = \frac{1}{q-q^{-1}}\left(C^{-n}\Phi_{m+n}^{(+)} - C^{-m}\Phi_{m+n}^{(-)}\right),\\
&\left[x_{m+1}^{\pm}, x_n^{\pm}\right]_{q^{\pm 2}} + \left[x_{n+1}^{\pm}, x_m^{\pm}\right]_{q^{\pm 2}} = 0
\end{aligned}$$

where $\Phi_{\pm r}^{(\pm)} = 0$ $(r < 0)$ and $\Phi_{\pm r}^{(\pm)}$ $(r \geq 0)$ is expressed in terms of $k^{\pm 1}$ and the h_s similarly to $\Phi_{i,r}^{(\pm)}$. In [25] it was shown that this algebra is isomorphic to the algebra [26], [27] defined by generators $e_i, f_i, t_i^{\pm 1}$ $(i = 0, 1)$ and relations

$$\begin{aligned}
&t_i^{\pm 1}t_i^{\mp 1} = 1, \quad t_it_j = t_jt_i,\\
&t_ie_jt_i^{-1} = q^{a_{ij}}e_j, \quad t_if_jt_i^{-1} = q^{-a_{ij}}f_j,\\
&[e_i, f_j] = \frac{\delta_{ij}}{q-q^{-1}}\left(t_i - t_i^{-1}\right),\\
&\left[e_i, \left[e_i, [e_i, e_{1-i}]_{q^2}\right]\right]_{q^{-2}} = 0,\\
&\left[f_i, \left[f_i, [f_i, f_{1-i}]_{q^2}\right]\right]_{q^{-2}} = 0.
\end{aligned}$$

We take the following correspondence of the generators:

$$\begin{aligned}
&e_1 = x_0^{+}, \quad f_1 = x_0^{-}, \quad t_1 = k,\\
&e_0 = Ck^{-1}x_1^{-}, \quad f_0 = x_{-1}^{+}kC^{-1}, \quad t_0 = Ck^{-1}.
\end{aligned} \tag{3.12}$$

Define an automorphism T_1 [28] of $U_q\left(\widehat{sl_2}\right)$ by

$$T_1\colon e_1 \mapsto -f_1 t_1,\quad f_1 \mapsto -t_1^{-1} e_1,\quad t_1 \mapsto t_1^{-1},\quad t_0 \mapsto t_0 t_1^2,$$
$$e_0 \mapsto \frac{1}{[2]}[e_1,[e_1,e_0]_{q^{-2}}],\quad f_0 \mapsto \frac{1}{[2]}[[f_0,f_1]_{q^2},f_1].$$

Let σ be the antiautomorphism of $U_q(\widehat{sl_2})$ determined by

$$\sigma\colon e_i \mapsto e_i,\quad f_i \mapsto f_i,\quad t_i \mapsto t_i^{-1}.$$

3.3. Automorphisms of $U_q(\hat{\mathcal{L}})$. The purpose of this subsection is to define q analogues of the automorphisms $\psi, \mathcal{Y}$ and $\tilde{\mathcal{Y}}$ of $\hat{\mathcal{L}}$, which we denote by the same letters.

Let $\mathcal{X}_j$ $(j = 0, 1)$ and $\mathcal{S}$ be the automorphisms of $U_q(\hat{\mathcal{L}})$ determined by

$$\mathcal{X}_j\colon x_{i,m}^{\pm} \mapsto (-1)^{j\delta_{ij}} x_{i,m\mp\delta_{ij}}^{\pm},\quad h_{i,r} \mapsto h_{i,r},\quad k_i \mapsto C^{-\delta_{ij}} k_i,\quad C \mapsto C,$$
$$\mathcal{S}\colon x_{i,m}^{\pm} \mapsto (-1)^m x_{1-i,m}^{\pm},\quad h_{i,r} \mapsto (-1)^r h_{1-i,r},\quad k_i \mapsto k_{1-i},\quad C \mapsto C$$

and let η be the antiautomorphism of $U_q(\hat{\mathcal{L}})$ determined by

$$\eta\colon x_{i,m}^{\pm} \mapsto x_{i,-m}^{\pm},\quad h_{i,r} \mapsto -C^r h_{i,-r},\quad k_i \mapsto k_i^{-1},\quad C \mapsto C.$$

Let further ρ_h and ρ_v be the homomorphisms $U_q\left(\widehat{sl_2}\right) \to U_q(\hat{\mathcal{L}})$ determined by

$$\rho_h\colon e_i \mapsto x_{i,0}^{+},\quad f_i \mapsto x_{i,0}^{-},\quad t_i \mapsto k_i,$$
$$\rho_v\colon x_m^{\pm} \mapsto x_{1,m}^{\pm},\quad h_r \mapsto h_{1,r},\quad k \mapsto k_1,\quad C \mapsto C.$$

Let $\mathcal{B}$ be the group defined by generators T, Y, S and relations

$$T^{-1} Y T^{-1} = Y^{-1},\quad S Y S^{-1} = Y^{-1},\quad S^2 = 1.$$

Let $\tilde{}$ denote the automorphism of $\mathcal{B}$ determined by

$$\tilde{T} = T,\quad \tilde{Y} = ST,\quad \tilde{S} = Y^{-1} T.$$

We can prove the following propositions.

Proposition 3.1. (1) *There exist automorphisms $\mathcal{T}_1$ and $\mathcal{T}_0$ of $U_q(\hat{\mathcal{L}})$ determined by*

$$\mathcal{T}_1 \circ \rho_v = \rho_v \circ T_1,\quad \mathcal{T}_1 \circ \rho_h = \rho_h \circ T_1,\quad \mathcal{T}_0 = \mathcal{S}\mathcal{T}_1\mathcal{S}^{-1}.$$

The inverse $\mathcal{T}_i^{-1}$ is given by $\eta \mathcal{T}_i \eta$.

(2) *The automorphisms* $\mathcal{T}_i$, $\mathcal{X}_i$ $(i = 0, 1)$ *and* $\mathcal{S}$ *satisfy*

$$\begin{aligned}
&\mathcal{X}_0\mathcal{X}_1 = \mathcal{X}_1\mathcal{X}_0,\\
&\mathcal{T}_i\mathcal{X}_{1-i} = \mathcal{X}_{1-i}\mathcal{T}_i,\\
&\mathcal{T}_i^{-1}\mathcal{X}_i\mathcal{T}_i^{-1} = \mathcal{X}_{1-i}^2\mathcal{X}_i^{-1},\\
&\mathcal{S}\mathcal{T}_i\mathcal{S}^{-1} = \mathcal{T}_{1-i}, \quad \mathcal{S}\mathcal{X}_i\mathcal{S}^{-1} = \mathcal{X}_{1-i}.
\end{aligned}$$

(3) *The homomorphism* $\mathcal{B} \to \mathrm{Aut}(U_q(\hat{\mathcal{L}}))$ *determined by* $T \mapsto \mathcal{T}_1, Y \mapsto \mathcal{X}_1\mathcal{X}_0^{-1}, S \mapsto \mathcal{S}$ *defines a* $\mathcal{B}$ *action on* $U_q(\hat{\mathcal{L}})$.

Proposition 3.2. (1) *There exists an automorphism* ψ *of* $U_q(\hat{\mathcal{L}})$ *determined by*

$$\psi \circ \rho_v = \rho_h, \quad \psi \circ \rho_h = \eta \circ \rho_v \circ \sigma.$$

The inverse ψ^{-1} *is given by* $\eta\psi\eta$.
(2) *The automorphism* ψ *and the* $\mathcal{B}$ *action in the preceding proposition satisfy*

$$\psi(x.u) = \tilde{x}.\psi(u) \quad \textit{for } x \in \mathcal{B} \quad \textit{and} \quad u \in U_q(\hat{\mathcal{L}}).$$

Set $\mathcal{Y} = \mathcal{X}_1\mathcal{X}_0^{-1}$ and $\tilde{\mathcal{Y}} = \mathcal{S}\mathcal{T}_1$. These automorphisms correspond to the actions of Y and $\tilde{Y}$, respectively. Note that $\mathcal{Y}$ maps as

$$\mathcal{Y}\colon x_{0,m}^{\pm} \mapsto x_{0,m\pm1}^{\pm}, \quad x_{1,m}^{\pm} \mapsto -x_{1,m\mp1}^{\pm}, \quad h_{i,r} \mapsto h_{i,r}.$$

Note also that $\mathcal{Y}, \tilde{\mathcal{Y}}$ and ψ satisfy

$$\psi \circ \mathcal{Y} = \tilde{\mathcal{Y}} \circ \psi \tag{3.13}$$

by part (2) of Proposition 3.2. By the above and Lemma 3.2 below, we can see that $\mathcal{Y}, \tilde{\mathcal{Y}}$ and ψ reduce to those in Section 2 in the case $q = 1$.

The above two propositions are proven as in [10] where the case $\gamma = q^{\pm1}$ was studied. Here we only note that to prove them we need the following lemma, which is also proven as in [10].

Lemma 3.1. *The algebra* $U_q(\hat{\mathcal{L}})$ *admits a presentation in terms of generators* $x_{i,0}^{\pm}, x_{i,1}^{\pm}, x_{i,-1}^{\pm}, k_i^{\pm1}$, $i = 0, 1$, $C^{\pm1}$ *and relations*

$$\begin{aligned}
&k_i^{\pm1}k_i^{\mp1} = C^{\pm1}C^{\mp1} = 1,\\
&C \textit{ central}, \quad k_ik_j = k_jk_i,\\
&k_ix_{j,m}^{\pm}k_i^{-1} = q^{\pm a_{ij}}x_{j,m}^{\pm},\\
&\left[x_{i,m}^{+}, x_{i,-m}^{-}\right] = \frac{C^mk_i - C^{-m}k_i^{-1}}{q - q^{-1}},
\end{aligned}$$

$$
\begin{aligned}
&\left[x^{+}_{i,\pm1}, x^{-}_{i,0}\right] = C\left[x^{+}_{i,0}, x^{-}_{i,\pm1}\right],\\
&\left[x^{\pm}_{i,0}, x^{\pm}_{i,-1}\right]_{q^{\pm2}} = \left[x^{\pm}_{i,1}, x^{\pm}_{i,0}\right]_{q^{\pm2}} = 0,\\
&\left[x^{+}_{i,-1}, X^{+}_{i,-2}\right]_{q^{2}} = \left[X^{-}_{i,2}, x^{-}_{i,1}\right]_{q^{-2}} = 0,\\
&\left[x^{+}_{i,m}, x^{-}_{1-i,n}\right] = 0,\\
&\left[I^{\pm}_{i}(0,1), x^{\mp}_{i,0}\right] = 0, \quad \left[I^{\pm}_{i}(-1,0), x^{\mp}_{1-i,0}\right] = 0, \quad \left[I^{\pm}_{i}(0,0), x^{\mp}_{i,0}\right] = 0,\\
&\left[I^{+}_{i}(-2,0), x^{-}_{1-i,0}\right] = 0, \quad \left[I^{-}_{i}(0,2), x^{+}_{i,0}\right] = 0,\\
&I^{\pm}_{i}(0,0) = I^{\pm}_{1-i}(0,0), \quad I^{\pm}_{i}(\mp1,\mp1) = I^{\pm}_{1-i}(\mp1,\mp1),\\
&\left[x^{\pm}_{i,0}, \left[x^{\pm}_{i,0}, \left[x^{\pm}_{i,0}, x^{\pm}_{1-i,0}\right]_{q^{-2}}\right]\right]_{q^{2}} = 0
\end{aligned}
$$

where

$$
\begin{aligned}
&X^{+}_{i,-2} = -\frac{k_i}{[2]}\left[\left[x^{-}_{i,0}, x^{+}_{i,-1}\right], x^{+}_{i,-1}\right]_{q^{-2}}, \quad X^{-}_{i,2} = -\left[x^{-}_{i,1}, \left[x^{-}_{i,1}, x^{+}_{i,0}\right]\right]_{q^{2}} \frac{k_i^{-1}}{[2]},\\
&I^{\pm}_{i}(m,n) = [2]\left[x^{\pm}_{i,m+1}, x^{\pm}_{1-i,n-1}\right]_{q^{\mp2}} + (\gamma+\gamma^{-1})\left[x^{\pm}_{1-i,n}, x^{\pm}_{i,m}\right]_{q^{\mp2}}
\end{aligned}
$$

and $x^{+}_{j,-2}$ *and* $x^{-}_{j,2}$ *in* $I^{\pm}_{i}(\mp1,\mp1)$, $I^{+}_{i}(-2,0)$ *and* $I^{-}_{i}(0,2)$ *should be replaced by* $X^{+}_{j,-2}$ *and* $X^{-}_{j,2}$, *respectively.*

3.4. Miscellaneous results. We will often need the following two lemmas in the next section.

Lemma 3.2. *Set* $\mathbf{x}^{\pm}_{i,m} = \psi(x^{\pm}_{i,m})$, $\mathbf{h}_{i,r} = \psi(h_{i,r})$ *and* $\mathbf{k}_i = \psi(k_i)$. *Then the following hold*:

$$
\begin{aligned}
&\mathbf{x}^{\pm}_{1,0} = x^{\pm}_{1,0}, \quad \mathbf{k}_1 = k_1, \quad \mathbf{x}^{+}_{1,-1} = x^{-}_{0,0}k_0, \quad \mathbf{x}^{-}_{1,1} = k_0^{-1}x^{+}_{0,0},\\
&\mathbf{x}^{+}_{0,0} = x^{-}_{1,-1}k_1 C, \quad \mathbf{x}^{-}_{0,0} = C^{-1}k_1^{-1}x^{+}_{1,1}, \quad \mathbf{k}_0 = C^{-1}k_1^{-1},\\
&\mathbf{x}^{+}_{0,1} = x^{+}_{0,-1}, \quad \mathbf{x}^{-}_{0,-1} = x^{-}_{0,1},\\
&\mathbf{h}_{1,1} = -[x^{+}_{0,0}, x^{+}_{1,0}]_{q^{-2}}, \quad \mathbf{h}_{0,1} = -[x^{+}_{1,1}, x^{+}_{0,-1}]_{q^{-2}},\\
&\mathbf{h}_{1,-1} = -[x^{-}_{1,0}, x^{-}_{0,0}]_{q^{2}}, \quad \mathbf{h}_{0,-1} = -[x^{-}_{0,1}, x^{-}_{1,-1}]_{q^{2}}.
\end{aligned}
$$

Proof. Follows from (3.12) and Proposition 3.2. □

Lemma 3.3. *The following relations hold in* $U_q(\hat{\mathcal{L}})$.

(1) $[2]\left[x^{\pm}_{i,m}, \left[x^{\pm}_{i,m\pm1}, x^{\pm}_{1-i,n}\right]_{q^{-2}}\right] + (\gamma+\gamma^{-1})\left[x^{\pm}_{i,m}, \left[x^{\pm}_{1-i,n\pm1}, x^{\pm}_{i,m}\right]_{q^{-2}}\right] = 0.$

(2) $\left[x^{\pm}_{i,m\pm1}, \left[x^{\pm}_{i,m\pm1}, x^{\pm}_{1-i,n}\right]_{q^{-2}}\right] + \left[x^{\pm}_{i,m}, \left[x^{\pm}_{1-i,n\pm2}, x^{\pm}_{i,m}\right]_{q^{-2}}\right] = 0.$

(3) $[2]\left[x^{\pm}_{1,m+1}, x^{\pm}_{0,n-1}\right]_{q^{\mp2}} + (\gamma+\gamma^{-1})\left[x^{\pm}_{0,n}, x^{\pm}_{1,m}\right]_{q^{\mp2}} = [2]\left[x^{\pm}_{0,n+1}, x^{\pm}_{1,m-1}\right]_{q^{\mp2}} + (\gamma+\gamma^{-1})\left[x^{\pm}_{1,m}, x^{\pm}_{0,n}\right]_{q^{\mp2}}.$

Proof. We shall show the claims for the $x^+_{j,l}$, the proof of those for the $x^-_{j,l}$ being similar. For this it is sufficient to prove the equalities with $m = 0$ since the equalities with $m \neq 0$ follow from these by applying $\mathcal{X}_i^{\pm 1}$.

By calculating

$$\left[\left[x^+_{i,0}, \left[x^+_{i,0}, \left[x^+_{i,0}, x^+_{1-i,n}\right]_{q^{\mp 2}}\right]\right]_{q^{\pm 2}}, x^-_{i,\pm 1}\right] = 0,$$

we obtain part (1) and

$$[2]\left[x^+_{i,-1}, \left[x^+_{i,0}, x^+_{1-i,n}\right]_{q^{-2}}\right] = (\gamma + \gamma^{-1})\left[x^+_{i,0}, \left[x^+_{i,0}, x^+_{1-i,n-1}\right]_{q^{-2}}\right].$$

Apply $\mathcal{X}_i^{-1}$ to the above equality. Then part (2) follows from the result and (1). Calculating the commutator of (1) with $x^-_{i,-1}$, we obtain (3). □

4. Quotient algebras of $U_q(\mathcal{L})_0$

4.1. The quotient algebra $\mathcal{A}$. For an algebra A and a family of elements $(a_j)_{j\in J}$ of A let $\langle a_j \mid j \in J\rangle$ denote the ideal of A generated by the elements a_j ($j \in J$). We assume that any subalgebra of an algebra A contains the identity element of A except in Proposition 4.4 below.

Hereafter we study $U_q(\mathcal{L}) := U_q(\hat{\mathcal{L}})/\langle C-1, k_1k_0-1\rangle$. We denote this algebra by $\mathcal{U}$. Let $\mathcal{U}^+$ and $\mathcal{U}^-$ be the subalgebras of $\mathcal{U}$ generated by the $x^+_{i,m}$ and the $x^-_{i,m}$, respectively, and $\mathcal{U}^0$ the subalgebra generated by k_1, k_1^{-1} and the $h_{i,r}$. The automorphisms of $U_q(\hat{\mathcal{L}})$ in Section 3.3 induce automorphisms of $\mathcal{U}$, which we denote by the same letters.

For $r \in \mathbf{Z} \setminus \{0\}$ and $i = 1, 2$, set

$$a_{i,r} = \frac{\gamma^{\varepsilon_i r} h_{1,r} + q^{-r} h_{0,r}}{\gamma^{\varepsilon_i r} - \gamma^{-\varepsilon_i r}} \quad \text{and} \quad \mathbf{a}_{i,r} = \psi(a_{i,r}),$$

so that

$$a_{1,r} + a_{2,r} = h_{1,r} \quad \text{and} \quad \gamma^{-r} a_{1,r} + \gamma^r a_{2,r} = -q^{-r} h_{0,r}.$$

Note that $a_{i,r} = a_i(0,r)$ and $\mathbf{a}_{i,r} = a_i(r,0)$ in the case $q = 1$ in the notation of Section 2.2.

For a subalgebra A of $\mathcal{U}$ and an integer m let

$$A_m = \left\{u \in A \;\middle|\; k_1 u k_1^{-1} = q^{2m} u\right\}.$$

Set $\mathcal{I} = \sum_{m>0} \mathcal{U}_{-m}\mathcal{U}_m$. Then $\mathcal{I}$ is an ideal of $\mathcal{U}_0$. Set $\mathcal{A} = \mathcal{U}_0/\mathcal{I}$. Clearly the automorphisms $\mathcal{X}_j, \mathcal{Y}, \tilde{\mathcal{Y}}$ and ψ of $\mathcal{U}$ induce automorphisms of $\mathcal{A}$, which we denote by the same symbols.

REMARK 2. If $\mathcal{U}$ admits a q analogue of the triangular decomposition (2.9), $\mathcal{U} \simeq U_q(\mathcal{N}^-) \otimes U_q(\mathcal{H}) \otimes U_q(\mathcal{N}^+)$, then $\mathcal{A} \simeq U_q(\mathcal{H})$. See also part (2) of Remark 3 below.

To study the algebra $\mathcal{A}$, we first prepare several lemmas.

Lemma 4.1. *In $\mathcal{A}$ the following equalities hold.*
(1) $x^+_{1,m}x^+_{0,n+2} - q(\gamma+\gamma^{-1})x^+_{1,m+1}x^+_{0,n+1} + q^2 x^+_{1,m+2}x^+_{0,n} = 0.$
(2) $x^-_{0,m}x^-_{1,n+2} - q^{-1}(\gamma+\gamma^{-1})x^-_{0,m+1}x^-_{1,n+1} + q^{-2} x^-_{0,m+2}x^-_{1,n} = 0.$

Proof. Follows from part (3) of Lemma 3.3. □

For $i = 1, 2$ and $r \in \mathbf{Z}_{>0}$ set

$$d_{i,r} = \frac{(q\gamma^{\varepsilon_i} - (q\gamma^{\varepsilon_i})^{-1})^2((q\gamma^{\varepsilon_i})^r - (q\gamma^{\varepsilon_i})^{-r})}{(\gamma^{\varepsilon_i} - \gamma^{-\varepsilon_i})(q-q^{-1})}\frac{[r]}{r}.$$

Define $\Psi_{i,r} \in \mathcal{U}$ ($i = 1, 2$, $r \in \mathbf{Z}$) by the generating series

$$\sum_{r\geq 0}\Psi_{i,\pm r}z^r = \exp\left(\left(q-q^{-1}\right)\sum_{r>0}(\gamma^{\varepsilon_i r} - \gamma^{-\varepsilon_i r})a_{i,\pm r}z^r\right)$$

and set $\mathbf{\Psi}_{i,r} = \psi(\Psi_{i,r})$.

Lemma 4.2. *In $\mathcal{A}$ the following equalities hold for $i = 1, 2$ and $r > 0$.*
(1) $[[a_{i,\pm r}, [a_{i,\pm 1}, \mathbf{a}_{i,\pm 1}]], \mathbf{a}_{i,\mp 1}] = d_{i,r}\Psi_{i,\pm(r+1)}$.
(2) $[[\mathbf{a}_{i,\pm r}, [\mathbf{a}_{i,\pm 1}, a_{i,\mp 1}]], a_{i,\pm 1}] = d_{i,r}\mathbf{\Psi}_{i,\pm(r+1)}$.

Proof. (1) Using Lemma 3.2 and Lemma 4.1, we find that

$$\begin{aligned}[a_{i,r}, [a_{i,1}, \mathbf{a}_{i,1}]] = &-\frac{\gamma^{\varepsilon_i r}[r]}{q^2 r}\frac{(q\gamma^{\varepsilon_i} - (q\gamma^{\varepsilon_i})^{-1})((q\gamma^{\varepsilon_i})^r - (q\gamma^{\varepsilon_i})^{-r})}{\gamma^{\varepsilon_i} - \gamma^{-\varepsilon_i}} \\ &\times (x^+_{1,r+1}x^+_{0,0} - q^{-1}\gamma^{\varepsilon_i}x^+_{1,r}x^+_{0,1})\end{aligned}$$

in $\mathcal{A}$. Utilizing this equality and the relations

$$\begin{aligned}\Phi^{(+)}_{1,r}x^+_{0,m} = &\, q^{-2}x^+_{0,m}\Phi^{(+)}_{1,r} \\ &+ (1-q^2)\sum_{n=1}^{r}\frac{\gamma^{n+1} - \gamma^{-(n+1)} - q^2(\gamma^{n-1} - \gamma^{-(n-1)})}{q^{n+2}(\gamma - \gamma^{-1})}x^+_{0,m+n}\Phi^{(+)}_{1,r-n}\end{aligned}$$

in $\mathcal{U}$ for $r > 0$, we obtain the claim for the upper sign after a little calculation. The proof of the equality for the lower sign is similar.

(2) The equality $\psi(\mathbf{a}_{i,\pm 1}) = q^{\mp 2}a_{i,\mp 1}$ in $\mathcal{A}$ can be easily checked, using

Lemma 3.2 and Lemma 4.1. Now the claim is obtained by applying ψ to part (1). □

Lemma 4.3. $\mathcal{U}_0 = \mathcal{U}^0\mathcal{U}_0^-\mathcal{U}_0^+ + \mathcal{I}$.

Proof. Let us denote the right hand side of the claim by J. Since $\mathcal{U} = \mathcal{U}^0\mathcal{U}^-\mathcal{U}^+, \mathcal{U}_0 = \sum_{m\in\mathbf{Z}}\mathcal{U}^0\mathcal{U}_m^-\mathcal{U}_{-m}^+$. For $n \geq 0$ and $m \in \mathbf{Z}$ set

$$\mathcal{U}^\pm(n) = \mathrm{Span}\left\{x_{i_1,l_1}^\pm \cdots x_{i_n,l_n}^\pm \;\middle|\; i_j = 0, 1,\ l_j \in \mathbf{Z},\ 1 \leq j \leq n\right\}$$

and $\mathcal{U}^\pm(n)_m = \mathcal{U}^\pm(n) \cap \mathcal{U}_m$. Then $\mathcal{U}^0\mathcal{U}_m^-\mathcal{U}_{-m}^+ = \sum_{r,s\geq 0}\mathcal{U}^0\mathcal{U}^-(r)_m\mathcal{U}^+(s)_{-m}$. Therefore to prove the claim it is sufficient to show that $\mathcal{U}^0\mathcal{U}^-(r)_m\mathcal{U}^+(s)_{-m} \subset J$ for any $r, s \geq 0$ if $m > 0$. This can be easily checked by induction on $r + s$, using the relation

$$\mathcal{U}^-(r)_m\mathcal{U}^+(s)_{-m} \subset \sum_{l,n>0}\mathcal{U}^0\mathcal{U}^-(r-l)_n\mathcal{U}^+(s-l)_{-n} + J \quad \text{if } m > 0.$$

This relation is proved by calculating $[x, y]$ ($x \in \mathcal{U}^-(r)_m$, $y \in \mathcal{U}^+(s)_{-m}$), using (3.5)–(3.8). □

Lemma 4.4. *Let* $^- : \mathcal{U}_0 \to \mathcal{U}_0/\mathcal{I}$ *be the quotient map. The subalgebra* $\overline{\mathcal{U}^0\mathcal{U}_0^\pm}$ *of* $\mathcal{A}$ *is generated by the elements* $k_1, k_1^{-1}, a_{i,r}$ *and* $\mathbf{a}_{i,s}$ ($i = 1, 2,\ r \neq 0,\ \pm s > 0$).

This lemma will be proven in Section 7.

Now we can prove several properties of $\mathcal{A}$.

Proposition 4.1. $\mathcal{A}$ *is generated by the elements* $k_1, k_1^{-1}, a_{i,r}, \mathbf{a}_{i,r}$ ($i = 1, 2,\ r \neq 0$).

Proof. By Lemma 4.3 $\mathcal{U}_0/\mathcal{I} = \overline{\mathcal{U}^0\mathcal{U}_0^-}\,\overline{\mathcal{U}^0\mathcal{U}_0^+}$. Therefore Lemma 4.4 shows that $\mathcal{A}$ is generated by the elements $k_1, k_1^{-1}, a_{i,r}$ and $\mathbf{a}_{i,r}$ ($i = 1, 2,\ r \neq 0$). □

For $i = 1, 2$ let $\mathcal{A}^i$ be the subalgebra of $\mathcal{A}$ generated by the elements $a_{i,r}, \mathbf{a}_{i,r}$ ($r \neq 0$). Then $\mathcal{A}^i$ is a q analogue of $U(\mathcal{H}_i)$ and the following proposition holds.

Proposition 4.2. (1) $\mathcal{A}^i$ *is generated by the elements* $a_{i,r}$ *and* $\mathbf{a}_{i,r}$ ($r = \pm 1$).
(2) $\mathcal{A}^1$ *and* $\mathcal{A}^2$ *commute with each other.*

Proof. (1) Follows from Lemma 4.2.

(2) By Lemma 3.2 and Lemma 4.1, $[a_{i,r}, \mathbf{a}_{j,s}] = 0$ ($i \neq j,\ r, s = \pm 1$). So the claim follows from part (1). □

Proposition 4.3. *In $\mathcal{A}$ the following hold.*

(1) $\mathcal{Y}(a_{i,r}) = a_{i,r}$, $\mathcal{Y}(\mathbf{a}_{i,r}) = (-q\gamma^{\varepsilon_i})^r \mathbf{a}_{i,r}$.

(2) $\tilde{\mathcal{Y}}(a_{i,r}) = (-q\gamma^{\varepsilon_i})^{-r} a_{i,r}$, $\tilde{\mathcal{Y}}(\mathbf{a}_{i,r}) = \mathbf{a}_{i,r}$.

(3) $\psi(\mathbf{a}_{i,r}) = q^{-2r} a_{i,-r}$.

(4) $\psi^2(a_{i,r}) = q^{-2r} a_{i,-r}$, $\psi^2(\mathbf{a}_{i,r}) = q^{-2r} \mathbf{a}_{i,-r}$.

Proof. The first equality of part (1) is immediate from the definition of $\mathcal{Y}$. The second equality of part (2) follows from this by applying ψ thanks to (3.13).

We shall show the second equality of part (1) and part (3) by induction on $|r|$. The rest of the claims follow from these. The case $|r| = 1$ is easily checked, using Lemma 3.2 and Lemma 4.1. Combining this case with Lemma 4.2 proves the case $|r| > 1$. □

4.2. The quotient algebras $\mathcal{A}_N$ and $\mathcal{A}_{\mathbf{r}}$. Define $\Lambda_r, \Lambda_{i,r} \in \mathcal{U}$ ($r \in \mathbf{Z}$, $i = 1, 2$) by the generating series

$$\sum_{r\geq 0} \Lambda_{\pm r} z^r = \exp\left(-\sum_{r>0} \frac{h_{1,\pm r}}{[r]} z^r\right) =: \Lambda^{\pm}(z), \tag{4.1}$$

$$\sum_{r\geq 0} \Lambda_{i,\pm r} z^r = \exp\left(-\sum_{r>0} \frac{a_{i,\pm r}}{[r]} z^r\right) =: \Lambda_i^{\pm}(z). \tag{4.2}$$

Set $\mathbf{\Lambda}_r = \psi(\Lambda_r)$ and $\mathbf{\Lambda}_{i,r} = \psi(\Lambda_{i,r})$. Set further $\Lambda_r^{(m)} = \tilde{\mathcal{Y}}^m(\Lambda_r)$ and $\mathbf{\Lambda}_r^{(m)} = \mathcal{Y}^m(\mathbf{\Lambda}_r)$ for $m \in \mathbf{Z}$. (We can show that $\mathbf{\Lambda}_r^{(m)} = \psi\left(\Lambda_r^{(-m)}\right)$ in $\mathcal{A}$.)

For a nonnegative integer N let $\mathcal{I}_N$ be the ideal of $\mathcal{U}_0$ generated by $\mathcal{I}$, $\sum_{m>N} \mathcal{U}_m \mathcal{U}_{-m}$ and $k_1 - q^N$. Set $\mathcal{A}_N = \mathcal{U}_0/\mathcal{I}_N$. Note that we can regard this as a quotient algebra of $\mathcal{A}$. By Proposition 4.1 this algebra is generated by the elements $a_{i,r}$ and $\mathbf{a}_{i,r}$ ($i = 1, 2$, $r \neq 0$). The automorphisms $\mathcal{Y}$, $\tilde{\mathcal{Y}}$ and ψ of $\mathcal{U}$ induce automorphisms of $\mathcal{A}_N$, which we denote by the same symbols.

First we prepare several lemmas.

Lemma 4.5. *In $\mathcal{A}_N$ the following hold.*

(1) $\Lambda_N^{(m)} \mathbf{a}_{i,r} = (q\gamma^{\varepsilon_i})^{-2r} \mathbf{a}_{i,r} \Lambda_N^{(m)}$.

(2) $\mathbf{\Lambda}_N^{(m)} a_{i,r} = (q\gamma^{\varepsilon_i})^{2r} a_{i,r} \mathbf{\Lambda}_N^{(m)}$.

To prove this lemma we need the following lemma, which is an immediate consequence of [29, Lemma 5.1] and [23, Proposition 4.3].

Lemma 4.6. *For $n \in \mathbf{Z}_{\geq 0}$ set $[n]! = [1][2]\cdots[n]$ and $\left(x^{\pm}_{i,m}\right)^{(n)} = \left(x^{\pm}_{i,m}\right)^n/[n]!$.*

(1)

$$(x^+_{1,0})^{(n+m)}(x^-_{1,1})^{(n)} \equiv (q^{-n(n-m-1)}/[m]!)\left(\sum_{j=0}^{n} q^{-j}\Lambda_{n-j}\left(\sum_{\substack{l_1,\ldots,l_m\geq 0\\ l_1+\cdots+l_m=j}} x^+_{1,l_1}\cdots x^+_{1,l_m}\right)\right)(-k_1)^n \mod \sum_{n\geq l>0}\mathcal{U}_{-l}\mathcal{U}_{l+m}$$

for $m, n \geq 0$. In particular

$$\left(x^+_{1,0}\right)^{(n+1)}\left(x^-_{1,1}\right)^{(n)} \equiv q^{-n(n-2)}\left(\sum_{j=0}^{n} q^{-j}\Lambda_{n-j}x^+_{1,j}\right)(-k_1)^n \mod \sum_{n\geq l>0}\mathcal{U}_{-l}\mathcal{U}_{l+1}, \tag{4.3}$$

$$\left(x^+_{1,0}\right)^{(n)}\left(x^-_{1,1}\right)^{(n)} \equiv q^{-n(n-1)}\Lambda_n(-k_1)^n \mod \sum_{n\geq l>0}\mathcal{U}_{-l}\mathcal{U}_{l}. \tag{4.4}$$

(2) *In $\mathcal{A}_N$ the following hold.*

(i) $\Lambda^{(m)}_{\pm r} = 0 \quad (r > N)$.

(ii) $\Lambda^{(m)}_{N}\Lambda^{(m)}_{-r} = \Lambda^{(m)}_{N-r}$ $(0 \leq r \leq N)$. *In particular* $\Lambda^{(m)}_{-N} = \left(\Lambda^{(m)}_{N}\right)^{-1}$.

Proof of Lemma 4.5. We shall show part (1) with $m = 0$. The rest of the claims follow from this by applying $\tilde{\mathcal{Y}}^m$ and $\mathcal{Y}^m\psi$ and using Proposition 4.3. Using the equalities

$$x^+_{1,m}\Lambda_r = \sum_{l=0}^{r}[l+1]\Lambda_{r-l}x^+_{1,m+l},$$
$$x^+_{0,m}\Lambda_r = \Lambda_r x^+_{0,m} - \theta(r\geq 1)\left(\gamma+\gamma^{-1}\right)\Lambda_{r-1}x^+_{0,m+1} + \theta(r\geq 2)\Lambda_{r-2}x^+_{0,m+2}$$

for $r \geq 0$ ($\theta(\,\cdot\,)$ is a step function) and Lemma 4.1, we obtain

$$x^+_{1,0}x^+_{0,0}\Lambda_N = \Lambda_N x^+_{1,0}x^+_{0,0} + \sum_{l=0}^{N-1} q^{-l}\Lambda_{N-1-l}\left([2]x^+_{1,l+1}x^+_{0,0} - \left(\gamma+\gamma^{-1}\right)x^+_{1,l}x^+_{0,1}\right) \tag{4.5}$$

in $\mathcal{A}$. By the equality $\left(x^+_{1,0}\right)^{(N+1)}\left(x^-_{1,1}\right)^{(N)}x^+_{0,0} = 0$ in $\mathcal{A}_N$ and (4.3),

$$0 = \Lambda_N x^+_{1,0}x^+_{0,0} + \sum_{l=0}^{N-1} q^{-(l+1)}\Lambda_{N-1-l}x^+_{1,l+1}x^+_{0,0}$$

in $\mathcal{A}_N$. Apply $\mathcal{Y}^2 - 1$ to the above equality. By summing the result and (4.5), we find that

$$(\mathbf{a}_{1,1} + \mathbf{a}_{2,1})\Lambda_N = \Lambda_N((q\gamma)^2\mathbf{a}_{1,1} + (q\gamma^{-1})^2\mathbf{a}_{2,1}).$$

in $\mathcal{A}_N$. Applying $\mathcal{Y}^l$ ($l \in \mathbf{Z}$) to the above and using Proposition 4.3, we obtain part (1) with $r = 1$. By Proposition 4.3 $\psi^2(h_{1,r}) = q^{-2r} h_{1,-r}$. This and part (2)-(ii) of Lemma 4.6 imply that $\psi^2(\Lambda_N) = q^{-2N}\Lambda_N^{-1}$. Therefore part (1) with $r = -1$ follows from the case $r = 1$ by applying ψ^2. The case $|r| > 1$ can be proven by using Lemma 4.2 (2). □

Lemma 4.7. *For integers r, s set*

$$\Pi_{r,s} = \Lambda_{1,r}\Lambda_{2,s} \quad \textit{and} \quad P_{r,s} = \Pi_{r,s}\Pi_{-r,-s}.$$

Then the following hold in $\mathcal{A}_N$.

(1) (i) $\Pi_{r,s} = 0,\ \Pi_{-r,-s} = 0$ ($r + s > N,\ r, s \geq 0$).
(ii) $\Pi_{r,N-r}\Pi_{-s,s-N} = 0$ ($0 \leq r \neq s \leq N$).

(2) (i) $P_{r,N-r} \in Z(\mathcal{A}_N)$ ($0 \leq r \leq N$).
(ii) $\sum_{r=0}^{N} P_{r,N-r} = 1$.
(iii) $P_{r,N-r}P_{s,N-s} = \delta_{r,s}P_{r,N-r}$ ($0 \leq r, s \leq N$).

Proof. Since $\Lambda^{\pm}(z) = \Lambda_1^{\pm}(z)\Lambda_2^{\pm}(z)$, Proposition 4.3 gives the equality

$$\Lambda_{\pm r}^{(m)} = (-q/\gamma)^{\mp rm}\sum_{j=0}^{r}\gamma^{\mp 2mj}\Pi_{\pm j,\pm(r-j)} \tag{4.6}$$

in $\mathcal{A}$ for $r \geq 0$ and any integer m. So part (2)-(i) of Lemma 4.6 gives (1)-(i). By part (2)-(ii) of Lemma 4.6 $\Lambda_N^{(m)}\Lambda_{-N}^{(n)} = \Lambda_N^{(m)}\bigl(\Lambda_N^{(n)}\bigr)^{-1}$. Lemma 4.5 and Proposition 4.1 show that this element commutes with all the generators of $\mathcal{A}_N$. Therefore $\Lambda_N^{(m)}\Lambda_{-N}^{(n)} \in Z(\mathcal{A}_N)$ for any m, n. This and (4.6) imply that $\pi_{r,s} := \Pi_{r,N-r}\Pi_{-s,s-N} \in Z(\mathcal{A}_N)$ for $0 \leq r, s \leq N$. So $\pi_{r,s}$ commutes with $\mathbf{\Lambda}_N$, while $\mathbf{\Lambda}_N\pi_{r,s} = \gamma^{4(r-s)}\pi_{r,s}\mathbf{\Lambda}_N$ by Lemma 4.5. These prove (1)-(ii) and (2)-(i) since $\mathbf{\Lambda}_N$ is invertible by Lemma 4.6 (2) and the use of the automorphism ψ. Part (2)-(ii) follows from (4.6) and $\Lambda_N^{(m)}\Lambda_{-N}^{(m)} = 1$, and the last claim follows from (1)-(ii) and (2)-(ii). □

Now we can derive the properties of $\mathcal{A}_N$ which we need. For $\mathbf{r} \in \mathbf{Z}^2_{\geq 0}$ set $\mathbf{P_r} = \psi(P_\mathbf{r})$. For a nonnegative integer N let Z_N be the set of pairs of nonnegative integers $\mathbf{r} = (r_1, r_2)$ such that $r_1 + r_2 = N$. For $\mathbf{r} \in Z_N$ set $\mathcal{A}_\mathbf{r} = \mathcal{A}_N P_\mathbf{r}\mathbf{P_r}$. Note that the automorphisms $\mathcal{Y}$ and $\tilde{\mathcal{Y}}$ preserve each $\mathcal{A}_\mathbf{r}$ by Proposition 4.3.

Proposition 4.4. (1) $\mathcal{A}_N = \bigoplus_{\mathbf{r}\in Z_N}\mathcal{A}_\mathbf{r}$ *is a direct sum of nonzero subalgebras with* $1_{\mathcal{A}_\mathbf{r}} = P_\mathbf{r}\mathbf{P_r}$.

(2) $\mathcal{A}_\mathbf{r} \simeq \mathcal{A}_N/\langle 1 - P_\mathbf{r}\rangle$ ($aP_\mathbf{r}\mathbf{P_r} \leftrightarrow \overline{a}$).

(3) *In $\mathcal{A}_\mathbf{r}$ the following hold*:
(i) $\Lambda_{i,\pm s} = 0,\ \mathbf{\Lambda}_{i,\pm s} = 0$ ($s > r_i,\ i = 1, 2$).
(ii) $\Lambda_{i,r_i}\Lambda_{i,-s} = \Lambda_{i,r_i-s},\ \mathbf{\Lambda}_{i,r_i}\mathbf{\Lambda}_{i,-s} = \mathbf{\Lambda}_{i,r_i-s}$ ($0 \leq s \leq r_i,\ i = 1, 2$).

(iii) $\Lambda_{i,r_i}\mathbf{a}_{j,s} = (q\gamma^{\varepsilon_i})^{-2s\delta_{i,j}}\mathbf{a}_{j,s}\Lambda_{i,r_i}$, $\mathbf{\Lambda}_{i,r_i}a_{j,s} = (q\gamma^{\varepsilon_i})^{2s\delta_{i,j}}a_{j,s}\mathbf{\Lambda}_{i,r_i}$.
Here for $x \in \mathcal{A}_N$ the element x in $\mathcal{A}_{\mathbf{r}}$ stands for $xP_{\mathbf{r}}\mathbf{P}_{\mathbf{r}} \in \mathcal{A}_{\mathbf{r}}$ or the image of x in $\mathcal{A}_N/\langle 1 - P_{\mathbf{r}}\rangle \simeq \mathcal{A}_{\mathbf{r}}$.

Proof. For $\mathbf{r} = (r_1, r_2)$ and $\mathbf{r}' = (r_1', r_2') \in Z_N$ set

$$\mathcal{A}_{\mathbf{r},\mathbf{r}'} = \mathcal{A}_N/\langle 1 - P_{\mathbf{r}}, 1 - \mathbf{P}_{\mathbf{r}'}\rangle.$$

First we shall show that part (3) for the $\Lambda_{i,s}$ holds in $\mathcal{A}_{\mathbf{r},\mathbf{r}'}$. Part (3) for the $\mathbf{\Lambda}_{i,s}$ with the r_i replaced by the r_i' is similarly proved. The elements $\Lambda_{i,\pm r_i}$ and $\Pi_{\pm\mathbf{r}}$ are invertible in $\mathcal{A}_{\mathbf{r},\mathbf{r}'}$. Noting this, we can see that (3)-(i) holds by part (1) of Lemma 4.7. By (3)-(i) and (4.6),

$$\Lambda_N^{(m)} = (-q)^{-Nm}\gamma^{m(r_2-r_1)}\Lambda_{1,r_1}\Lambda_{2,r_2}. \tag{4.7}$$

Noting this and substituting (4.6) into part (2)-(ii) of Lemma 4.6, we obtain

$$\sum_{s=0}^{r}\gamma^{2ms}\Lambda_{1,r_1}\Lambda_{1,-s}\Lambda_{2,r_2}\Lambda_{2,-(r-s)} = \sum_{s=r-r_2}^{r_1}\gamma^{2ms}\Lambda_{1,r_1-s}\Lambda_{r_2-(r-s)}$$

for $0 \le r \le N$ and any integer m. This proves (3)-(ii). Part (3)-(iii) follows from (4.7), Lemma 4.5 and part (2) of Proposition 4.2.

By Lemma 4.7 and a similar result for the $\mathbf{P}_{\mathbf{r}}$, the sum $1 = \sum_{\mathbf{r},\mathbf{r}'\in Z_N} P_{\mathbf{r}}\mathbf{P}_{\mathbf{r}'}$ is a decomposition of 1 into a sum of orthogonal central idempotents if we allow some of the $P_{\mathbf{r}}\mathbf{P}_{\mathbf{r}'}$ to be 0. So $\mathcal{A}_N = \bigoplus_{\mathbf{r},\mathbf{r}'\in Z_N}\mathcal{A}_N P_{\mathbf{r}}\mathbf{P}_{\mathbf{r}'}$ is a direct sum of subalgebras. Moreover $\langle 1 - P_{\mathbf{r}}, 1 - \mathbf{P}_{\mathbf{r}'}\rangle = \mathcal{A}_N(1 - P_{\mathbf{r}}) + \mathcal{A}_N(1 - \mathbf{P}_{\mathbf{r}'}) = \bigoplus_{(\mathbf{s},\mathbf{s}')\neq(\mathbf{r},\mathbf{r}')}\mathcal{A}_N P_{\mathbf{s}}\mathbf{P}_{\mathbf{s}'}$. Therefore $\mathcal{A}_{\mathbf{r},\mathbf{r}'} \simeq \mathcal{A}_N P_{\mathbf{r}}\mathbf{P}_{\mathbf{r}'}$. By part (3)-(iii) for $\mathcal{A}_{\mathbf{r},\mathbf{r}'}$ we obtain

$$\begin{aligned}\Lambda_{i,r_i}\mathbf{\Lambda}_{i,r_i'} &= (q\gamma^{\varepsilon_i})^{-2r_i'}\mathbf{\Lambda}_{i,r_i'}\Lambda_{i,r_i} \quad \text{(by the first equality)}\\ &= (q\gamma^{\varepsilon_i})^{-2r_i}\mathbf{\Lambda}_{i,r_i'}\Lambda_{i,r_i} \quad \text{(by the second equality)}\end{aligned}$$

in $\mathcal{A}_{\mathbf{r},\mathbf{r}'}$. This imlplies that $1 = 0$ in $\mathcal{A}_{\mathbf{r},\mathbf{r}'}$ if $\mathbf{r} \neq \mathbf{r}'$. Therefore $\mathcal{A}_{\mathbf{r},\mathbf{r}'} = 0$ and $P_{\mathbf{r}}\mathbf{P}_{\mathbf{r}'} = 0$ in $\mathcal{A}_N$ if $\mathbf{r} \neq \mathbf{r}'$. By this $\langle 1 - P_{\mathbf{r}}, 1 - \mathbf{P}_{\mathbf{r}}\rangle = \langle 1 - P_{\mathbf{r}}\rangle$. The fact that $P_{\mathbf{r}}\mathbf{P}_{\mathbf{r}} \neq 0$ follows from the result that there exists a nontrivial $\mathcal{A}_{\mathbf{r}}$ module (See Proposition 4.5 below). This completes the proof. □

Corollary 4.1. *$\mathcal{A}_{\mathbf{r}}$ is generated by the elements $\Lambda_{i,s}, \Lambda_{i,r_i}^{-1}, \mathbf{\Lambda}_{i,s}$ and $\mathbf{\Lambda}_{i,r_i}^{-1}$* $(0 \le s \le r_i,\ i = 1, 2)$.

Proof. By the definition of the $\Lambda_{i,s}$ and the $\mathbf{\Lambda}_{i,s}$, the elements $a_{i,r}$ and $\mathbf{a}_{i,r}$ are expressed in terms of them. Therefore the claim follows from part (3) of the proposition and Proposition 4.1. □

The following corollary is immediate.

Corollary 4.2. *For an $\mathcal{A}_N$ module V and $\mathbf{r} \in Z_N$ set $V_{\mathbf{r}} = P_{\mathbf{r}}\mathbf{P}_{\mathbf{r}}V$. Then $V = \bigoplus_{\mathbf{r}\in Z_N} V_{\mathbf{r}}$ is a direct sum of $\mathcal{A}_N$ modules and the $\mathcal{A}_N$ module structure on V induces an $\mathcal{A}_{\mathbf{r}}$ module structure on $V_{\mathbf{r}}$.*

4.3. The homomorphism $\boldsymbol{\phi}_{\mathbf{r}}\colon \boldsymbol{\mathcal{A}}_{\mathbf{r}} \to \boldsymbol{C}_{\mathbf{1}}^{r_1} \otimes \boldsymbol{C}_{\mathbf{2}}^{r_2}$. In this subsection, we first obtain a representation of $\mathcal{U}$ which induces a representation of $\mathcal{A}_{\mathbf{r}}$ when restricted to the highest weight space. Then, using this, we prove the existence of a homomorphism from $\mathcal{A}_{\mathbf{r}}$ to some algebra $C_1^{r_1} \otimes C_2^{r_2}$.

Set $p_\alpha^{1/2} = q\gamma^{\varepsilon_\alpha}$ for $\alpha = 1, 2$. For a sequence $\mathbf{a} = (\alpha_1, \dots, \alpha_m)$ of integers 1 and 2, define the F algebra $C_{\mathbf{a}}$ to be the vector space $F(y_1, \dots, y_m)\otimes F\bigl[x_1, x_1^{-1}, \dots, x_m, x_m^{-1}\bigr]$ with multiplication rule

$$\left(f \otimes \prod_i x_i^{l_i}\right)\left(g \otimes \prod_i x_i^{n_i}\right) = fg' \otimes \prod_i x_i^{l_i+n_i}$$

where $g'(y_1, \dots, y_m) = g\bigl(p_{\alpha_1}^{l_1} y_1, \dots, p_{\alpha_m}^{l_m} y_m\bigr)$. For simplicity we shall write $f \prod_i x_i^{l_i}$ for $f \otimes \prod_i x_i^{l_i} \in C_{\mathbf{a}}$. For $1 \le i < m$ we identify $C_{(\alpha_1,\dots,\alpha_i)} \otimes C_{(\alpha_{i+1},\dots,\alpha_m)}$ with a subalgebra of $C_{\mathbf{a}}$ via the correspondence

$$f \prod_{j=1}^{i} x_j^{l_j} \otimes g \prod_{j=1}^{m-i} x_j^{n_j} \mapsto f(y_1, \dots, y_i)g(y_{i+1}, \dots, y_m) \prod_{j=1}^{i} x_j^{l_j} \prod_{j=1}^{m-i} x_{i+j}^{n_j}.$$

Let $V_{\mathbf{a}} = C_{\mathbf{a}} \otimes (F^2)^{\otimes m}$. We shall write av for $a \otimes v \in V_{\mathbf{a}}$. We shall also write af for $a\otimes f \in C_{\mathbf{a}}\otimes\mathrm{End}((F^2)^{\otimes m})$ and regard this as an element of $\mathrm{End}(V_{\mathbf{a}})$ by letting a act on $C_{\mathbf{a}}$ by left multiplication. We denote the canonical basis of F^2 by v_1 and v_2. Let $E_{ij}^{(l)}$ signify the matrix unit E_{ij} acting on the l-th factor of the tensor product $(F^2)^{\otimes m}$ and set $H_l = E_{11}^{(l)} - E_{22}^{(l)}$.

Proposition 4.5. *Let $\mathbf{a} = (\alpha_1, \dots, \alpha_N)$ be a sequence of integers* 1 *and* 2.
(1) *The following assignment defines a $\mathcal{U}$ module structure on $V_{\mathbf{a}}$:*

$$x_{1,m}^+ \mapsto \sum_{i=1}^{N} \prod_{i<j} \frac{q^{H_j} y_i - q^{-H_j} y_j}{y_i - y_j} y_i^m E_{12}^{(i)},$$

$$x_{1,m}^- \mapsto \sum_{i=1}^{N} \prod_{j<i} \frac{q^{H_j} y_i - q^{-H_j} y_j}{y_i - y_j} y_i^m E_{21}^{(i)},$$

$$x_{0,m}^+ \mapsto \sum_{i=1}^{N} \prod_{i<j} \frac{q^{-H_j} p_{\alpha_i}^{1/2} y_i - \left(q p_{\alpha_j}^{-1/2}\right)^{H_j} y_j}{p_{\alpha_i}^{1/2} y_i - p_{\alpha_j}^{-H_j/2} y_j} p_{\alpha_i}^{m/2} y_i^m x_i E_{21}^{(i)},$$

$$
\begin{aligned}
x^-_{0,m} &\mapsto \sum_{i=1}^{N} \prod_{j<i} \frac{q^{-H_j} p_{\alpha_i}^{-1/2} y_i - \left(q p_{\alpha_j}^{-1/2}\right)^{H_j} y_j}{p_{\alpha_i}^{-1/2} y_i - p_{\alpha_j}^{-H_j/2} y_j} p_{\alpha_i}^{-m/2} y_i^m x_i^{-1} E_{12}^{(i)},\\
h_{1,r} &\mapsto \frac{[r]}{r} \sum_{i=1}^{N} y_i^r \left(q^{-r} E_{11}^{(i)} - q^r E_{22}^{(i)}\right),\\
h_{0,r} &\mapsto \frac{[r]}{r} \sum_{i=1}^{N} y_i^r \left(q^{-r} p_{\alpha_i}^{r/2} E_{22}^{(i)} - q^r p_{\alpha_i}^{-r/2} E_{11}^{(i)}\right),\\
k_1 &\mapsto q^{\sum_{i=1}^N H_i}.
\end{aligned}
$$

(2) *For $i = 1, 2$ let r_i be the number of i among $\alpha_1, \dots, \alpha_N$ and set $\mathbf{r} = (r_1, r_2)$. Let $w = v_1 \otimes v_1 \otimes \cdots \otimes v_1 \in (F^2)^{\otimes N}$. Then the above $\mathcal{U}$ module structure induces an $\mathcal{A}_{\mathbf{r}}$ module strucutre on $W_{\mathbf{a}} := C_{\mathbf{a}} \otimes w$.*

Proof. (1) The relations except (3.10) are easily checked. Since relations (3.10) follow from (3.11) and those already proved, it suffices to check (3.11). This can be done by the often used technique as follows. Let $X^\pm_{i,0}$ and $K_i^{\pm 1}$ be the images of $x^\pm_{i,0}$ and $k_i^{\pm 1}$ under the map in the proposition. For $i = 0, 1$ the formulas

$$
e.g = X^+_{i,0} g - K_i g K_i^{-1} X^+_{i,0}, \quad f.g = \left(X^-_{i,0} g - g X^-_{i,0}\right) K_i, \quad t^{\pm 1}.g = K_i^{\pm 1} g K_i^{\mp 1}
$$

for $g \in \mathrm{End}(V_{\mathbf{a}})$ define a $U_q(sl_2)$ module strucure on $\mathrm{End}(V_{\mathbf{a}})$. (The above formulas are the same as those in adjoint representations. See, for example, [30, Section 4.18].) Since $f.X^+_{1-i,0} = 0$, $t.X^+_{1-i,0} = q^{-2} X^+_{1-i,0}$ and e is nilpotent, we obtain $e^3.X^+_{1-i,0} = 0$, which is equivalent to (3.11) for the $x^+_{j,0}$. The relations for the $x^-_{j,0}$ can be checked similarly.

(2) Clearly $\mathcal{U}_0$ preserves $W_{\mathbf{a}}$ and $\mathcal{I}_N$ annihilates it. Since

$$
\Lambda_{i,\pm l}(u \otimes w) = (-q)^{\mp l} \sum_{\substack{1 \le j_1 < \cdots < j_l \le N \\ \alpha_{j_1} = \cdots = \alpha_{j_l} = i}} y_{j_1}^{\pm 1} \cdots y_{j_l}^{\pm 1} u \otimes w \tag{4.8}
$$

for $l > 0$ and $u \in C_{\mathbf{a}}$, $\Lambda_{i,r_i} \Lambda_{i,-r_i} = 1$ on $W_{\mathbf{a}}$. Therefore the claim follows from part (2) of Proposition 4.4. □

REMARK 3. (1) The coproduct

$$
\begin{aligned}
&\Delta(x^+_{i,m}) = \sum_{l \ge 0} x^+_{i,m-l} \otimes \Phi^{(+)}_{i,l} + 1 \otimes x^+_{i,m}, \quad \Delta(x^-_{i,m}) = x^-_{i,m-l} \otimes 1 + \sum_{l \ge 0} \Phi^{(-)}_{i,-l} \otimes x^-_{i,m+l},\\
&\Delta(h_{i,r}) = h_{i,r} \otimes 1 + 1 \otimes h_{i,r}, \quad \Delta(k_i) = k_i \otimes k_i
\end{aligned}
$$

by Drinfel'd defines a $\mathcal{U}$ module structure on tensor products of $V_{(1)}$ and $V_{(2)}$. This

action contains infinite sums, but we can sum them up to obtain the expression in the proposition.
(2) Let A be the subalgebra of $\mathcal{U}_0$ generated by k_1, k_1^{-1}, $h_{i,r}$ and $\mathbf{h}_{i,r}$ ($i = 0, 1$, $r \neq 0$). By Proposition 4.2 $a_{i,r}$ and $\mathbf{a}_{1-i,s}$ commute in $\mathcal{A}$, while they generally do not commute on representations in the above proposition. Therefore the surjective homomorphism $A \to \mathcal{A}$ ($h_{i,r} \mapsto h_{i,r}, \mathbf{h}_{i,r} \mapsto \mathbf{h}_{i,r}, k_1 \mapsto k_1$) is not an isomorphism.

For $\mathbf{a} = (\overbrace{\alpha, \dots, \alpha}^{m})$ ($\alpha = 1$ or 2) set $C_\alpha^m = C_{\mathbf{a}}$ and define elements e_l and D_l ($l \geq 0$) of C_α^m by

$$e_l = \sum_{\substack{I\subset\{1,2,\dots,m\}\\ |I|=l}} \prod_{i\in I} y_i \quad \text{and} \quad D_l = \sum_{\substack{I\subset\{1,2,\dots,m\}\\ |I|=l}} \prod_{\substack{i\in I\\ j\notin I}} \frac{q y_i - q^{-1} y_j}{y_i - y_j} \prod_{i\in I} x_i. \tag{4.9}$$

Proposition 4.6. *For $\mathbf{r} = (r_1, r_2) \in \mathbf{Z}_{\geq 0}^2$ there exists a homomorphism $\phi_{\mathbf{r}} \colon \mathcal{A}_{\mathbf{r}} \to C_1^{r_1} \otimes C_2^{r_2}$ determined by*

$$\Lambda_{1,l} \mapsto e_l \otimes 1, \quad \Lambda_{2,l} \mapsto 1 \otimes e_l, \quad \boldsymbol{\Lambda}_{1,l} \mapsto D_l \otimes 1, \quad \boldsymbol{\Lambda}_{2,l} \mapsto 1 \otimes D_l \quad (l > 0).$$

To prove this proposition, we need the following lemma.

Lemma 4.8. *Let $\mathbf{a}$ and w be the ones in* Proposition 4.5. *For $1 \leq i_1 < i_2 < \cdots < i_n \leq N$ let $w_{i_1,i_2,\dots,i_n}$ denote w with the v_1's in the $i_1, \dots, i_n$ th factors replaced by the v_2's. For $1 \leq i \neq j \leq N$ set $c_{i,j} = (q y_i - q^{-1} y_j)/(y_i - y_j)$. Then for $u \in C_{\mathbf{a}}$ the following hold.*

(1)
$$\left(x_{0,m}^+\right)^{(n)} (u \otimes w) = \sum_{1\leq i_1<i_2<\cdots<i_n\leq N} \prod_{k=1}^{n} \prod_{\substack{i_k<j\\ j\neq i_l\,(k<l\leq n)}} \frac{q^{-1} p_{\alpha_{i_k}}^{1/2} y_{i_k} - q p_{\alpha_j}^{-1/2} y_j}{p_{\alpha_{i_k}}^{1/2} y_{i_k} - p_{\alpha_j}^{-1/2} y_j} \times \prod_{k=1}^{n} p_{\alpha_{i_k}}^{m/2} y_{i_k}^m x_{i_k} u \otimes w_{i_1,i_2,\dots,i_n}.$$

(2)
$$\left(x_{1,m}^+\right)^{(n)} \left(u \otimes w_{i_1,i_2,\dots,i_n}\right) = \prod_{k=1}^{n} \left(y_{i_k}^m \prod_{\substack{i_k<j\\ j\neq i_l\,(k<l\leq n)}} c_{i_k,j} \right) u \otimes w.$$

Proof. (1) The claim can be proven by induction on n, using the equality

$$\sum_{i=1}^{n} \prod_{\substack{j=1\\ j\neq i}}^{n} c_{i,j} = [n].$$

(2) A little calculation gives

$$\begin{aligned}
&\left(x_{1,m}^+\right)^n \left(u \otimes w_{i_1,i_2,\ldots,i_n}\right) \\
&= \sum_{\sigma\in S_n} \prod_{k=1}^{n} \left(y_{i_k}^m \prod_{i_{\sigma(k)}<j} c_{i_{\sigma(k)},j} \prod_{\substack{\sigma(k)<l \\ l\neq\sigma(m)\,(1\le m<k)}} \left(c_{i_l,i_{\sigma(k)}}/c_{i_{\sigma(k)},i_l}\right) \right) u \otimes w, \\
&= \prod_{k=1}^{n} \left(y_{i_k}^m \prod_{\substack{i_k<j \\ j\neq i_l(k<l\le n)}} c_{i_k,j} \right) \prod_{1\le k<l\le n} c_{i_l,i_k} \left(\sum_{\tau\in S_n} \prod_{\substack{1\le k<l\le n \\ \tau(k)>\tau(l)}} c_{i_k,i_l}/c_{i_l,i_k} \right) u \otimes w.
\end{aligned}$$

(In the first equality, the k-th $x_{1,m}^+$ from the right flips v_2 in the $i_{\sigma(k)}$-th factor of $w_{i_1,i_2,\ldots,i_n}$ to v_1.) Therefore the claim follows from the equality

$$\sum_{\tau\in S_n} \prod_{\substack{1\le k<l\le n \\ \tau(k)>\tau(l)}} c_{k,l} \prod_{\substack{1\le k<l\le n \\ \tau(k)<\tau(l)}} c_{l,k} = [n]!.$$

This can be easily proven by induction on n, using the well known equality

$$[n]! = \sum_{\sigma\in S_n} q^{2l(\sigma)-n(n-1)/2}$$

where $l(\,\cdot\,)$ is the length function. $\square$

Proof of Proposition 4.6. Let $\mathbf{a}$ and w be the ones in Proposition 4.5. Define $G \in \mathrm{End}(W_{\mathbf{a}})$ by

$$g(y_1,\ldots,y_N)\prod_{i=1}^{N} x_i^{n_i} \otimes w \mapsto g(y_1,\ldots,y_N) \prod_{\substack{1\le i<j\le N \\ \alpha_i=\alpha_j}} \prod_{k=0}^{n_j-n_i-1} \frac{p_{\alpha_j}^k y_j - y_i}{q p_{\alpha_j}^k y_j - q^{-1} y_i} \prod_{i=1}^{N} x_i^{n_i} \otimes w$$

where the product $\prod_{k=0}^{n_j-n_i-1} z_k$ should be understood as 1 if $n_j = n_i$ and as $1/\prod_{k=n_j-n_i}^{-1} z_k$ if $n_j < n_i$ (the z_k stand for the fractions in the above equation). Then $G^{-1}(y_i G(u\otimes w)) = y_i u \otimes w$ and

$$G^{-1}(x_i G(u\otimes w)) = \left(\prod_{\substack{j<i \\ \alpha_j=\alpha_i}} c_{i,j} \right) x_i \left(\prod_{\substack{i<j \\ \alpha_j=\alpha_i}} c_{j,i}^{-1} \right) u \otimes w \tag{4.10}$$

for any $u \in C_{\mathbf{a}}$.

By Lemma 4.6 and Lemma 3.2,

$$\begin{aligned}\mathbf{\Lambda}_n &\equiv q^{n(n-1)}\left(\mathbf{x}_{1,0}^{+}\right)^{(n)}\left(\mathbf{x}_{1,1}^{-}\right)^{(n)}\left(-\mathbf{k}_1^{-1}\right)^{n} \mod \mathcal{I},\\ &= \left(-q^{-2}\right)^{n}\left(x_{1,0}^{+}\right)^{(n)}\left(x_{0,0}^{+}\right)^{(n)}\end{aligned} \tag{4.11}$$

for $n \geq 0$. Apply $\mathcal{Y}^m$ to the above and let the result act on $u \otimes w$ ($u \in C_{\mathbf{a}}$). Then, thanks to the equality $\mathbf{\Lambda}_n = \sum_{0\leq l\leq n}\mathbf{\Lambda}_{1,l}\mathbf{\Lambda}_{2,n-l}$ and Proposition 4.3, we obtain

$$(-q^2)^n(q/\gamma)^{mn}\sum_{l=0}^{n}\gamma^{2ml}\mathbf{\Lambda}_{1,l}\mathbf{\Lambda}_{2,n-l}(u\otimes w) = (x_{1,-m}^{+})^{(n)}(x_{0,m}^{+})^{(n)}(u\otimes w).$$

Lemma 4.8 implies that the r.h.s. of the above equality is equal to

$$\sum_{1\leq i_1<\cdots<i_n\leq N}\prod_{k=1}^{n}p_{\alpha_{i_k}}^{m/2}\left(\prod_{k=1}^{n}\prod_{\substack{i_k<j\\ j\neq i_l\,(k<l\leq n)\\ \alpha_j=\alpha_{i_k}}}c_{i_k,j}\right)\prod_{k=1}^{n}x_{i_k}\left(\prod_{k=1}^{n}\prod_{\substack{i_k<j\\ j\neq i_l\,(k<l\leq n)\\ \alpha_j=\alpha_{i_k}}}c_{j,i_k}\right)u\otimes w.$$

This and (4.10) give

$$\begin{aligned}&(-q^2)^n(q/\gamma)^{mn}\sum_{l=0}^{n}\gamma^{2ml}G^{-1}(\mathbf{\Lambda}_{1,l}\mathbf{\Lambda}_{2,n-l}G(u\otimes w))\\ &=\sum_{\substack{I\subset\{1,2,\ldots,N\}\\ |I|=n}}\prod_{i\in I}p_{\alpha_i}^{m/2}\prod_{\substack{i\in I\\ j\notin I\\ \alpha_i=\alpha_j}}c_{i,j}\prod_{i\in I}x_i u\otimes w\end{aligned}$$

for any integer m and $0\leq n\leq N$.

Now let $\mathbf{a} = (\overbrace{1,\ldots,1}^{r_1},\overbrace{2,\ldots,2}^{r_2})$. Set $J_i = \{1,2,\ldots,r_i\}$ for $i = 1,2$ and let $x'_i = x_{r_1+i}$ and $c'_{i,j} = c_{r_1+i,r_1+j}$ for $1\leq i, j\leq r_2$. Then the r.h.s. of the above equality is equal to

$$(q/\gamma)^{mn}\sum_{l=0}^{n}\gamma^{2ml}\sum_{\substack{I_1\subset J_1,|I_1|=l\\ I_2\subset J_2,|I_2|=n-l}}\prod_{\substack{i\in I_1\\ j\in J_1\setminus I_1}}c_{i,j}\prod_{i\in I_1}x_i\prod_{\substack{i\in I_2\\ j\in J_2\setminus I_2}}c'_{i,j}\prod_{i\in I_2}x'_i u\otimes w.$$

By this and (4.8) we find that there exists an $\mathcal{A}_{\mathbf{r}}$ module structure on $C_1^{r_1}\otimes C_2^{r_2}$ ($\subset C_{\mathbf{a}}\simeq W_{\mathbf{a}}$) such that the elements $\Lambda_{1,l}$, $\Lambda_{2,l}$, $\mathbf{\Lambda}_{1,l}$ and $\mathbf{\Lambda}_{2,l}$ ($l>0$) act by left multiplications of $(-q)^{-l}e_l\otimes 1$, $(-q)^{-l}1\otimes e_l$, $(-q^2)^{-l}D_l\otimes 1$ and $(-q^2)^{-l}1\otimes D_l$, respectively. This and Corollary 4.1 prove the claim. □

4.4. Main results. Now we can prove a q analogue of Theorem 2.1. For $i = 1,2$ let $\mathcal{A}_{\mathbf{r}}^i$ be the subalgebra of $\mathcal{A}_{\mathbf{r}}$ generated by the elements $a_{i,s}$ and $\mathbf{a}_{i,s}$ ($s\neq 0$).

Theorem 4.1. (1) *$\mathcal{A}_{\mathbf{r}} \simeq \mathcal{A}^1_{\mathbf{r}} \otimes \mathcal{A}^2_{\mathbf{r}}$ $(a_1a_2 \leftrightarrow a_1 \otimes a_2)$ as algebras.*
(2) *For $i = 1, 2$ the homomorphism $\mathcal{A}^i_{\mathbf{r}} \to C_i^{r_i}$ determined by*

$$\Lambda_{i,l} \mapsto e_l, \quad \mathbf{\Lambda}_{i,l} \mapsto D_l \quad (0 < l \le r_i)$$

is injective. Therefore $\mathcal{A}^i_{\mathbf{r}}$ depends only on r_i out of r_1 and r_2.

For $i = 1, 2$ define $d_j^{(i)}$ $(1 \le j \le r_i)$ to be the element of $\mathrm{End}(F(y_1, \dots, y_{r_i}))$ such that $d_j^{(i)} f(y_1, \dots, y_{r_i}) = f(y_1, \dots, p_i y_j, \dots, y_{r_i})$.

Corollary 4.3. *For $i = 1, 2$ $\mathcal{A}^i_{\mathbf{r}}$ is isomorphic to the subalgebra of $\mathrm{End}(F(y_1, \dots, y_{r_i}))$ generated by $e_l, \mathcal{D}_l^{(i)}$ $(0 < l \le r_i)$, $e_{r_i}^{-1}$ and $\mathcal{D}_{r_i}^{-1}$. Here the e_l are the ones in* (4.9) (*with $m = r_i$*) *regarded as multiplication operators and the $\mathcal{D}_l^{(i)}$ denote the D_l with the x_j replaced by the $d_j^{(i)}$.*

To prove Theorem 4.1 and its corollary, we need the following two lemmas. The proof of the first lemma will be given in Section 7.

Lemma 4.9. *Let $\mathcal{A}'_{\mathbf{r}}$ be the subalgebra of $\mathcal{A}_{\mathbf{r}}$ generated by the elements $a_{i,l}$ and $\mathbf{h}_{1,l}$ $(i = 1, 2,\ l > 0)$. Then $\phi_{\mathbf{r}}|_{\mathcal{A}'_{\mathbf{r}}}$ is injective.*

Lemma 4.10. *Set $\Lambda^{\mathbf{m}} = \prod_{i=1,2} \Lambda_{i,r_i}^{m_i}$ and $\mathbf{\Lambda}^{\mathbf{m}} = \prod_{i=1,2} \mathbf{\Lambda}_{i,r_i}^{m_i} \in \mathcal{A}_{\mathbf{r}}$ for $\mathbf{m} = (m_1, m_2) \in \mathbf{Z}^2$. Then for any element u of $\mathcal{A}_{\mathbf{r}}$ there exist an element v of $\mathcal{A}'_{\mathbf{r}}$ and $\mathbf{m}_1,\ \mathbf{m}_2, \mathbf{m}_3 \in \mathbf{Z}^2$ such that $u = \mathbf{\Lambda}^{\mathbf{m}_1} \Lambda^{\mathbf{m}_2} v \Lambda^{\mathbf{m}_3}$.*

Proof. Fixing $\mathbf{r}$, for $i = 1, 2$ let $\mathcal{A}'_i$ be the subalgebra of $\mathcal{A}_{\mathbf{r}}$ generated by the elements $a_{i,l}$ and $\mathbf{a}_{i,l}$ $(l > 0)$. By Proposition 4.4 (3) and Corollary 4.1 for any element u of $\mathcal{A}_{\mathbf{r}}$ there exist $\mathbf{m}, \mathbf{n} \in \mathbf{Z}^2$ such that $\mathbf{\Lambda}^{\mathbf{m}} \Lambda^{\mathbf{n}} u \in \mathcal{A}'_1 \mathcal{A}'_2$. Therefore it is sufficient to show that for $u_i \in \mathcal{A}'_i$ $(i = 1, 2)$ there exist $\mathbf{m}, \mathbf{n} \in \mathbf{Z}^2$ such that $u_1 \in \Lambda^{\mathbf{m}} \mathcal{A}'_{\mathbf{r}}$ and $u_2 \in \mathcal{A}'_{\mathbf{r}} \Lambda^{\mathbf{n}}$. We shall show the claim for u_2, the proof of that for u_1 being similar.

Define an automorphism κ of $\mathcal{A}_{\mathbf{r}}$ by $\kappa(u) = \Lambda_{1,r_1} u \Lambda_{1,r_1}^{-1}$ $(u \in \mathcal{A}_{\mathbf{r}})$ and let $s = (q\gamma)^{-2}$. Any element $u \in \mathcal{A}'_2$ is written as a linear combination of monomials in $a_{2,l}$ and $\mathbf{a}_{2,l}$ $(l > 0)$. Let $\hat{u}$ denote u with the $\mathbf{a}_{2,l}$ replaced by the $\mathbf{h}_{1,l}$. ($\hat{u}$ depends on the expression of u.) Then, since $\mathbf{h}_{1,l} = \mathbf{a}_{1,l} + \mathbf{a}_{2,l}$, $\hat{u}$ is expanded as $\hat{u} = \sum_{m=0}^{M} u_m$ where $u_0 = u$ and u_m satisfies $\kappa(u_m) = s^m u_m$. So u is a scalar multiple of $\prod_{m=1}^{M} (\kappa - s^m)(\hat{u})$. Since $\hat{u}$ and Λ_{1,r_1} are in $\mathcal{A}'_{\mathbf{r}}$, this proves the claim. $\square$

Proof of Theorem 4.1. Since $\phi_{\mathbf{r}}(\Lambda_{i,r_i})$ and $\phi_{\mathbf{r}}(\mathbf{\Lambda}_{i,r_i})$ $(i = 1, 2)$ are invertible, Lemma 4.9 and Lemma 4.10 show that $\phi_{\mathbf{r}}$ is injective. If we identify $C_i^{r_i}$ with a subalgebra of $C_1^{r_1} \otimes C_2^{r_2}$ naturally, then $\phi_{\mathbf{r}}(\mathcal{A}^i_{\mathbf{r}}) \subset C_i^{r_i}$. This and the injectivity of $\phi_{\mathbf{r}}$ prove part (2).

By Propositions 4.1 and 4.2, $\mathcal{A}^1_{\mathbf{r}}$ and $\mathcal{A}^2_{\mathbf{r}}$ commute and $\mathcal{A}_{\mathbf{r}} = \mathcal{A}^1_{\mathbf{r}}\mathcal{A}^2_{\mathbf{r}}$. So $m\colon \mathcal{A}^1_{\mathbf{r}} \otimes \mathcal{A}^2_{\mathbf{r}} \to \mathcal{A}_{\mathbf{r}}$ $(a_1 \otimes a_2 \mapsto a_1a_2)$ is a surjective homomorphism. Part (2) proves that the composite map $\phi_{\mathbf{r}} \circ m\colon \mathcal{A}^1_{\mathbf{r}} \otimes \mathcal{A}^2_{\mathbf{r}} \to C_1^{r_1} \otimes C_2^{r_2}$ is injective. This shows the injectivity of m. $\square$

Proof of Corollary 4.3. The homomorphism $\mathcal{C}_i^{r_i} \to \mathrm{End}(F(y_1, \ldots, y_{r_i}))$ $(y_j \mapsto y_j,\ x_j \mapsto \mathcal{D}_j^{(i)})$ is injective and $\mathcal{A}^i_{\mathbf{r}}$ is generated by the elements $\Lambda_{i,s}, \Lambda_{i,r_i}^{-1}, \boldsymbol{\Lambda}_{i,s}$ and $\boldsymbol{\Lambda}_{i,r_i}^{-1}$ $(0 \le s \le r_i)$. Therefore the claim follows from part (2) of Theorem 4.1. $\square$

5. A comparison with Reference [2]

In this section we compare our result with the work [2] of M. Varagnolo and E. Vasserot. Since their results are restricted to the quantum toroidal algebra of type sl_n $(n \ge 3)$, we extend part of them to our case in the first two subsections. We fix $\alpha \in \{1, 2\}$ and set $\gamma_\alpha = \gamma^{\varepsilon_\alpha}$.

5.1. A relation between representations of $U_q(sl_{3,\mathrm{tor}})$ and those of $U_q(\hat{\mathcal{L}})$. Let $(b_{ij})_{0\le i,j\le 2}$ be the Cartan matrix for $\widehat{sl_3}$. For $0 \le i, j \le 2$ set $\xi_{ij} = 1$ if $(i, j) \ne (2, 0), (0, 2)$ and $\xi_{20} = \xi_{02}^{-1} = \gamma_\alpha^2/q$. $U_q(sl_{3,\mathrm{tor}})$ [1] [2] is defined to be the F algebra defined by generators $x^\pm_{i,m}, h_{i,r}, k_i^{\pm 1}$ and $C^{\pm 1}$ $(0 \le i \le 2, m \in \mathbf{Z}, r \in \mathbf{Z} \setminus \{0\})$ and relations

$$k_i^{\pm1}k_i^{\mp1} = C^{\pm1}C^{\mp1} = 1, \tag{5.1}$$

$$C^{\pm1} \text{ central}, \quad \left[k_i, k_j\right] = \left[k_i, h_{j,r}\right] = 0, \tag{5.2}$$

$$\xi_{ij}^r \left[h_{i,r}, h_{j,s}\right] = \delta_{r+s,0}\frac{\left[rb_{ij}\right]}{r}\frac{C^r - C^{-r}}{q - q^{-1}}, \tag{5.3}$$

$$k_i x^\pm_{j,m} k_i^{-1} = q^{\pm b_{ij}} x^\pm_{j,m}, \tag{5.4}$$

$$\xi_{ij}^r \left[h_{i,r}, x^\pm_{j,m}\right] = \pm\frac{\left[rb_{ij}\right]}{r} C^{(r\mp|r|)/2} x^\pm_{j,r+m}, \tag{5.5}$$

$$\left[x^+_{i,m}, x^-_{j,n}\right] = \frac{\delta_{ij}}{q - q^{-1}}\left(C^{-n}\Phi^{(+)}_{i,m+n} - C^{-m}\Phi^{(-)}_{i,m+n}\right), \tag{5.6}$$

$$\left[x^\pm_{i,m+1}, x^\pm_{i,n}\right]_{q^{\pm2}} + \left[x^\pm_{i,n+1}, x^\pm_{i,m}\right]_{q^{\pm2}} = 0, \tag{5.7}$$

$$\xi_{ij}\left[x^\pm_{i,m+1}, x^\pm_{j,n}\right]_{q^{\mp1}} + \left[x^\pm_{j,n+1}, x^\pm_{i,m}\right]_{q^{\mp1}} = 0 \quad (i \ne j), \tag{5.8}$$

$$x^\pm_{i,m_1}x^\pm_{i,m_2}x^\pm_{j,n} - [2]x^\pm_{i,m_1}x^\pm_{j,n}x^\pm_{i,m_2} + x^\pm_{j,n}x^\pm_{i,m_1}x^\pm_{i,m_2} + (m_1 \leftrightarrow m_2) = 0 \quad (i \ne j) \tag{5.9}$$

where $\Phi^{(\pm)}_{i,\pm r} = 0$ $(r < 0)$ and $\Phi^{(\pm)}_{i,\pm r}$ $(r \ge 0)$ is expressed in terms of $k_i^{\pm1}$ and the $h_{i,s}$ as in the case $U_q(\hat{\mathcal{L}})$. Set $Q' = \mathbf{Z}\alpha_1 \oplus \mathbf{Z}\alpha_2$ and let $\alpha_0 = -\alpha_1 - \alpha_2$. By assigning $\pm\alpha_i$ to $x^\pm_{i,m}$ and 0 to $k_i^{\pm1}$, $h_{i,r}$ and $C^{\pm1}$, $U_q(sl_{3,\mathrm{tor}})$ is endowed with a structure of Q' graded algebras. We denote the homogeneous subspace of degree α of it by $U_q(sl_{3,\mathrm{tor}})_\alpha$.

Set $A = \sum_{m\in\mathbf{Z}} U_q(sl_{3,\mathrm{tor}})_{m\alpha_1}$ and

$$J = \sum_{\substack{l,m\in\mathbf{Z}\\ n\in\mathbf{Z}_{>0}}} U_q(sl_{3,\mathrm{tor}})_{l\alpha_1-n\alpha_2} U_q(sl_{3,\mathrm{tor}})_{m\alpha_1+n\alpha_2}.$$

Then A is a subalgebra of $U_q(sl_{3,\mathrm{tor}})$ and J is an ideal of A.

Proposition 5.1. *There exists a homomorphism $U_q(\hat{\mathcal{L}}) \to A/J$ determined by*

$$x^{\pm}_{1,m} \mapsto x^{\pm}_{1,m}, \quad h_{1,r} \mapsto h_{1,r}, \quad k_1 \mapsto k_1, \quad C \mapsto C, \quad k_0 \mapsto k_0k_2,$$
$$x^{+}_{0,m} \mapsto \gamma_\alpha^{-m} x^{+}_{2,0}x^{+}_{0,m}, \quad x^{-}_{0,m} \mapsto \gamma_\alpha^{-m} x^{-}_{0,m}x^{-}_{2,0}, \quad h_{0,r} \mapsto \gamma_\alpha^{-r} h_{0,r} + \gamma_\alpha^{r} h_{2,r}.$$

Proof. By (5.8)

$$x^{+}_{2,m}x^{+}_{0,n} = \gamma_\alpha^{-2m} x^{+}_{2,0}x^{+}_{0,m+n} \quad \text{and} \quad x^{-}_{0,n}x^{-}_{2,m} = \gamma_\alpha^{-2m} x^{-}_{0,m+n}x^{-}_{2,0}$$

in A/J. Noting this, the relations of $U_q(\hat{\mathcal{L}})$ except (3.11) with $(i, j) = (0, 1)$ can be easily checked. Set $X = x^{+}_{2,0}x^{+}_{0,0}$ and $H = x^{+}_{2,0}x^{+}_{1,0}x^{+}_{0,0}$. Then $X^2 = \left(x^{+}_{2,0}\right)^2\left(x^{+}_{0,0}\right)^2 \big/ [2]$ and $X^3 = \left(x^{+}_{2,0}\right)^3\left(x^{+}_{0,0}\right)^3 \big/ [3]!$ in A/J. Using these, we find that

$$\left[X^2, x^{+}_{1,0}\right] = [2][X, H], \quad [2]\left[X^3, x^{+}_{1,0}\right] = [3]\left[X^2, H\right]$$

in A/J. The remaining relation for the $x^{+}_{l,0}$ follows from the above equalities. □

Corollary 5.1. *Suppose that V is a $U_q(sl_{3,\mathrm{tor}})$ module. If W is an A submodule of V satisfying $JW = 0$, then W can be regarded as a $U_q(\hat{\mathcal{L}})$ module via the homomorphism in the proposition.*

5.2. The double affine Hecke algebra $\ddot{H}_r$ and representations of $\mathcal{U}$. For an integer $r \geq 2$ define H_r to be the F algebra with generators $T_i^{\pm1}$ $(1 \leq i \leq r-1)$ and relations

$$T_i^{\pm1}T_i^{\mp1} = 1, \quad (T_i - q)(T_i + q^{-1}) = 0, \tag{5.10}$$
$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad T_iT_j = T_jT_i \quad (|i-j| > 1). \tag{5.11}$$

Let $\ddot{H}_r$ be the F algebra [31] defined by generators $T_i^{\pm1}$, $X_j^{\pm1}$ and $Y_j^{\pm1}$ $(1 \leq i \leq r-1,$ $1 \leq j \leq r)$ and relations (5.10), (5.11) and

$$X_i^{\pm1}X_i^{\mp1} = 1, \quad Y_i^{\pm1}Y_i^{\mp1} = 1,$$
$$X_iX_j = X_jX_i, \quad Y_iY_j = Y_jY_i,$$
$$T_iX_iT_i = X_{i+1}, \quad T_i^{-1}Y_iT_i^{-1} = Y_{i+1},$$

$$
\begin{aligned}
&T_i X_j = X_j T_i, \quad T_i Y_j = Y_j T_i \quad (j \neq i, i+1),\\
&(X_1 \cdots X_r) Y_1 = p_\alpha Y_1 (X_1 \cdots X_r),\\
&X_2 Y_1^{-1} X_2^{-1} Y_1 = T_1^2
\end{aligned}
$$

where p_α is the one in Section 4.3. We shall regard $\ddot{H}_r$ modules as H_r modules via the homomorphism $H_r \to \ddot{H}_r$ $(T_i \mapsto T_i)$.

Let $v_1, \dots, v_n$ be the canonical basis of F^n and define $t \in \mathrm{End}(F^n \otimes F^n)$ by

$$
t(v_i \otimes v_j) = \begin{cases} q v_i \otimes v_j & \text{if } i = j,\\ v_j \otimes v_i & \text{if } i < j,\\ v_j \otimes v_i + (q - q^{-1}) v_i \otimes v_j & \text{if } i > j. \end{cases}
$$

Then, as is well known, the tensor product $(F^n)^{\otimes r}$ is endowed with a structure of H_r modules by $T_i = 1^{\otimes i-1} \otimes t \otimes 1^{\otimes r-i-1}$.

In the sl_n case set $K_i = q^{E_{ii} - E_{i+1\,i+1}}$ $(1 \leq i \leq n-1)$ and $K_0 = q^{E_{nn} - E_{11}} \in \mathrm{End}(F^n)$. Define the following elements of $\mathrm{End}((F^n)^{\otimes r})$:

$$
\begin{aligned}
&E_i = \sum_{j=1}^{r} 1^{\otimes j-1} \otimes E_{i\,i+1} \otimes K_i^{\otimes r-j}, \quad F_i = \sum_{j=1}^{r} \left(K_i^{-1}\right)^{\otimes j-1} \otimes E_{i+1\,i} \otimes 1^{\otimes r-j},\\
&E_0^{(j)} = 1^{\otimes j-1} \otimes E_{n1} \otimes K_0^{\otimes r-j}, \quad F_0^{(j)} = \left(K_0^{-1}\right)^{\otimes j-1} \otimes E_{1n} \otimes 1^{\otimes r-j}
\end{aligned}
$$

where $1 \leq i \leq n-1$ and $1 \leq j \leq r$.

The following proposition is a special case of [2].

Proposition 5.2. *For a right $\ddot{H}_r$ module M there exists a $U_q(sl_{3,\mathrm{tor}})$ module structure on $M \otimes_{H_r} (F^3)^{\otimes r}$ determined by*

$$
\begin{aligned}
&k_i(m \otimes u) = m \otimes \left(K_i^{\otimes r}\right) u \quad (i = 0, 1, 2), \quad C(m \otimes u) = m \otimes u,\\
&x_{i,0}^{+}(m \otimes u) = m \otimes E_i u, \quad x_{i,0}^{-}(m \otimes u) = m \otimes F_i u \quad (i = 1, 2),\\
&x_{0,0}^{+}(m \otimes u) = \sum_j m X_j \otimes E_0^{(j)} u, \quad x_{0,0}^{-}(m \otimes u) = \sum_j m X_j^{-1} \otimes F_0^{(j)} u,\\
&y_0^{+}(m \otimes u) = \sum_j m Y_j^{-1} \otimes E_0^{(j)} u, \quad y_0^{-}(m \otimes u) = \sum_j m Y_j \otimes F_0^{(j)} u
\end{aligned}
$$

where $y_0^{+} = (k_1 k_2)^{-1} \left[x_{1,1}^{-}, x_{2,0}^{-}\right]_q$ and $y_0^{-} = \left[x_{2,0}^{+}, x_{1,-1}^{+}\right]_{q^{-1}} k_1 k_2$.

By applying Corollary 5.1 to the A submodule $\sum_{i_1, \dots, i_r = 1,2} M \otimes (v_{i_1} \otimes \cdots \otimes v_{i_r})$ of the $U_q(sl_{3,\mathrm{tor}})$ module in Proposition 5.2, we obtain the following proposition.

Proposition 5.3. *For a right $\ddot{H}_r$ module M there exists a $\mathcal{U}$ module structure on $M \otimes_{H_r} (F^2)^{\otimes r}$ determined by*

$$k_1(m \otimes u) = m \otimes \left(K_1^{\otimes r}\right) u,$$
$$x_{1,0}^+(m \otimes u) = m \otimes E_1 u, \quad x_{1,0}^-(m \otimes u) = m \otimes F_1 u,$$
$$x_{0,0}^+(m \otimes u) = \sum_j mX_j \otimes E_0^{(j)} u, \quad x_{0,0}^-(m \otimes u) = \sum_j mX_j^{-1} \otimes F_0^{(j)} u,$$
$$y_0^+(m \otimes u) = \sum_j mY_j^{-1} \otimes E_0^{(j)} u, \quad y_0^-(m \otimes u) = \sum_j mY_j \otimes F_0^{(j)} u$$

where $y_0^+ = k_1^{-1} x_{1,1}^-$ and $y_0^- = x_{1,-1}^+ k_1$.

5.3. A comparison with Reference [2]. Let $\mathbf{C}(\gamma)[y_1^{\pm 1}, \dots, y_r^{\pm 1}]$ be the ring of Laurent polynomials in $y_1, \dots, y_r$. Define $\tau_1, \dots, \tau_{r-1}, \omega \in \mathrm{End}(\mathbf{C}(\gamma)[y_1^{\pm 1}, \dots, y_r^{\pm 1}])$ by

$$\tau_i = \frac{q y_i - q^{-1} y_{i+1}}{y_i - y_{i+1}} (i, i+1) - (q - q^{-1}) \frac{y_{i+1}}{y_i - y_{i+1}},$$
$$\omega f(y_1, \dots, y_r) = f(p_\alpha y_r, y_1, \dots, y_{r-1}).$$

It is well known [31] that there exists a right $\ddot{H}_r$ module structure on $\mathbf{C}(\gamma)[y_1^{\pm 1}, \dots, y_r^{\pm 1}]$ such that

$$m.T_i = \tau_i m, \quad m.Y_j = y_j^{-1} m, \quad m.X_j = \tau_j^{-1} \cdots \tau_{r-1}^{-1} \omega \tau_1 \cdots \tau_{j-1} m.$$

We denote this right $\ddot{H}_r$ module by M.

Proposition 5.4. *Set $\mathbf{r} = (r, 0)$ if $\alpha = 1$ and $\mathbf{r} = (0, r)$ if $\alpha = 2$. Letting $w = v_1 \otimes \cdots \otimes v_1$, set $V = M \otimes_{H_r} \mathbf{C}(\gamma) w$.*
(1) *The $\mathcal{U}$ module strucutre on $M \otimes_{H_r} (F^2)^{\otimes r}$ induces an $\mathcal{A}_{\mathbf{r}}$ module structure on V.*
(2) *V is isomorphic to $\mathbf{C}(\gamma)[y_1^{\pm 1}, \dots, y_r^{\pm 1}]^{S_r}$ as vector spaces.*
(3) *If we identify V with $\mathbf{C}(\gamma)[y_1^{\pm 1}, \dots, y_r^{\pm 1}]^{S_r}$, then $\Lambda_{\alpha,l}$ and $\boldsymbol{\Lambda}_{\alpha,l}$ $(l > 0)$ act on V as $(-q^{-2})^l e_l$ and $(-q^{-2})^l \mathcal{D}_l^{(\alpha)}$, respectively.*

Proof. (1) Clearly $\mathcal{U}_0$ preserves V and $\mathcal{I}_r$ annihilates it. So the $\mathcal{U}$ module structure induces an $\mathcal{A}_r$ module structure on V. Suppose that we have shown that $(*)$ $a_{\beta,l} = 0$ on V for $l > 0$ and $\beta \neq \alpha$. Then $P_{\mathbf{s}} = 0$ on V if $\mathbf{s} \in Z_r$ and $\mathbf{s} \neq \mathbf{r}$. Hence we obtain the claim by Corollary 4.2.

By (4.4)

$$\Lambda_n \equiv \left(-q^{-2}\right)^n \left(x_{1,0}^+\right)^{(n)} \left(y_0^+\right)^{(n)} \mod \mathcal{I}.$$

Using this and (4.11), we can easily obtain

$$\Lambda_l(m \otimes w) = (-q^{-2})^l \sum_{1 \le i_1 < \cdots < i_l \le r} m Y_{i_1}^{-1} \cdots Y_{i_l}^{-1} \otimes w, \tag{5.12}$$
$$\mathbf{\Lambda}_l(m \otimes w) = (-q^{-2})^l \sum_{1 \le i_1 < \cdots < i_l \le r} m X_{i_1} \cdots X_{i_l} \otimes w$$

for $l > 0$. The first equality gives

$$h_{1,l}(m \otimes w) = \frac{q^{-2l}[l]}{l} m \left(\sum_{i=1}^{r} Y_i^{-l} \right) \otimes w$$

for $l > 0$. Since $(X_1 \cdots X_r) Y_j = p_\alpha Y_j (X_1 \cdots X_r)$ in $\ddot{H}_r$ for any j, we find that $\mathbf{\Lambda}_r h_{1,l} = p_\alpha^l h_{1,l} \mathbf{\Lambda}_r$ for $l > 0$ on V. This and Lemma 4.5 prove $(*)$.

(2) Set

$$\Xi = \sum_{\sigma \in S_r} q^{l(\sigma)} T_\sigma$$

where $T_\sigma = T_{i_1} \cdots T_{i_n}$ for a reduced expression $\sigma = s_{i_1} \cdots s_{i_n}$ $(s_i = (i, i+1))$. Then $\Xi T_i = T_i \Xi = q \Xi$ for any i and

$$M = \operatorname{Im} \Xi \oplus \operatorname{Ker} \Xi, \quad \operatorname{Ker} \Xi = \sum \operatorname{Im}(T_i - q)$$

by Propositions 1.1–1.2 of [32]. Therefore

$$M \otimes_{H_r} \mathbf{C}(\gamma) w \simeq M \Big/ \sum \operatorname{Im}(T_i - q) \simeq \operatorname{Im} \Xi = \mathbf{C}\left[y_1^{\pm 1}, \ldots, y_r^{\pm 1}\right]^{S_r}.$$

(3) The action of $\sum_{1 \le i_1 < \cdots < i_l \le r} X_{i_1} \cdots X_{i_l}$ on M is equal to $\mathcal{D}_l^{(\alpha)}$ when restricted to $\mathbf{C}[y_1^{\pm 1}, \ldots, y_r^{\pm 1}]^{S_r}$ [31]. By $(*)$ and a similar result for the $\mathbf{a}_{i,s}$, $\Lambda_l = \Lambda_{\alpha,l}$ and $\mathbf{\Lambda}_l = \mathbf{\Lambda}_{\alpha,l}$ on V for $l > 0$. Therefore we obtain the claim from (5.12). □

6. Proof of Proposition 2.1

6.1. The subalgebra K of L. Let K be the $\mathbf{C}(\gamma)$ Lie algebra generated by $x_{i,m}^+$ and $h_{i,r}$ $(i = 0, 1,\ m \in \mathbf{Z}_{\ge 0},\ r \in \mathbf{Z}_{>0})$ with defining relations

$$[h_{i,r}, h_{j,s}] = 0, \tag{6.1}$$
$$\left[h_{i,r}, x_{i,m}^+\right] = 2x_{i,m+r}^+, \tag{6.2}$$
$$\left[h_{i,r}, x_{1-i,m}^+\right] = -(\gamma^r + \gamma^{-r}) x_{1-i,m+r}^+, \tag{6.3}$$
$$\left[x_{i,m}^+, x_{i,n}^+\right] = 0, \tag{6.4}$$
$$\left[x_{i,m_1}^+, \left[x_{i,m_2}^+, \left[x_{i,m_3}^+, x_{1-i,n}^+\right]\right]\right] = 0, \tag{6.5}$$

$$[x^+_{i,m+2}, x^+_{1-i,n}] - (\gamma + \gamma^{-1})[x^+_{i,m+1}, x^+_{1-i,n+1}] + [x^+_{i,m}, x^+_{1-i,n+2}] = 0, \tag{6.6}$$

$$2[x^+_{i,0}, [x^+_{i,1}, x^+_{1-i,m}]] - (\gamma + \gamma^{-1})[x^+_{i,0}, [x^+_{i,0}, x^+_{1-i,m+1}]] = 0, \tag{6.7}$$

$$[x^+_{i,1}, [x^+_{i,1}, x^+_{1-i,m}]] - [x^+_{i,0}, [x^+_{i,0}, x^+_{1-i,m+2}]] = 0. \tag{6.8}$$

This Lie algebra is endowed with a structure of $Q \oplus \mathbf{Z}$ graded Lie algebras by assigning (α_i, m) to $x^+_{i,m}$ and $(0, r)$ to $h_{i,r}$ as before.

Define a graded subalgebra $\mathcal{K}$ of the $Q \oplus \mathbf{Z}$ graded Lie algebra $\mathcal{L}$ by

$$\mathcal{K} = \bigoplus_{k>0,\, l\geq 0} \mathbf{C}(\gamma) f(k,l) \oplus E \oplus \bigoplus_{k,l\geq 0} \mathbf{C}(\gamma) e(k,l) \tag{6.9}$$

where

$$E = \bigoplus_{r>0} \mathbf{C}(\gamma)(\epsilon_1(r,0) - \epsilon_2(r,0)) \oplus \bigoplus_{\substack{r\geq 0,\, s>0\\ i=1,2}} \mathbf{C}(\gamma)\epsilon_i(r,s). \tag{6.10}$$

Proposition 6.1. *There is an isomorphism of $Q\oplus\mathbf{Z}$ graded Lie algebras $K \to \mathcal{K}$ determined by*

$$x^+_{1,m} \mapsto e(0,m), \quad h_{1,r} \mapsto \epsilon_1(0,r) - \epsilon_2(0,r),$$
$$x^+_{0,m} \mapsto \gamma^{-m} f(1,m), \quad h_{0,r} \mapsto \gamma^r \epsilon_2(0,r) - \gamma^{-r}\epsilon_1(0,r).$$

This proposition will be proved in the next subsection. We shall identify K with $\mathcal{K}$ by the above correspondence.

6.2. Proofs of Proposition 2.1 and Proposition 6.1. Let $\mathfrak{t}$ be the Lie algebra over $\mathbf{C}(\gamma)$ defined by generators $x^\pm_{i,m}$ and $h_{i,r}$ ($i = 0,1$, $m, r \in \mathbf{Z}_{\geq 0}$) and relations (2.2)–(2.8) involving only the generators, i.e.,

$$[h_{i,r}, h_{j,s}] = 0, \tag{6.11}$$

$$[h_{i,r}, x^\pm_{i,m}] = \pm 2 x^\pm_{i,m+r}, \tag{6.12}$$

$$[h_{i,r}, x^\pm_{1-i,m}] = \mp(\gamma^r + \gamma^{-r}) x^\pm_{1-i,m+r}, \tag{6.13}$$

$$[x^+_{i,m}, x^-_{j,n}] = \delta_{ij} h_{i,m+n}, \tag{6.14}$$

$$[x^\pm_{i,m}, x^\pm_{i,n}] = 0, \tag{6.15}$$

$$[x^\pm_{i,m_1}, [x^\pm_{i,m_2}, [x^\pm_{i,m_3}, x^\pm_{1-i,n}]]] = 0. \tag{6.16}$$

Lemma 6.1. *Let $\mathfrak{t}^+$ be the subalgebra of $\mathfrak{t}$ generated by the elements $x^+_{i,m}$. Then $\mathfrak{t}^+$ is defined by generators $x^+_{i,m}$ ($i = 0,1$, $m \geq 0$) and relations* (6.4)–(6.8).

Proof. Let $\tilde{\mathfrak{t}}$ be the Lie algebra over $\mathbf{C}(\gamma)$ defined by generators $x^\pm_{i,m}$ and $h_{i,r}$ ($i = 0,1$, $m, r \geq 0$) and relations (6.11)–(6.14). Let $\tilde{\mathfrak{t}}^+$ and $\tilde{\mathfrak{t}}^-$ be the subalgebras of

$\tilde{\mathfrak{t}}$ generated by the $x_{i,m}^+$ and $x_{i,m}^-$, respectively, and $\tilde{\mathfrak{t}}^0$ the subalgebra generated by the $h_{i,r}$. Then $\tilde{\mathfrak{t}} = \tilde{\mathfrak{t}}^- \oplus \tilde{\mathfrak{t}}^0 \oplus \tilde{\mathfrak{t}}^+$, and $\tilde{\mathfrak{t}}^+$ and $\tilde{\mathfrak{t}}^-$ are the free Lie algebras on the $x_{i,m}^+$ and the $x_{i,m}^-$. Let I^+ and I^- be the ideals of $\tilde{\mathfrak{t}}$ generated by the elements on the left hand side of (6.15)–(6.16) with the upper sign and the lower sign, respectively. We shall show that $(*)$ I^+ is the ideal of $\tilde{\mathfrak{t}}^+$ generated by the elements

$$\begin{aligned}
&r_i(m,n) = \left[x_{i,m}^+, x_{i,n}^+\right],\\
&s_i(m_1,m_2,m_3;n) = \mathrm{Sym}_{m_1,m_2,m_3}\left[x_{i,m_1}^+, \left[x_{i,m_2}^+, \left[x_{i,m_3}^+, x_{1-i,n}^+\right]\right]\right],\\
&t_{1,i}(m) = 2\left[x_{i,0}^+, \left[x_{i,1}^+, x_{1-i,m}^+\right]\right] - (\gamma+\gamma^{-1})\left[x_{i,0}^+, \left[x_{i,0}^+, x_{1-i,m+1}^+\right]\right],\\
&t_{2,i}(m) = \left[x_{i,1}^+, \left[x_{i,1}^+, x_{1-i,m}^+\right]\right] - \left[x_{i,0}^+, \left[x_{i,0}^+, x_{1-i,m+2}^+\right]\right],\\
&t_{3,i}(m,n) = \left[x_{i,m+2}^+, x_{1-i,n}^+\right] - (\gamma+\gamma^{-1})\left[x_{i,m+1}^+, x_{1-i,n+1}^+\right] + \left[x_{i,m}^+, x_{1-i,n+2}^+\right]
\end{aligned}$$

with $i = 0, 1$ and $m, n, m_1, m_2, m_3 \geq 0$. A similar result for I^- also holds. If these are proven, we obtain $\mathfrak{t} = \tilde{\mathfrak{t}}/(I^- + I^+) = \mathfrak{t}^- \oplus \mathfrak{t}^0 \oplus \mathfrak{t}^+$ with $\mathfrak{t}^\pm \simeq \tilde{\mathfrak{t}}^\pm/I^\pm$.

Let J be the ideal of $\tilde{\mathfrak{t}}$ generated by the elements $r_i(m,n)$ ($i = 0, 1,\ m, n \geq 0$). For $i = 0, 1$ let S_i be the linear span of the elements $s_i(m_1,m_2,m_3;n)$ ($m_1, m_2, m_3, n \geq 0$) and I_i the ideal of $\tilde{\mathfrak{t}}$ generated by S_i. Let T_i be the ideal of $\tilde{\mathfrak{t}}^+$ generated by the elements $s_i(m_1,m_2,m_3;n), t_{1,i}(m), t_{2,i}(m)$ and $t_{3,i}(m,n)$ ($m_1, m_2, m_3, m, n \geq 0$). As is easily shown, J is the ideal of $\tilde{\mathfrak{t}}^+$ generated by the elements $r_i(m,n)$. So $(*)$ is equivalent to $I_0 + I_1 + J = T_0 + T_1 + J$. Since $\mathrm{ad}\big(U(\tilde{\mathfrak{t}}^0)\big)S_i = S_i$, I_i is equal to $\mathrm{ad}\big(U(\tilde{\mathfrak{t}}^+)\big)\,\mathrm{ad}\big(U(\tilde{\mathfrak{t}}^-)\big)S_i$. Therefore, to prove the lemma, it is sufficient to show that $\mathrm{ad}\big(U(\tilde{\mathfrak{t}}^+)\big)\,\mathrm{ad}\big(U(\tilde{\mathfrak{t}}^-)\big)S_i \equiv T_i \mod J$ ($i = 0, 1$).

A little calculation shows that

$$\mathrm{ad}\left(U\left(\tilde{\mathfrak{t}}^-\right)\right) S_i \equiv S_i + S_i^1 + S_i^2 \mod J$$

where

$$S_i^1 = \sum_{l\geq 0} \mathrm{ad}\left(x_{i,l}^-\right) S_i \quad \text{and} \quad S_i^2 = \sum_{l\geq 0} \mathrm{ad}\left(x_{i,l}^-\right) S_i^1.$$

Letting $\{k\} = (\gamma^k - \gamma^{-k})/(\gamma - \gamma^{-1})$ for $k \in \mathbf{Z}$, set

$$\begin{aligned}
&u_i(k,l,n) = 2\left[x_{i,k}^+, \left[x_{i,l}^+, x_{1-i,n}^+\right]\right] - (\gamma^{k-l} + \gamma^{l-k})\left[x_{i,0}^+, \left[x_{i,0}^+, x_{1-i,k+l+n}^+\right]\right],\\
&w_i(k,l) = \left[x_{i,k}^+, x_{1-i,l}^+\right] - \{k\}\left[x_{i,1}^+, x_{1-i,k+l-1}^+\right] + \{k-1\}\left[x_{i,0}^+, x_{1-i,k+l}^+\right]
\end{aligned}$$

for $i = 0, 1$ and integers k, l, n. Then we can prove the following:

$$\begin{aligned}
S_i^1 &\equiv \mathrm{Span}\{u_i(k,l,n) \mid k, l, n \geq 0\} \mod J,\\
S_i^2 &\equiv \mathrm{Span}\{w_i(k,l) \mid k, l \geq 0, k + l \geq 1\} \mod J.
\end{aligned}$$

Since

$$\begin{aligned}&36 \times u_i(k,l,n)\\ &\equiv 3\left[s_i(k,0,0;n), x^-_{i,l}\right] - 2\left[s_i(0,0,0;n), x^-_{i,k+l}\right] + (\gamma^l + \gamma^{-l})\left[s_i(0,0,0;n+l), x^-_{i,k}\right],\end{aligned}$$

the inclusion $\supset$ (modulo J) of the first equality holds. As for the second equality, the inclusion $\supset$ follows from the equalities

$$w_i(k+1,l) - \{2\}\frac{w_i(k,l+1)}{2} = \frac{\left[u_i(1,k,l), x^-_{i,0}\right]}{4} \quad \text{and} \quad w_i(0,l) = 0.$$

The reverse inclusions are proved by direct calculations.

Now we can conclude that

$$\begin{aligned}S_i^1 &\equiv \mathrm{Span}\{t_{1,i}(m), t_{2,i}(m) \mid m \geq 0\} \mod \left(J + \mathrm{ad}\left(U\left(\tilde{\mathfrak{t}}^+\right)\right) S_i^2\right),\\ S_i^2 &\equiv \mathrm{Span}\{t_{3,i}(m,n) \mid m,n \geq 0\} \mod J\end{aligned}$$

since

$$\begin{aligned}u_i(k,l,n) &\equiv -(\{l\}\{k-1\} + \{k\}\{l-1\})t_{1,i}(k+l+n-1)\\ &\qquad + 2\{k\}\{l\}t_{2,i}(k+l+n-2) \mod \left(J + \mathrm{ad}\left(U\left(\tilde{\mathfrak{t}}^+\right)\right) S_i^2\right),\\ t_{1,i}(m) &= u_i(0,1,m), \quad t_{2,i}(m) = \frac{u_i(1,1,m)}{2},\\ t_{3,i}(m,n) &= w_i(m+2,n) - (\gamma+\gamma^{-1})w_i(m+1,n+1) + w_i(m,n+2),\\ w_i(0,l) &= w_i(1,l) = 0.\end{aligned}$$

This proves $\mathrm{ad}\left(U\left(\tilde{\mathfrak{t}}^+\right)\right)\mathrm{ad}\left(U\left(\tilde{\mathfrak{t}}^-\right)\right) S_i \equiv T_i \mod J$. □

The following lemma is proven in the same way as Lemma 6.1.

Lemma 6.2. *Let K^+ be the subalgebra of K generated by the elements $x^+_{i,m}$. Then K^+ is defined by generators $x^+_{i,m}$ ($i = 0,1$, $m \geq 0$) and relations* (6.4)–(6.8). *Therefore $K^+ \simeq \mathfrak{t}^+$.*

Let $\mathfrak{g}$ be the Lie algebra over $\mathbf{C}$ defined by generators $x^+_{i,l}$ ($i = 0,1$, $l \geq 0$) and relations (6.4)–(6.8) with $\gamma = 1$. This Lie algebra is endowed with a $Q \oplus \mathbf{Z}$ graded Lie algebras similarly to K.

Lemma 6.3. *The dimension of the homogeneous subspace of degree* (α, n) *of* $\mathfrak{g}$ *is given by*

$$\dim \mathfrak{g}(\alpha, n) = \begin{cases} 1 & \text{if } \alpha \text{ is a positive real root of } \widehat{sl_2} \text{ and } n \geq 0 \\ 2 - \delta_{n,0} & \text{if } \alpha = m\delta \ (m > 0) \text{ and } n \geq 0 \\ 0 & \text{otherwise} \end{cases} .$$

Proof. Let $\mathfrak{t}_{\gamma=1}$ be the Lie algebra over $\mathbf{C}$ defined by generators $x^{\pm}_{i,l}, h_{i,r}$ ($i = 0, 1$, $l, r \in \mathbf{Z}_{\geq 0}$) and relations (6.11)–(6.16) with $\gamma = 1$. This Lie algebra is endowed with a structure of $Q \oplus \mathbf{Z}$ graded Lie algebras similarly to L. Let $\mathfrak{t}^{+}_{\gamma=1}$ be the subalgebra of $\mathfrak{t}_{\gamma=1}$ generated by the $x^{+}_{i,l}$.

Set $B = \mathbf{C}[s, t, t^{-1}]$ and $\Omega_B = B\,ds \oplus B\,dt$. Let $\mathcal{G}$ be the Lie algebra $sl_2(\mathbf{C}) \otimes B \oplus \Omega_B/dB$ with the multiplication rule

$$[x_1 \otimes b_1 + c_1, x_2 \otimes b_2 + c_2] = [x_1, x_2] \otimes b_1 b_2 + \mathrm{tr}(x_1 x_2)\overline{(db_1)b_2} \quad (c_1, c_2 \in \Omega_B/dB)$$

where $^{-}\colon \Omega_B \to \Omega_B/dB$ is the quotient map. Then as in [3] we can easily show
(1) $\mathcal{G}$ is the universal covering of $sl_2(\mathbf{C}) \otimes B$ and $\mathfrak{t}_{\gamma=1}$ is isomorphic to $\mathcal{G}$ via the correspondence

$$x^{+}_{1,n} \leftrightarrow s^n \otimes E_{12}, \quad x^{-}_{1,n} \leftrightarrow s^n \otimes E_{21}, \quad x^{+}_{0,n} \leftrightarrow s^n t \otimes E_{21}, \quad x^{-}_{0,n} \leftrightarrow s^n t^{-1} \otimes E_{12}.$$

(2) $\dim \mathfrak{t}_{\gamma=1}(\alpha, n) = \begin{cases} 1 & \text{if } \alpha \text{ is a real root of } \widehat{sl_2} \text{ and } n \geq 0 \\ 0 & \text{if } \alpha \neq 0 \text{ is not a root of } \widehat{sl_2}. \end{cases}$

(3) $\mathfrak{t}_{\gamma=1}(m\delta, n) = \mathbf{C}s^n t^m \otimes (E_{11} - E_{22}) \oplus \begin{cases} \mathbf{C}\overline{s^{n-1}t^m ds} & \text{if } m \neq 0 \text{ and } n \geq 1 \\ \mathbf{C}\overline{s^n t^{-1} dt} & \text{if } m = 0 \text{ and } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$ under the identification $\mathfrak{t}_{\gamma=1} = \mathcal{G}$.

(4) $\mathfrak{t}^{+}_{\gamma=1} = \sum \mathfrak{t}_{\gamma=1}(\alpha, n)$ where α and n are summed over positive roots of $\widehat{sl_2}$ and nonnegative integers, respectively.

These imply that $\dim \mathfrak{t}^{+}_{\gamma=1}(\alpha, n)$ satisfies the equality for $\dim \mathfrak{g}(\alpha, n)$ in the lemma. Since $\mathfrak{t}^{+}_{\gamma=1} \simeq \mathfrak{g}$ by Lemma 6.1 (for the case $\gamma = 1$), this completes the proof. $\sqcap$

Now we are in a position to prove Proposition 6.1 and Proposition 2.1.

Proof of Proposition 6.1. Clearly the assignment in the proposition defines a surjective homomorphism $F\colon K \to \mathcal{K}$ and $F(K(\alpha, n)) = \mathcal{K}(\alpha, n)$. Therefore it is sufficient to show that $(*)$ $\dim K(\alpha, n) \leq \dim \mathcal{K}(\alpha, n)$ for any α and n.

Let K^0 and K^+ be the subalgebras of K generated by the $h_{i,r}$ and the $x^{+}_{i,l}$, respectively. Then $K = K^0 \oplus K^+$, $K^0 = \bigoplus_{n>0} K(0, n)$ with $K(0, n) = \sum_{i=0,1} \mathbf{C}(\gamma) h_{i,n}$ and $K^+ = \bigoplus_{\alpha \neq 0} K(\alpha, n)$. By Lemma 6.2 $\mathfrak{g}$ is defined by the same generators and relations as those of K^+ with $\gamma = 1$. Therefore

$$\dim_{\mathbf{C}(\gamma)} K(\alpha, n) = \dim_{\mathbf{C}(\gamma)} K^+(\alpha, n) \leq \dim_{\mathbf{C}} \mathrm{g}(\alpha, n)$$

for $\alpha \neq 0$ by specialization $\gamma \to 1$. By Lemma 6.3 and (6.9) $\dim_{\mathbf{C}} \mathrm{g}(\alpha, n) = \dim_{\mathbf{C}(\gamma)} \mathcal{K}(\alpha, n)$ for $\alpha \neq 0$ and by (6.10)

$$\dim_{\mathbf{C}(\gamma)} \mathcal{K}(0, n) = \begin{cases} 2 & \text{if } n > 0, \\ 0 & \text{if } n \leq 0. \end{cases}$$

So we obtain $(*)$. □

Corollary 6.1 (of the proof). $\dim_{\mathbf{C}(\gamma)} K^+(m\delta, n) = 2$ *for* $m, n > 0$.

Proof of Proposition 2.1. (1) There exists a surjective homomorphism of $Q \oplus \mathbf{Z}$ graded Lie algebras $\hat{L} \to \hat{\mathcal{L}}$ determined by the assignment in the proposition. Therefore it is sufficient to show that $\dim \hat{L}(\alpha, n) \leq \dim \hat{\mathcal{L}}(\alpha, n)$ $(*)$ for any α and n.

Let $\hat{L}^+$ and $\hat{L}^-$ be the subalgebras of $\hat{L}$ generated by the $x^+_{i,l}$ and the $x^-_{i,l}$, respectively, and $\hat{L}^0$ the subalgebra generated by the $h_{i,r}$ and c. As in [3] we can easily show that $\hat{L} = \hat{L}^- \oplus \hat{L}^0 \oplus \hat{L}^+$ and that

$$\dim \hat{L}(\alpha, n) = \begin{cases} 1 & \text{if } \alpha \text{ is a real root of } \widehat{sl_2} \\ 0 & \text{if } \alpha \neq 0 \text{ is not a root of } \widehat{sl_2} \end{cases}.$$

Since $\hat{L}(0, n) = \sum_{i=0,1} \mathbf{C}(\gamma) h_{i,n} + \mathbf{C}(\gamma) c \delta_{n,0}$, $\dim \hat{L}(0, n) \leq 2 + \delta_{n,0}$. On the other hand $\dim \hat{\mathcal{L}}(\alpha, n)$ satisfies the above equality for $\dim \hat{L}(\alpha, n)$ and

$$\hat{\mathcal{L}}(m\delta, n) = \begin{cases} \bigoplus_{i=1,2} \mathbf{C}(\gamma) \epsilon_i(m, n) & \text{if } (m, n) \neq (0, 0), \\ \mathbf{C}(\gamma) h \oplus \bigoplus_{i=1,2} \mathbf{C}(\gamma) c_i & \text{if } (m, n) = (0, 0). \end{cases}$$

By these we can see that $(*)$ holds unless $\alpha = m\delta$ $(m \neq 0)$. We shall show $(*)$ for $\alpha = m\delta$ $(m > 0)$ in the next paragraph. The proof of the case $m < 0$ is similar.

Fix $m > 0$. Note that $\hat{L}(m\delta, n) \subset \hat{L}^+$. For an integer l let $\hat{L}^+_l$ be the subalgebra of $\hat{L}$ generated by the elements $x^+_{i,k}$ $(i = 0, 1,\ k \geq l)$ and set $\hat{L}_{n,l} = \hat{L}(m\delta, n) \cap \hat{L}^+_l$. Then

$$\hat{L}_{n,l} \subset \hat{L}_{n,l-1} \quad \text{and} \quad \hat{L}(m\delta, n) = \bigcup_l \hat{L}_{n,l}. \tag{6.17}$$

Let ι be the automorphism of $\hat{L}$ determined by $x^{\pm}_{i,k} \mapsto x^{\pm}_{i,k\mp 1}$, $h_{i,r} \mapsto h_{i,r} - \delta_{r,0} c$ and $c \mapsto c$. Then $\hat{L}_{n,-k} = \iota^k(\hat{L}_{n+2km,0})$. Clearly there exists a homomorphism $\mathrm{t} \to \hat{L}$ such that $x^{\pm}_{i,k} \mapsto x^{\pm}_{i,k}$ and $h_{i,r} \mapsto h_{i,r}$. So by Lemma 6.2 we can see that the assignment

$x^+_{i,k} \mapsto x^+_{i,k}$ defines a homomorphism $K^+ \to \hat{L}$. This homomorphism and Corollary 6.1 imply that $\dim \hat{L}_{n,0} \leq \dim K^+(m\delta, n) = 2$ for $n > 0$. The last two results prove that $\dim \hat{L}_{n,-k} \leq 2$ if $n + 2km > 0$. From this and (6.17) we can conclude that $\hat{L}(m\delta, n) = \hat{L}_{n,l}$ if $l \ll 0$ and that $\dim \hat{L}(m\delta, n) \leq 2 = \dim \hat{\mathcal{L}}(m\delta, n)$. □

7. Proofs of Lemma 4.4 and Lemma 4.9

7.1. Proofs of Lemma 4.4 and Lemma 4.9.

7.1.1. First we prepare two lemmas for the algebras $\mathcal{U}$ and $\mathcal{A}$.

Lemma 7.1. *For $m = 0, 1$ let $\mathcal{U}^{\pm}[m]$ be the subalgebra of $\mathcal{U}^{\pm}$ generated by $x^{\pm}_{0,\mp m}$ and $x^{\pm}_{1,\pm m}$. Then for $i = 0, 1$ the elements $\mathbf{h}_{i,r}$ are in $\mathcal{U}^+[1 - i]$ if $r > 0$ and in $\mathcal{U}^-[1 - i]$ if $r < 0$.*

Proof. Since $\mathbf{h}_{1,\pm r} = \rho_h(h_{\pm r})$, the case $i = 1$ follows from [25]. (See also (7.1) below.) From the equality $h_{0,r} = (-1)^r \mathcal{S}\mathcal{Y}^{-1} h_{1,r}$ we obtain $\mathbf{h}_{0,r} = (-1)^r \mathcal{Y}^{-1}\mathcal{S}\mathbf{h}_{1,r}$ by Proposition 3.2 (2). Therefore the equalities $\mathcal{Y}^{-1}\mathcal{S}x^{\pm}_{1,0} = x^{\pm}_{0,\mp 1}$ and $\mathcal{Y}^{-1}\mathcal{S}x^{\pm}_{0,0} = -x^{\pm}_{1,\pm 1}$ and the case $i = 1$ prove the case $i = 0$. □

Lemma 7.2. *Let $\tilde{\mathcal{A}}$ be the subalgebra of $\mathcal{A}$ generated by $k_1, k_1^{-1}, a_{i,r}$ and $\mathbf{a}_{i,s}$ ($i = 1, 2,\ r \neq 0,\ s > 0$). Then $\mathcal{X}^{\pm 1}_j(\tilde{\mathcal{A}}) \subset \tilde{\mathcal{A}}$ for $j = 0, 1$.*

Proof. Part (2) of Lemma 4.2 implies that $\mathbf{a}_{i,s}$ ($s > 0$) is in the subalgebra of $\mathcal{A}$ generated by $a_{i,1}, a_{i,-1}$ and $\mathbf{a}_{i,1}$. So $\tilde{\mathcal{A}}$ is generated by $k_1, k_1^{-1}, h_{i,r}$ and $\mathbf{h}_{i,1}$ ($i = 0, 1$, $r \neq 0$). Therefore the claim follows from $\mathcal{X}^{\pm 1}_j(k_1) = k_1$, $\mathcal{X}^{\pm 1}_j(h_{i,r}) = h_{i,r}$ and

$$(-1)^{\delta_{j,1}}\mathcal{X}^{\pm 1}_j(\mathbf{h}_{i,1}) = \left[[2]h_{j,\mp 1} + (\gamma + \gamma^{-1})h_{1-j,\mp 1}, \mathbf{h}_{i,1}\right] \big/ ([2]^2 - (\gamma + \gamma^{-1})^2). \qquad \square$$

7.1.2. Next we introduce several notations. Set $R = \{f(q)/g(q) \mid f, g \in \mathbf{C}(\gamma)[q], g(1) \neq 0\} \subset F$. Let $U_q(K)$ (resp. $U^R_q(K)$) be the F algebra (resp. R algebra) defined by generators $X^+_{i,m}$ and $H_{i,r}$ ($i = 0, 1$, $m \in \mathbf{Z}_{\geq 0}$, $r \in \mathbf{Z}_{>0}$) and relations

$$[H_{i,r}, H_{j,s}] = 0,$$

$$\left[H_{i,r}, X^+_{i,m}\right] = \frac{[2r]}{r} X^+_{i,r+m},$$

$$\left[H_{i,r}, X^+_{1-i,m}\right] = -\frac{[r](\gamma^r + \gamma^{-r})}{r} X^+_{1-i,r+m},$$

$$\left[X^+_{i,m+1}, X^+_{i,n}\right]_{q^2} + \left[X^+_{i,n+1}, X^+_{i,m}\right]_{q^2} = 0,$$

$$\mathrm{Sym}_{m_1,m_2,m_3}\left[X^+_{i,m_1}, \left[X^+_{i,m_2}, \left[X^+_{i,m_3}, X^+_{1-i,n}\right]_{q^{-2}}\right]\right]_{q^2} = 0,$$

$$\begin{aligned}
&[2]\left[X^+_{i,m+2}, X^+_{1-i,l}\right]_{q^{-2}} + (\gamma+\gamma^{-1})\left[X^+_{1-i,l+1}, X^+_{i,m+1}\right]_{q^{-2}}\\
&= [2]\left[X^+_{1-i,l+2}, X^+_{i,m}\right]_{q^{-2}} + (\gamma+\gamma^{-1})\left[X^+_{i,m+1}, X^+_{1-i,l+1}\right]_{q^{-2}},\\
&[2]\left[X^+_{i,0}, \left[X^+_{i,1}, X^+_{1-i,n}\right]_{q^{-2}}\right] + (\gamma+\gamma^{-1})\left[X^+_{i,0}, \left[X^+_{1-i,n+1}, X^+_{i,0}\right]_{q^{-2}}\right] = 0,\\
&\left[X^+_{i,1}, \left[X^+_{i,1}, X^+_{1-i,n}\right]_{q^{-2}}\right] + \left[X^+_{i,0}, \left[X^+_{1-i,n+2}, X^+_{i,0}\right]_{q^{-2}}\right] = 0.
\end{aligned}$$

Assigning (α_i, m) to $X^+_{i,m}$ and $(0, r)$ to $H_{i,r}$, these algebras are endowed with structures of $Q \oplus \mathbf{Z}$ graded algebras. We denote the homogeneous subspace of degree (α, l) of $U_q(K)$ by $U_q(K)(\alpha, l)$ and set

$$U_q(K)_m = \sum_{n,l\in\mathbf{Z}} U_q(K)(m\alpha_1 + n\delta, l)$$

for any integer m. Define $U^R_q(K)(\alpha, l)$ and $U^R_q(K)_m$ similarly.

Set $I_F = \sum_{m>0} U_q(K)_{-m} U_q(K)_m$ and define I_R similarly. Let $I_{q=1} = \sum_{m>0} U(K)_{-m} U(K)_m$. Then $U(K)_0/I_{q=1} \simeq U(E)$. Noting this, we shall often denote $U_q(K)_0/I_F$ and $U^R_q(K)_0/I_R$ by $U_q(E)$ and $U^R_q(E)$, respectively, by abuse of notation. Since I_F is a graded ideal of the $\mathbf{Z} \oplus \mathbf{Z}$ graded algebra $U_q(K)_0$, $U_q(E) = \bigoplus_{m,n\in\mathbf{Z}_{\geq 0}} U_q(E)(m\delta, n)$. A similar formula holds also for $U^R_q(E)$.

Define polynomials in (commutative) variables a_m ($m \in \mathbf{Z}_{>0}$) with coefficients in R, $g_r(a_1, \ldots, a_{r-1})$ ($r > 0$), by the generating series

$$\left(\exp\left((q-q^{-1})\sum_{m>0} a_m z^m\right) - 1\right) \Big/ (q-q^{-1}) = \sum_{r>0}(a_r + (q-q^{-1})g_r(a_1, \ldots, a_{r-1}))z^r$$

Using these polynomials, we define $\mathbf{H}_{1,r} \in U^R_q(K)_0$ ($r > 0$) by

$$\begin{aligned}
&\mathbf{H}_{1,1} = -\left[X^+_{0,0}, X^+_{1,0}\right]_{q^{-2}},\\
&\mathbf{H}_{1,r+1} = \frac{r}{[2r]}\left[\left[\mathbf{H}_{1,r}, X^+_{0,0}\right], X^+_{1,0}\right]_{q^{-2}} - (q-q^{-1})g_{r+1}(\mathbf{H}_{1,1}, \ldots, \mathbf{H}_{1,r}) \quad (r > 0)
\end{aligned} \tag{7.1}$$

where the $\mathbf{H}_{1,s}$ are ordered in $g_{r+1}(\mathbf{H}_{1,1}, \ldots, \mathbf{H}_{1,r})$ so that $\mathbf{H}_{1,s}$ is to the left of $\mathbf{H}_{1,t}$ if $s < t$. Note that the elements $\mathbf{h}_{1,r}$ of $\mathcal{U}$ defined in Section 3 satisfy the above equations with $X^+_{i,0}$ replaced by $x^+_{i,0}$ ($i = 0, 1$). Further we define $\Lambda_{i,r} \in U^R_q(K)_0$ ($i = 1, 2$, $r > 0$) by (4.2) with the $h_{i,s}$ replaced by the $H_{i,s}$. We also define $\mathbf{H}_{1,r}, \Lambda_{i,r} \in U_q(K)_0$ ($r > 0$, $i = 1, 2$) in the same way.

In this section we denote the elements $x^+_{i,m}$, $h_{i,r}$ and $\Lambda_{i,r}$ of $U(K)$ by $\dot{x}^+_{i,m}$, $\dot{h}_{i,r}$ and $\dot{\Lambda}_{i,r}$. Further we set $\dot{\mathbf{h}}_{1,r} = \epsilon_1(r, 0) - \epsilon_2(r, 0) \in U(E)$ for $r > 0$. We signify the automorphism $\tilde{\mathcal{Y}}$ of $U(E)$ by $\tilde{\mathcal{Y}}_{q=1}$.

7.1.3. The purpose of this subsubsection is to prove Lemma 7.3 and Lemma 7.5 below.

Assigning $(\pm\alpha_i, m)$ to $x^{\pm}_{i,m}$, $(0, r)$ to $h_{i,r}$ and $(0, 0)$ to $k_1^{\pm 1}$, $\mathcal{U}$ is endowed with a structure of $Q \oplus \mathbf{Z}$ graded algebras. The quotient algebras $\mathcal{A}$ and $\mathcal{A}_{\mathbf{r}}$ of $\mathcal{U}_0$ are $\mathbf{Z} \oplus \mathbf{Z}$ graded for the same reason as $U_q(E) = U_q(K)_0/I_F$. Let $\mathcal{A}'$ be the subalgebra of $\mathcal{A}$ generated by the elements $h_{i,l}, \mathbf{h}_{1,l}$ $(i = 0, 1,\ l > 0)$ and $\mathcal{A}'_{\mathbf{r}}$ as before. Then these algebras are graded subalgebras of $\mathcal{A}$ and $\mathcal{A}_{\mathbf{r}}$, respectively. We denote their homogeneous subspaces of degree (m, n) by $\mathcal{A}'(m\delta, n)$ and $\mathcal{A}'_{\mathbf{r}}(m\delta, n)$.

For $\mathbf{r} = (r_1, r_2) \in \mathbf{Z}^2_{\geq 0}$ let $I_{\mathbf{r}}(E)$ be the $\tilde{\mathcal{Y}}^{\pm 1}_{q=1}$ invariant ideal of $U(E)$ generated by the elements $\dot{\Lambda}_{i,n}$ $(n > r_i,\ i = 1, 2)$ and

$$\sum_{j=0}^{n} \dot{\Lambda}_{n-j} \left(\sum_{\substack{l_1,\dots,l_m \geq 0 \\ l_1+\dots+l_m = j}} \left\| \dot{x}^+_{1,l_1} \cdots \dot{x}^+_{1,l_m} \left(\dot{x}^+_{0,0}\right)^m \right\| \right) \quad (m, n \geq 0,\ \ m + n > r_1 + r_2).$$

Lemma 7.3. *For $m, n \in \mathbf{Z}_{\geq 0}$ the following hold.*
(1) $\dim_F \mathcal{A}'_{\mathbf{r}}(m\delta, n) \geq \dim_F \phi_{\mathbf{r}}(\mathcal{A}'_{\mathbf{r}}(m\delta, n)) \geq \dim_{\mathbf{C}(\gamma)}(U(E)/I_{\mathbf{r}}(E))(m\delta, n)$.
(2) $\dim_F \mathcal{A}'(m\delta, n) \geq \dim_{\mathbf{C}(\gamma)} U(E)(m\delta, n)$.

To prove this lemma, we need the following lemma, which will be proven in Section 7.2.

Lemma 7.4. (1) $I_{\mathbf{r}}(E) = U(E) \cap I_{\mathbf{r}}$.
(2) $I_{\mathbf{r}}(E) = \bigoplus_{m+n>\min(r_1,r_2)} I_{\mathbf{r}}(E) \cap U(E)(m\delta, n)$.

Proof of Lemma 7.3. (1) Since $U(E)/I_{\mathbf{r}}(E)$ is generated by the elements $\dot{h}_{i,l}$ and $\dot{\mathbf{h}}_{1,l}$ $(i = 0, 1,\ l > 0)$, there exists a family $\{f_i\}_{i\in I}$ of polynomials in noncommutative variables with coefficients in $\mathbf{C}(\gamma)$ such that the vectors $v_i := f_i(\dot{h}_{1,1}, \dots, \dot{h}_{0,1}, \dots, \dot{\mathbf{h}}_{1,1}, \dots)$ are a basis of the $\mathbf{C}(\gamma)$ vector space $(U(E)/I_{\mathbf{r}}(E))(m\delta, n)$ for each m and n. Replacing the elements $\dot{h}_{i,l}$ and $\dot{\mathbf{h}}_{1,l}$ by $h_{i,l}$ and $\mathbf{h}_{1,l}$ in $\mathcal{A}'_{\mathbf{r}}$, the f_i define the vectors $\tilde{v}_i \in \mathcal{A}'_{\mathbf{r}}(m\delta, n)$. Part (1) of Lemma 7.4 implies that the map $U(E)/I_{\mathbf{r}}(E) \to U(\mathcal{H})/I_{\mathbf{r}}$ $(\bar{u} \mapsto \bar{u})$ is injective. Therefore the vectors $(\varphi_{1,r_1} \otimes \varphi_{2,r_2})(v_i) \in \mathcal{C}_{\gamma}^{\otimes r_1} \otimes \mathcal{C}_{\gamma^{-1}}^{\otimes r_2}$ are linearly independent over $\mathbf{C}(\gamma)$ by Theorem 2.1. This implies that the vectors $\phi_{\mathbf{r}}(\tilde{v}_i)$ are linearly independent over F since $\phi_{\mathbf{r}}(\tilde{v}_i)|_{q=1} = (\varphi_{1,r_1} \otimes \varphi_{2,r_2})(v_i)$.

(2) By part (1) $\dim_F \mathcal{A}'(m\delta, n) \geq \dim_{\mathbf{C}(\gamma)}(U(E)/I_{\mathbf{r}}(E))(m\delta, n)$ for any $\mathbf{r}$. By part (2) of Lemma 7.4 we obtain the claim from this by letting r_1 and r_2 sufficiently large. □

Define B to be the subalgebra of $\mathcal{U}$ generated by $x^+_{i,m}$ and $h_{i,r}$ $(i = 0, 1,\ m \geq 0,\ r > 0)$. This algebra inherits a structure of $Q \oplus \mathbf{Z}$ graded algebras from $\mathcal{U}$. Let $^-: \mathcal{U}_0 \to \mathcal{U}_0/\mathcal{I}$ $(= \mathcal{A})$ be the quotient map as before. Let further T denote the ideal of

$U_q^R(E)$ consisiting of all the torsion elements of the R module $U_q^R(E)$.

Lemma 7.5. (1) $\overline{B_0} = \mathcal{A}' \simeq U_q(E)$. *In particular* $\overline{B_0} \subset \tilde{\mathcal{A}}$.
(2) $U_q^R(E)/T$ *is isomorphic to the* R *subalgebra of* $\mathcal{A}$ *generated by the elements* $h_{i,r}$ *and* $\mathbf{h}_{1,r}$ $(i = 0, 1,\ r > 0)$.

Proof. (1) We regard $\mathbf{C}(\gamma)$ as an R module by letting $f(q)$ act as $f(1)$. Clearly there exists a $\mathbf{C}(\gamma)$ algebra homomorphism $U(K) \to U_q^R(K) \otimes_R \mathbf{C}(\gamma)$ determined by $\dot{x}^+_{i,m} \mapsto X^+_{i,m} \otimes 1$ and $\dot{h}_{i,r} \mapsto H_{i,r} \otimes 1$. This map induces a surjective homomorphism

$$U(E) \simeq U(K)_0/I_{q=1} \to U_q^R(K)_0/I_R \otimes_R \mathbf{C}(\gamma) \quad (\dot{h}_{i,r} \mapsto H_{i,r} \otimes 1,\ \dot{\mathbf{h}}_{1,r} \mapsto \mathbf{H}_{1,r} \otimes 1).$$

Since

$$U_q^R(K)_0/I_R \otimes_R F \simeq U_q(K)_0/I_F = U_q(E) \tag{7.2}$$

and each $\big(U_q^R(K)_0/I_R\big)(m\delta, n)$ is finitely generated over R, we obtain

$$\dim_{\mathbf{C}(\gamma)} U(E)(m\delta, n) \geq \dim_F U_q(E)(m\delta, n)$$

by specialization argument.

By part (1) of Lemma 7.1 $\mathcal{A}'$ is a graded subalgebra of the $\mathbf{Z} \oplus \mathbf{Z}$ graded algebra $\overline{B_0}$. By Lemma 3.3 there exists a homomorphism $U_q(K) \to B$ $(X^+_{i,m} \mapsto x^+_{i,m},\ H_{i,r} \mapsto h_{i,r})$, which induces a surjective homomorphism $U_q(E) \to \overline{B_0}$. Therefore

$$\dim_F U_q(E)(m\delta, n) \geq \dim_F \overline{B_0}(m\delta, n) \geq \dim_F \mathcal{A}'(m\delta, n).$$

By the above two inequalities and part (2) of Lemma 7.3, we can conculde that $\mathcal{A}' = \overline{B_0} \simeq U_q(E)$.

(2) Set $C = U_q^R(E)/T$. Then $C = \bigoplus_{m,n} C(m\delta, n)$ since T is graded, and each $C(m\delta, n)$ is a free R module of finite rank. Let C' be the subalgebra of C generated by the elements $H_{i,r}$ and $\mathbf{H}_{1,r}$ $(i = 0, 1,\ r > 0)$. Since R is a principal ideal domain, there exist an R basis $v_1, \ldots, v_M$ of $C(m\delta, n)$, a nonnegative integer $N \leq M$ and $a_1(q), \ldots, a_N(q) \in R$ such that $a_1(q)v_1, \ldots, a_N(q)v_N$ is an R basis of $C' \cap C(m\delta, n)$ for each m and n. Let μ be the $\mathbf{C}(\gamma)$ linear map $C' \otimes_R \mathbf{C}(\gamma) \xrightarrow{i\otimes 1} C \otimes_R \mathbf{C}(\gamma)$ where i is the inclusion map. Let ν be the composite map $U(E) \to U_q^R(E) \otimes_R \mathbf{C}(\gamma) \xrightarrow{\bar{}\otimes 1} C \otimes_R \mathbf{C}(\gamma)$ where the first map is the one in part (1) and $\bar{}$ is the quotient map. Since ν is surjective and $U(E)$ is generated by the elements $\dot{h}_{i,r}$ and $\dot{\mathbf{h}}_{1,r}$ $(i = 0, 1,\ r > 0)$, μ is surjective. This implies that $N = M$ and $a_i(1) \neq 0$ $(1 \leq i \leq N)$. Hence $C = C'$.

The R module C is free and F is the fraction field of R. Therefore the map $C \to C \otimes_R F$ $(c \mapsto c \otimes 1)$ is injective and the map $U_q^R(E) \otimes_R F \xrightarrow{\bar{}\otimes 1} C \otimes_R F$ is bijective. These and (7.2) imply that $C \to U_q(E)$ $(H_{i,r} \mapsto H_{i,r},\ \mathbf{H}_{1,r} \mapsto \mathbf{H}_{1,r})$ is an injective homomorphism of R algebras. So C can be identified with the R subalgebra of $U_q(E)$ generated by the elements $H_{i,r}$ and $\mathbf{H}_{1,r}$ ($i = 0, 1,\ r > 0$). Therefore part (1) proves the claim. $\square$

7.1.4. Now we can give the proofs of Lemma 4.4 and Lemma 4.9.

Proof of Lemma 4.4. We shall show the claim for $\overline{\mathcal{U}^0\mathcal{U}_0^+}$, the proof for the case $\overline{\mathcal{U}^0\mathcal{U}_0^-}$ being similar. For any $x \in \overline{\mathcal{U}_0^+}$ there exists $m \geq 0$ such that $(\mathcal{X}_0\mathcal{X}_1)^{-m}(x) \in \overline{B_0}$. So $\overline{\mathcal{U}_0^+} \subset \tilde{\mathcal{A}}$ by Lemma 7.2 and Lemma 7.5. Since $\overline{\mathcal{U}^0} \subset \tilde{\mathcal{A}}$, we obtain the inclusion $\overline{\mathcal{U}^0\mathcal{U}_0^+} \subset \tilde{\mathcal{A}}$. The reverse inclusion follows from Lemma 7.1. $\square$

Proof of Lemma 4.9. Set $\tilde{U}_q^R = U_q^R(E)/T$. By Proposition 4.3 $\tilde{\mathcal{Y}}(\mathbf{h}_{1,r}) = \mathbf{h}_{1,r}$ and $\tilde{\mathcal{Y}}(h_{i,r}) = \sum_j c_{ij}^r h_{j,r}$ ($c_{ij}^r \in R$) in $\mathcal{A}$. So Lemma 7.5 implies that the automorphism $\tilde{\mathcal{Y}}$ of $\mathcal{A}$ defines automorphisms of $U_q(E)$ and $\tilde{U}_q^R$, which we denote by the same letter.

Let $I_{\mathbf{r}}^F(E)$ (resp. $I_{\mathbf{r}}^R(E)$) be the $\tilde{\mathcal{Y}}^{\pm 1}$ invariant ideal of $U_q(E)$ (resp. $\tilde{U}_q^R(E)$) generated by $\Lambda_{i,n}$ ($n > r_i,\ i = 1, 2$) and the images of the following elements in $U_q(E)$ (resp. $\tilde{U}_q^R(E)$):

$$\sum_{j=0}^{n} q^{-j}\Lambda_{n-j}\left(\sum_{\substack{l_1,\dots,l_m\geq 0\\ l_1+\cdots+l_m=j}} X_{1,l_1}^+\cdots X_{1,l_m}^+\left(X_{0,0}^+\right)^m\right) \tag{7.3}$$

where $m, n \geq 0$ and $m + n > r_1 + r_2$. Let g be the composite map $U(E) \xrightarrow{\nu} \tilde{U}_q^R(E) \otimes_R \mathbf{C}(\gamma) \xrightarrow{\bar{\ }\otimes 1} \tilde{U}_q^R(E)/I_{\mathbf{r}}^R(E) \otimes_R \mathbf{C}(\gamma)$ where ν is the map in the proof of Lemma 7.5 (2). The automorphism $\tilde{\mathcal{Y}}$ of $\tilde{U}_q^R(E)$ defines an automorphism of $\tilde{U}_q^R(E)/I_{\mathbf{r}}^R(E) \otimes_R \mathbf{C}(\gamma)$ naturally, which we denote by the same symbol. This satisfies $\tilde{\mathcal{Y}}^{\pm 1} \circ g = g \circ \tilde{\mathcal{Y}}_{q=1}^{\pm 1}$. So $\operatorname{Ker} g$ is invariant under $\tilde{\mathcal{Y}}_{q=1}^{\pm 1}$. Further g annihilates the generators of $I_{\mathbf{r}}(E)$. Therefore $I_{\mathbf{r}}(E) \subset \operatorname{Ker} g$ and g induces a surjective homomorphism $U(E)/I_{\mathbf{r}}(E) \to \tilde{U}_q^R(E)/I_{\mathbf{r}}^R(E) \otimes_R \mathbf{C}(\gamma)$.

By Lemma 7.5 $\tilde{U}_q^R(E)$ can be identified with an R subalgebra of $U_q(E)$ and $\tilde{U}_q^R(E) \otimes_R F \simeq U_q(E)$ ($u \otimes c \leftrightarrow uc$). Since $I_{\mathbf{r}}^F(E) = F I_{\mathbf{r}}^R(E)$, we find that $\tilde{U}_q^R(E)/I_{\mathbf{r}}^R(E) \otimes_R F \simeq U_q(E)/I_{\mathbf{r}}^F(E)$.

If we identify $U_q(E)$ with $\mathcal{A}'$ by Lemma 7.5, the element (7.3) is a scalar multiple of $(x_{1,0}^+)^{m+n}(x_{1,1}^-)^n(x_{0,0}^+)^m$ by part (1) of Lemma 4.6. So it vanishes in $\mathcal{A}'_{\mathbf{r}}$ (regarded as a quotient algebra of $\mathcal{A}'$) if $m + n > r_1 + r_2$. This and Proposition 4.4 (3) imply that the composite map $U_q(E) \xrightarrow{\sim} \mathcal{A}' \to \mathcal{A}'_{\mathbf{r}}$ induces a surjective homomorphism $U_q(E)/I_{\mathbf{r}}^F(E) \to \mathcal{A}'_{\mathbf{r}}$.

From the above we obtain the inequality $\dim_{\mathbf{C}(\gamma)}(U(E)/I_{\mathbf{r}}(E))(m\delta, n) \geq \dim_F \mathcal{A}'_{\mathbf{r}}(m\delta, n)$ by specialization $q \to 1$. Combining this with part (1) of Lemma 7.3, we get $\dim_F \phi_{\mathbf{r}}(\mathcal{A}'_{\mathbf{r}}(m\delta, n)) = \dim_F \mathcal{A}'_{\mathbf{r}}(m\delta, n)$. $\square$

7.2. Proof of Lemma 7.4. To complete the proofs of Lemma 4.4 and Lemma 4.9, we shall show Lemma 7.4.

Set $Z' = \{(k, l) \in \mathbf{Z}^2 \mid k \geq 0,\ l > 0\}$ and $Z = \{(u, 0) \mid u \in \mathbf{Z}_{>0}\} \cup Z'$. Let $\iota = (1, 0)$ and $\mathbf{1} = (0, 1)$. Fix a total order $\succeq$ on $\mathbf{Z}^2$ such that

$$u_1\iota \succ u_2\iota \quad \text{if} \quad u_1 > u_2 > 0 \quad \text{and} \quad u\iota \succ \mathbf{k} \quad \text{for} \quad u \in \mathbf{Z}_{>0} \quad \text{and} \quad \mathbf{k} \in Z'.$$

Hereafter most of the time we consider the elements $A^{r_i}_{i,M}(\mathbf{k}_1, \ldots, \mathbf{k}_M)$ for $\mathbf{k}_1, \ldots, \mathbf{k}_M \in Z$. We shall denote them simply by $A_{i,M}(\mathbf{k}_1, \ldots, \mathbf{k}_M)$ since they are independent of r_i.

For $M \geq 1$ and $\mathbf{k}_i = (k_i, l_i) \in Z$ $(i = 1, \ldots, M)$ set

$$v_M(\mathbf{k}_1, \ldots, \mathbf{k}_M) = \left\| e(\mathbf{0})^M f(\mathbf{k}_M) \cdots f(\mathbf{k}_1) \right\| \big/ M!.$$

Lemma 7.6. *Let $\mathbf{r} = (r_1, r_2) \in \mathbf{Z}^2_{\geq 0}$. For $i = 1, 2$ the following hold in $U(\mathcal{H})$:*

(1) *$A^{r_i}_{i,M}(\mathbf{k}_1, \ldots, \mathbf{k}_M)$ is symmetric in the variables $\mathbf{k}_1, \ldots, \mathbf{k}_M$.*

(2) $A^{r_i}_{i,M}(\mathbf{k}_1, \ldots, \mathbf{k}_M) = A^{r_i}_{i,M-1}(\mathbf{k}_1, \ldots, \mathbf{k}_{M-1})A^{r_i}_{i,1}(\mathbf{k}_M) - \sum_{i=1}^{M-1} p^{\varepsilon_i l_i k_M} A^{r_i}_{i,M-1}(\mathbf{k}_1, \ldots, \mathbf{k}_i + \mathbf{k}_M, \ldots, \mathbf{k}_{M-1})$.

(3) $A^{r_i}_{i,M}(\mathbf{k}_1, \ldots, \mathbf{k}_{M-1}, \mathbf{0}) = (r_i + 1 - M)A^{r_i}_{i,M-1}(\mathbf{k}_1, \ldots, \mathbf{k}_{M-1})$.

(4) $n!\Lambda_{i,n} = (-1)^n A_{i,n}(\mathbf{1}, \ldots, \mathbf{1})$ *for* $n > 0$.

(5) *For* $\mathbf{k}_j = (k_j, l_j) \in Z$ $(1 \leq j \leq M)$

$$v_M(\mathbf{k}_1, \ldots, \mathbf{k}_M) = \sum_{I \sqcup J = \{1,\ldots,M\}} \gamma^{2\sum_\alpha k_{j_\alpha} l_{j_\alpha}} A_{1,a}(\mathbf{k}_{i_1}, \ldots, \mathbf{k}_{i_a}) A_{2,b}(\mathbf{k}_{j_1}, \ldots, \mathbf{k}_{j_b})$$

where $I = \{i_1, \ldots, i_a\}$ $(i_1 < \cdots < i_a)$ and $J = \{j_1, \ldots, j_b\}$ $(j_1 < \cdots < j_b)$ in the summand.

(6) *The elements $A^{r_i}_{i,M}(\mathbf{k}_1, \ldots, \mathbf{k}_M)$ $(M \geq 0,\ \mathbf{k}_1 \succeq \cdots \succeq \mathbf{k}_M,\ \mathbf{k}_1, \ldots, \mathbf{k}_M \in \mathbf{Z}^2 \setminus \{0\})$ form a basis of $U(\mathcal{H}_i)$.*

Proof. The claims except parts (5) and (6) were proved in [19]. The equality in (5) modulo $U(\mathcal{H})(h - (r_1 + r_2))$ was proved for any nonnegative integers r_1 and r_2 in the same reference. Since $A^{r_i}_{i,M}(\mathbf{k}_1, \ldots, \mathbf{k}_M)$ is independent of r_i for $\mathbf{k}_j \in Z$ $(\forall j)$ and $\bigcap_{r \geq 0} U(\mathcal{H})(h - r) = \{0\}$, we obtain (5).

For $n \geq 1$ set

$$U(\mathcal{H}_i)[n] = \sum_{0 \leq m \leq n} \sum_{\mathbf{k}_1, \ldots, \mathbf{k}_m \in \mathbf{Z}^2 \setminus \{\mathbf{0}\}} \mathbf{C}(\gamma) a_i(\mathbf{k}_1) \cdots a_i(\mathbf{k}_m).$$

Then by the definition of $A^{r_i}_{i,M}$ and part (3)

$$A^{r_i}_{i,M}(\mathbf{k}_1, \ldots, \mathbf{k}_M) \equiv a_i(\mathbf{k}_1) \cdots a_i(\mathbf{k}_M) \mod U(\mathcal{H}_i)[M - 1]$$

for $M \geq 2$ and $\mathbf{k}_1, \ldots, \mathbf{k}_M \in \mathbf{Z}^2 \setminus \{0\}$. So we obtain (6) by the PBW theorem. $\square$

For $t, M_1, M_2 \geq 0$, $u_1, \ldots, u_t \in \mathbf{Z}_{>0}$ and $\mathbf{k}_1, \ldots, \mathbf{k}_{M_1}, \mathbf{m}_1, \ldots, \mathbf{m}_{M_2} \in Z'$, set

$$\begin{aligned} &C_{t,M_1,M_2}\left(u_1, \ldots, u_t; \mathbf{k}_1, \ldots, \mathbf{k}_{M_1}; \mathbf{m}_1, \ldots, \mathbf{m}_{M_2}\right) \\ &= \sum_{I \sqcup J = \{1,\ldots,t\}} A_{1,a+M_1}\left(u_{i_1}\iota, \ldots, u_{i_a}\iota, \mathbf{k}_1, \ldots, \mathbf{k}_{M_1}\right) A_{2,b+M_2}\left(u_{j_1}\iota, \ldots, u_{j_b}\iota, \mathbf{m}_1, \ldots, \mathbf{m}_{M_2}\right) \end{aligned}$$

with the notation in part (5) of Lemma 7.6.

Lemma 7.7.

$$U(E) = \oplus \bigoplus_{\substack{u_1 \geq \cdots \geq u_t > 0 \\ \mathbf{k}_1 \succeq \cdots \succeq \mathbf{k}_{M_1} \\ \mathbf{m}_1 \succeq \cdots \succeq \mathbf{m}_{M_2}}} \mathbf{C}(\gamma) C_{t,M_1,M_2}(u_1, \ldots, u_t; \mathbf{k}_1, \ldots, \mathbf{k}_{M_1}; \mathbf{m}_1, \ldots, \mathbf{m}_{M_2})$$

where the first sum is taken over nonnegative integers t, M_1, M_2.

Proof. $U(E)$ is the subalgebra of $U(\mathcal{H})$ generated by the elements $A_{1,1}(u,0) + A_{2,1}(u,0)$ $(u > 0)$ and $A_{i,1}(\mathbf{k})$ $(i = 1, 2,\ \mathbf{k} \in Z')$. By the definition of the $A_{i,M}(\mathbf{k}_1, \ldots, \mathbf{k}_M)$,

$$A_{1,M_1}\left(\mathbf{k}_1, \ldots, \mathbf{k}_{M_1}\right) A_{2,M_2}\left(\mathbf{m}_1, \ldots, \mathbf{m}_{M_2}\right) \in U(E)$$

for $\mathbf{k}_1, \ldots, \mathbf{k}_{M_1}, \mathbf{m}_1, \ldots, \mathbf{m}_{M_2} \in Z'$. Multiplying this by $A_{1,1}(u,0) + A_{2,1}(u,0)$ $(u > 0)$ repeatedly, we obtain the inclusion $\supset$. The reverse inclusion follows from the fact that the r.h.s. contains 1 and is preserved by multiplication by the generators of $U(E)$. The directness of the sum on the r.h.s. follows from Lemma 7.6 (6). $\square$

Lemma 7.8. *$A_{i,M}(\mathbf{k}_1, \ldots, \mathbf{k}_M) \in I_{\mathbf{r}}(E)$ if $M > r_i$ and $\mathbf{k}_1, \ldots, \mathbf{k}_M \in Z'$.*

Proof. First we show that $A_{i,M}(l_1\mathbf{1}, \ldots, l_M\mathbf{1}) \in I_{\mathbf{r}}(E)$ if $M > r_i$ and $l_j > 0$ $(\forall j)$ by induction on $t := \#\{j \mid l_j \neq 1\}$. The case $t = 0$ follows from part (4) of Lemma 7.6. Suppose that the claim is proved up to t. Then by calculating $A_{i,1}((l-1)\mathbf{1})A_{i,M}(l_1\mathbf{1}, \ldots, l_t\mathbf{1}, \mathbf{1}, \ldots, \mathbf{1})$ we find that

$$(M-t)A_{i,M}(l_1\mathbf{1}, \ldots, l_t\mathbf{1}, l\mathbf{1}, \mathbf{1}, \ldots, \mathbf{1}) + A_{i,M+1}(l_1\mathbf{1}, \ldots, l_t\mathbf{1}, (l-1)\mathbf{1}, \mathbf{1}, \ldots, \mathbf{1}) \in I_{\mathbf{r}}(E)$$

for $l > 1$ by (2.11) and Lemma 7.6 (1). Using this, we can prove the case $t+1$ by induction on l.

Next for $M > r_i$ we show that $A_{i,M}(\mathbf{k}_1, \ldots, \mathbf{k}_t, \mathbf{1}, \ldots, \mathbf{1}) \in I_{\mathbf{r}}(E)$ if $\mathbf{k}_j = (k_j, l_j) \in Z'$ $(1 \leq j \leq t)$ by induction on t. The case $t = M$ proves the claim. If $t = 0$, or $t > 0$ and $k_j = 0$ $(1 \leq j \leq t)$, then the claim has already been proved. Suppose that $t > 0$ and $k_j > 0$ for some $j \in \{1, \ldots, t\}$ and that the claim is proved up to $t-1$. Then by (2.11) and Lemma 7.6

$$(M - t + 1)\left(1 - p^{\varepsilon_i k_j}\right) A_{i,M}(\mathbf{k}_1, \ldots, \mathbf{k}_t, \mathbf{1}, \ldots, \mathbf{1})$$

$$\equiv \left[A_{i,1}\left(k_j, l_j - 1\right) + \delta_{l_j,1} A_{1-i,1}\left(k_j, l_j - 1\right), A_{i,M}\left(\mathbf{k}_1, \ldots, \hat{\mathbf{k}}_j, \ldots, \mathbf{k}_t, \mathbf{1}, \ldots, \mathbf{1}\right)\right]$$
$$\equiv 0 \mod I_{\mathbf{r}}(E)$$

where ˆ denotes omission. This proves the case t. □

Lemma 7.9. *If $M_1 > r_1$ or $M_2 > r_2$ or $M_1 + M_2 + t > r_1 + r_2$, then*

$$C_{t,M_1,M_2}(u_1, \ldots, u_t; \mathbf{k}_1, \ldots, \mathbf{k}_{M_1}; \mathbf{m}_1, \ldots, \mathbf{m}_{M_2}) \in I_{\mathbf{r}}(E)$$

for $u_1, \ldots, u_t \in \mathbf{Z}_{>0}$ and $\mathbf{k}_1, \ldots, \mathbf{k}_{M_1}, \mathbf{m}_1, \ldots, \mathbf{m}_{M_2} \in Z'$.

Proof. First we prove the claim for $M_1 > r_1$ or $M_2 > r_2$. By Lemma 7.8 we can see that the claim holds if $t = 0$. By multiplying this by $A_{1,1}(u, 0) + A_{2,1}(u, 0)$ $(u > 0)$ repeatedly, we obtain the case $t > 0$.

Next we consider the case $M_1 + M_2 + t > r_1 + r_2$. Set $N = M_1 + M_2$. By Lemma 4.6 (1) and Lemma 7.6 (5)

$$\begin{aligned}
&(-1)^N \frac{N!}{t!} \times \sum_{j=0}^{N} \Lambda_{N-j} \sum_{\substack{l_1,\ldots,l_t \geq 0 \\ l_1+\cdots+l_t = j}} \left\| x^+_{1,l_1} \cdots x^+_{1,l_t} (x^+_{0,0})^t \right\| \\
&= \frac{\left\| \left(x^+_{1,0}\right)^{N+t} \left(x^-_{1,1}\right)^N \left(x^+_{0,0}\right)^t \right\|}{(N+t)!} = v_{N+t}(\overbrace{\iota, \ldots, \iota}^{t}, \overbrace{\mathbf{1}, \ldots, \mathbf{1}}^{N}) \\
&= \sum_{m=0}^{N} \binom{N}{m} w_m \in I_{\mathbf{r}}(E)
\end{aligned} \tag{7.4}$$

where $w_m = C_{t,m,N-m}(1, \ldots, 1; \mathbf{1}, \ldots, \mathbf{1}; \mathbf{1}, \ldots, \mathbf{1})$. Since the ideal $I_{\mathbf{r}}(E)$ is invariant under the automorphisms $\tilde{\mathcal{Y}}^{\pm 1}$, each w_m is in $I_{\mathbf{r}}(E)$. Therefore

$$C_{t,M_1,M_2}(1, \ldots, 1; \mathbf{1}, \ldots, \mathbf{1}; \mathbf{1}, \ldots, \mathbf{1}) \in I_{\mathbf{r}}(E).$$

Using this and calculating

$$\left[A_{i,1}((l-1)\mathbf{1}), C_{t,M_1,M_2}(1, \ldots, 1; l_1\mathbf{1}, \ldots, l_a\mathbf{1}, \mathbf{1}, \ldots, \mathbf{1}; n_1\mathbf{1}, \ldots, n_b\mathbf{1}, \mathbf{1}, \ldots, \mathbf{1})\right]_{p^{\varepsilon_i(l-1)}},$$

we can show that

$$C_{t,M_1,M_2}(1, \ldots, 1; l_1\mathbf{1}, \ldots, l_{M_1}\mathbf{1}; n_1\mathbf{1}, \ldots, n_{M_2}\mathbf{1}) \in I_{\mathbf{r}}(E)$$

for $l_1, \ldots, l_{M_1}, n_1, \ldots, n_{M_2} > 0$ as in the first part of the proof of Lemma 7.8. Now the calculations of commutators such as

$$\left[A_{1,1}\left(k_j - 1, l_j\right), C_{t+1,M_1-1,M_2}\left(1, \ldots, 1; \mathbf{k}_1, \ldots, \mathbf{k}_{j-1}, \mathbf{k}_{j+1}, \ldots, \mathbf{k}_{M_1}; \mathbf{m}_1, \ldots, \mathbf{m}_{M_2}\right)\right]$$

with $k_j > 0$ prove that

$$C_{t,M_1,M_2}\left(1,\dots,1;\mathbf{k}_1,\dots,\mathbf{k}_{M_1};\mathbf{m}_1,\dots,\mathbf{m}_{M_2}\right)\in I_{\mathbf{r}}(E)$$

for $\mathbf{k}_j=(k_j,l_j)$, $\mathbf{m}_j=(m_j,n_j)\in Z'$ $(\forall j)$ by induction on M_1+M_2. Finally the repeated multiplication by $A_{1,1}(u,0)+A_{2,1}(u,0)$ $(u>0)$ proves the claim as in the first part of the proof of Lemma 7.8. □

Now we can give the

Proof of Lemma 7.4. (1) Recall that the multiplication map $U(\mathcal{H}_1)\otimes U(\mathcal{H}_0)\otimes U(\mathcal{H}_2)\to U(\mathcal{H})$ is an isomorphism of algebras. This proves that $I_{\mathbf{r}}\cap U(\mathcal{H}_1)U(\mathcal{H}_2)$ is equal to $U(\mathcal{H}_1)I^2_{r_2}+I^1_{r_1}U(\mathcal{H}_2)$. Therefore, by (2.11) and Lemma 7.6,

$$I_{\mathbf{r}}\cap U(\mathcal{H}_1)U(\mathcal{H}_2)=\sum_{\substack{M_1>r_1 \text{ or } M_2>r_2\\ \mathbf{k}_1,\dots,\mathbf{k}_{M_1},\mathbf{m}_1,\dots,\mathbf{m}_{M_2}\in\mathbf{Z}^2\setminus\{0\}}}\mathbf{C}(\gamma)A^{r_1}_{1,M_1}(\mathbf{k}_1,\dots,\mathbf{k}_{M_1})A^{r_2}_{2,M_2}(\mathbf{m}_1,\dots,\mathbf{m}_{M_2}).$$

Since $U(E)\subset U(\mathcal{H}_1)U(\mathcal{H}_2)$, the above equality, Lemma 7.7 and Lemma 7.6 (6) imply that

$$U(E)\cap I_{\mathbf{r}}=\oplus_{t,M_1,M_2}\bigoplus_{\substack{u_1\ge\cdots\ge u_t>0\\ \mathbf{k}_1\succeq\cdots\succeq\mathbf{k}_{M_1}\\ \mathbf{m}_1\succeq\cdots\succeq\mathbf{m}_{M_2}}}\mathbf{C}(\gamma)C(u_1,\dots,u_t;\mathbf{k}_1,\dots,\mathbf{k}_{M_1};\mathbf{m}_1,\dots,\mathbf{m}_{M_2})\tag{7.5}$$

where $\mathbf{k}_j,\mathbf{m}_j\in Z'$ for all j and the first sum is taken over t,M_1,M_2 such that $M_1>r_1$ or $M_2>r_2$ or $M_1+M_2+t>r_1+r_2$. Therefore Lemma 7.9 proves the inclusion $I_{\mathbf{r}}(E)\supset U(E)\cap I_{\mathbf{r}}$.

By (2.11) $\tilde{\mathcal{Y}}(A^r_{i,M}(\mathbf{k}_1,\dots,\mathbf{k}_M))=(-\gamma)^{-\varepsilon_i\sum_j l_j}A^r_{i,M}(\mathbf{k}_1,\dots,\mathbf{k}_M)$. So $U(E)\cap I_{\mathbf{r}}$ is invariant under $\tilde{\mathcal{Y}}^{\pm1}$. Equation (7.4) implies that the generators of $I_{\mathbf{r}}(E)$ are in $U(E)\cap J_{\mathbf{r}}$. Therefore Proposition 2.2 proves the reverse inclusion.

(2) The element $C(u_1,\dots,u_t;\mathbf{k}_1,\dots,\mathbf{k}_{M_1};\mathbf{m}_1,\dots,\mathbf{m}_{M_2})$ is in $U(\mathcal{H})(m\delta,n)$ with $m=\sum_j u_j+\sum_j k_j+\sum_j m_j$ and $n=\sum_j l_j+\sum_j n_j$. Since $u_j>0$ and $\mathbf{k}_j,\mathbf{m}_j\in Z'$, $m\ge t$ and $n\ge M_1+M_2$. Therefore (7.5) proves the claim. □

8. Proof of Proposition 2.2

Lemma 8.1. *Let A be the $\mathbf{C}(\gamma)$ algebra of polynomials in variables a_m $(m\in\mathbf{Z}\setminus\{0\})$. Define $\lambda_m\in A$ $(m\in\mathbf{Z})$ by the generating series*

$$\sum_{m\ge0}\lambda_{\pm m}z^m=\exp\left(-\sum_{r>0}\frac{a_{\pm r}}{r}z^r\right).$$

For a nonnegative integer r let A_r be the quotient of A by the ideal generated by the elements $\lambda_{\pm m} = 0$ $(m > r)$ and $\lambda_k\lambda_{-r} - \lambda_{k-r}$ $(0 \leq k \leq r)$. Then

$$A_r \simeq \mathbf{C}(\gamma)\left[y_1^{\pm 1}, \dots, y_r^{\pm 1}\right]^{S_r} \quad (a_l \leftrightarrow \zeta_l,\ \lambda_m \leftrightarrow (-1)^m e_m)$$

where $\zeta_l = \sum_{i=1}^{r} y_i^l$ and $e_{\pm m} = \sum_{1\leq i_1<\cdots<i_m\leq r} y_{i_1}^{\pm 1}\cdots y_{i_m}^{\pm 1}$ for $l \in \mathbf{Z}\setminus\{0\}$ and $m \in \mathbf{Z}_{\geq 0}$.

Proof. The claim easily follows from the fact that the algebra of symmetric polynomials in n variables is isomorphic to the algebra of polynomials in n variables. $\square$

Proof of Proposition 2.2. The homomorphism φ_{i,r_i} in Theorem 2.1 maps $\Lambda_{i,\pm m}$ to $(-1)^m e_{\pm m} \in \mathbf{C}(\gamma)\left[y_1^{\pm 1}, \dots, y_{r_i}^{\pm 1}\right] \subset \mathcal{C}_{\gamma^{\varepsilon_i}}^{\otimes r_i}$ with the notation of Lemma 8.1. Since the e_m satisfy $e_{\pm n} = 0$ $(n > r_i)$ and $e_k e_{-r_i} = e_{k-r_i}$ $(0 \leq k \leq r_i)$ and φ_{i,r_i} is injective, we find that

$$\Lambda_{i,\pm n} = 0 \quad (n > r_i) \quad \text{and} \quad \Lambda_{i,k}\Lambda_{i,-r_i} = \Lambda_{i,k-r_i} \quad (0 \leq k \leq r_i)$$

in $U(\mathcal{H})/I_{\mathbf{r}}$ for $i = 1, 2$. By Lemma 5 of [19] $\|e(\mathbf{k}_1)\cdots e(\mathbf{k}_M)f(\mathbf{m}_M)\cdots f(\mathbf{m}_1)\|$ is a linear combination of $A_{1,M_1}^{r_1}(\mathbf{l}_1, \dots, \mathbf{l}_{M_1})A_{2,M_2}^{r_2}(\mathbf{l}'_1, \dots, \mathbf{l}'_{M_2})$ with $M_1 + M_2 = M$ and $\mathbf{l}_j, \mathbf{l}'_j \in \mathbf{Z}^2$ mod $U(\mathcal{H})(h - (r_1 + r_2))$. This implies that

$$\|e(\mathbf{k}_1)\cdots e(\mathbf{k}_M)f(\mathbf{m}_M)\cdots f(\mathbf{m}_1)\| \in I_{\mathbf{r}} \quad \text{if } M > r_1 + r_2$$

since $A_{i,M}^{r_i}(\mathbf{k}_1, \dots, \mathbf{k}_M) \in I_{\mathbf{r}}$ if $M > r_i$ by (2.11). Therefore all the generators of $J_{\mathbf{r}}$ vanish in $U(\mathcal{H})/I_{\mathbf{r}}$.

If we set $\zeta_0 = r$, then $\sum_{j=0}^{r}(-1)^j\zeta_{l+j}e_{r-j} = 0$ in $\mathbf{C}\left[y_1^{\pm 1}, \dots, y_r^{\pm 1}\right]^{S_r}$ for $l \in \mathbf{Z}$. So, if we set $a_0 = r$, then $\sum_{j=0}^{r} a_{l+j}\lambda_{r-j} = 0$ in A_r for any integer l by Lemma 8.1. Therefore, considering the homomorphism $A_{r_i} \to U(\mathcal{H})/J_{\mathbf{r}}$ $(a_m \mapsto a_i(0, m))$, we find that $\sum_{j=0}^{r_i} A_{i,1}^{r_i}(0, l + j)\Lambda_{i,r_i-j} = 0$ in $U(\mathcal{H})/J_{\mathbf{r}}$ for any l. Noting this and the fact that the Λ_{i,r_i} are invertible in $U(\mathcal{H})/J_{\mathbf{r}}$, we can prove that all the generators of $I_{\mathbf{r}}$ vanish in $U(\mathcal{H})/J_{\mathbf{r}}$ as in Section 5.2 of [19]. $\square$

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