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ON THE CAUCHY PROBLEM FOR 2 × 2 WEAKLY HYPERBOLIC SYSTEMS

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Abstract

We prove results on existence and uniqueness of solutions of the Cauchy Problem for 2×2 weakly hyperbolic systems. The results follow from an extension for systems of the work by O. Oleinik done in the scalar case.

1. Introduction

We consider the non-characteristic Cauchy problem

$$PU = I\partial_t U - A(x,t)\partial_x U + A_1(x,t)U = f(x,t) \tag{1.1}$$

$$U(x,0) = \Phi(x), \tag{1.2}$$

on $G = \{0 \le t \le T, x \in \mathbb{R}\}$.

Here A and A_1 are 2×2 real valued matrices, $f = (f_1, f_2)$ and $\Phi = (\phi_1, \phi_2)$ are vector-valued functions. Assume that

$$(a_{11} + a_{22})^2 - 4\det(A) \ge 0. \tag{1.3}$$

Under this hypothesis the system (1.1) is called weakly hyperbolic (see Kreiss-Lorenz [5]).

For the data we suppose the following regularity conditions. Let $k \ge 2$, $p \ge -1$ and assume that:

1) $A \in C^1(G, M_2(\mathbb{R}))$, the derivatives $\partial_x^l \partial_t^\rho$ of A and A_1 are bounded on G for $l+\rho \le k-1$ and for $\rho = 0$, $l \le k+1$.
2) The traces at $t = 0$ of the derivatives $\partial_x^l \partial_t^\rho$ of A and A_1, with $\rho \le p+1$ and $\rho + l \le p + k + 3$, are bounded.
3) For some $0 < t_0 < T$, the derivatives $\partial_x^l \partial_t^\rho$ of A and A_1, with $l \le k+1$, $\rho \le p+2$, are bounded for $0 \le t \le t_0$.
4) $\phi_j \in H^{k+p+4}(G)$ has compact support, for $j = 1, 2$.
5) For $j = 1, 2$, f_j has compact support in x and regularity to be described later; namely that the respective norms appearing at (2.4) are finite, with k replaced by $k+1$.

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As done by Nishitani (see [9]), without loss of generality we can assume that $\mathrm{tr}(A) = 0$ (trace of A). In fact, we perform a local change of coordinates in a neighborhood of $t = 0$ leaving the lines $t = \text{const.}$ invariant, that is,

$$\varphi(x,t) = (\varphi_0(x,t), t) = (y,s),$$

where $\varphi_0 : G \to \mathbb{R}^2 \in C^1$ is the unique solution of the equation

$$2\partial_t\varphi_0 - (a_{11} + a_{22})\partial_x\varphi_0 = 0 \tag{1.4}$$

with $\varphi_0(x, 0) = x$. The system (1.1) is transformed into

$$I\partial_s\tilde{U} + \tilde{A}(y,s)\partial_y\tilde{U} + \tilde{A}_1(y,s)\tilde{U} = \tilde{f}(y,s) \tag{1.5}$$

where

$$\tilde{A}(y,s) = \begin{pmatrix} \partial_t\varphi_0 - a_{11}\partial_x\varphi_0 & -a_{12}\partial_x\varphi_0 \\ -a_{21}\partial_x\varphi_0 & \partial_t\varphi_0 - a_{22}\partial_x\varphi_0 \end{pmatrix}.$$

From (1.4) we have $\mathrm{tr}(\tilde{A}) = 0$. Hence

$$\mathrm{tr}(\tilde{A})^2 - 4\det\tilde{A} = -4\det\tilde{A} = ((a_{11} + a_{22})^2 - 4\det A)(\partial_x\varphi_0)^2 \geq 0,$$

by (1.3). Therefore (1.3) is valid for (1.5).

So, from now on, we may assume

$$A(x,t) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix} \quad \text{and} \quad a(x,t) = -\det A \geq 0 \tag{1.6}$$

Let $B = (b_{ij})$ be given by

$$B = \begin{pmatrix} -\partial_t a_{11} + a_{11}\partial_x a_{11} + a_{21}\partial_x a_{12} & -a_{11}\partial_x a_{12} - \partial_t a_{12} + a_{12}\partial_x a_{11} \\ -\partial_t a_{21} + a_{11}\partial_x a_{21} - a_{21}\partial_x a_{11} & \partial_t a_{11} + a_{11}\partial_x a_{11} + a_{12}\partial_x a_{21} \end{pmatrix} + AA_1.$$

Our first uniqueness result is:

Theorem 1.1. *Assume that*

$$\begin{aligned} \alpha t\left[b_{11}^2 + b_{12}^2\right] &\leq \theta a + \partial_t a, \\ \alpha t\left[b_{21}^2 + b_{22}^2\right] &\leq \theta a + \partial_t a. \end{aligned} \tag{1.7}$$

Then the problem (1.1)–(1.2) *has at most one solution* $U \in H^k(G, \mathbb{R}^2)$. *Here* θ *is some constant,* $\alpha(t) = \alpha_1\chi_{[0,t_0]} + \alpha_2\chi_{[t_0,T]}$, *where* t_0, α_1 *and* α_2 *are positive constants,* χ *is a characteristic function and* $\alpha_1/2 > (2p + 6)^{-1}$.

REMARK 1.1. Theorem 1.1 also holds for semi-linear 2×2 systems of first order if k is replaced by $k+1$, since we may expand a nonlinear f into a first order Taylor expansion at $(x, t, 0, 0)$ and write $f(x, t, U) = A_1(x, t)U$.

Let

$$\tilde{B} = \begin{pmatrix} \partial_t a_{11} + a_{11}\partial_x a_{11} + a_{21}\partial_x a_{12} & -a_{11}\partial_x a_{12} + \partial_t a_{12} + a_{12}\partial_x a_{11} \\ \partial_t a_{21} + a_{11}\partial_x a_{21} - a_{21}\partial_x a_{11} & -\partial_t a_{11} + a_{11}\partial_x a_{11} + a_{12}\partial_x a_{21} \end{pmatrix} + A_1 A.$$

We prove an existence theorem:

Theorem 1.2. *Assume that*

$$\begin{aligned} \alpha t\left(\left(\tilde{b}_{11}\right)^2 + \left(\tilde{b}_{12}\right)^2\right) &\leq \theta a + \partial_t a, \\ \alpha t\left(\left(\tilde{b}_{22}\right)^2 + \left(\tilde{b}_{21}\right)^2\right) &\leq \theta a + \partial_t a, \end{aligned} \tag{1.8}$$

Then there exists a solution $U \in H^{k-1}(G, \mathbb{R}^2)$ *of the Cauchy problem* (1.1)–(1.2). *Here* θ *is a constant,* $\alpha(t) = \alpha_1 \chi_{[0,t_0]} + \alpha_2 \chi_{[t_0,T]}$, *where* t_0, α_1 *and* α_2 *are positive constants,* χ *is a characteristic function and* $\alpha_1/2 > (2p+6)^{-1}$.

From Theorems 1.1 and 1.2 we have the following consequences:

Corollary 1.1. *If* $A = A(t)$, $A_1 = 0$ *and* (1.7) *holds, then there exists a unique solution* $U \in H^{k-1}(G, \mathbb{R}^2)$ *of the Cauchy problem* (1.1)–(1.2).

This follows since in this case the conditions (1.7) and (1.8) coincide.

The next corollary can be thought of as a generalization of the result of Colombini and Spagnolo (see [1]).

Corollary 1.2. *If the data of problem* (1.1)–(1.2) *are sufficiently regular and*

$$a^{(j)}(0) \neq 0 \quad \text{for some } j \leq 2, \tag{1.9}$$

where, in (1.6), $a = a(t)$, *then there exists a unique solution* $U \in H^{k-1}(G, \mathbb{R}^2)$ *of the Cauchy problem* (1.1)–(1.2) *in a neighborhood at* $t = 0$.

This follows because the hypothesis (1.9) implies both (1.7) and (1.8) in a neighborhood at $t = 0$.

If we assume

$$a_{12}(x,t) \neq 0 \quad \text{and} \quad a_{21}(x,t) \neq 0 \quad \forall (x,t), \tag{1.10}$$

we have

Theorem 1.3. *Assume*

$$\alpha t\left[\frac{a_{21}\partial_t a_{11} - a_{11}\partial_t a_{21} + a_{21}\,\mathrm{tr}(AA_1)}{a_{21}}\right]^2 \le \theta a + \partial_t a,$$
$$\alpha t\left[\frac{-a_{12}\partial_t a_{11} - a_{11}\partial_t a_{12} + a_{12}\,\mathrm{tr}(AA_1)}{a_{12}}\right]^2 \le \theta a + \partial_t a, \tag{1.11}$$

Then there exists a unique solution $U \in H^{k-1}(G, \mathbb{R}^2)$ *of* (1.1)–(1.2). *Here* θ *is a constant,* $\alpha(t) = \alpha_1\chi_{[0,t_0]} + \alpha_2\chi_{[t_0,T]}$, *with* t_0, α_1 *and* α_2 *positive constants,* χ *is a characteristic function and* $\alpha_1/2 > (2p+6)^{-1}$.

That is, (1.7) and (1.8) are replaced by (1.11).

REMARK 1.2. Condition (1.7) is not, in general, necessary for the conclusion of Theorem 1.1. In fact, if A is symmetric then uniqueness follows independently of lower order terms (see Cossi-dos Santos Filho [2]). For example take g given by $g(t) = e^{-1/t}$ for $t > 0$ and $g(t) = 0$ for $t \le 0$, the symmetric system

$$A(t) = \begin{pmatrix} g & 0 \\ 0 & -g \end{pmatrix}$$

is weakly hyperbolic. For $A_1 = 0$ we obtain

$$B(t) = \begin{pmatrix} -g'(t) & 0 \\ 0 & g'(t) \end{pmatrix}$$

Thus for any choice of θ and α

$$\alpha t\left[b_{11}^2 + b_{12}^2\right] = \alpha t(g')^2 > \theta(g)^2 + 2gg' = \theta a + a_t$$

for $t > 0$ small enough. Then (1.7) never holds near $t = 0$.

REMARK 1.3. From the example given by C. Min-You, see [6], in order to have existence of solution in the Cauchy problem for weakly hyperbolic operators, in spaces of functions with finite degree of regularity, some conditions must be imposed on the lower order terms. This justifies that, in general, conditions like (1.7), (1.8) and (1.11) cannot be removed if we are to have well posedness for the Cauchy problem. More precisely, consider

$$u_{tt} - t^2 u_{xx} = bu_x, \quad t > 0,\ 0 \le x \le 1,$$

with the initial condition

$$u|_{t=0} = \mu(x), \quad u_t|_{t=0} = 0,$$

where $b = 4n + 1$, $n \geq 0$ is an integer. The unique solution has the form

$$u(x,t) = \sum_{l=0}^{n} \frac{\sqrt{\pi}t^{2l}}{l!(n-l)!\Gamma(l+1/2)} \partial_x^l \mu \left(x + \frac{1}{2}t^2 \right).$$

Now for the first order system in the form (1.1) associated to this second order scalar differential equation, we have $a(t) = t^2$, $\tilde{b}_{11} = 0 = \tilde{b}_{12}$, $\tilde{b}_{21} = 2t$, and $\tilde{b}_{22} = 4n + 1$. In Theorem 1.2 condition (1.8) takes the form $\alpha t\big[(\tilde{b}_{21})^2 + (\tilde{b}_{22})^2\big] \leq \theta a + a_t$, which holds for $\alpha \leq 2/(4n+1)^2$. So $\alpha > (2p+6)^{-1}$, hence $p = p(n)$ tends to infinity with n; here p measures the degree of regularity of the initial value.

This paper is organized in the following way:

In Section 2 the notation is established and we also state an extension, namely Theorem 2.1, of O. Oleinik's theorem (see Theorem 1 of [11]) for 2×2 systems of second order partial differential equations uncoupled in its principal part. Then we prove results on uniqueness and non-uniqueness, analogous to the ones in the scalar case presented by Colombini and Spagnolo (see [1]). The non-uniqueness result follows from Nakane (see [7] and [8]). Again, as in [11], we consider a regularization of the data and perturb the original system so that it becomes strictly hyperbolic. The basic lemma (Lemma 2.1) for the proof of Theorem 2.1 is then stated.

In Sections 3 to 5 inequalities to be used in the proof of Lemma 2.1 are derived.

In Section 6, the proof of Lemma 2.1 is then established. Also we prove Theorem 2.1.

In Section 7 theorems 1.1, 1.2 and 1.3 are obtained from Theorem 2.1.

Finally, in Section 8 a result similar to Theorem 5.2 of Nishitani and Spagnolo (see [10]), is proved.

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2. Extension of Oleinik's theorem: statement

We consider the non-characteristic Cauchy problem

$$LU = I\partial_t^2 U - \partial_x(aI\partial_x U) + B\partial_x U + C\partial_t U + DU = f \tag{2.1}$$

$$U|_{t=0} = \Phi, \quad U_t|_{t=0} = \Psi, \tag{2.2}$$

on G. Here $B(x,t) = (b_{ij}(x,t))$, $C(x,t) = (c_{ij}(x,t))$ and $D(x,t) = (d_{ij}(x,t))$ are 2×2 real valued matrices, with $a(x,t) \geq 0$, and $U(x,t) = (u_1(x,t), u_2(x,t))$, $f(x,t) = (f_1(x,t), f_2(x,t))$, $\Phi(x) = (\phi_1(x), \phi_2(x))$ and $\Psi(x) = (\psi_1(x), \psi_2(x))$ are vector-valued functions. Under these hypotheses the system (2.1) is called weakly hyperbolic.

Now we introduce the notation:

$$G_\tau = \{0 \le t \le \tau, x \in \mathbb{R}\}, \quad (\Phi, \Psi)_{t=\tau} = \int_{\mathbb{R}} [\phi_1\psi_1 + \phi_2\psi_2]\, dx,$$

$$[U, V]_{G_\tau} = \int_{G_\tau} [u_1 v_1 + u_2 v_2]\, dt\, dx,$$

$$\|U\|_{\tau;k} = \left\{ \sum_{\rho+l \le k} \left(\partial_x^l \partial_t^\rho U, \partial_x^l \partial_t^\rho U\right)_{t=\tau} \right\}^{1/2},$$

$$\|U\|_{G_\tau;k} = \left\{ \int_0^\tau \|U\|^2_{\sigma;k}\, d\sigma \right\}^{1/2},$$

$$\|U\|_{\tau;q,k} = \left\{ \sum_{\rho \le q, \rho+l \le k} \left(\partial_x^l \partial_t^\rho U, \partial_x^l \partial_t^\rho U\right)_{t=\tau} \right\}^{1/2},$$

$$\|U\|_{G_\tau;q,k} = \left\{ \int_0^\tau \|U\|^2_{\sigma;q,k}\, d\sigma \right\}^{1/2}.$$

By $H^k(G)$ we denote the class of functions obtained by closing the set of infinitely differentiable functions in $G = G_T$ with compact support in x with respect to the norm $\|U\|_{G_T;k}$.

Our first goal is to obtain sufficient conditions under which the Cauchy problem (2.1)–(2.2) is well posed. It is an extension for systems of the form (2.1) of an earlier work by Oleinik ([11], see Theorem 1) for the scalar case.

For the data we require some regularity conditions. Let $k \ge 2$, $p \ge -1$ and assume that:

1) The derivatives $\partial_x^l \partial_t^\rho$ of $a, \partial_x a, B, C, \partial_t C, D,$ with $l + \rho \le k - 2$ and for $\rho = 0$, $l \le k$, are bounded in G. Moreover the derivatives $\partial_x^l \partial_t^\rho$ of $a, \partial_x a, B, C, D$ with $\rho \le p$ and $\rho+l \le p+k+2$, are bounded at $t = 0$ and the derivatives $\partial_x^l \partial_t^\rho$ of the same functions are bounded for $0 \le t \le t_0$, with $\rho \le p + 1$ and $l \le k$.

2) f, Φ and Ψ are compactly supported in x.

Under these hypotheses we will prove:

Theorem 2.1. *Assume that, for the coefficients of* (2.1), *the inequalities*

$$\begin{aligned} \alpha t \left((b_{11})^2 + (b_{12})^2\right) &\le \theta a + \partial_t a, \\ \alpha t \left((b_{22})^2 + (b_{21})^2\right) &\le \theta a + \partial_t a, \end{aligned} \tag{2.3}$$

hold in G. *Then there exists a unique solution* $U \in H^k(G, \mathbb{R}^2)$ *of the Cauchy problem*

(2.1)–(2.2) *and the estimate*

$$\|U\|^2_{\tau;k} \le M_1 \left\{ \|\Phi\|^2_{0;k+p+4} + \|\Psi\|^2_{0;k+p+3} + \|f\|^2_{G_\tau;0,k} + \|f\|^2_{\tau;k-2} + \sum_{\rho\le p} \|f\|^2_{0;\rho,p+k+2-\rho} + \max_{0\le\sigma\le t_0} \|f\|^2_{\sigma;p+1,k} \right\}, \tag{2.4}$$

holds, provided the norms of f, Φ, Ψ *on the right of* (2.4) *are finite. Here* θ *is constant,* $\alpha(t) = \alpha_1\chi_{[0,t_0]} + \alpha_2\chi_{[t_0,T]}$, *where* t_0, α_1 *and* α_2 *are positive constants with* $\alpha_1/2 > (2p+6)^{-1}$ *and* M_1 *is a constant depending on the coefficients of the systems* (2.1) *and on their derivatives indicated above.*

REMARK 2.1. If $k > (n/2) + r$, by the Sobolev lema $H^k \subset C^r$. In the case $n = 2$, if $k > 3$, then the classical solution to the Cauchy problem (2.1)–(2.2) exists.

As a consequence of Theorem 2.1 we obtain three corollaries; they are generalizations of results of Colombini and Spagnolo (see [1]).

Corollary 2.1. *If the data of problem* (2.1)–(2.2) *are sufficiently regular, with* $a = a(t)$ *in* (2.1) *and* (1.9) *holds, then there exists a unique solution* $U \in H^k(G, \mathbb{R}^2)$ *of the Cauchy problem* (2.1)–(2.2) *in a neighborhood of* $t = 0$.

Proof. Assume that $a \in C^2$ in a neighborhood at $t = 0$. We will prove only one inequality of (2.3), since the other will follows in the same way. Consider

$$\begin{aligned} f(x,t) &= \theta a(t) + a'(t) - \alpha t \left(b_{11}^2 + b_{12}^2\right) \\ &\ge \theta a(t) + a'(t) - \alpha t \left\|b_{11}^2 + b_{12}^2\right\|_\infty = h(t). \end{aligned}$$

We have $h(0) = \theta a(0) + a'(0) > 0$ if $a(0) \neq 0$ or $a'(0) \neq 0$, since $a \ge 0$. Hence $f \ge h \ge 0$ in a neighborhood at $t = 0$.

If $a(0) = 0 = a'(0)$ and $a''(0) \neq 0$, hence $a''(0) > 0$, since $a \ge 0$. In this case, $h(0) = 0$ and $h'(0) = a''(0) - \alpha \left\|b_{11}^2 + b_{12}^2\right\|_\infty$. If $\left\|b_{11}^2 + b_{12}^2\right\|_\infty = 0$, hence $f \ge h \ge 0$ in a neighborhood at $t = 0$, since $h'(0) > 0$. If $\left\|b_{11}^2 + b_{12}^2\right\|_\infty > 0$, then we take $0 < \alpha < a''(0) \big/ \left(\left\|b_{11}^2 + b_{12}^2\right\|_\infty\right)$ and $h'(0) > 0$. Therefore, $f \ge h \ge 0$ in a neighborhood at $t = 0$. □

We say that the strong uniqueness property holds for an operator if for all the operators having the same principal parts the uniqueness is true.

In the next two corollaries, we suppose that B, C and D are diagonal matrices in (2.1).

Corollary 2.2. *If in* (2.1) $a = a(t)$ *has a zero of finite order* $k \geq 3$ *at* $t = 0$, *then the Cauchy problem* (2.1)–(2.2) *does not have the strong uniqueness property*.

Proof. The proof follows from results for the scalar case as one can see in [7] and [8]. □

Summing up, from the two previous corollaries, with B, C and D diagonal matrices, we have:

Corollary 2.3. *The system* (2.1) *with* $a = a(t)$ *has the strong uniqueness property if only if the condition* (1.9) *holds in a neighborhood of* $t = 0$.

Now we consider a regularization of the data of problem (2.1)–(2.2). For this goal we take $0 \leq \varphi,\ \tilde{\varphi} \in C_0^\infty(\mathbb{R})$ such that

$$\int \varphi(x)\,dx = 1, \quad \int \tilde{\varphi}(\sigma)\,d\sigma = 1 \quad \text{and} \quad \operatorname{supp}(\varphi) \subset [-1, 1], \quad \operatorname{supp}(\tilde{\varphi}) \subset [0, 1].$$

With $\epsilon > 0$ we consider the functions $\varphi_\epsilon(x) = \epsilon^{-1}\varphi(x/\epsilon)$ and $\tilde{\varphi}_\epsilon(\sigma) = \epsilon^{-1}\tilde{\varphi}(\sigma/\epsilon)$. Let

$$P_\epsilon[v](x,t) = \int_{\mathbb{R}} \varphi_\epsilon(x-y)v(y,t)\,dy,$$
$$Q_\epsilon[v](x,t) = \int_{\mathbb{R}} \tilde{\varphi}_\epsilon(\tau - t)P_\epsilon[v](x,\tau)\,d\tau.$$

Now we consider the following functions:

$$\begin{aligned} &\phi_\epsilon^j = P_\epsilon[\phi_j], \quad \psi_\epsilon^j = P_\epsilon[\psi_j], \quad f_\epsilon^j = Q_\epsilon[f_j], \\ &a_\epsilon = Q_\epsilon[a], \quad b_\epsilon^{ij} = Q_\epsilon[b_{ij}], \quad c_\epsilon^{ij} = Q_\epsilon[c_{ij}], \quad d_\epsilon^{ij} = Q_\epsilon[d_{ij}]. \end{aligned} \tag{2.5}$$

These functions are well defined in $G^\epsilon = \{0 \leq t \leq T - \epsilon, x \in \mathbb{R}\}$. Since condition (2.3) of Theorem (2.1) is satisfied, the inequalities

$$\begin{aligned} \alpha t \left(\left(b_\epsilon^{11}\right)^2 + \left(b_\epsilon^{12}\right)^2 \right) &\leq \theta a_\epsilon + \partial_t a_\epsilon \\ \alpha t \left(\left(b_\epsilon^{22}\right)^2 + \left(b_\epsilon^{21}\right)^2 \right) &\leq \theta a_\epsilon + \partial_t a_\epsilon \end{aligned} \tag{2.6}$$

hold in G^ϵ. For the proof of this we apply the operator Q_ϵ to the both sides of inequality (2.3) to obtain

$$Q_\epsilon\left[\alpha t b_{11}^2\right] + Q_\epsilon\left[\alpha t b_{12}^2\right] \leq \theta Q_\epsilon[a] + Q_\epsilon\left[\partial_t a\right],$$
$$Q_\epsilon\left[\alpha t b_{22}^2\right] + Q_\epsilon\left[\alpha t b_{21}^2\right] \leq \theta Q_\epsilon[a] + Q_\epsilon\left[\partial_t a\right].$$

By the Cauchy-Schwarz inequality

$$\alpha t(b_\epsilon^{ij})^2 = \alpha t(Q_\epsilon[b_{ij}])^2 = \alpha t\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \tilde{\varphi}_\epsilon(\tau - t)\varphi_\epsilon(x-y)b_{ij}(y,\tau)\,dy\,d\tau\right)^2$$
$$\leq \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \tilde{\varphi}_\epsilon(\tau - t)\varphi_\epsilon(x-y)\alpha t b_{ij}^2(y,\tau)\,dy\,d\tau\right).1 = Q_\epsilon\left[\alpha t b_{ij}^2\right].$$

Also,

$$Q_\epsilon[a_t](x,t) = \int_0^1\int_{-1}^1 \tilde{\varphi}(\sigma)\varphi(z)\partial_t a(x-\epsilon z, t+\epsilon\sigma)\,dz\,d\sigma = \partial_t Q_\epsilon[a](x,t),$$

therefore condition (2.6) holds for $t \leq T - \epsilon$.

Before stating the basic lemma, the following elementary remark is in order: The system

$$L_\epsilon = I\partial_t^2 - \begin{pmatrix} \epsilon & 0 \\ 0 & \dfrac{\epsilon}{2} \end{pmatrix}\partial_x^2 - \partial_x(a_\epsilon I\partial_x) + B_\epsilon\partial_x + C_\epsilon\partial_t + D_\epsilon \tag{2.7}$$

is strictly hyperbolic in $G^\epsilon = \{0 \leq t \leq T-\epsilon, x \in \mathbb{R}\}$.

For the regularized strictly hyperbolic problem we have:

Lemma 2.1. *Let $U_\epsilon(x,t)$ be the solution of the Cauchy problem*

$$L_\epsilon U_\epsilon = f_\epsilon, \tag{2.8}$$

$$U_\epsilon|_{t=0} = \Phi_\epsilon(x), \quad \partial_t U_\epsilon|_{t=0} = \Psi_\epsilon(x), \tag{2.9}$$

in $G^\epsilon = \{0 \leq t \leq T-\epsilon, x \in \mathbb{R}\}$. Then for $0 \leq \tau \leq T-\epsilon$ the following inequality holds:

$$\begin{aligned}\|U_\epsilon\|_{\tau;k}^2 \leq M_2\Bigg\{&\|\Phi_\epsilon\|_{0;k+p+4}^2 + \|\Psi_\epsilon\|_{0;k+p+3}^2 + \|f_\epsilon\|_{G_\tau;0,k}^2 + \|f_\epsilon\|_{\tau;k-2}^2\\ &+\sum_{\rho\leq p}\|f_\epsilon\|_{0;\rho,p+k+2-\rho}^2 + \max_{0\leq\sigma\leq t_0}\|f_\epsilon\|_{\sigma;p+1,k}^2\Bigg\}.\end{aligned} \tag{2.10}$$

Before the long proof of this lemma, which runs from Sections 3 to 6, we make two remarks and define some auxiliary functions that will be used in the proof of the lemma.

REMARK 2.2. The constant M_2 depends on the maximum modulus of derivatives $\partial_x^l\partial_t^\rho$ of $a_\epsilon, \partial_x a_\epsilon, B_\epsilon, C_\epsilon, \partial_t C_\epsilon, D_\epsilon$, in G^ϵ, for $l+\rho \leq k-2$ and for $\rho = 0$, $l \leq k$, and moreover, on the derivatives $\partial_x^l\partial_t^\rho$ of $a_\epsilon, \partial_x a_\epsilon, B_\epsilon, C_\epsilon, D_\epsilon$, for $\rho \leq p$ and $\rho+l \leq p+k+2$

at $t = 0$, as well as the derivatives $\partial_x^l \partial_t^\rho$ of these functions with $\rho \leq p+1$, $l \leq k$, taken for $0 \leq t \leq t_0$.

REMARK 2.3. Since $\Phi_\epsilon, \Psi_\epsilon, f_\epsilon$ have compact support in x and the system is strictly hyperbolic, then the same holds for U_ϵ.

Let us consider the function $V_p = (v_1, v_2)$ given by

$$V_p = \Phi_\epsilon + \Psi_\epsilon t + \frac{t^2}{2!}\partial_t^2 U_\epsilon|_{t=0} + \cdots + \frac{t^{p+2}}{(p+2)!}\partial_t^{p+2} U_\epsilon|_{t=0}, \tag{2.11}$$

where the derivatives of U_ϵ at $t = 0$ are expressed by means of equation (2.8) and the equations obtained from it by differentiation with respect to t, account being taken of the initial conditions (2.9). By induction we can prove that V_p depends on the derivatives $\partial_x^l \partial_t^\rho$ of $a_\epsilon, \partial_x a_\epsilon, B_\epsilon, C_\epsilon, D_\epsilon, f_\epsilon$, with $l + \rho \leq p$, at $t = 0$, and the derivatives ∂_x^l of Φ_ϵ and Ψ_ϵ, with $l \leq p+2$ and $l \leq p+1$, respectively.

For $U = U_\epsilon - V_p$, we have the equation

$$L_\epsilon(U) = f_\epsilon - L_\epsilon(V_p) = \mathcal{F}(x,t). \tag{2.12}$$

We have $\partial_t^{(j)} \mathcal{F}(x,t)|_{t=0} = 0$, $0 \leq j \leq p$. In fact, by Taylor expansion up to order $p+3$ in $t = 0$ of U_ϵ we have

$$U_\epsilon(x,t) = \sum_0^{p+2} \frac{\partial_t^{(j)} U_\epsilon(x,0)}{j!} t^j + O(t^{p+3}).$$

Hence

$$U = U_\epsilon - V_p = O(t^{p+3}),$$

which proves our claim.

With $U = (u_1, u_2)$, let

$$W = (w_1, w_2) = \left(\int_t^\tau u_1(x,\sigma)\, d\sigma, \int_t^\tau u_2(x,\sigma)\, d\sigma \right).$$

Multiply (2.12) by $We^{\theta t}$ and integrate over G_τ to obtain

$$\left[L_\epsilon(U), We^{\theta t}\right]_{G_\tau} = \left[\mathcal{F}, We^\theta t\right]_{G_\tau}. \tag{2.13}$$

We need to estimate all the derivatives $\partial_x^l \partial_t^\rho U$, $l + \rho \leq k$. In the next three section we estimates U, $\partial_x^l U$, $l \leq k$ and $\partial_x^l \partial_t^\rho U$, for $l + \rho \leq k$, respectively.

3. Estimate for U

In this section we prove one inequality for U:

Lemma 3.1. *For* $0 \le \tau \le t_0$ *we have*

$$(U, Ue^{\theta\tau})_{t=\tau} \le M_3 \tau^{2p+6} \|\mathcal{F}\|^2_{p+1,t_0}, \tag{3.1}$$

M_3 is a constant depending on the maximum modulus of $\partial_x B_\epsilon, C_\epsilon, \partial_t C_\epsilon, D_\epsilon$ and t_0. Here $\|\mathcal{F}\|^2_{p+1,t_0} = \sum_1^2 \max_{0\le\sigma\le t_0} \big(\partial_t^{(p+1)}\mathcal{F}_j, \partial_t^{(p+1)}\mathcal{F}_j\big)\big|_{t=\sigma}$.

Proof. We will prove the lemma in three steps. In the first and second steps we estimate the left- and right-hand sides of (2.13), respectively. In the third step we prove the estimative (3.1), by using the Gronwall's lemma.

STEP 1. Using integration by parts we write each term of the left-hand side of (2.13) in order to have the smallest order of derivative of U possible. To achieve this we use the fact that $\partial_t^{(\rho)} u_j|_{t=0} = 0$; $\forall \rho \le p+2$, $w_j(x,\tau) = 0$ and that u_j and w_j have compact support in x. For $0 \le t \le t_0$, we have:

$$\left[\partial_t^2 U, We^{\theta t}\right]_{G_\tau} = \frac{1}{2}(U, Ue^{\theta t})_{t=\tau} + \left[Ue^{\theta t}, \theta^2 W - \frac{3}{2}\theta U\right]_{G_\tau}, \tag{3.2}$$

$$\left[\epsilon_j \partial_x^2 u_j, e^{\theta t} w_j\right]_{G_\tau} = -\frac{\epsilon_j}{2}\left(\partial_x w_j, e^{\theta t}\partial_x w_j\right)_{t=0} - \frac{\epsilon_j}{2}\left[\partial_x w_j, \theta e^{\theta t}\partial_x w_j\right]_{G_\tau}, \tag{3.3}$$

$$\left[\partial_x(a_\epsilon U_x), We^{\theta t}\right]_{G_\tau} = -\frac{1}{2}(W_x, a_\epsilon W_x)|_{t=0} - \frac{1}{2}\left[W_x, (\theta a_\epsilon + \partial_t a_\epsilon) W_x e^{\theta t}\right]_{G_\tau}. \tag{3.4}$$

We further obtain

$$\begin{aligned}\left[C_\epsilon \partial_t U, We^{\theta t}\right]_{G_\tau} &= \sum_1^2 \left[u_j, c_\epsilon^{jj} e^{\theta t} u_j - \partial_t(c_\epsilon^{jj} e^{\theta t}) w_j\right]_{G_\tau} \\ &\quad + \left[u_2, c_\epsilon^{12} e^{\theta t} u_1 - \partial_t(c_\epsilon^{12} e^{\theta t}) w_1\right]_{G_\tau} + \left[u_1, c_\epsilon^{21} e^{\theta t} u_2 - \partial_t(c_\epsilon^{21} e^{\theta t}) w_2\right]_{G_\tau}.\end{aligned} \tag{3.5}$$

By definition the term

$$\begin{aligned}\left[B_\epsilon \partial_x U, We^{\theta t}\right]_{G_\tau} &= \sum_1^2 \left[b_\epsilon^{jj} \partial_x u_j, w_j e^{\theta t}\right]_{G_\tau} \\ &\quad + \left[b_\epsilon^{12} \partial_x u_2, w_1 e^{\theta t}\right]_{G_\tau} + \left[b_\epsilon^{21} \partial_x u_1, w_2 e^{\theta t}\right]_{G_\tau}.\end{aligned} \tag{3.6}$$

For the first two terms in the right side of (3.6) we have

$$\left|\left[b_\epsilon^{jj}\partial_x u_j, w_j e^{\theta t}\right]_{G_\tau}\right| \le M_4\tau^2\left[u_j, t^{-1}u_j e^{\theta t}\right]_{G_\tau} + \frac{1}{2}\alpha\left[tb_\epsilon^{jj}\partial_x w_j, b_\epsilon^{jj}\partial_x w_j e^{\theta t}\right]_{G_\tau} + \frac{1}{2\alpha}\left[u_j, t^{-1}u_j e^{\theta t}\right]_{G_\tau}. \tag{3.7}$$

where M_4 is a constant depending on $\sup_{G_\tau}|\partial_x B_\epsilon|$. In fact, take $u = u_j$, $w = w_j$, $b = b_\epsilon^{jj}$ and integrating by parts in x we obtain

$$\left[bu_x, we^{\theta t}\right]_{G_\tau} = -\left[u\partial_x b, we^{\theta t}\right]_{G_\tau} - \left[b\partial_x w, ue^{\theta t}\right]_{G_\tau}.$$

From $xy \le (1/2)(\alpha x^2 + (1/\alpha)y^2)$ it follows that

$$\begin{aligned}\left|\left[bw_x, ue^{\theta t}\right]_{G_\tau}\right| &= \left|\int_G (t\alpha)^{1/2} bw_x e^{\theta t/2} ue^{\theta t/2}(t\alpha)^{-1/2}\,dx\,dt\right| \\ &\le \frac{1}{2}\int_G t\alpha b^2 w_x^2 e^{\theta t} + (t\alpha)^{-1}u^2 e^{\theta t}\,dx\,dt \\ &= \frac{1}{2}\alpha\left[tbw_x, bw_x e^{\theta t}\right]_{G_\tau} + \frac{1}{2\alpha}\left[u, t^{-1}ue^{\theta t}\right]_{G_\tau}\end{aligned}$$

On the other hand,

$$\begin{aligned}&\left|\int_0^\tau u(x,t)b_x(x,t)e^{\theta t}w(x,t)\,dt\right| \\ &= \left|\int_0^\tau u(x,t)b_x(x,t)e^{\theta t}\int_t^\tau u(x,\sigma)\,d\sigma\,dt\right| \\ &= \left|\int_0^\tau\int_0^\sigma u(x,t)\left(\frac{\sigma}{t}\right)^{1/2} e^{\theta(t-\sigma)/2}u(x,\sigma)\left(\frac{t}{\sigma}\right)^{1/2} e^{\theta(t+\sigma)/2}b_x\,dt\,d\sigma\right| \\ &\le \frac{1}{2}\int_0^\tau\int_0^\sigma u^2(x,t)\sigma t^{-1}e^{\theta(t-\sigma)}|b_x| + u^2(x,\sigma)t\sigma^{-1}e^{\theta(t+\sigma)}|b_x|\,dt\,d\sigma \\ &\le M_4\tau^2\int_0^\tau u^2(x,t)t^{-1}e^{\theta t}\,dt.\end{aligned}$$

Hence (3.7) holds.

For the mixed terms in (3.6) we will prove the inequalities:

$$\left[b_\epsilon^{21}\partial_x u_1, w_2 e^{\theta t}\right]_{G_\tau} \le \frac{1}{2}\alpha\left[tb_\epsilon^{21}\partial_x w_2, b_\epsilon^{21}\partial_x w_2 e^{\theta t}\right]_{G_\tau} + \frac{1}{2\alpha}\left[u_1, t^{-1}u_1 e^{\theta t}\right]_{G_\tau} + \frac{1}{2}\left[u_1, \partial_x(b_\epsilon^{21})^2 u_1 e^{\theta t}\right]_{G_\tau} + M_5\tau^2\left[u_2, t^{-1}u_2 e^{\theta t}\right]_{G_\tau} \tag{3.8}$$

and

$$\left[b_\epsilon^{12}\partial_x u_2, w_1 e^{\theta t}\right]_{G_\tau} \le \frac{1}{2}\alpha\left[t b_\epsilon^{12}\partial_x w_1, b_\epsilon^{12}\partial_x w_1 e^{\theta t}\right]_{G_\tau} + \frac{1}{2\alpha}\left[u_2, t^{-1}u_2 e^{\theta t}\right]_{G_\tau} + \frac{1}{2}\left[u_2, (\partial_x b_\epsilon^{12})^2 u_2 e^{\theta t}\right]_{G_\tau} + M_6\tau^2\left[u_1, t^{-1}u_1 e^{\theta t}\right]_{G_\tau}. \tag{3.9}$$

In fact, since U has compact support in x, integration by parts in x yields

$$\left[b_\epsilon^{21}\partial_x u_1, w_2 e^{\theta t}\right]_{G_\tau} = -\int_G u_1\partial_x(b_\epsilon^{21}w_2)e^{\theta t}\,dx\,dt.$$

Using the elementary inequality for real numbers, $yz \le (1/2)(y^2+z^2)$, we have

$$\left[u_1, \partial_x(w_2)b_\epsilon^{21}e^{\theta t}\right]_{G_\tau} \le \frac{1}{2}\alpha\left[t b_\epsilon^{21}\partial_x w_2, b_\epsilon^{21}\partial_x w_2 e^{\theta t}\right]_{G_\tau} + \frac{1}{2\alpha}\left[u_1, t^{-1}u_1 e^{\theta t}\right]_{G_\tau}.$$

On the other hand

$$\left[u_1\partial_x b_\epsilon^{21}, w_2 e^{\theta t}\right]_{G_\tau} \le \frac{1}{2}\left[u_1, (\partial_x b_\epsilon^{21})^2 e^{\theta t}u_1\right]_{G_\tau} + \frac{1}{2}\left[w_2, w_2 e^{\theta t}\right]_{G_\tau}.$$

By Cauchy-Schwarz's inequality and $2xy \le x^2+y^2$ we have

$$\begin{aligned}
\left[w_2, w_2 e^{\theta t}\right]_{G_\tau} &= \int_{\mathbb{R}}\int_0^\tau \left(\int_t^\tau u_2(x,\sigma)\,d\sigma\right)\left(\int_t^\tau u_2(x,\gamma)\,d\gamma\right)e^{\theta t}\,dt\,dx \\
&\le \int_{\mathbb{R}}\int_0^\tau\int_t^\tau \left(\int_t^\tau u_2^2(x,\sigma)e^{\theta t}\,d\sigma\right)^{1/2}\left(\int_t^\tau u_2^2(x,\gamma)e^{\theta t}\,d\sigma\right)^{1/2} d\gamma\,dt\,dx \\
&\le \frac{1}{2}\int_{\mathbb{R}}\int_0^\tau\int_t^\tau \left\{\int_t^\tau u_2^2(x,\sigma)e^{\theta t}\,d\sigma + \int_t^\tau u_2^2(x,\gamma)e^{\theta t}d\sigma\right\} d\gamma\,dt\,dx \\
&\le M_5\tau^2\int_{\mathbb{R}}\int_0^\tau u_2^2(x,t)t^{-1}e^{\theta t}\,dt.
\end{aligned}$$

Hence (3.8) follows. In a similar way we obtain (3.9).

STEP 2. Now we estimate $[\mathcal{F}, We^{\theta t}]_{G_\tau}$.

a) Integrating by parts in t we obtain

$$\begin{aligned}
\left[\mathcal{F}_j, w_j e^{\theta t}\right]_{G_\tau} &= \int_{\mathbb{R}}\int_0^\tau \mathcal{F}_j(x,t)\partial_t\left\{-\int_t^\tau w_j(x,s)e^{\theta s}\,ds\right\}dt\,dx \\
&= -\int_{\mathbb{R}}\left\{\mathcal{F}_j W_1^j\Big|_0^\tau - \int_0^\tau \partial_t(\mathcal{F}_j)W_1^j\,dt\right\}dx
\end{aligned}$$

where $W_1^j = \int_t^\tau w_j e^{\theta s}\,ds$. Since $\mathcal{F}_j(x,0) = W_1^j(x,\tau) = 0$,

$$\left[\mathcal{F}_j, w_j e^{\theta t}\right]_{G_\tau} = \int_{\mathbb{R}}\int_0^\tau \partial_t(\mathcal{F}_j)W_1^j\,dt\,dx.$$

More generally, from $\partial_t^{(l)}\mathcal{F}_j(x,0) = W_\nu^j(x,\tau) = 0$, $\forall l \le p$, we get

$$\left[\mathcal{F}_j, w_j e^{\theta t}\right]_{G_\tau} = \left[\partial_t^{p+1}\mathcal{F}_j, W_{p+1}^j\right],$$

where $W_0 = We^{\theta t}$ and

$$W_{\nu+1}^j = \int_t^\tau W_\nu^j(x,\sigma)\,d\sigma, \quad \nu = 0,1,\ldots,p.$$

b) The functions W_{p+1}^j satisfy the following estimate

$$\left|W_{p+1}^j\right|^2 \le \tau^{2p+3}e^{2\theta T}\int_0^\tau u_j^2(x,\sigma)\,d\sigma.$$

Indeed, by Cauchy-Schwarz's inequality

$$\begin{aligned}\left|W_0^j\right|^2 &\le e^{2\theta T}\left(\int_t^\tau |u_j(x,\sigma)|\,d\sigma\right)^2 \le e^{2\theta T}\int_t^\tau \left|u_j^2(x,\sigma)\right|d\sigma \int_t^\tau 1\,d\sigma \\ &\le \tau e^{2\theta T}\int_0^\tau |u_j(x,\sigma)|^2\,d\sigma.\end{aligned}$$

Assume that the estimative holds for W_q^j, by Cauchy-Schwarz

$$\begin{aligned}|W_{q+1}^j|^2 &= \left|\int_t^\tau W_q^j(x,\sigma)\,d\sigma\right|^2 \le \int_t^\tau |W_q^j(x,\sigma)|^2\,d\sigma\int_t^\tau 1\,d\sigma \\ &\le \tau\int_t^\tau\left[\tau^{2(q-1)+3}e^{2\theta T}\int_0^\tau u_j^2(x,\sigma')\,d\sigma'\right]d\sigma \\ &\le \tau^2\tau^{2(q-1)+3}e^{2\theta T}\int_0^\tau u_j^2(x,\sigma')\,d\sigma' \le \tau^{2q+3}e^{2\theta T}\int_0^\tau u_j^2(x,\sigma')\,d\sigma',\end{aligned}$$

hence the claim follows.

c) It follows from b) that

$$\left|\left[\mathcal{F}, We^{\theta t}\right]_{G_\tau}\right| \le \sum_1^2 \delta\left[u_j, t^{-1}u_j e^{\theta t}\right]_{G_\tau} + \frac{e^{\theta t}\tau^{2p+6}}{4\delta}\|\mathcal{F}\|_{p+1,t_0}^2, \tag{3.10}$$

where δ = constant and

$$2(\alpha^{-1}+\delta) < 2p+6, \quad \|\mathcal{F}\|_{p+1,t_0}^2 = \sum_1^2 \max_{0\le\sigma\le t_0}\left(\partial_t^{(p+1)}\mathcal{F}_j, \partial_t^{(p+1)}\mathcal{F}_j\right)\Big|_{t=\sigma}.$$

Indeed, by Cauchy-Schwarz's inequality we obtain

$$\left|\left[\mathcal{F}_j, w_j e^{\theta t}\right]_{G_\tau}\right| \le \int_{G_\tau} \left|\partial_t^{(p+1)}(\mathcal{F}_j) W_{p+1}^j\right| dx\, dt$$
$$\le \left(\int_{G_\tau} \left|\partial_t^{(p+1)}\mathcal{F}_j\right|^2 dx\, dt\right)^{1/2} \left(\int_{G_\tau} \left|W_{p+1}^j\right|^2 dx\, dt\right)^{1/2}.$$

But

$$\int_{G_\tau} \left|W_{p+1}^j\right|^2 dx\, dt \le \int_{G_\tau} \tau^{2p+3} e^{2\theta T} \int_0^\tau u_j^2(x,\sigma)\, d\sigma\, dx\, dt$$
$$= \tau^{2p+3} e^{2\theta T} \tau \int_{\mathbb{R}} \int_0^\tau u_j^2(x,\sigma)\sigma e^{\theta\sigma}(\sigma e^{\theta\sigma})^{-1}\, d\sigma\, dx$$
$$\le \tau^{2p+3} e^{2\theta T} \tau^2 \int_{\mathbb{R}} \int_0^\tau u_j^2(x,\sigma) e^{\theta\sigma}\sigma^{-1}\, d\sigma\, dx$$

On the other hand,

$$\int_{G_\tau} \left|\partial_t^{(p+1)}\mathcal{F}_j\right|^2 dx\, dt \le \int_{\mathbb{R}} \max_{0\le\sigma\le t_0} \left|\partial_t^{(p+1)}\mathcal{F}_j(x,\sigma)\right|^2 \tau\, dx$$
$$\le \tau \max_{0\le\sigma\le t_0} \left(\partial_t^{(p+1)}\mathcal{F}_j, \partial_t^{(p+1)}\mathcal{F}_j\right)\Big|_{t=\sigma} = \tau \|\mathcal{F}_j\|_{p+1,t_0}^2.$$

Hence, using the fact that $xy \le \delta x^2 + y^2/(4\delta)$, we obtain

$$\left|\left[\mathcal{F}_j, w_j e^{\theta t}\right]_{G_\tau}\right| \le \left(\tau \|\mathcal{F}_j\|_{p+1,t_0}^2\right)^{1/2} \left(\tau^{2p+3} e^{2\theta T} \tau^2 \left[u_j, t^{-1} u_j e^{\theta t}\right]\right)^{1/2}$$
$$\le \delta \left[u_j, t^{-1} u_j e^{\theta t}\right] + \frac{\tau^{2p+6} e^{\theta T}}{4\delta} \|\mathcal{F}_j\|_{p+1,t_0}^2.$$

Then (3.10) follows.

STEP 3. From (3.2) to (3.10) and by the hypothesis (2.6), with θ large enough, we deduce from (2.13) that for $\tau \le t_0$

$$\left(U, U e^{\theta t}\right) \le 2\left(\delta + \alpha^{-1}\right)\left[U, t^{-1} U e^{\theta t}\right]_{G_\tau}$$
$$+ M_7 \tau \left[U, t^{-1} U e^{\theta t}\right]_{G_\tau} + M_8 \tau^{2p+6} \|\mathcal{F}\|_{p+1,t_0}^2$$

where the constant M_7 depends on the maximum modulus in G_{t_0} of the $\partial_x B_\epsilon, C_\epsilon$, $\partial_t C_\epsilon, D_\epsilon$ and on T, and $M_8 = e^{\theta T}/(4\delta)$. If we take $y(\tau) = [U, t^{-1} U e^{\theta t}]_{G_\tau}$ and use the fact that $\partial_t^\rho U|_{t=0} = 0$, $\forall \rho \le p+2$, then it follows

$$\tau y'(\tau) = \tau \frac{d}{d\tau}\left(\int_{\mathbb{R}} \int_0^\tau \sum_1^2 u_j^2 t^{-1} e^{\theta t}\, dt\, dx\right) = \left(U, U e^{\theta t}\right)_{t=\tau}.$$

Therefore, we have the inequality

$$\tau y'(\tau) \leq 2(\delta + \alpha^{-1})y(\tau) + M_7 \tau y(\tau) + M_8 \tau^{2p+6} \|\mathcal{F}\|^2_{p+1,t_0}. \tag{3.11}$$

Using a technique similar to the one in proof of Gronwall's lemma, namely multiplying (3.11) by $e^{-[2(\alpha^{-1}+\delta)\ln\tau + M_7\tau]}$, we obtain

$$\begin{aligned}\left(y(\tau)e^{-[2(\alpha^{-1}+\delta)\ln\tau+M_7\tau]}\right)' &= \left[y'(\tau) - \left(\frac{2(\alpha^{-1}+\delta)}{\tau} + M_7\right) y(\tau)\right] e^{-[2(\alpha^{-1}+\delta)\ln\tau+M_7\tau]} \\ &\leq M_8 \tau^{2p+6-1} \|\mathcal{F}\|^2_{p+1,t_0} e^{-[2(\alpha^{-1}+\delta)\ln\tau+M_7\tau]}.\end{aligned}$$

Integrating in τ it follows that

$$y(\tau)\tau^{-2(\alpha^{-1}+\delta)}e^{-M_7\tau} \leq \int_0^\tau M_8 s^{2p+6-1} \|\mathcal{F}\|^2_{p+1,t_0} e^{-M_7 s} s^{-2(\alpha^{-1}+\delta)}\, ds.$$

Since $2(\alpha^{-1}+\delta) < 2p+6$ we have $2p+6-2(\alpha^{-1}+\delta)-1 > -1$, hence

$$\begin{aligned} y(\tau) &\leq \tau^{2(\alpha^{-1}+\delta)} e^{M_7\tau} \|\mathcal{F}\|^2_{p+1,t_0} M_9 s^{2p+6-2(\alpha^{-1}-\delta)}\Big|_0^\tau \\ &= M_{10}\tau^{2p+6}\|\mathcal{F}\|^2_{p+1,t_0}. \end{aligned} \tag{3.12}$$

So from (3.11) and (3.12) we have

$$\begin{aligned}(U, Ue^{\theta\tau})_{t=\tau} &= 2(\delta+\alpha^{-1})M_{10}\tau^{2p+6}\|\mathcal{F}\|^2_{p+1,t_0} + M_7\tau M_{10}\tau^{2p+6}\|\mathcal{F}\|^2_{p+1,t_0} \\ &+ M_8\tau^{2p+6}\|\mathcal{F}\|^2_{p+1,t_0}.\end{aligned}$$

□

4. Estimates for $\partial_x^l U$, $l \leq k$

Lemma 4.1. *For $l \leq k$ and $0 \leq \tau \leq t_0$ we have*

$$\left(\partial_x^l U, \partial_x^l U e^{\theta\tau}\right)_{t=\tau} \leq E_l \tau^{2p+6} \sum_{\beta\leq k} \left\|\partial_x^\beta \mathcal{F}\right\|^2_{p+1,t_0}, \tag{4.1}$$

where the constant E_l depends on the maximum modulus of the derivatives ∂_x^l of $a_\epsilon, \partial_x a_\epsilon, B_\epsilon, C_\epsilon, \partial_t C_\epsilon, D_\epsilon$, $l \leq k$.

Proof. For the proof we will use induction over k. With θ_1 a positive constant to be chosen below we consider the equality

$$\left[\partial_x^k L_\epsilon(U), \partial_x^k W e^{\theta_1 t}\right]_{G_\tau} = \left[\partial_x^k \mathcal{F}, \partial_x^k W e^{\theta_1 t}\right]_{G_\tau}. \tag{4.2}$$

We will prove the lemma in two steps. In the first step, we transform the integrals in (4.2) by integration by parts in the same manner as was done in the derivation of (3.1). In the second step the estimate (4.1) follows by using Gronwall's lemma and the induction hypothesis.

STEP 1. We have:

$$\left|\left[\partial_x^k \mathcal{F}, \partial_x^k W e^{\theta t}\right]_{G_\tau}\right| \le \sum_1^2 \delta \left[\partial_x^k u_j, t^{-1}\partial_x^k u_j e^{\theta t}\right]_{G_\tau} + M_{11}\tau^{2p+6}\left\|\partial_x^k \mathcal{F}\right\|^2_{p+1,t_0}, \tag{4.3}$$

$$\left[\partial_x^k \partial_t^2 U, \partial_x^k W e^{\theta_1 t}\right]_{G_\tau} = \frac{1}{2}\left(\partial_x^k U, \partial_x^k U e^{\theta_1 t}\right)_{t=\tau} + \left[\partial_x^k U e^{\theta_1 t}, \theta_1^2 \partial_x^k W - \frac{3}{2}\theta_1 \partial_x^k U\right]_{G_\tau}, \tag{4.4}$$

$$\begin{aligned}\left[\epsilon_j \partial_x^k \partial_x^2 u_j, e^{\theta_1 t}\partial_x^k w_j\right]_{G_\tau} &= -\frac{\epsilon_j}{2}\left(\partial_x^{k+1} w_j, e^{\theta_1 t}\partial_x^{k+1} w_j\right)_{t=0} \\ &\quad - \frac{\epsilon_j}{2}\left[\partial_x^{k+1} w_j, \theta_1 e^{\theta_1 t}\partial_x^{k+1} w_j\right]_{G_\tau},\end{aligned} \tag{4.5}$$

$$\begin{aligned}\left[\partial_x^k (a_\epsilon u_x)_x, e^{\theta_1 t}\partial_x^k w\right]_{G_\tau} &= -\frac{1}{2}\left(a_\epsilon \partial_x^k w_x, e^{\theta_1 t}\partial_x^k w_x\right)_{t=0} \\ &\quad - \frac{1}{2}\left[(\theta_1 a_\epsilon + \partial_t a_\epsilon)\partial_x^k w_x, e^{\theta_1 t}\partial_x^k w_x\right]_{G_\tau} \\ &\quad + \sum_{1\le\beta\le k} C_\beta \left[\partial_x^\beta (a_\epsilon)\partial_x^{k-\beta} u_x, e^{\theta_1 t}\partial_x^k w_x\right]_{G_\tau},\end{aligned} \tag{4.6}$$

where C_β are constants. Let us estimate the last sum of equation (4.6). For $\beta = 1$ the integral

$$\left[\partial_x (a_\epsilon)\partial_x^{k-1} u_x, e^{\theta_1 t}\partial_x^k w_x\right]_{G_\tau}$$

can be estimated using the Glaeser ([3]) inequality, namely, for each $t \in [0, T]$

$$|\partial_x (a_\epsilon)(x,t)|^2 \le M a_\epsilon (x,t)$$

with $M = \sup_G \left|\partial_x^2 a_\epsilon\right|$. Using the inequality $xy \le (x^2/4) + y^2$ we get

$$\left|\left[\partial_x (a_\epsilon)\partial_x^{k-1} u_x, e^{\theta_1 t}\partial_x^k w_x\right]_{G_\tau}\right| \le |A_1| + M\left[a_\epsilon \partial_x^k w_x, e^{\theta_1 t}\partial_x^k w_x\right]_{G_\tau}.$$

Here, as well as below, we denote by A_j integrals admitting the estimate

$$|A_j| \le N_j \sum_{\beta\le k} \tau \left[\partial_x^\beta U, t^{-1}\partial_x^\beta U e^{\theta_1 t}\right]_{G_\tau},$$

where the N_j are constants depending on the coefficients in equation (2.8). We use integration by parts to transform the integrals in the last term of (4.6) which correspond to $\beta \ge 2$. We have

$$\begin{aligned}&\sum_{2\le\beta\le k} C_\beta \left[\partial_x^\beta (a_\epsilon)\partial_x^{k-\beta} u_x, e^{\theta_1 t}\partial_x^k w_x\right]_{G_\tau} \\ &= -\sum_{2\le\beta\le k} C_\beta \left[(\partial_x^\beta (a_\epsilon)\partial_x^{k-\beta} u_x)_x, e^{\theta_1 t}\partial_x^k w\right]_{G_\tau} = A_2,\end{aligned}$$

where N_2 depends on the maximum of the modulus of derivatives ∂_x^l, for $l \leq k$ of $a_\epsilon, \partial_x a_\epsilon$.

By definition

(4.7)

$$\left[\partial_x^k(B_\epsilon \partial_x U), e^{\theta_1 t}\partial_x^k W\right]_{G_\tau} = \sum_1^2 \left[\partial_x^k\left(b_\epsilon^{jj}\partial_x u_j\right), e^{\theta_1 t}\partial_x^k w_j\right]_{G_\tau}$$
$$+ \left[\partial_x^k\left(b_\epsilon^{21}\partial_x u_1\right), e^{\theta_1 t}\partial_x^k w_2\right]_{G_\tau} + \left[\partial_x^k\left(b_\epsilon^{12}\partial_x u_2\right), e^{\theta_1 t}\partial_x^k w_1\right]_{G_\tau}$$

For the first two terms in the right side of (4.7) we have

$$\left|\left[\partial_x^k\left(b_\epsilon^{jj}\partial_x u_j\right), e^{\theta_1 t}\partial_x^k w_j\right]_{G_\tau}\right| \leq \left|\left[b_\epsilon^{jj}\partial_x^k\partial_x u_j, e^{\theta_1 t}\partial_x^k w_j\right]_{G_\tau} + A_3\right|$$
$$\leq \left|\left[b_\epsilon^{jj}\partial_x^k u_j, e^{\theta_1 t}\partial_x^{k+1} w_j\right]_{G_\tau} + A_4\right|$$
$$\leq |A_4| + \frac{\alpha}{2}\left[t b_\epsilon^{jj}\partial_x^{k+1} w_j, e^{\theta_1 t} b_\epsilon^{jj}\partial_x^{k+1} w_j\right]_{G_\tau}$$
$$+ \frac{1}{2\alpha}\left[\partial_x^k u_j, t^{-1}e^{\theta_1 t}\partial_x^k u_j\right]_{G_\tau}$$

where N_4 depends on the maximum of the modulus of derivatives ∂_x^l, for $l \leq k$ of b_ϵ^{jj}. For the mixed terms in (4.7) we have

$$\left[\partial_x^k(b_\epsilon^{12}\partial_x u_2), e^{\theta_1 t}\partial_x^k w_1\right]_{G_\tau} = \left[b_\epsilon^{12}\partial_x^{k+1} u_2, e^{\theta_1 t}\partial_x^k w_1\right]_{G_\tau} + \sum_1^k \left[\partial_x^\beta(b_\epsilon^{12})\partial_x^{k+1-\beta} u_2, e^{\theta_1 t}\partial_x^k w_1\right]_{G_\tau}$$

As before we obtain

$$\left[b_\epsilon^{12}\partial_x^{k+1} u_2, e^{\theta_1 t}\partial_x^k w_1\right]_{G_\tau} \leq \frac{\alpha}{2}\left[t b_\epsilon^{12}\partial_x^{k+1} w_1, b_\epsilon^{12} e^{\theta_1 t}\partial_x^{k+1} w_1\right]_{G_\tau} + \frac{1}{2\alpha}\left[\partial_x^k u_2, t^{-1}e^{\theta_1 t}\partial_x^k u_2\right]_{G_\tau}$$
$$+ \frac{1}{2}\left[\partial_x^k u_2, e^{\theta_1 t}(\partial_x b_\epsilon^{12})^2\partial_x^k u_2\right]_{G_\tau} + \frac{1}{2}\left[\partial_x^k w_1, e^{\theta_1 t}\partial_x^k w_1\right]_{G_\tau}.$$

Again

$$\sum_1^k \left[\partial_x^\beta(b_\epsilon^{12})\partial_x^{k+1-\beta} u_2, e^{\theta_1 t}\partial_x^k w_1\right]_{G_\tau} \leq |A_5|,$$

where N_5 depending on $\partial_x^l b_\epsilon^{12}$ for $l \leq k$. Since we can prove an analogous inequality for the term $\left[\partial_x^k(b_\epsilon^{21}\partial_x u_1), e^{\theta_1 t}\partial_x^k w_2\right]_{G_\tau}$ in (4.7), we obtain

$$
\begin{aligned}
\left[\partial_x^k(B_\epsilon \partial_x U), e^{\theta_1 t}\partial_x^k W\right]_{G_\tau} \leq & \sum_1^2 \frac{\alpha}{2}\left[t b_\epsilon^{jj}\partial_x^{k+1} w_j, e^{\theta_1 t} b_\epsilon^{jj}\partial_x^{k+1} w_j\right]_{G_\tau} \\
& + \sum_1^2 \frac{1}{\alpha}\left[\partial_x^k u_j, t^{-1} e^{\theta_1 t}\partial_x^k u_j\right]_{G_\tau} \\
& + \frac{\alpha}{2}\left[t b_\epsilon^{12}\partial_x^{k+1} w_1, b_\epsilon^{12} e^{\theta_1 t}\partial_x^{k+1} w_1\right]_{G_\tau} \\
& + \frac{\alpha}{2}\left[t b_\epsilon^{21}\partial_x^{k+1} w_2, b_\epsilon^{21} e^{\theta_1 t}\partial_x^{k+1} w_2\right]_{G_\tau} + |A_6|.
\end{aligned} \tag{4.8}
$$

where N_6 depends on $\partial_x^l B_\epsilon$, for $l \leq k$.

By definition

$$
\begin{aligned}
& \left[\partial_x^k(C_\epsilon \partial_t U), \partial_x^k W e^{\theta_1 t}\right]_{G_\tau} \\
& = \sum_1^2 \left[\partial_x^k(c_\epsilon^{jj}\partial_t u_j), \partial_x^k w_j e^{\theta_1 t}\right]_{G_\tau} \\
& \quad + \left[\partial_x^k(c_\epsilon^{12}\partial_t u_2), \partial_x^k w_1 e^{\theta t}\right]_{G_\tau} + \left[\partial_x^k(c_\epsilon^{21}\partial_t u_1), \partial_x^k w_2 e^{\theta t}\right]_{G_\tau}
\end{aligned} \tag{4.9}
$$

For the first two terms in right-hand side of (4.9) we obtain

$$
\begin{aligned}
& \left[\partial_x^k(c_\epsilon^{jj}\partial_t u_j), \partial_x^k w_j e^{\theta_1 t}\right]_{G_\tau} \\
& = \left[\partial_x^k(\partial_t(c_\epsilon^{jj} u_j) - u_j \partial_t c_\epsilon^{jj}), e^{\theta_1 t}\partial_x^k w_j\right]_{G_\tau} \\
& = \left[\partial_x^k(c_\epsilon^{jj} u_j), e^{\theta_1 t}(\partial_x^k u_j - \theta_1 \partial_x^k w_j)\right]_{G_\tau} - \left[\partial_x^k(u_j \partial_t c_\epsilon^{jj}), e^{\theta_1 t}\partial_x^k w_j\right]_{G_\tau} = A_7.
\end{aligned}
$$

For the mixed term in (4.9) we have

$$
\begin{aligned}
\left[\partial_x^k(c_\epsilon^{12}\partial_t u_2), \partial_x^k w_1 e^{\theta_1 t}\right]_{G_\tau} = & \left[\partial_x^k(c_\epsilon^{12} u_2), e^{\theta_1 t}(\partial_x^k u_2 - \theta_1 \partial_x^k w_1)\right]_{G_\tau} \\
& - [\partial_x^k(u_2 \partial_t c_\epsilon^{12}), e^{\theta_1 t}\partial_x^k w_1]_{G_\tau} = A_8.
\end{aligned}
$$

We can prove an analogous equality for the other mixed term in (4.9). Therefore,

$$
\left[\partial_x^k(C_\epsilon \partial_t U), \partial_x^k W e^{\theta t}\right]_{G_\tau} = A_9, \tag{4.10}
$$

where N_9 depends on the derivatives ∂_x^l of C_ϵ and $\partial_t C_\epsilon$ for $l \leq k$.

Finally

$$
\left[\partial_x^k(D_\epsilon U), e^{\theta_1 t}\partial_x^k U\right]_{G_\tau} = A_{10}, \tag{4.11}
$$

where N_{10} depends on $\partial_x^l D_\epsilon$ for $l \leq k$.

STEP 2. From (4.3) to (4.11) as well as the condition (2.5) with $\theta = \theta_1 - 2M$, by choosing the constant θ_1 sufficiently large, and by using the induction hypotheses (4.1), we deduce from (4.2) that

$$(4.12)\quad \left(\partial_x^k U, \partial_x^k U e^{\theta_1 t}\right)_{t=\tau} \leq 2(\delta + \alpha^{-1}) y_k(\tau) + \tau K y_k(\tau) + \tau^{2p+6} \tilde{K} \sum_{\beta \leq k} \left\| \partial_x^\beta \mathcal{F} \right\|^2_{p+1,t_0},$$

where $y_k(\tau) = \left[\partial_x^k U, t^{-1} \partial_x^k U e^{\theta_1 t}\right]_{G_\tau}$ and the constants $K, \tilde{K}$ depends on the maximum of the modulus of derivatives ∂_x^l of $a_\epsilon, \partial_x a_\epsilon$, B_ϵ, $C_\epsilon, \partial_t C_\epsilon, D_\epsilon$ for $l \leq k$, as well as on the derivatives ∂_x^l of a_ϵ and $\partial_x a_\epsilon$, for $l \leq 2$ when $k = 1$. Since

$$\tau y_k'(\tau) = \left(\partial_x^k U, \partial_x^k U e^{\theta_1 t}\right)_{t=\tau},$$

it follows from (4.12) and from Gronwall's lemma that

$$y_k(\tau) \leq \tilde{\tilde{K}} \tau^{2p+6} \sum_{\beta \leq k} \left\| \partial_x^\beta \mathcal{F} \right\|^2_{p+1,t_0}.$$

Therefore, for $l \leq k$ we have

$$\left(\partial_x^l U, \partial_x^l U e^{\theta_1 t}\right)_{t=\tau} \leq 2\left(\delta + \alpha^{-1}\right) \tilde{\tilde{K}} \tau^{2p+6} \sum_{\beta \leq k} \left\| \partial_x^\beta \mathcal{F} \right\|^2_{p+1,t_0} + \tau K \tilde{\tilde{K}} \tau^{2p+6} \sum_{\beta \leq k} \left\| \partial_x^\beta \mathcal{F} \right\|^2_{p+1,t_0} + \tau^{2p+6} \tilde{K} \sum_{\beta \leq k} \left\| \partial_x^\beta \mathcal{F} \right\|^2_{p+1,t_0} \leq E_l \tau^{2p+6} \sum_{\beta \leq k} \left\| \partial_x^\beta \mathcal{F} \right\|^2_{p+1,t_0}. \qquad \square$$

5. Estimates for $\partial_x^l \partial_t^\rho U$, $\rho + l \leq k$

Lemma 5.1. *For $l \leq k - 1$ and $0 \leq \tau \leq t_0$ we have*

$$(5.1)\quad \left(\partial_x^l U_t, e^{-\theta_2 t} \partial_x^l U_t\right)_{t=\tau} \leq M_{12} \left\{ \tau^{2p+6} \sum_{\beta \leq k} \| \partial_x^\beta \mathcal{F} \|^2_{p+1,t_0} + \sum_{\gamma \leq k-1} \left[\partial_x^\gamma \mathcal{F}, \partial_x^\gamma \mathcal{F}\right]_{G_\tau} \right\}$$

where M_{12} depends of derivatives ∂_x^l of $a_\epsilon, \partial_x a_\epsilon, B_\epsilon, C_\epsilon, \partial_t C_\epsilon, D_\epsilon$ for $l \leq k$.

Proof. In order to estimate the derivatives $\partial_x^l U_t$ for $l \leq k - 1$, we consider the equality

$$(5.2)\quad \left[\partial_x^l L_\epsilon(U), e^{-\theta_2 t} \partial_x^l U_t\right]_{G_\tau} = \left[\partial_x^l \mathcal{F}, e^{-\theta_2 t} \partial_x^l U_t\right]_{G_\tau},$$

where θ_2 = constant > 0. The proof consists of two steps. In the first step we estimate the right-hand side of (5.2) and using integration by parts we write each term on the left-hand side of (5.2) in order to have the smallest possible order of derivative of U. In the second step the estimates (5.1) follows from Gronwall's lemma.

STEP 1. Integrating by parts we obtain

$$\left[\partial_x^l \mathcal{F}, e^{-\theta_2 t}\partial_x^l U_t\right]_{G_\tau} \leq \frac{1}{2}\left(\left[\partial_x^l \mathcal{F}, e^{-\theta_2 t}\partial_x^l \mathcal{F}\right]_{G_\tau} + \left[\partial_x^l U_t, e^{-\theta_2 t}\partial_x^l U_t\right]_{G_\tau}\right), \tag{5.3}$$

$$\left[\partial_x^l \partial_t^2 U, e^{-\theta_2 t}\partial_x^l U_t\right]_{G_\tau} = \frac{1}{2}\left(\partial_x^l U_t, e^{-\theta_2 t}\partial_x^l U_t\right)_{t=\tau} + \frac{1}{2}\left[\partial_x^l U_t, \theta_2 e^{-\theta_2 t}\partial_x^l U_t\right]_{G_\tau}, \tag{5.4}$$

$$\begin{aligned}\left[\epsilon_j \partial_x^l \partial_x^2 u_j, e^{-\theta_2 t}\partial_x^l \partial_t u_j\right]_{G_\tau} = &-\frac{\epsilon_j}{2}\left(\partial_x^{l+1} u_j, e^{-\theta_2 t}\partial_x^{l+1} u_j\right)_{t=\tau} \\ &-\frac{\epsilon_j}{2}\left[\partial_x^{l+1} u_j, \theta_2 e^{-\theta_2 t}\partial_x^{l+1} u_j\right]_{G_\tau},\end{aligned} \tag{5.5}$$

$$\begin{aligned}\left[\partial_x^l (a_\epsilon U_x)_x, e^{-\theta_2 t}\partial_x^l U_t\right]_{G_\tau} = &-\frac{1}{2}\left(a_\epsilon \partial_x^{l+1} U, e^{-\theta_2 t}\partial_x^{l+1} U\right)_{t=\tau} \\ &+ \frac{1}{2}\left[\partial_x^{l+1} U, (\partial_t a_\epsilon - \theta_2 a_\epsilon)\, e^{-\theta_2 t}\partial_x^{l+1} U\right]_{G_\tau} \\ &+ \sum_{1\leq\gamma\leq l} \tilde{\beta}_\gamma [(\partial_x^\gamma (a_\epsilon)\partial_x^{l-\gamma} U_x)_x, e^{-\theta_2 t}\partial_x^l U_t]_{G_\tau}\end{aligned} \tag{5.6}$$

where $\tilde{\beta}_\gamma$ are constants.

By definition

$$\begin{aligned}\left[\partial_x^l (B_\epsilon \partial_x U), \partial_x^l U_t e^{-\theta_2 t}\right]_{G_\tau} = &\sum_1^2 \left[\partial_x^l (b_\epsilon^{jj}\partial_x u_j), e^{-\theta_2 t}\partial_x^l \partial_t u_j\right]_{G_\tau} \\ &+ \left[\partial_x^l (b_\epsilon^{12}\partial_x u_2), e^{-\theta_2 t}\partial_x^l \partial_t u_1\right]_{G_\tau} \\ &+ \left[\partial_x^l (b_\epsilon^{21}\partial_x u_1), e^{-\theta_2 t}\partial_x^l \partial_t u_2\right]_{G_\tau}.\end{aligned} \tag{5.7}$$

For the first two terms in the right-hand side of (5.7) we obtain

$$\begin{aligned}\left[\partial_x^l\left(b_\epsilon^{jj}\partial_x u_j\right), e^{-\theta_2 t}\partial_x^l \partial_t u_j\right]_{G_\tau} &\leq \frac{1}{4}\left[\partial_x^l\left(b_\epsilon^{jj}\partial_x u_j\right), e^{-\theta_2 t}\partial_x^l\left(b_\epsilon^{jj}\partial_x u_j\right)\right]_{G_\tau} \\ &\quad + \left[\partial_x^l \partial_t u_j, e^{-\theta_2 t}\partial_x^l \partial_t u_j\right]_{G_\tau} \\ &\leq \left[\partial_x^l \partial_t u_j, e^{-\theta_2 t}\partial_x^l \partial_t u_j\right]_{G_\tau} + M_{13}\sum_{\gamma\leq l+1}\left[\partial_x^\gamma u_j, e^{-\theta_2 t}\partial_x^\gamma u_j\right]_{G_\tau}.\end{aligned}$$

where M_{13} depends on the derivatives $\partial_x^l b_\epsilon^{jj}$, for $l \leq k-1$. The mixed terms in the right-hand side of (5.7) have analogous inequalities to the last estimates, therefore

$$\begin{aligned}\left[\partial_x^l (B_\epsilon \partial_x U), \partial_x^l U_t e^{-\theta_2 t}\right]_{G_\tau} \leq &\left[\partial_x^l \partial_t U, e^{-\theta_2 t}\partial_x^l \partial_t U\right]_{G_\tau} \\ &+ M_{14}\sum_{\gamma\leq l+1}\left[\partial_x^\gamma U, e^{-\theta_2 t}\partial_x^\gamma U\right]_{G_\tau},\end{aligned} \tag{5.8}$$

where M_{14} depends of derivatives $\partial_x^l B_\epsilon$ for $l \leq k-1$.

Finally

$$\left[\partial_x^l(C_\epsilon \partial_t U), e^{-\theta_2 t}\partial_x^l \partial_t U\right]_{G_\tau} \leq M_{15} \sum_{\gamma \leq l} \left[\partial_x^\gamma U_t, e^{-\theta_2 t}\partial_x^\gamma U_t\right]_{G_\tau}, \tag{5.9}$$

$$\left[\partial_x^l(D_\epsilon U), e^{-\theta_2 t}\partial_x^l \partial_t U\right]_{G_\tau} \leq M_{16}\left[\partial_x^l U, e^{-\theta_2 t}\partial_x^l U\right]_{G_\tau} + \left[\partial_x^l U_t, e^{-\theta_2 t}\partial_x^l U_t\right], \tag{5.10}$$

where M_{15} and M_{16} depend on the derivatives $\partial_x^l C_\epsilon$ and $\partial_x^l D_\epsilon$, $l \leq k-1$, respectively.

STEP 2. Using (5.3) through (5.10) together with (4.1), and choosing the constant θ_2 sufficiently large, we obtain from (5.2), using induction on l, that for $l \leq k-1$

$$\begin{aligned}\left(\partial_x^l U_t, e^{-\theta_2 t}\partial_x^l U_t\right)_{t=\tau} \leq M_{17} &\left\{\left[\partial_x^l U_t, e^{-\theta_2 t}\partial_x^l U_t\right]_{G_\tau} + \tau^{2p+6}\sum_{\beta \leq k}\left\|\partial_x^\beta \mathcal{F}\right\|^2_{p+1,t_0}\right\} \\ &+ \frac{1}{2}\sum_{\gamma \leq k-1}\left[\partial_x^\gamma \mathcal{F}, \partial_x^\gamma \mathcal{F}\right]_{G_\tau}\end{aligned} \tag{5.11}$$

where M_{17} depends of derivatives ∂_x^l of $a_\epsilon, \partial_x a_\epsilon, B_\epsilon$ $C_\epsilon, \partial_t C_\epsilon, D_\epsilon$, $l \leq k$. By Gronwall's lemma the estimates (5.1) follows from (5.11). □

Lemma 5.2. *For $\rho \geq 0$, $l + \rho \leq k-2$ and $0 \leq \tau \leq t_0$ we have*

$$\begin{aligned}&\left(\partial_x^l \partial_t^{\rho+2} U, \partial_x^l \partial_t^{\rho+2} U\right)_{t=\tau} \\ &\leq M_{18}\left\{\tau^{2p+6}\sum_{\beta \leq k}\left\|\partial_x^\beta \mathcal{F}\right\|^2_{p+1,t_0} + \sum_{\gamma \leq k-1}\left[\partial_x^\gamma \mathcal{F}, \partial_x^\gamma \mathcal{F}\right]_{G_\tau}\right\} \\ &\quad + \sum_{l+\rho \leq k-2}\left(\partial_x^l \partial_t^\rho f_\epsilon, \partial_x^l \partial_t^\rho f_\epsilon\right)_{t=\tau}\end{aligned} \tag{5.12}$$

where M_{18} depends on the derivatives $\partial_x^l \partial_t^\rho$ of $a_\epsilon, \partial_x a_\epsilon, B_\epsilon, C_\epsilon, \partial_t C_\epsilon, D_\epsilon$ in G for $l + \rho \leq k-2$ and for $\rho = 0$, $l \leq k$.

Proof. To prove this we apply the operator $\partial_x^l \partial_t^\rho$ to the equation (2.12) and then we obtain equations which give the derivatives $\partial_x^l \partial_t^{\rho+2}$ expressed in terms of the derivatives estimated above. □

6. Proof of Lemma 2.1 and the Theorem 2.1

First of all we prove Lemma 2.1 and then conclude the proof of Theorem 2.1.

Proof of Lemma 2.1. The proof will be a consequence of the following two steps. In the first step, we will use the estimates already proved to obtain (2.10) for $\tau \leq t_0$. In the second step we will prove the estimative (2.10) for $t_0 \leq \tau \leq T$.

STEP 1. For $U = U_\epsilon - V_p$, we have

$$\|U_\epsilon\|^2_{\tau;k} = \sum_{\rho+l\le k} \left(\partial_x^l \partial_t^\rho U_\epsilon, \partial_x^l \partial_t^\rho U_\epsilon\right)_{t=\tau} = \sum_{\rho+l\le k} \left(\partial_x^l \partial_t^\rho (U+V_p), \partial_x^l \partial_t^\rho (U+V_p)\right)_{t=\tau}$$
$$= \sum_{\rho+l\le k} \left\{ \left(\partial_x^l \partial_t^\rho U, \partial_x^l \partial_t^\rho U\right)_{t=\tau} + 2\left(\partial_x^l \partial_t^\rho U, \partial_x^l \partial_t^\rho V_P\right)_{t=\tau} + \left(\partial_x^l \partial_t^\rho V_p, \partial_x^l \partial_t^\rho V_p\right)_{t=\tau} \right\}.$$

We have two cases to consider:
a) For $\rho = 0$ and $l \le k$, by (4.1) and (2.12) we obtain

$$\left(\partial_x^l U, \partial_x^l U\right)_{t=\tau} \le E_l \tau^{2p+6} \sum_{\beta\le k} \left\|\partial_x^\beta \mathcal{F}\right\|^2_{p+1,t_0}$$
$$= E_l \tau^{2p+6} \sum_{\beta\le k} \max_{0\le\sigma\le t_0} \left(\partial_x^\beta \partial_t^{(p+1)}(f_\epsilon - L_\epsilon V_p), \partial_x^\beta \partial_t^{(p+1)}(f_\epsilon - L_\epsilon V_p)\right)\Big|_{t=\sigma}.$$

But

$$\max_{0\le\sigma\le t_0} \left(\partial_x^\beta \partial_t^{(p+1)} L_\epsilon V_p, \partial_x^\beta \partial_t^{(p+1)} L_\epsilon V_p\right)\Big|_{t=\sigma}$$
$$\le M_{19} \left\{ \|\Phi_\epsilon\|^2_{0;k+p+4} + \|\Psi_\epsilon\|^2_{0;k+p+3} + \sum_{\rho\le p} \|f_\epsilon\|^2_{0;\rho,p+k+2-\rho} \right\},$$

where M_{19} depends on the derivatives $\partial_x^l \partial_t^\rho$ of $a_\epsilon, \partial_x a_\epsilon, B_\epsilon, C_\epsilon, D_\epsilon$ for $\rho \le p$ and $\rho + l \le p + k + 2$ at $t = 0$, and on the derivatives $\partial_x^l \partial_t^\rho$ of these functions with $\rho \le p + 1$ and $l \le k$ in $0 \le \tau \le t_0$. Therefore,

$$\left(\partial_x^l U, \partial_x^l U\right)_{t=\tau}$$
$$\le M_{20} \left\{ \|\Phi_\epsilon\|^2_{0;k+p+4} + \|\Psi_\epsilon\|^2_{0;k+p+3} + \sum_{\rho\le p} \|f_\epsilon\|^2_{0;\rho,p+k+2-\rho} + \max_{0\le\sigma\le t_0} \|f_\epsilon\|^2_{\sigma;p+1,k} \right\}, \tag{6.1}$$

where M_{20} is a constant depending on E_l and M_{19}.
b) The other terms are $\partial_x^l U_t$, $l \le k-1$ and $\partial_x^l \partial_t^{\rho+2}$, $l + \rho \le k - 2$, which are bounded by the right-hand side of (5.12). But

$$\sum_{\gamma\le k-1} \left[\partial_x^\gamma \mathcal{F}, \partial_x^\gamma \mathcal{F}\right]_{G_\tau}$$
$$\le \tau M_{21} \left\{ \|f_\epsilon\|^2_{G_\tau;0,k-1} + \|\Phi_\epsilon\|^2_{0;k+p+4} + \|\Psi_\epsilon\|^2_{0;k+p+3} + \sum_{\rho\le p} \|f_\epsilon\|^2_{0;\rho,p+k+2-\rho} \right\},$$

where the constant M_{21} depends on the derivatives $\partial_x^l \partial_t^\rho$ of $a_\epsilon, \partial_x a_\epsilon, B_\epsilon, C_\epsilon, D_\epsilon$ for $\rho \le p$ and $\rho + l \le p + k + 2$ at $t = 0$, as well as the derivatives ∂_x^l of these functions for $l \le k$ at $0 \le t \le t_0$.

Putting together a) and b), we have (2.10) for $\tau \leq t_0$.

STEP 2. We can obtain the estimates (2.10) for $t_0 \leq \tau \leq T-\epsilon$ in a similar way. Namely, to estimate $(U,U)_{t=\tau}$ we consider the equation (2.13) and transform, using integration by parts, its terms on the left-hand side in the same way as for $\tau \leq t_0$. But instead of (3.7), (3.8) and (3.9) we consider the inequalities

$$\begin{aligned}\left|\left[b_\epsilon^{jj}\partial_x u_j, w_j e^{\theta t}\right]_{G_\tau}\right| \leq M_{22}\left[u_j, u_j e^{\theta t}\right]_{G_\tau} + \frac{\alpha}{2}\left[t b_\epsilon^{jj}\partial_x w_j, b_\epsilon^{jj}\partial_x w_j e^{\theta t}\right]_{G_{t_0}} \\ + \frac{1}{2\alpha}\left[u_j, t^{-1}u_j e^{\theta t}\right]_{G_{t_0}} + \frac{\lambda}{2}\left[b_\epsilon^{jj}\partial_x w_j, b_\epsilon^{jj}\partial_x w_j e^{\theta t}\right]_{G_{\tau-t_0}} \\ + \frac{1}{2\lambda}\left[u_j, u_j e^{\theta t}\right]_{G_{\tau-t_0}},\end{aligned} \tag{6.2}$$

$$\begin{aligned}\left[b_\epsilon^{21}\partial_x u_1, w_2 e^{\theta t}\right]_{G_\tau} \leq \frac{\alpha}{2}\left[t b_\epsilon^{21}\partial_x w_2, b_\epsilon^{21}\partial_x w_2 e^{\theta t}\right]_{G_{t_0}} + \frac{1}{2\alpha}\left[u_1, t^{-1}u_1 e^{\theta t}\right]_{G_{t_0}} \\ + \frac{\lambda}{2}\left[b_\epsilon^{21}\partial_x w_2, b_\epsilon^{21}\partial_x w_2 e^{\theta t}\right]_{G_{\tau-t_0}} + \frac{1}{2\lambda}\left[u_1, u_1 e^{\theta t}\right]_{G_{\tau-t_0}} \\ + \frac{1}{2}\left[u_1, \partial_x (b_\epsilon^{21})^2 u_1 e^{\theta t}\right]_{G_\tau} + \frac{1}{2}\left[w_2, w_2 e^{\theta t}\right]_{G_\tau},\end{aligned} \tag{6.3}$$

and

$$\begin{aligned}\left[b_\epsilon^{12}\partial_x u_2, w_1 e^{\theta t}\right]_{G_\tau} \leq \frac{\alpha}{2}\left[t b_\epsilon^{12}\partial_x w_1, b_\epsilon^{12}\partial_x w_1 e^{\theta t}\right]_{G_{t_0}} + \frac{1}{2\alpha}\left[u_2, t^{-1}u_2 e^{\theta t}\right]_{G_{t_0}} \\ + \frac{\lambda}{2}\left[b_\epsilon^{12}\partial_x w_1, b_\epsilon^{12}\partial_x w_1 e^{\theta t}\right]_{G_{\tau-t_0}} + \frac{1}{2\lambda}\left[u_2, u_2 e^{\theta t}\right]_{G_{\tau-t_0}} \\ + \frac{1}{2}\left[u_2, \partial_x (b_\epsilon^{12})^2 u_2 e^{\theta t}\right]_{G_\tau} + \frac{1}{2}\left[w_1, w_1 e^{\theta t}\right]_{G_\tau}.\end{aligned} \tag{6.4}$$

where $\lambda \leq \alpha t$ is a constant for $t \geq t_0$, and we use the estimate (3.12) for $y(t_0) = \left[U, t^{-1}U e^{\theta t}\right]_{G_{t_0}}$. We estimate the right-hand side of (2.13) using the inequality

$$\left|[\mathcal{F}, e^{\theta t}W]_{G\tau}\right| \leq [\mathcal{F}, e^{\theta t}\mathcal{F}] + M_{23}[U, e^{\theta t}U]_{G\tau},$$

where the constant M_{23} depends only on T. Setting $z(\tau) = [U, U e^{\theta t}]_{G_\tau}$ and choosing the constant θ sufficiently large, for $\tau \geq t_0$ we obtain from (2.13) the inequality

$$z'(\tau) \leq M_{24} z(\tau) + M_{25}\|\mathcal{F}\|^2_{p+1,t_0} + M_{26}[\mathcal{F},\mathcal{F}]_{G_{T-\epsilon}}, \tag{6.5}$$

where M_{24}, M_{25} and M_{26} depends on the maximum of the absolute values of B_ϵ, C_ϵ, $\partial_t C_\epsilon$, D_ϵ and on t_0. Since $z'(\tau) = \left(U, e^{\theta t}U\right)_{t=\tau}$, hence by (6.5) we have the inequality

$$\left(U, e^{\theta t}U\right)_{t=\tau} \leq M_{30}\left\{\|\mathcal{F}\|^2_{p+1,t_0} + [\mathcal{F},\mathcal{F}]_{G_{T-\epsilon}}\right\}. \tag{6.6}$$

In a similar way we estimate $\left(\partial_x^l U, \partial_x^l U\right)_{t=\tau}$, $l \leq k$ and $\tau \geq t_0$. The derivatives $\partial_x^l U_t$, $l \leq k-1$, and $\partial_x^l \partial_t^{\rho+2}$, $\rho \geq 0$, $l+\rho+2 \leq k$, for $\tau \geq t_0$ are estimated in the same way as for $\tau \leq t_0$.

From the estimates for U and the relation $U = U_\epsilon - V_p$, there follows (2.10) for U_ϵ. □

Now we will prove Theorem 2.1.

Proof of Theorem 2.1. We will consider three steps. In the first step, we obtain a function U as limit of the functions U_ϵ obtained in the Lemma 2.1. In the second and third steps we prove the existence and the uniqueness, respectively.

STEP 1. Consider, in the domain G^ϵ, the Cauchy problem for the systems (2.8) with coefficients defined by (2.5) and initial conditions by (2.9). Since for $\epsilon > 0$ the system (2.8) is strictly hyperbolic, the problem (2.8)–(2.9) has a solution $U_\epsilon \in C^\infty(G^\epsilon, \mathbb{R}^2)$ and by (2.6) the estimates (2.10) hold. Hence, $U_\epsilon \in H^k(G^\epsilon, \mathbb{R}^2)$ is a bounded sequence, from Rellich theorem, it has a subsequence U_{ϵ_j} that converges for a function U in $H^t(G, \mathbb{R}^2)$, for every $t < s$. On the other hand, since $H^k(G)$ is a reflexive space and U_{ϵ_j} is a bounded sequence, it has a subsequence $U_{\epsilon_{jl}}$ weakly converging to a function $V \in H^k(G, \mathbb{R}^2)$ and the estimates (2.10) hold for a limiting function V. Therefore, by uniqueness of the limit we have $U = V \in H^k(G, \mathbb{R}^2)$.

STEP 2. Let U be given in Step 1:

a) Since $U_\epsilon|_{t=0} = \Phi_\epsilon \to \Phi$ on $H^{k+p+4}(\mathbb{R}, \mathbb{R}^2)$ and $U_{\epsilon_j} \to U$ in $H^{k-1}(G, \mathbb{R}^2)$, then $U|_{t=0} = \Phi$ in $H^k(\mathbb{R}, \mathbb{R}^2)$. It is also true that $U_t|_{t=0} = \Psi$ in $H^k(\mathbb{R}, \mathbb{R}^2)$.

b) Now $U \in H^k(G, \mathbb{R}^2)$ satisfies $LU = f$ in $\mathcal{D}'(G, \mathbb{R}^2)$. Indeed, for $\varphi \in C_c^\infty(G)$ we have

$$\left\langle L_{\epsilon_j} U_{\epsilon_j}, \varphi \right\rangle = \left\langle f_{\epsilon_j}, \varphi \right\rangle \to \langle f, \varphi \rangle,$$

since $f_\epsilon \to f$ in $L^2(G)$.

On the other hand, we have $\left\langle L_{\epsilon_j} U_{\epsilon_j}, \varphi \right\rangle \to \langle LU, \varphi \rangle$. Indeed, let's consider only two typical terms of the left hand side. For the first term we have

$$\left\langle \partial_t^2 U_{\epsilon_j}, \varphi \right\rangle = \left\langle U_{\epsilon_j}, \partial_t^2 \varphi \right\rangle \to \left\langle U, \partial_t^2 \varphi \right\rangle = \left\langle \partial_t^2 U, \varphi \right\rangle,$$

since $\langle U_{\epsilon_j}, \varphi \rangle \to \langle U, \varphi \rangle, \forall \varphi \in C_c^\infty \subset H^{1-k}(G)$. For the second term,

$$\begin{aligned} \left\langle \partial_x(a_{\epsilon_j} \partial_x U_{\epsilon_j}) - \partial_x(a \partial_x U), \varphi \right\rangle &= -\left\langle a_{\epsilon_j} \partial_x U_{\epsilon_j} - a \partial_x U, \partial_x \varphi \right\rangle \\ &= -\left\langle a_{\epsilon_j} \partial_x U_{\epsilon_j} - a \partial_x U_{\epsilon_j} + a \partial_x U_{\epsilon_j} - a \partial_x U, \partial_x \varphi \right\rangle \\ &= \left\langle U_{\epsilon_j} - U, \partial_x(a \partial_x \varphi) \right\rangle - \left\langle (a_{\epsilon_j} - a) \partial_x U_{\epsilon_j}, \partial_x \varphi \right\rangle. \end{aligned}$$

Since $\left\| \partial_x^l a \right\|_\infty \leq M, \forall l \leq k$, hence $\partial_x(a \partial_x \varphi) \in H^{1-k}$ and $\left\langle U_{\epsilon_j} - U, \partial_x(a \partial_x \varphi) \right\rangle \to 0$. Again, we have

$$\left\langle \left(a_{\epsilon_j} - a\right) \partial_x U_{\epsilon_j}, \partial_x \varphi \right\rangle \leq \left\| \partial_x U_{\epsilon_j} \right\|_{L^2} \left\| (a_{\epsilon_j} - a) \partial_x \varphi \right\|_{L^2} \to 0.$$

Indeed, by hypothesis, $\partial_x^l a \in L^\infty, \forall l \leq k$ hence $a \chi_{S(\varphi)} \in L^\infty \cap L^p, \forall p$. Therefore

$(a_{\epsilon_j} - a)\chi_{S(\varphi)} \to 0$ in $L^p, \forall 1 \leq p < \infty$, where $\chi_{S(\varphi)}$ is the characteristic function in $S(\varphi)$. Since $k \geq 2$, we have $\|\partial_x U_{\epsilon_j}\|_{L^2} \leq \|U_{\epsilon_j}\|_k \leq M$, $\forall \epsilon_j$, and the claim follows.

On the same way, we have $\langle B_{\epsilon_j} \partial_x U_{\epsilon_j}, \varphi\rangle \to \langle B\partial_x U, \varphi\rangle$, $\langle C_{\epsilon_j} \partial_t U_{\epsilon_j}, \varphi\rangle \to \langle C\partial_t U, \varphi\rangle$, and $\langle D_{\epsilon_j} U_{\epsilon_j}, \varphi\rangle \to \langle DU, \varphi\rangle$, $\forall \varphi \in C_c^\infty(G)$.

Hence $U \in H^k(G, \mathbb{R}^2)$ verifies $LU = f$ in $\mathcal{D}'(G, \mathbb{R}^2)$. Since $f_j \in L^2([0,T], H^k(\mathbb{R}))$, the equality holds in $L^2([0,T], H^{k-2}(\mathbb{R}))$.

STEP 3. The uniqueness of the solution for the Cauchy problem (2.1)–(2.2) follows from the estimates (3.1), which remains valid for the limit function U. □

7. On the Cauchy problem for weakly hyperbolic 2×2 systems of first order

First we prove Theorem 1.1:

Proof of Theorem 1.1. The proof will be done in two steps. In the first step we apply an operator (transpose of co-factor operator of principal part) to the left hand side of system (1.1) obtaining a system of the form (2.1). In the second step, we show that the theorem follows from Theorem 2.1.

STEP 1. Set

$$Q = I\partial_t + A\partial_x. \tag{7.1}$$

The matrix $A(x,t)$ enjoys a very good property, namely, $A^2(x,t) = a(x,t)I$. Hence we obtain a second order system

$$(Q \circ P)U = I\partial_t^2 U - \partial_x(aI\partial_x U) + B\partial_x U + C\partial_t U + DU = Qf, \tag{7.2}$$

where $C = A_1$, $D = QA_1$, and

$$B = \begin{pmatrix} -\partial_t a_{11} + a_{11}\partial_x a_{11} + a_{21}\partial_x a_{12} & -a_{11}\partial_x a_{12} - \partial_t a_{12} + a_{12}\partial_x a_{11} \\ -\partial_t a_{21} + a_{11}\partial_x a_{21} - a_{21}\partial_x a_{11} & \partial_t a_{11} + a_{11}\partial_x a_{11} + a_{12}\partial_x a_{21} \end{pmatrix} + AA_1.$$

With $\Psi = (\psi_1, \psi_2)$ and $\psi_j \in C_c^\infty(G)$, we consider the initial conditions

$$U(x,0) = \Phi(x), \quad (\partial_t U)(x,0) = \Psi(x), \tag{7.3}$$

for the problem 7.2.

STEP 2. By (1.7) the Cauchy problem (7.2)–(7.3) satisfies the hypotheses of Theorem 2.1. Hence, there is a unique solution $U \in H^k(G, \mathbb{R}^2)$ of (1.1)–(1.2). Indeed, if there are two distinct solutions $U_1, U_2 \in H^k(G, \mathbb{R}^2)$ of (1.1)–(1.2), then $U_1, U_2 \in H^k(G, \mathbb{R}^2)$ will be distinct solutions of (7.2)–(7.3), which is in contradiction with Theorem 2.1. □

Now we prove Theorem 1.2:

Proof of Theorem 1.2. Applying the operator Q to the right-hand side of system for P, we obtain the second order system

$$I\partial_t^2 U - \partial_x(aI\partial_x U) + \tilde{B}\partial_x U + A_1\partial_t U = f, \tag{7.4}$$

where

$$\tilde{B} = \begin{pmatrix} \partial_t a_{11} + a_{11}\partial_x a_{11} + a_{21}\partial_x a_{12} & -a_{11}\partial_x a_{12} + \partial_t a_{12} + a_{12}\partial_x a_{11} \\ \partial_t a_{21} + a_{11}\partial_x a_{21} - a_{21}\partial_x a_{11} & -\partial_t a_{11} + a_{11}\partial_x a_{11} + a_{12}\partial_x a_{21} \end{pmatrix} + A_1 A,$$

with the initial conditions

$$U(x,0) = 0, \quad (\partial_t U)(x,0) = \Phi(x). \tag{7.5}$$

From (1.8), by Theorem 2.1, there exists a unique solution $U \in H^k(G,\mathbb{R}^2)$ of the problem (7.4)–(7.5). Hence we have that $QU \in H^{k-1}(G,\mathbb{R}^2)$ is a solution of the Cauchy problem (1.1)–(1.2). □

As in Nishitani ([9]) we prove Theorem 1.3, where instead of Q given in (7.1), we take the operator

$$Q = I\partial_t + A\partial_x - A_x + ({}^{\text{co}}A_1)^t + E; \tag{7.6}$$

here $({}^{\text{co}}A_1)^t$ is the transpose of co-factor matrix of A_1 and $A_x = \partial_x A$. Hence

$$\begin{aligned} P \circ Q = I\partial_t^2 - aI\partial_x^2 + (A_t - AE + \operatorname{tr}(AA_1)I)\partial_x \\ + (A_1 + ({}^{\text{co}}A_1)^t + E - A_x)\partial_t + (P(E + ({}^{\text{co}}A_1)^t - A_x))I, \end{aligned}$$

since $A_1 A - A({}^{\text{co}}A_1)^t = A_1 A + ({}^{\text{co}}(A_1 A))^t = \operatorname{tr}(AA_1)I$. With $E = (e_{ij})$, the matrix $A_t - AE + \operatorname{tr}(AA_1)I$ takes the form

$$\begin{pmatrix} \partial_t a_{11} - a_{11}e_{11} - a_{12}e_{21} + \operatorname{tr}(AA_1) & \partial_t a_{12} - a_{11}e_{12} - a_{12}e_{22} \\ \partial_t a_{21} + a_{11}e_{21} - a_{21}e_{11} & -\partial_t a_{11} + a_{11}e_{22} - a_{21}e_{12} + \operatorname{tr}(AA_1) \end{pmatrix} \tag{7.7}$$

Now we determine e_{ij} so that (7.7) is a diagonal matrix. Since

$$a_{12}(x,t) \neq 0 \quad \text{and} \quad a_{21}(x,t) \neq 0 \quad \forall (x,t),$$

we take $e_{12} = 0$, $e_{21} = 0$ and

$$e_{11} = \frac{\partial_t a_{21}}{a_{21}}, \quad e_{22} = \frac{\partial_t a_{12}}{a_{12}}.$$

We set

$$Y(x,t) = a_{21}\partial_t a_{11} - a_{11}\partial_t a_{21} + a_{21}\,\mathrm{tr}(AA_1),$$
$$Z(x,t) = -a_{12}\partial_t a_{11} + a_{11}\partial_t a_{12} + a_{12}\,\mathrm{tr}(AA_1).$$

These choices are summarized in:

Lemma 7.1. *Let Q be given by* (7.6), *with E as above. Then*

$$P \circ Q = \partial_t^2 - aI\partial_x^2 + B\partial_x + R\partial_t + S$$

where

$$B = \mathrm{diag}\left(\frac{Y(t,x)}{a_{21}}, \frac{Z(t,x)}{a_{12}}\right), \quad R = E - A_x + A_1 + ({}^{\mathrm{co}}A_1)^t$$

and $S = P(E) + P(({}^{\mathrm{co}}A_1)^t - A_x)$.

We next obtain

Lemma 7.2. *Let*

$$\tilde{Q} = I\partial_t + A\partial_x + A_x + ({}^{\mathrm{co}}A_1)^t + \tilde{E},$$

with

$$\tilde{E} = -\,\mathrm{diag}\left(\frac{\partial_t a_{12}}{a_{12}}, \frac{\partial_t a_{21}}{a_{21}}\right).$$

Then

$$\tilde{Q} \circ P = \partial_t^2 - \partial_x(aI\partial_x) + \tilde{B}\partial_x + \tilde{R}\partial_t + \tilde{S},$$

with

$$\tilde{B} = \mathrm{diag}\left(\frac{Z(t,x)}{a_{12}}, \frac{Y(t,x)}{a_{21}}\right), \quad \tilde{R} = \tilde{E} + A_x + A_1 + ({}^{\mathrm{co}}A_1)^t, \quad \tilde{S} = \tilde{Q}(A_1).$$

Proof. Note that $a_x = A_x A + AA_x$ and $AA_1 - ({}^{\mathrm{co}}A_1)^t A = \mathrm{tr}(AA_1)I$. Then the proof follows from a computation similar to the one presented in the proof of Lemma 7.1. □

Summing up we have:

Proof of Theorem 1.3. By condition (1.11) and lemmas 7.1 and 7.2, the result follows from Theorem 2.1. □

8. On the Nishitani-Spagnolo's result

As in [10], we consider

$$A = A(x) \quad \text{and} \quad A_1 = A_1(x)$$

then we take $E = 0$ and so the condition (1.11) is written as

$$\alpha t[\mathrm{tr}(AA_1)]^2 \leq \theta a(x) \tag{8.1}$$

Suppose

$$A_1(x) = \begin{pmatrix} \delta_1 & \beta_1 \\ \beta_2 & \delta_2 \end{pmatrix}$$

with derivatives of all orders bounded on $\mathbb{R}$. The condition presented by Nishitani-Spagnolo (see [10]) for the Cauchy problem (1.1)–(1.2) to be well posed are:

$$(a_{12}a_{21})(x) \geq 0, \quad |(a_{12}\beta_2)(x)| \leq M\sqrt{(a_{12}a_{21})(x)}, \quad |(a_{21}\beta_1)(x)| \leq M\sqrt{(a_{12}a_{21})(x)} \tag{8.2}$$

From the following proposition and Example 8.1 we see that (8.2) is more restrictive than (8.1).

Proposition 8.1. (8.2) *implies* (8.1).

Proof. We have

$$\begin{aligned}\mathrm{tr}(AA_1)^2 &= (a_{11}(\delta_1 - \delta_2) + a_{12}\beta_2 + a_{21}\beta_1)^2 \\ &\leq \frac{3}{2}\left[\left\|(\delta_1 - \delta_2)^2\right\|_\infty a_{11}^2 + (a_{12}\beta_2)^2 + (a_{21}\beta_1)^2\right]\end{aligned}$$

But $a = a_{11}^2 + a_{12}a_{21}$ and $a_{12}(x)a_{21}(x) \geq 0$, then $a_{11}^2 \leq a_{11}^2 + a_{12}a_{21} = a$. Using the last two inequalities of (8.2) it follows that $\mathrm{tr}(AA_1) \leq Ma(x)$. □

EXAMPLE 8.1. Consider the function $b \in C^\infty$, $b \not\equiv 0$ and with bounded derivatives. Take

$$A(x) = \begin{pmatrix} b(x) & b(x) \\ -b(x) & -b(x) \end{pmatrix}.$$

then we have $a(x) = -\det A = 0$. For $\delta_1 + \beta_2 = \beta_1 + \delta_2$ we have that $A_1(x)$ satisfies $\mathrm{tr}(AA_1) = 0$, then (8.1) holds. However, the conditions (8.2) are not satisfied, because there exist x such that $a_{12}(x)a_{21}(x) = -b^2(x) < 0$.

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