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SECOND TERM OF ASYMPTOTICS FOR KdVB EQUATION WITH LARGE INITIAL DATA

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Abstract

We consider the Cauchy problem for the Korteweg-de Vries-Burgers equation. Global existence and smoothing effect for the Cauchy problem to the Korteweg-de Vries-Burgers equation was proved previously for the case of large initial data. Recently the first term of the large time asymptotics was obtained without restriction on the size of the initial data. The aim of the present paper is to obtain the second term of the large time asymptotic behavior of solutions to the Cauchy problem for the Korteweg-de Vries-Burgers equation in the case of the initial data of arbitrary size.

1. Introduction

We consider the Korteweg-de Vries-Burgers equation

$$u_t + uu_x - u_{xx} + u_{xxx} = 0, \quad x \in \mathbf{R}, \ t > 0 \tag{1.1}$$

(here the subscripts x, t denote the differentiation with respect to the spatial and time coordinates, respectively). Korteweg-de Vries-Burgers equation (1.1) is a nonlinear model taking into account the simplest dispersive and dissipative processes, thus it is applicable in many fields of Physics and Technology (see, e.g. [17], [20], and references cited therein). Global existence and smoothing effect for the Cauchy problem to the Korteweg-de Vries-Burgers equation was proved in [17], [3]: there exists a unique solution $u(t,x) \in \mathbf{C}^{\infty}((0,\infty); \mathbf{H}^{\infty}(\mathbf{R}))$ to the Cauchy problem for the Korteweg-de Vries-Burgers equation (1.1) with initial data $u_0 \in \mathbf{H}^s(\mathbf{R})$, $s > -1/2$. Here and below we denote the Sobolev space by $\mathbf{H}^s(\mathbf{R}) = \{\phi(x) \in \mathbf{L}^2(\mathbf{R})\colon \|\langle i\partial_x\rangle^s \phi(x)\|_{\mathbf{L}^2} < \infty\}$, $\langle x\rangle = \sqrt{1+x^2}$, $\mathbf{L}^p(\mathbf{R})$ is the usual Lebesgue space, $\mathbf{L}^{1,1}(\mathbf{R}) = \{\phi(x) \in \mathbf{L}^1(\mathbf{R})\colon \|\langle x\rangle\phi(x)\|_{\mathbf{L}^1} < \infty\}$ is the weighted Lebesgue space. We now mention some results on the large time asymptotic behavior of solutions to (1.1). Everywhere below we suppose that the total mass of the initial data $M = \int_{\mathbf{R}} u_0(x)\,dx \neq 0$. (For the case $M = 0$, see paper [3]). In paper [16] it was proved that for small initial data $u_0 \in \mathbf{L}^{1,1}(\mathbf{R}) \cap \mathbf{H}^7(\mathbf{R})$, the solutions

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of (1.1) have the asymptotics

$$u(t) = t^{-1/2} f_M((\cdot)t^{-1/2}) + O(t^{-(1/2)-\gamma}) \tag{1.2}$$

as $t \to \infty$, where $\gamma \in (0, 1/2)$ and

$$f_M(x) = -2\partial_x \log\left(\cosh\frac{M}{4} - \sinh\left(\frac{M}{4}\right)\mathrm{Erf}\left(\frac{x}{2}\right)\right)$$

is the self-similar solution for the Burgers equation [2]

$$u_t + uu_x - u_{xx} = 0, \quad x \in \mathbf{R}, \ t > 0, \tag{1.3}$$

defined by the total mass

$$M = \int_{\mathbf{R}} u_0(x)\,dx$$

of the initial data. Here

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\,dy$$

is the error function. The conditions on the initial data were generalized in paper [15], where it was proved that the solution of (1.1) with small initial data $u_0 \in \mathbf{L}^1(\mathbf{R}) \cap \mathbf{L}^2(\mathbf{R})$ have asymptotics

$$u(t) = t^{-1/2} f_M((\cdot)t^{-1/2}) + o(t^{-1/2}) \tag{1.4}$$

as $t \to \infty$. Recently in paper [13] it was proved that if the initial data $u_0 \in \mathbf{H}^s(\mathbf{R}) \cap \mathbf{L}^1(\mathbf{R})$, where $s > -1/2$, then there exists a unique solution $u(t,x) \in \mathbf{C}^\infty((0,\infty); \mathbf{H}^\infty(\mathbf{R}))$ to the Cauchy problem for the Korteweg-de Vries-Burgers equation (1.1), which have asymptotics (1.4). Moreover if additionally the initial data $u_0(x) \in \mathbf{L}^{1,1}(\mathbf{R})$, then the asymptotics (1.2) is true. In paper [18], it was proved that for small initial data $u_0 \in \mathbf{L}^{1,1}(\mathbf{R}) \cap \mathbf{H}^5(\mathbf{R})$ such that $u_0'' \in \mathbf{L}^1(\mathbf{R})$ the solutions of (1.1) have the following two terms of the large time asymptotics

$$u(t) = t^{-1/2} f_M((\cdot)t^{-1/2}) + \frac{\log t}{t} \widetilde{f}_M((\cdot)t^{-1/2}) + O\left(\frac{\sqrt{\log t}}{t}\right) \tag{1.5}$$

as $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$, where

$$\widetilde{f}_M(x) = -\frac{(f_M(x) - x/2)\,e^{-x^2/4}}{2\sqrt{\pi}\,H(x)} \int_{\mathbf{R}} H(y) f_M^3(y)\,dy$$

with

$$H(x) = \cosh \frac{M}{4} - \sinh \frac{M}{4} \operatorname{Erf}\left(\frac{x}{2}\right).$$

In the present paper we will extend this result for the case, when the initial data have an arbitrary size.

Some other results for dissipative equations with critical nonlinearities were shown in papers [1], [4], [5], [6], [7], [8], [10], [12], [21].

The aim of the present paper is to obtain the second term of the large time asymptotic behavior of solutions to the Cauchy problem for equation (1.1) in the case of the initial data of arbitrary size.

Theorem 1. *Let $u_0(x) \in \mathbf{H}^s(\mathbf{R}) \cap \mathbf{L}^{1,1}(\mathbf{R})$, where $s > -1/2$, and $M = \int_{\mathbf{R}} u_0(x)\,dx \neq 0$. Then the solution $u(t,x)$ to the Cauchy problem for the Korteweg-de Vries-Burgers equation* (1.1) *with the initial condition $u_0(x)$ has asymptotics* (1.5) *as $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$.*

REMARK 1. In the case of zero total mass $M = 0$ the main term of the asymptotics is the same as that for the linear heat equation, and the second term of the asymptotics can be found by a similar approach.

REMARK 2. The term $\sqrt{\log t}$ in the estimate of the remainder in formula (1.5) comes from estimate (2.18) of Lemma 3 below. It could be removed by a more delicate consideration.

REMARK 3. We believe that the third term of the asymptotics can be found by a similar method.

Below we denote the Sobolev spaces

$$\mathbf{H}^k(\mathbf{R}) = \{\phi \in \mathbf{L}^2(\mathbf{R});\ \|\langle i\partial_x\rangle^k \phi\|_{\mathbf{L}^2} < \infty\},$$

where $\langle x\rangle = \sqrt{1+x^2}$, the usual Lebesgue space is $\mathbf{L}^p(\mathbf{R})$, $1 \le p \le \infty$ and

$$\mathbf{L}^{1,1}(\mathbf{R}) = \{\phi(x) \in \mathbf{L}^1(\mathbf{R}) \colon \|\langle x\rangle \phi(x)\|_{\mathbf{L}^1} < \infty\}$$

is the weighted Lebesgue space.

2. Proof of Theorem 1

Recently in paper [13] it was proved that if the initial data $u_0 \in \mathbf{H}^s(\mathbf{R}) \cap \mathbf{L}^1(\mathbf{R})$, where $s > -1/2$, then there exists a unique solution $u(t,x) \in \mathbf{C}^\infty((0,\infty); \mathbf{H}^\infty(\mathbf{R}))$ to the Cauchy problem for the Korteweg-de Vries-Burgers equation (1.1), which have the following optimal time decay estimates

$$\left\|\partial_x^k u(t)\right\|_{\mathbf{L}^p} \le C\,\langle t\rangle^{-k/2-(1/2)(1-1/p)}, \tag{2.1}$$

for $t > 0$, where $1 \le p \le \infty$, $k = 0, 1, 2, 3$. From now on C denotes various positive constants.

Denote the Green operator

$$\mathcal{G}(t-\tau)\phi(\tau) = \int_{\mathbf{R}} G(t-\tau, x-y)\phi(\tau, y)\,dy,$$

where the Green function

$$G(t,x) = (2\pi)^{-1/2}\int_{\mathbf{R}} e^{ix\xi - t\xi^2 + it\xi^3}\,d\xi.$$

We write the following integral equation associated with the Cauchy problem for the Korteweg-de Vries-Burgers equation (1.1)

$$u(t,x) = \mathcal{G}(t)u_0 - \frac{1}{2}\int_0^t \partial_x \mathcal{G}(t-\tau)u^2(\tau)\,d\tau. \tag{2.2}$$

We use the following notation $\langle t\rangle = \sqrt{1+t^2}$, $\{t\} = t/\langle t\rangle$. First we give estimates for the Green operator $\mathcal{G}$ (see [13]).

Lemma 1. *The estimates are true*

$$\left\|x^n \partial_x^k G(t)\right\|_{\mathbf{L}^p} \le C\{t\}^{-k/2-(1/2)(1-1/p)}\langle t\rangle^{(n-k)/2-(1/2)(1-1/p)}$$

for all $t > 0$, $k, n \ge 0$, $2 \le p \le \infty$ *and*

$$\left\|x^n \partial_x^k G(t)\right\|_{\mathbf{L}^p} \le C\{t\}^{-k/2-1/4}\langle t\rangle^{(n-k)/2-(1/2)(1-1/p)}$$

for all $t > 0$, $k, n \ge 0$, $1 \le p \le 2$.

Proof. We have by the Plancherel theorem

$$\begin{aligned}\left\|x^n \partial_x^k G(t)\right\|_{\mathbf{L}^p} &\le C\left\|\partial_\xi^n\left(\xi^k e^{-t\xi^2+it\xi^3}\right)\right\|_{\mathbf{L}^{p/(p-1)}} \le C\langle t\rangle^{n/2}\left\||\xi|^k e^{-Ct\xi^2}\right\|_{\mathbf{L}^{p/(p-1)}} \\ &\le C\{t\}^{-k/2-(1/2)(1-1/p)}\langle t\rangle^{(n-k)/2-(1/2)(1-1/p)}\end{aligned}$$

for all $t > 0$, $k, n \ge 0$, $2 \le p \le \infty$. Therefore by the Cauchy inequality we get

$$\left\|x^n \partial_x^k G(t)\right\|_{\mathbf{L}^1} = \int_{|x|\le\sqrt{\langle t\rangle}} \left|x^n \partial_x^k G(t,x)\right| dx + \int_{|x|\ge\sqrt{\langle t\rangle}} |x|^{-1}\left|x^{n+1}\partial_x^k G(t,x)\right| dx$$

$$\le C\langle t\rangle^{1/4}\left\|x^n\partial_x^k G(t)\right\|_{\mathbf{L}^2}+C\langle t\rangle^{-1/4}\left\|x^{n+1}\partial_x^k G(t)\right\|_{\mathbf{L}^2}$$
$$\le C\{t\}^{-k/2-1/4}\langle t\rangle^{(n-k)/2}.$$

The second estimate of the lemma follows now by the Hölder inequality. Lemma 1 is proved. □

Now let us prove the estimate of the $\mathbf{L}^{1,1}$-norm of solutions of the Cauchy problem (1.1)

$$\|u(t)\|_{\mathbf{L}^{1,1}}\le C\langle t\rangle^{1/2}. \tag{2.3}$$

We multiply equation (1.1) by $|x|\,\mathrm{sign}(u(t,x))$ and integrate with respect to x over $\mathbf{R}$ to get

$$\int_{\mathbf{R}} u_t(t,x)\,|x|\,\mathrm{sign}(u(t,x))\,dx+\int_{\mathbf{R}}|x|\,u(t,x)u_x(t,x)\,\mathrm{sign}(u(t,x))\,dx$$
$$=\int_{\mathbf{R}} u_{xx}(t,x)\,|x|\,\mathrm{sign}(u(t,x))\,dx-\int_{\mathbf{R}} u_{xxx}(t,x)\,|x|\,\mathrm{sign}(u(t,x))\,dx.$$

We have

$$\int_{\mathbf{R}} u_t(t,x)\,|x|\,\mathrm{sign}(u(t,x))\,dx=\int_{\mathbf{R}}\frac{\partial}{\partial t}\,|u(t,x)|\,|x|\;dx=\frac{d}{dt}\,\|u(t)\|_{\mathbf{L}^{1,1}},$$
$$\int_{\mathbf{R}} u(t,x)u_x(t,x)\,|x|\,\mathrm{sign}(u(t,x))\,dx=\int_{\mathbf{R}}|x|\,\frac{\partial}{\partial x}(|u(t,x)|\,u(t,x))\,dx$$
$$=\int_{\mathbf{R}}\mathrm{sign}(x)\,|u(t,x)|\,u(t,x)\,dx$$
$$\le\|u(t)\|^2_{\mathbf{L}^2}\le C\langle t\rangle^{-1/2},$$

and

$$\int_{\mathbf{R}} u_{xx}(t,x)\,|x|\,\mathrm{sign}(u(t,x))\,dx = -2\sum_{u(t,\chi_i)=0}|x|\,|u_x(t,\chi_i)|$$
$$-\int_{\mathbf{R}}\mathrm{sign}(x)\frac{\partial}{\partial x}\,|u(t,x)|\;dx$$
$$= -2\sum_{u(t,\chi_i)=0}|x|\,|u_x(t,\chi_i)|+2\,|u(t,0)|$$
$$\le 2\,\|u(t)\|_{\mathbf{L}^\infty}\le C\langle t\rangle^{-1/2}.$$

Therefore we get

$$\frac{d}{dt}\,\|u(t)\|_{\mathbf{L}^{1,1}}\le C\langle t\rangle^{-1/2}-\int_{\mathbf{R}} u_{xxx}(t,x)\,|x|\,\mathrm{sign}(u(t,x))\,dx \tag{2.4}$$

$$\leq C\langle t\rangle^{-1/2} + \|u_{xxx}(t)\|_{\mathbf{L}^{1,1}}.$$

Next we give estimates for the norm $\|u(t)\|_{\mathbf{L}^{2,1}} \equiv \|\langle x\rangle u(t,x)\|_{\mathbf{L}^2}$.

Multiplying equation (1.1) by $2x^2u$ and integrating with respect to $x \in \mathbf{R}$ we get

$$\partial_t \int_{\mathbf{R}} x^2u^2\,dx + \frac{2}{3}\int_{\mathbf{R}} x^2\partial_x u^3\,dx - 2\int_{\mathbf{R}} x^2uu_{xx}\,dx + 2\int_{\mathbf{R}} x^2uu_{xxx}\,dx = 0. \tag{2.5}$$

Since

$$\begin{aligned}\int_{\mathbf{R}} x^2\partial_x u^3\,dx &= -2\int_{\mathbf{R}} xu^3\,dx \leq C\,\|u(t)\|_{\mathbf{L}^{2,1}}\,\|u(t)\|_{\mathbf{L}^2}\,\|u(t)\|_{\mathbf{L}^\infty}\\ &\leq C\langle t\rangle^{-3/4}\,\|u(t)\|_{\mathbf{L}^{2,1}},\\ \int_{\mathbf{R}} x^2uu_{xx}\,dx &= -\int_{\mathbf{R}} x^2(u_x)^2\,dx + \int_{\mathbf{R}} u^2\,dx\\ &= -\|u_x(t)\|^2_{\mathbf{L}^{2,1}} + \|u(t)\|^2_{\mathbf{L}^2}\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbf{R}} x^2uu_{xxx}\,dx &= -\int_{\mathbf{R}} x^2u_xu_{xx}\,dx - 2\int_{\mathbf{R}} xuu_{xx}\,dx\\ &= 3\int_{\mathbf{R}} x(u_x)^2\,dx \leq 3\,\|u_x(t)\|_{\mathbf{L}^{2,1}}\,\|u_x(t)\|_{\mathbf{L}^2}\\ &\leq C\langle t\rangle^{-3/4}\,\|u_x(t)\|_{\mathbf{L}^{2,1}}\end{aligned}$$

we get

$$\begin{aligned}\frac{d}{dt}\,\|u(t)\|^2_{\mathbf{L}^{2,1}} &\leq C\langle t\rangle^{-3/4}\,\|u(t)\|_{\mathbf{L}^{2,1}} + C\langle t\rangle^{-1/2}\\ &\quad + C\langle t\rangle^{-3/4}\,\|u_x(t)\|_{\mathbf{L}^{2,1}} - 2\,\|u_x(t)\|^2_{\mathbf{L}^{2,1}}\\ &\leq C\langle t\rangle^{-3/4}\,\|u(t)\|_{\mathbf{L}^{2,1}} + C\langle t\rangle^{-1/2},\end{aligned}$$

hence integrating we see that

$$\|u(t)\|_{\mathbf{L}^{2,1}} \leq C\langle t\rangle^{1/4}. \tag{2.6}$$

In the next lemma we obtain the estimates for the norm $\|u_{xxx}(t)\|_{\mathbf{L}^{1,1}}$. Denote $\mathbf{W}^3_1(\mathbf{R}) = \{\phi \in \mathbf{L}^1(\mathbf{R});\ \|\langle i\partial_x\rangle^3\phi\|_{\mathbf{L}^1} < \infty\}$.

Lemma 2. *Let the initial data* $u_0 \in \mathbf{H}^2(\mathbf{R}) \cap \mathbf{W}^3_1(\mathbf{R})$ *and estimate* (2.1) *be valid. Then the estimate is true*

$$\|u_{xxx}(t)\|_{\mathbf{L}^{1,1}} \leq C\langle t\rangle^{-1} \tag{2.7}$$

for all $t > 0$.

Proof. First we need to estimate the norm $\|u_x(t)\|_{\mathbf{L}^{1,1}}$. By the integral equation (2.2) we have

$$\begin{aligned}
\|u_x(t)\|_{\mathbf{L}^{1,1}} &\leq \|\partial_x \mathcal{G}(t)u_0\|_{\mathbf{L}^{1,1}} \\
&+ \int_0^{t/2} \Big(\left\|\partial_x^2 G(t-\tau)\right\|_{\mathbf{L}^{1,1}} \|u(\tau)\|_{\mathbf{L}^2}^2 \\
&\qquad + \left\|\partial_x^2 G(t-\tau)\right\|_{\mathbf{L}^1} \|u(\tau)\|_{\mathbf{L}^2} \|u(\tau)\|_{\mathbf{L}^{2,1}} \Big)\, d\tau \\
&+ \int_{t/2}^{t} (\|\partial_x G(t-\tau)\|_{\mathbf{L}^{1,1}} \|u(\tau)\|_{\mathbf{L}^2} \|u_x(\tau)\|_{\mathbf{L}^2} \\
&\qquad + \|\partial_x G(t-\tau)\|_{\mathbf{L}^1} \|u(\tau)\|_{\mathbf{L}^{2,1}} \|u_x(\tau)\|_{\mathbf{L}^2})\, d\tau
\end{aligned}$$

whence by estimate

$$\left\|\partial_x^k G(t)\right\|_{\mathbf{L}^{1,1}} \leq C\{t\}^{-1/4-k/2}\langle t\rangle^{1-k/2}$$

we get

$$\begin{aligned}
\|u_x(t)\|_{\mathbf{L}^{1,1}} &\leq C + C\int_0^{t/2} \left(\langle t-\tau\rangle^{-1/2}\langle\tau\rangle^{-1/2} + \langle t-\tau\rangle^{-1}\right) d\tau \\
&\quad + \int_{t/2}^{t} \{t-\tau\}^{-3/4} \left(\langle\tau\rangle^{-1} + \langle t-\tau\rangle^{-1/2}\,\langle\tau\rangle^{-1/2}\right) d\tau \leq C.
\end{aligned}$$

In the same manner by the integral equation (2.2) we have

$$\begin{aligned}
\|u_{xxx}(t)\|_{\mathbf{L}^{1,1}} &\leq \left\|\partial_x^3 \mathcal{G}(t)u_0\right\|_{\mathbf{L}^{1,1}} \\
&+ C\int_0^{t/2} \left(\|\partial_x^4 G(t-\tau)\|_{\mathbf{L}^{1,1}} \|u(\tau)\|_{\mathbf{L}^2}^2 + \|\partial_x^4 G(t-\tau)\|_{\mathbf{L}^1} \|u(\tau)\|_{\mathbf{L}^2} \|u(\tau)\|_{\mathbf{L}^{2,1}}\right) d\tau \\
&+ C\int_{t/2}^{t} (\|\partial_x G(t-\tau)\|_{\mathbf{L}^{1,1}} (\|u(\tau)\|_{\mathbf{L}^2} \|u_{xxx}(\tau)\|_{\mathbf{L}^2} + \|u_x(\tau)\|_{\mathbf{L}^2} \|u_{xx}(\tau)\|_{\mathbf{L}^2}) \\
&\qquad + \|\partial_x G(t-\tau)\|_{\mathbf{L}^1} (\|u(\tau)\|_{\mathbf{L}^{2,1}} \|u_{xxx}(\tau)\|_{\mathbf{L}^2} + \|u_x(\tau)\|_{\mathbf{L}^{1,1}} \|u_{xx}(\tau)\|_{\mathbf{L}^\infty}))\, d\tau,
\end{aligned}$$

hence we obtain

$$\begin{aligned}
\|u_{xxx}(t)\|_{\mathbf{L}^{1,1}} &\leq C\langle t\rangle^{-1} + C\int_0^{t/2} \left(\langle t-\tau\rangle^{-3/2}\,\langle\tau\rangle^{-1/2} + \langle t-\tau\rangle^{-2}\right) d\tau \\
&\quad + \int_{t/2}^{t} \{t-\tau\}^{-3/4} \left(\langle\tau\rangle^{-2} + \langle t-\tau\rangle^{-1/2}\,\langle\tau\rangle^{-3/2}\right) d\tau \leq C\langle t\rangle^{-1}.
\end{aligned}$$

Thus the estimate of the lemma is true. Lemma 2 is proved. □

Integration of inequality (2.4) yields

$$\|u(t)\|_{\mathbf{L}^{1,1}} \leq \|u_0\|_{\mathbf{L}^{1,1}} + C\langle t\rangle^{1/2} \leq C\langle t\rangle^{1/2}.$$

Therefore estimate (2.3) is true for all $t \geq 0$.

Now we obtain the second term of the large time asymptotics as $t \to \infty$ of solutions $u(t,x)$ to the Cauchy problem for the Korteweg-de Vries-Burgers equation (1.1). We take the initial time $T > 0$ to be sufficiently large and define $v(t,x)$ as a solution to the Cauchy problem for the Burgers equation with $u(t,x)$ as the initial data

$$\begin{cases} v_t + vv_x - v_{xx} = 0, & t > T, \; x \in \mathbf{R}, \\ v(T,x) = u(T,x), & x \in \mathbf{R}. \end{cases} \tag{2.8}$$

By the Hopf-Cole [14] transformation $v(t,x) = -2(\partial/\partial x)\log Z(t,x)$ equation (2.8) is converted to the heat equation $Z_t = Z_{xx}$. Therefore we obtain

$$Z(t,x) = \int_{\mathbf{R}} dy G_0(t, x-y) \exp\left(-\frac{1}{2}\int_{-\infty}^{y} u(T,\xi)\, d\xi\right), \tag{2.9}$$

where $G_0(t,x) = (4\pi t)^{-1/2} e^{-x^2/4t}$ is the Green function for the heat equation. Note that the following estimates are true

$$\|v(t)\|_{\mathbf{L}^p} \leq C\langle t\rangle^{-k/2-(1/2)(1-1/p)} \tag{2.10}$$

for all $t > T$, $1 \leq p \leq \infty$, $k = 0, 1, 2$.

Consider now the difference $w(t,x) = u(t,x) - v(t,x)$ for $t > T$. By (1.1) and (2.8) we get the Cauchy problem

$$\begin{cases} w_t + \dfrac{\partial}{\partial x}(vw) + \dfrac{1}{2}\dfrac{\partial}{\partial x} w^2 - w_{xx} + w_{xxx} + v_{xxx} = 0, & t > T, \; x \in \mathbf{R}, \\ w(T,x) = 0, & x \in \mathbf{R}. \end{cases} \tag{2.11}$$

We have the estimates (see [13])

$$\|Z(t)\|_{\mathbf{L}^\infty} + \|Z^{-1}(t)\|_{\mathbf{L}^\infty} \leq C \tag{2.12}$$

for all $t \geq T$ and

$$\|w(t)\|_{\mathbf{L}^p} \leq C\,\langle t\rangle^{-k/2-(1/2)(1-1/p)-\gamma} \tag{2.13}$$

for all $t \geq T$, $2 \leq p \leq \infty$, $k = 0, 1, 2$, where $\gamma \in (0, 1/2)$.

Following the heuristic considerations of paper [18] we easily see that the main term of the asymptotic expansion of $w(t,x)$ as $t \to \infty$ is determined by the linear

Cauchy problem

$$\begin{cases} \varphi_t + \dfrac{\partial}{\partial x}(\varphi v) - \varphi_{xx} + v_{xxx} = 0, & t > T, \ x \in \mathbf{R}, \\ \varphi(T,x) = 0, & x \in \mathbf{R}. \end{cases} \tag{2.14}$$

To eliminate the second term from (2.14), let us integrate (2.14) with respect to x and make the substitution

$$\int_{-\infty}^{x} \varphi(t,y)\,dy = \frac{s(t,x)}{Z(t,x)},$$

where $Z(t,x)$ is defined by (2.9). We obtain

$$\begin{cases} s_t - s_{xx} + Z v_{xx} = 0, & t > T, \ x \in \mathbf{R}, \\ s(T,x) = 0, & x \in \mathbf{R}. \end{cases} \tag{2.15}$$

It is easy to integrate (2.15) to get

$$s(t,x) = -\int_T^t \mathcal{G}_0(t-\tau) Z(\tau) v_{xx}(\tau)\,d\tau. \tag{2.16}$$

In the following lemma we evaluate the large time asymptotics of the solution $\varphi(t)$ of linear problem (2.14)

$$\varphi(t) = \partial_x\left(\frac{s(t)}{Z(t)}\right) = -Z^{-1}(t)\int_T^t (\partial_x \mathcal{G}_0(t-\tau) + 2v(t)\mathcal{G}_0(t-\tau)) Z(\tau) v_{xx}(\tau)\,d\tau. \tag{2.17}$$

Lemma 3. *Let $u(T,x) \in \mathbf{H}^2(\mathbf{R}) \cap \mathbf{L}^{1,1}(\mathbf{R})$. Then the asymptotics*

$$\varphi(t) = t^{-1}\widetilde{f}_M(\chi)\log t + O\left(t^{-1}\sqrt{\log t}\right) \tag{2.18}$$

is valid as $t \to \infty$ uniformly with respect to $\chi = x/\sqrt{t} \in \mathbf{R}$, where

$$\widetilde{f}_M(x) = -\frac{1}{4\sqrt{\pi}H(x)} e^{-x^2/4}\left(f_M(x) - \frac{x}{2}\right)\int_{\mathbf{R}} H(y) f_M^3(y)\,dy,$$
$$f_M(x) = -2\partial_x \log H(x),$$
$$H(x) = \cosh\frac{M}{4} - \sinh\frac{M}{4}\,\mathrm{Erf}\left(\frac{x}{2}\right).$$

Proof. Let us represent the integral with respect to τ in (2.17) as the sum of three parts ($t > T + e$)

$$\int_T^t d\tau = \int_T^{T+1} + \int_{t/\log t}^{t} + \int_{T+1}^{t/\log t} \equiv I_1 + I_2 + I_3. \tag{2.19}$$

For all $x \in \mathbf{R}$ and $t > T$ we have

$$0 < C_1 < Z(t, x) < C_2,$$

and for each $t > 0$ the following inequalities hold:

$$\begin{aligned} &\left\| \partial_x^l Z(t, \cdot) \right\|_{\mathbf{L}^p} \le C \langle t \rangle^{-l/2+1/(2p)}, \\ &\left\| \partial_x^l G_0(t, \cdot) \right\|_{\mathbf{L}^p} \le C t^{-l/2+1/(2p)-1/2}, \\ &\left\| \partial_x^l v(t, \cdot) \right\|_{\mathbf{L}^p} \le C \langle t \rangle^{-l/2-1/2+1/(2p)} \end{aligned} \tag{2.20}$$

for all $l = 1, 2, 3,\ 1 \le p \le \infty$. By using these inequalities, we readily estimate the first two integrals in representation (2.19)

$$|I_1| \le \frac{C}{t} \int_T^{T+1} \|v_{xx}(\tau)\|_{\mathbf{L}^1}\, d\tau = O(t^{-1}) \tag{2.21}$$

as $t \to \infty$, and

$$\begin{aligned} |I_2| &\le \int_{t/\log t}^{t} d\tau\, \|v_{xx}(\tau)\|_{\mathbf{L}^\infty} \left(\|\partial_x G_0(t-\tau)\|_{\mathbf{L}^1} + \|v(\tau)\|_{\mathbf{L}^\infty} \|G_0(t-\tau)\|_{\mathbf{L}^1} \right) \\ &\le C \int_{t/\log t}^{t} \tau^{-3/2} \left((t-\tau)^{-1/2} + t^{-1/2} \right) d\tau = O\left(t^{-1} \sqrt{\log t} \right), \end{aligned} \tag{2.22}$$

since changing variables of integration $\tau = t - y^2$ we have

$$\begin{aligned} \int_{t/\log t}^{t} \tau^{-3/2} (t-\tau)^{-1/2}\, d\tau &= 2 \int_0^{\sqrt{t - t/\log t}} \left(\sqrt{t} - y \right)^{-3/2} \left(\sqrt{t} + y \right)^{-3/2} dy \\ &\le C t^{-3/4} \int_0^{\sqrt{t - t/\log t}} \left(\sqrt{t} - y \right)^{-3/2} dy \\ &= C t^{-1} \left(1 - \sqrt{1 - \frac{1}{\log t}} \right)^{-1/2} = O\left(t^{-1} \sqrt{\log t} \right) \end{aligned}$$

as $t \to \infty$. In the third integral I_3 we integrate by parts with respect to y to obtain

$$\begin{aligned} I_3 \; = \; & \int_{T+1}^{t/\log t} d\tau \int_0^\infty dy \Lambda_y(x, y, t, \tau) \int_y^\infty F(q, \tau)\, dq \\ & - \int_{T+1}^{t/\log t} d\tau \int_{-\infty}^0 dy \Lambda_y(x, y, t, \tau) \int_{-\infty}^y F(q, \tau)\, dq \\ & + \int_{T+1}^{t/\log t} d\tau \Lambda(x, 0, t, \tau) \int_{\mathbf{R}} F(\tau, y)\, dy \\ \equiv \; & I_4 + I_5 + I_6, \end{aligned}$$

where $F(\tau, y) = -Z(\tau, y)v_{yy}(\tau, y)$ and

$$\Lambda(x, y, t, \tau) = Z^{-1}(t, x)(\partial_x G_0(t-\tau, x-y) + v(t, x)G_0(t-\tau, x-y)).$$

Since

$$\sup_{T+1\le\tau\le t/\log t}\sup_{x\in\mathbf{R}}\sup_{y\in\mathbf{R}}|\Lambda_y(x, y, t, \tau)| \le Ct^{-3/2},$$
$$\left\| x\partial_x^l Z(t, x)\right\|_{\mathbf{L}^1(\mathbf{R})} \le C\langle t\rangle^{1-l/2}, \qquad l = 1, 2, 3,$$

and therefore

$$\|xv_{xx}(t)\|_{\mathbf{L}^1(\mathbf{R})} \le C\langle t\rangle^{-1/2},$$

we obtain

$$\begin{aligned}|I_4| &\le \int_{T+1}^{t/\log t} d\tau \int_0^\infty dy\left|\Lambda_y(x, y, t, \tau)\right| \int_y^\infty Z(q, \tau)\left|v_{qq}(q, \tau)\right| dq\\ &\le Ct^{-3/2}\int_{T+1}^{t/\log t} d\tau\int_0^\infty dy\int_y^\infty \left|v_{qq}(q, \tau)\right| dq\\ &\le Ct^{-3/2}\int_{T+1}^{t/\log t} d\tau \int_0^\infty q\left|v_{qq}(q, \tau)\right| dq\\ &\le Ct^{-3/2}\int_{T+1}^{t/\log t}\frac{d\tau}{\sqrt{\tau}} \le Ct^{-3/2}\sqrt{\frac{t}{\log t}} = O(t^{-1}).\end{aligned} \tag{2.23}$$

The integral I_5 can be estimated similarly. Since

$$\partial_x^l G_0(t-\tau, x) = \partial_x^l G_0(t, x) + O(t^{-1/2-l/2}\log^{-1} t), \qquad l = 0, 1,$$

for $T + 1 \le \tau \le t/\log t$, we derive the estimate

$$I_6 = \frac{1}{H(\chi)}\left(\partial_x G_0(t, x) + \frac{f_M(\chi)}{\sqrt{t}}G_0(t, x)\right)\int_{T+1}^{t/\log t} d\tau\int_{\mathbf{R}} F(\tau, y)\,dy + O(t^{-1}), \tag{2.24}$$

from the estimate

$$\partial_x^l Z(t, x) = t^{-1/2}\left(\frac{d^l H(\chi)}{d\chi^l} + O(t^{-1/2})\right), \qquad l = 0, 1, 2, 3. \tag{2.25}$$

Since

$$\partial_x Z = -2Zv,$$

integration by parts yields, by virtue of (2.25),

$$\int_{\mathbf{R}} F(\tau, y)\,dy = -\frac{1}{2}\int_{\mathbf{R}} v^3(\tau, y)Z(\tau, y)\,dy$$

$$= -\frac{1}{2\tau}\int_{\mathbf{R}} f_M^3(y)H(y)\,dy + O(\tau^{-3/2}).$$

Then from (2.21)–(2.24) we obtain (2.18). Lemma 3 is proved.

It follows from (2.11) and (2.14) that the remainder $\psi(t,x) = w(t,x) - \varphi(t,x)$ is the solution to the Cauchy problem

$$\begin{cases} \psi_t + \dfrac{\partial}{\partial x}(v\psi) - \psi_{xx} + \dfrac{1}{2}\dfrac{\partial}{\partial x}w^2 + w_{xxx} = 0, & t > T,\ x \in \mathbf{R}, \\ \psi(T,x) = 0, & x \in \mathbf{R}. \end{cases} \tag{2.26}$$

To eliminate the second term from (2.26) as above we integrate this equation with respect to x and introduce the new unknown function

$$r(t,x) = Z(t,x)\int_{-\infty}^{x} \psi(t,y)\,dy.$$

Then we obtain

$$\begin{cases} r_t - r_{xx} + F = 0, & t > T,\ x \in \mathbf{R}, \\ r(T,x) = 0, & x \in \mathbf{R}, \end{cases} \tag{2.27}$$

where

$$F = \frac{1}{2}Zw^2 + Zw_{xx}.$$

In view of (2.13) and (2.12) we find

$$\begin{aligned} \|F(t)\|_{\mathbf{L}^p} &\le C\,\|w\|_{\mathbf{L}^\infty}\,\|w\|_{\mathbf{L}^p} + C\,\|w_{xx}\|_{\mathbf{L}^p} \\ &\le C\langle t\rangle^{-1-2\gamma+1/(2p)} + C\langle t\rangle^{-3/2-\gamma+1/(2p)} \le C\langle t\rangle^{-1-2\gamma+1/(2p)} \end{aligned} \tag{2.28}$$

for all $t \ge T$, $1 \le p \le \infty$. Using the integral equation associated with (2.27) we obtain

$$\begin{aligned} \|r(t)\|_{\mathbf{L}^p} \le &\int_T^{(t+T)/2} d\tau\,\|G(t-\tau)\|_{\mathbf{L}^p}\,\|F(\tau)\|_{\mathbf{L}^1} \\ &+ \int_{(t+T)/2}^{t} d\tau\,\|G(t-\tau)\|_{\mathbf{L}^1}\,\|F(\tau)\|_{\mathbf{L}^p}, \end{aligned}$$

hence in view of estimate (2.28) we find

$$\begin{aligned} \|r(t)\|_{\mathbf{L}^p} \le\ & C\int_T^{(t+T)/2} (t-\tau)^{-1/2+1/(2p)}\langle\tau\rangle^{-1/2-2\gamma}\,d\tau \\ &+ C\int_{(t+T)/2}^{t} \{t-\tau\}^{-3/4}\langle\tau\rangle^{-1-2\gamma+1/(2p)}\,d\tau \le C\langle t\rangle^{-1/2+1/(2p)} \end{aligned}$$

for all $t \geq T$, $1 \leq p \leq \infty$, if we take $\gamma \in (1/4, 1/2)$. In the same manner we have

$$\begin{aligned}\|r_x(t)\|_{\mathbf{L}^p} &\leq \int_T^{(t+T)/2} d\tau \, \|\partial_x G(t-\tau)\|_{\mathbf{L}^p} \, \|F(\tau)\|_{\mathbf{L}^1} \\ &\quad + \int_{(t+T)/2}^t d\tau \, \|\partial_x G(t-\tau)\|_{\mathbf{L}^1} \, \|F(\tau)\|_{\mathbf{L}^p} \\ &\leq C \int_T^{(t+T)/2} (t-\tau)^{-1+1/(2p)} \langle \tau \rangle^{-1/2-2\gamma} d\tau \\ &\quad + C \int_{(t+T)/2}^t \{t-\tau\}^{-3/4} (t-\tau)^{-1/2} \langle \tau \rangle^{-1-2\gamma+1/(2p)} \, d\tau \leq C \langle t \rangle^{-1+1/(2p)}\end{aligned}$$

for all $t \geq T$, $1 \leq p \leq \infty$. Using the identity

$$\psi = Z^{-1} \left(r_x + \frac{1}{2} r v \right)$$

we obtain the estimate

$$\|u(t) - v(t) - \varphi(t)\|_{\mathbf{L}^\infty} \leq C \langle t \rangle^{-1}$$

for all $t \geq T$. When $u_0 \in \mathbf{L}^{1,1}$, then

$$v(t,x) = t^{-1/2} f_M(xt^{-1/2}) + O(t^{-1})$$

for $t \to \infty$ and in view of Lemma 3, the asymptotics of the theorem follows. Theorem 1 is proved. □

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