Favini, A., Lorenzi, A., Tanabe, H. and Yagi, A.
Osaka J. Math.
42 (2005), 385–406

AN L^p-APPROACH TO SINGULAR LINEAR PARABOLIC EQUATIONS IN BOUNDED DOMAINS

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(Received August 19, 2002)

Abstract

Singular means here that the parabolic equation is *not* in normal form neither can it be reduced to such a form. For this class of problems, following the operator approach used in [1], we prove global in time existence and uniqueness theorems related to (spatial) L^p-spaces. Various improvements to [2], [3] are given.

1. Introduction

In this paper we will consider the following boundary value problem

$$D_t[m(x)u(x,t)] + A(x, D_x)u(x,t) = f(x,t), \qquad \forall (x,t) \in \Omega \times [0,\tau], \tag{1.1}$$

$$u(x,t) = 0, \qquad \forall (x,t) \in \partial\Omega \times [0,\tau], \tag{1.2}$$

$$m(x)u(x,t) \to m(x)u_0(x), \qquad \text{for a.e. } x \in \Omega, \ \text{ as } t \to 0+, \tag{1.3}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain with a boundary of class C^2, while $A(x, D_x)$ is the following second-order uniformly elliptic operator in *divergence form*

$$A(x, D) = -\sum_{i,j=1}^{n} D_{x_i}[a_{i,j}(x)D_{x_j}] + a_0(x). \tag{1.4}$$

Moreover, $0 \not\equiv m \in L^\infty(\Omega)$ is a non-negative function which need not to be bounded away from 0. Consequently, our parabolic equation is, in general, *singular*.

Particular cases of (1.1) are discussed in the monograph [3], pp.74–80. See also [2]. Note that in [3], p.80, the restriction $p \in (2, +\infty)$ should be made.

Using the theoretical results in [3] and the fundamental approach in [4] we can develop an L^p-theory, $p \in (1, +\infty)$, also in the present degenerate case*. The keystone in order to apply the results in [1] and [3], Theorem 3.28, p.69, to (1.1)–(1.4)

2000 *Mathematics Subject Classification* : Primary: 35K20, Secondary: 47D03.

Work partially supported by the Italian Ministero dell'Istruzione, dell'Università e della Ricerca (progetto FIRB 2001, *Analisi di equazioni a derivate parziali, lineari e non lineari*: *aspetti metodologici*, *modellistica*, *applicazioni*) and by University of Bologna Funds for selected research topics.

*We note that in this case the initial condition (1.3) should be more correctly meant as the following L^p-limit: $\|m(\cdot)u(\cdot,t) - m(\cdot)u_0(\cdot)\|_{L^p(\Omega)} \to 0$ as $t \to 0+$.

consists in showing an operator estimate of the form

$$\|L(\lambda M + L)^{-1}\|_{\mathcal{L}(X)} \leq C(1 + |\lambda|)^{1-\beta}, \qquad \forall \lambda \in \Sigma_\alpha, \tag{1.5}$$

where $X = L^p(\Omega)$, $0 < \beta \leq \alpha \leq 1$, $\alpha + \beta > 1$,

$$\Sigma_\alpha = \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \geq -c(1 + |\operatorname{Im}\lambda|)^\alpha\}, \qquad (c > 0), \tag{1.6}$$

and

$$\mathcal{D}(L) = \mathcal{D}(L_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \qquad Lu(x) = A(x, D)u(x), \qquad u \in \mathcal{D}(L), \tag{1.7}$$

$$\mathcal{D}(M) = L^p(\Omega), \qquad Mu(x) = m(x)u(x). \tag{1.8}$$

We in fact show that (1.5) holds with α=1, $\beta = 1/p$, $p \in (1, +\infty)$.

Moreover, when m is ρ-regular, i.e.

$$m \in C^1(\overline{\Omega}), \qquad |\nabla m(x)| \leq C_1 m(x)^\rho, \qquad \forall x \in \overline{\Omega}, \quad \text{for some } \rho \in (0, 1], \tag{1.9}$$

C_1 being a positive constant, we can improve the index β in estimate (1.5) from $\beta = 1/p$ to

$$\beta = \begin{cases} (2-\rho)^{-1}, & \text{if} \quad p \in (1, 2), \ \rho \in (2-p, 1], \\ 2[p(2-\rho)]^{-1}, & \text{if} \quad p \in [2, +\infty), \ \rho \in (0, 1]. \end{cases} \tag{1.10}$$

The result proved in this paper will be applied, in a subsequent paper, to identify the *unknown* kernel k in the integro-differential singular equation of parabolic type

$$\begin{aligned} D_t[m(x)u(x,t)] + A(x, D_x)u(x,t) &= \int_0^t k(t-s)B(x, D_x)u(x,s)\,\mathrm{d}s + f(x,t), \\ &\qquad \forall (x,t) \in \Omega \times [0, \tau], \end{aligned} \tag{1.11}$$

$B(x, D_x)$ being a linear second-order differential operator.

We stress that the present paper was originated by a requirement of additional smoothness of solution u of (1.11) needed to recover the unknown kernel k. This occurrence is in accordance with the well-known fact that inverse problems usually force deeper, and sometimes, unexpected insights in *direct problems*.

2. Solving the spectral problem $(\lambda M + L)u = f$

The basic aim of this section consists in showing that estimate (1.5) holds when the linear operators M and L are defined by (1.7) and (1.8), respectively. To this aim we assume that the coefficients $a_{i,j}$ and a_0 satisfy the properties

$$a_{i,j} \in C^1(\overline{\Omega}), \quad a_0 \in C(\overline{\Omega}), \quad a_{i,j} = a_{j,i}, \quad i, j = 1, \ldots, n, \tag{2.1}$$

$$c_0|\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \leq c_1|\xi|^2, \quad \forall x \in \overline{\Omega}, \forall \xi \in \mathbf{R}^n, \quad a_0(x) \geq \gamma, \quad \forall x \in \overline{\Omega}, \tag{2.2}$$

c_0, c_1 and γ being three positive constants.

A remarkable result by Okazawa [4, p.702] provides, for any $u \in \mathcal{D}(L)$,

$$\begin{aligned} &\operatorname{Re}((L-a_0)u, u|u|^{p-2}) \\ &\geq \begin{cases} c_0 \int_\Omega |u|^{p-2}|\nabla u|^2 \,\mathrm{d}x \geq 0, & \text{if } p \in [2,\infty), \\ c_0(p-1)\int_\Omega (|u|^2+\delta)^{(p-2)/2}|\nabla u|^2\,\mathrm{d}x \geq 0, & \text{if } p \in (1,2), \end{cases} \end{aligned} \tag{2.3}$$

$$|\operatorname{Im}(Lu, u|u|^{p-2})| \leq \frac{|p-2|}{2\sqrt{p-1}} \operatorname{Re}((L-a_0)u, u|u|^{p-2}), \tag{2.4}$$

where the brackets denote

$$(f,g) = \int_\Omega f(x)\overline{g(x)}\,\mathrm{d}x, \qquad f \in L^p(\Omega), \quad g \in L^{p'}(\Omega), \quad \frac{1}{p}+\frac{1}{p'} = 1,$$

$u|u|^{p-2}$ is assumed to vanish whenever u does, and $\delta > 0$ is arbitrary.

REMARK 2.1. It is important to observe that bound (2.4) holds even in the degenerate elliptic case (cf. [4, p. 702] and the following Lemma 3.3).

From (2.4) we immediately deduce the estimate

$$\begin{aligned} &|\operatorname{Im}(Lu, u|u|^{p-2})| + \frac{|p-2|}{2\sqrt{p-1}}\int_\Omega a_0(x)|u(x)|^p\,\mathrm{d}x \\ &\leq \frac{|p-2|}{2\sqrt{p-1}} \operatorname{Re}(Lu, u|u|^{p-2}). \end{aligned} \tag{2.5}$$

Consider now the spectral problem

$$u \in \mathcal{D}(L), \qquad \lambda m u + Lu = f \in L^p(\Omega). \tag{2.6}$$

Taking the real and imaginary parts of the scalar product of both sides in (2.6) with $u|u|^{p-2}$, we get

$$\operatorname{Re}\lambda \int_\Omega m|u|^p\,\mathrm{d}x + \operatorname{Re}(Lu, u|u|^{p-2}) = \operatorname{Re}\int_\Omega f\overline{u}|u|^{p-2}\,\mathrm{d}x, \tag{2.7}$$

$$\operatorname{Im}\lambda \int_\Omega m|u|^p\,\mathrm{d}x + \operatorname{Im}(Lu, u|u|^{p-2}) = \operatorname{Im}\int_\Omega f\overline{u}|u|^{p-2}\,\mathrm{d}x. \tag{2.8}$$

From (2.8) we deduce the inequalities

$$|\operatorname{Im}\lambda|\int_\Omega m|u|^p\,\mathrm{d}x \le |\operatorname{Im}(Lu,u|u|^{p-2})| + \left|\operatorname{Im}\int_\Omega f\overline{u}|u|^{p-2}\,\mathrm{d}x\right|. \tag{2.9}$$

Multiply then both sides in (2.9) by a positive constant k and add the obtained inequality to equation (2.7). From (2.5) we get

$$\begin{aligned}(\operatorname{Re}\lambda + k|\operatorname{Im}\lambda|)\int_\Omega m|u|^p\,\mathrm{d}x &+ \Big(1 - k\frac{|p-2|}{2\sqrt{p-1}}\Big)\operatorname{Re}(Lu,u|u|^{p-2}) \\ \le \operatorname{Re}\int_\Omega f\overline{u}|u|^{p-2}\,\mathrm{d}x &+ k\left|\operatorname{Im}\int_\Omega f\overline{u}|u|^{p-2}\,\mathrm{d}x\right| \le (1+k)\|f\|_p\|u\|_p^{p-1}.\end{aligned} \tag{2.10}$$

Choose now $k = k_1(p)$ so small as to satisfy

$$h_1(p) =: 1 - k_1(p)\frac{|p-2|}{2\sqrt{p-1}} > 0, \qquad \forall p \in (1,+\infty). \tag{2.11}$$

Observe that

$$\begin{aligned}\operatorname{Re}(Lu,u|u|^{p-2}) &= \operatorname{Re}((L-a_0)u,u|u|^{p-2}) \\ &\quad + \frac{1}{2}\operatorname{Re}(a_0u,u|u|^{p-2}) + \frac{1}{2}\operatorname{Re}(a_0u,u|u|^{p-2}) \\ &\ge \operatorname{Re}((L-a_0)u,u|u|^{p-2}) + \frac{\gamma}{2}\|u\|_p^p + \frac{\gamma}{2\|m\|_\infty}\int_\Omega m|u|^p\,\mathrm{d}x,\end{aligned} \tag{2.12}$$

since $m(x) \le \|m\|_\infty$ implies

$$\frac{m(x)}{\|m\|_\infty}\,\frac{a_0(x)}{2} \le \frac{a_0(x)}{2}.$$

In view of (2.11), (2.12) and (2.3), we obtain from (2.10) that

$$\begin{aligned}&\Big(\operatorname{Re}\lambda + k_1(p)|\operatorname{Im}\lambda| + \frac{\gamma h_1(p)}{2\|m\|_\infty}\Big)\int_\Omega m|u|^p\,\mathrm{d}x \\ &+ \frac{\gamma h_1(p)}{2}\|u\|_p^p + h_1(p)\operatorname{Re}((L-a_0)u,u|u|^{p-2}) \le [k_1(p)+1]\|f\|_p\|u\|_p^{p-1}.\end{aligned} \tag{2.13}$$

Introduce now the sector

$$\Sigma_1 = \left\{\mu \in \mathbf{C} : \operatorname{Re}\mu + \frac{k_1(p)}{2}|\operatorname{Im}\mu| + \frac{\gamma h_1(p)}{4\|m\|_\infty} \ge 0\right\}.$$

Then, for $\lambda \in \Sigma_1$,

$$\operatorname{Re}((L-a_0)u,u|u|^{p-2}) \le \frac{k_1(p)+1}{h_1(p)}\|f\|_p\|u\|_p^{p-1}, \tag{2.14}$$

$$\|u\|_p \leq \frac{2(k_1(p)+1)}{\gamma h_1(p)}\|f\|_p. \tag{2.15}$$

Consequently,

$$\left(\operatorname{Re}\lambda + k_1(p)|\operatorname{Im}\lambda| + \frac{\gamma h_1(p)}{2\|m\|_\infty}\right)\int_\Omega m|u|^p \, dx \leq C_1(p)\|f\|_p^p.$$

We now need a simple proposition. For the proof see Section 6.

Proposition 2.1. *Let $k > 0$ and $\varepsilon > 0$ be two positive constants, and let $\Sigma_{k,\varepsilon}$ be a sectorial domain given by*

$$\Sigma_{k,\varepsilon} = \left\{\mu \in \mathbf{C} : \operatorname{Re}\mu + \frac{k}{2}|\operatorname{Im}\mu| + \frac{\varepsilon}{2} \geq 0\right\}.$$

Then it holds that

$$|\lambda| + 1 \leq \left(\frac{2}{k} + \frac{2}{\varepsilon} + 1\right)(\operatorname{Re}\lambda + k|\operatorname{Im}\lambda| + \varepsilon), \quad \lambda \in \Sigma_{k,\varepsilon}.$$

Since $\Sigma_1 = \Sigma_{k_1(p),h_1(p)(2\|m\|_\infty)^{-1}}$, this proposition then yields

$$\left(|\lambda|+1\right)\int_\Omega m|u|^p \, dx \leq C_2(p)\|f\|_p\|u\|_p^{p-1} \leq C_3(p)\|f\|_p^p, \quad \lambda \in \Sigma_1. \tag{2.16}$$

To show that $(\lambda M + L)^{-1}$ is a bounded operator on $L^p(\Omega)$ for $\lambda \in \Sigma_1$, it now suffices to verify that $\mathcal{R}(\lambda M + L) = L^p(\Omega)$. But this is verified by the usual techniques without difficulty. In fact, for each $\lambda \in \Sigma_1$, we already know that $\mathcal{R}(\Lambda + \lambda M + L) = L^p(\Omega)$ provided $\Lambda > 0$ is a sufficiently large number. Let $0 \leq \theta \leq 1$ be a parameter, and consider the family of closed linear operators $A(\theta) = \theta\Lambda + \lambda M + L$, $0 \leq \theta \leq 1$. Then the desired result is obtained by the following proposition the proof of which will be given in the final section.

Proposition 2.2. *Let $A(\theta)$, $0 \leq \theta \leq 1$, be a family of closed linear operators acting on a Banach space X with constant domain $\mathcal{D}(A(\theta)) \equiv \mathcal{D}$. Assume that the family satisfies the conditions*

$$\delta\|u\| \leq \|A(\theta)u\|, \quad u \in \mathcal{D}, \tag{2.17}$$

$$\|\{A(\theta) - A(\theta')\}u\| \leq N|\theta - \theta'|\|u\|, \quad u \in \mathcal{D} \tag{2.18}$$

with some constants $\delta > 0$ and $N > 0$ independent of $\theta, \theta' \in [0,1]$. Then, $\mathcal{R}(A(1)) = X$ implies $\mathcal{R}(A(\theta)) = X$ for every $\theta \in [0,1)$.

We can now summarize the results proved in this section in Theorem 2.1.

Theorem 2.1. *Let L and M be the linear operators defined by* (1.7) *and* (1.8), *the coefficients $a_{i,j}$ $i, j = 1, \dots, n$, a_0 enjoying properties* (2.1) *and* (2.2) *and m being a non-negative function in $L^\infty(\Omega)$. Then the spectral equation $\lambda Mu + Lu = f$, with $f \in L^p(\Omega)$, admits, for any $\lambda \in \Sigma_1 = \{\mu \in \mathbf{C} : \operatorname{Re}\mu + (k_1(p)/2)|\operatorname{Im}\mu| + \gamma h_1(p)/(4\|m\|_\infty) \geq 0\}$ and $p \in (1, +\infty)$, a unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ satisfying the estimates*

$$\|u\|_p \leq C_4(p)\|f\|_p, \qquad \|Mu\|_p \leq C_5(p)|\lambda|^{-1/p}\|f\|_p, \qquad \lambda \in \Sigma_1,$$
$$\|Lu\|_p \leq C_6(p)(1 + |\lambda|^{1/p'})\|f\|_p, \qquad \lambda \in \Sigma_1.$$

3. The case when m is ρ-regular and $p \in [2, +\infty)$

We will show that when the multiplier m is more regular, i.e. it satisfies (1.9), our β can be chosen larger than $1/p$. We recall that all the previous estimates (2.6)–(2.16) hold for any $p \in (1, +\infty)$.

First of all we need the following lemma concerning the computation of the gradient of the function $\overline{u}|u|^{p-2}$ when $p \in [2, +\infty)$. For this purpose we need some lemmata.

Lemma 3.1. *Let $u \in W_0^{1,p}(\Omega)$ with $p \in [2, +\infty)$. Then the function $\overline{u}|u|^{p-2}$ belongs to $W_0^{1,p}(\Omega)$ and the following formulae hold*

$$\begin{aligned} D_{x_j}\overline{u}|u|^{p-2} &= |u|^{p-2}D_{x_j}\overline{u} + (p-2)g_p(u)\operatorname{Re}(g_p(u)D_{x_j}u), \\ &\quad \text{a.e. in } \Omega, \quad j = 1, \dots, n, \end{aligned} \tag{3.1}$$

where

$$g_p(u)(x) = \begin{cases} \overline{u(x)}|u(x)|^{(p-4)/2}, & \text{if } u(x) \neq 0, \\ 0, & \text{if } u(x) = 0. \end{cases} \tag{3.2}$$

Proof. Let ϕ be any function in $C_0^\infty(\Omega)$. Then the following equalities hold:

$$\begin{aligned} &\langle D_{x_j}\phi, \overline{u}|u|^{p-2}\rangle = \lim_{\varepsilon\to 0+}\langle D_{x_j}\phi, \overline{u}(|u|^2 + \varepsilon)^{(p-2)/2}\rangle \\ &= -\lim_{\varepsilon\to 0+}\langle \phi, (|u|^2+\varepsilon)^{(p-2)/2}D_{x_j}\overline{u} + \frac{p-2}{2}\overline{u}(|u|^2+\varepsilon)^{(p-4)/2}(\overline{u}D_{x_j}u + uD_{x_j}\overline{u})\rangle \\ &= -\lim_{\varepsilon\to 0+}\langle \phi, (|u|^2+\varepsilon)^{(p-2)/2}D_{x_j}\overline{u} + (p-2)\overline{u}(|u|^2+\varepsilon)^{(p-4)/2}\operatorname{Re}(\overline{u}D_{x_j}u)\rangle \\ &= -\lim_{\varepsilon\to 0+}\langle \phi, (|u|^2+\varepsilon)^{(p-2)/2}D_{x_j}\overline{u} + (p-2)\overline{u}(|u|^2+\varepsilon)^{(p-4)/4}\operatorname{Re}(\overline{u}(|u|^2+\varepsilon)^{(p-4)/4}D_{x_j}u)\rangle \\ &= -\langle \phi, |u|^{p-2}D_{x_j}\overline{u} + (p-2)g_p(u)\operatorname{Re}(g_p(u)D_{x_j}u)\rangle. \end{aligned} \tag{3.3}$$

We have used here the relation $\lim_{\varepsilon\to 0+}\overline{u}(x)(|u(x)|^2+\varepsilon)^{(p-4)/4} = g_p(u)(x)$, which takes advantage of the assumption $p \in [2, +\infty)$. □

REMARK 3.1. From definition (3.2) we easily deduce the identity

$$|g_p(u)(x)| = |u(x)|^{(p-2)/2}. \tag{3.4}$$

We can now prove the following Lemma 3.2.

Lemma 3.2. *Let $(b_{i,j})_{i,j=1,\dots,n}$ be a matrix of functions in $C^1(\overline{\Omega};\mathbf{R})$ such that*

$$b_{i,j} = b_{j,i} \quad i, j = 1, \dots, n, \tag{3.5}$$

$$c_0|\xi|^2\mu(x) \le \sum_{i,j=1}^{n} b_{i,j}(x)\xi_i\xi_j \le c_1|\xi|^2\mu(x), \qquad \forall x \in \overline{\Omega},\ \forall \xi \in \mathbf{R}^n, \tag{3.6}$$

where $\mu \in C(\overline{\Omega})$ is a non-negative function and c_0, c_1 are two positive constants. Then for any $p \in [2, +\infty)$, the linear operator $K = -\sum_{i,j=1}^{n} D_{x_i}[b_{i,j}(x)D_{x_j}]$ with $\mathcal{D}(K) = \mathcal{D}(L)$ (cf. (1.7)) *satisfies the relations*

$$\begin{aligned} &c_0\left(\int_\Omega \mu|u|^{p-2}|Du|^2\,\mathrm{d}x + \int_\Omega \mu \sum_{j=1}^{n}[\mathrm{Re}(g_p(u)D_{x_j}u)]^2\,\mathrm{d}x\right)\\ &\le \mathrm{Re}(Ku, \overline{u}|u|^{p-2}) \le c_1\left(\int_\Omega \mu|u|^{p-2}|Du|^2\,\mathrm{d}x + \int_\Omega \mu \sum_{j=1}^{n}[\mathrm{Re}(g_p(u)D_{x_j}u)]^2\,\mathrm{d}x\right), \end{aligned} \tag{3.7}$$

$$\mathrm{Im}(Ku, \overline{u}|u|^{p-2}) = (p-2)\int_\Omega \sum_{i,j=1}^{n} b_{i,j}[\mathrm{Re}(g_p(u)D_{x_i}u)][\mathrm{Im}(g_p(u)D_{x_j}u)]\,\mathrm{d}x. \tag{3.8}$$

Proof. From Lemma 3.1 and an integration by parts we easily deduce the identity

$$\begin{aligned} (Ku, \overline{u}|u|^{p-2}) &= \int_\Omega \sum_{i,j=1}^{n} b_{i,j}D_{x_j}uD_{x_i}(\overline{u}|u|^{p-2})\,\mathrm{d}x\\ &= \int_\Omega \sum_{i,j=1}^{n} |u|^{p-2}b_{i,j}D_{x_j}uD_{x_i}\overline{u}\,\mathrm{d}x\\ &\quad + (p-2)\int_\Omega \sum_{i,j=1}^{n} b_{i,j}g_p(u)D_{x_j}u\,\mathrm{Re}(g_p(u)D_{x_i}u)\,\mathrm{d}x. \end{aligned} \tag{3.9}$$

Relations (3.7) and (3.8) follow immediately from (3.9) taking the real and the imaginary parts. □

Lemma 3.3. *Under the assumptions in the statement of* Lemma 3.2 *operator K satisfies inequalities* (2.3) *and* (2.4) *with K in the place of $L - a$.*

Proof. This lemma has essentially been proved in [4], although a slight modification is needed in its proof. For any $\varepsilon > 0$ define $a_{i,j} = b_{i,j} + \varepsilon\delta_{i,j}$, $i, j = 1, \dots, n$, and set $K_\varepsilon = K - \varepsilon\Delta$. Since the matrix $(a_{i,j})_{i,j=1,\dots,n}$ is uniformly positive definite, from (2.3) and (2.4), with $u \in \mathcal{D}(L_0)$, we obtain the inequalities

$$0 \le \operatorname{Re}(K_\varepsilon u, u|u|^{p-2}) = \operatorname{Re}(Ku, u|u|^{p-2}) + \varepsilon \operatorname{Re}(-\Delta u, u|u|^{p-2}), \tag{3.10}$$

$$\begin{aligned} |\operatorname{Im}(K_\varepsilon u, u|u|^{p-2})| &= |\operatorname{Im}(Ku, u|u|^{p-2}) + \varepsilon \operatorname{Im}(-\Delta u, u|u|^{p-2})| \\ &\le \frac{|p-2|}{2\sqrt{p-1}}[\operatorname{Re}(Ku, u|u|^{p-2}) + \varepsilon \operatorname{Re}(-\Delta u, u|u|^{p-2})]. \end{aligned} \tag{3.11}$$

Taking the limit as $\varepsilon \to 0+$ in (3.10) and (3.11), we easily deduce that K satisfies (2.3) and (2.4). □

We shall use also the following identity

$$\begin{aligned} &(Lu, m^{p-1}u|u|^{p-2}) = (m^{p-1}Lu, u|u|^{p-2}) \\ &= (K_0 u, u|u|^{p-2}) + (p-1)\left(m^{p-2}\sum_{i,j=1}^{n} a_{i,j} D_{x_i} m D_{x_j} u, u|u|^{p-2}\right), \quad u \in \mathcal{D}(L), \end{aligned} \tag{3.12}$$

where

$$K_0 = -\sum_{i,j=1}^{n} D_{x_i}[m(x)^{p-1} a_{i,j}(x) D_{x_j}] + m(x)^{p-1} a_0(x).$$

Let now u be a solution to equation (2.6). Taking the scalar product of both sides in (2.6) with $m^{p-1}u|u|^{p-2}$ and using (3.12), we easily get the equalities

$$\begin{aligned} &(f, m^{p-1}u|u|^{p-2}) = (\lambda m u + Lu, m^{p-1}u|u|^{p-2}) \\ &= \lambda\|Mu\|_p^p + (K_0 u, u|u|^{p-2}) + (p-1)\left(m^{p-2}\sum_{i,j=1}^{n} a_{i,j} D_{x_i} m D_{x_j} u, u|u|^{p-2}\right). \end{aligned} \tag{3.13}$$

Taking the real and imaginary parts in (3.13) and using (2.4) with $L - a_0$ replaced by $K = K_0 - m^{p-1}a_0$, we easily deduce the inequalities

$$\begin{aligned} &\operatorname{Re}\lambda\|Mu\|_p^p + \gamma\int_\Omega m^{p-1}|u|^p\,\mathrm{d}x + \operatorname{Re}((K_0 - m^{p-1}a_0)u, u|u|^{p-2}) \\ &\le |(f, m^{p-1}u|u|^{p-2})| + (p-1)\left|\left(m^{p-2}\sum_{i,j=1}^{n} a_{i,j} D_{x_i} m D_{x_j} u, u|u|^{p-2}\right)\right|, \end{aligned} \tag{3.14}$$

$$\begin{aligned} |\operatorname{Im}\lambda|\|Mu\|_p^p &\le |\operatorname{Im}((K_0 - m^{p-1}a_0)u, u|u|^{p-2})| \\ &\quad + |(f, m^{p-1}u|u|^{p-2})| + (p-1)\left|\left(m^{p-2}\sum_{i,j=1}^{n} a_{i,j} D_{x_i} m D_{x_j} u, u|u|^{p-2}\right)\right| \end{aligned}$$

$$\leq \frac{|p-2|}{2\sqrt{p-1}} \operatorname{Re}((K_0 - m^{p-1}a_0)u, u|u|^{p-2}) + |(f, m^{p-1}u|u|^{p-2})|$$

$$+ (p-1)\left|\left(m^{p-2}\sum_{i,j=1}^{n} a_{i,j}D_{x_i}mD_{x_j}u, u|u|^{p-2}\right)\right|. \tag{3.15}$$

Multiply now by $k_1(p)$ (cf. (2.11)) the first and last sides in (3.15) and add to the first and last sides in (3.14). We get the estimate

$$\begin{aligned}
&[\operatorname{Re}\lambda + k_1(p)|\operatorname{Im}\lambda| + \gamma\|m\|_\infty^{-1}]\|Mu\|_p^p \\
&\quad + \left(1 - k_1(p)\frac{|p-2|}{2\sqrt{p-1}}\right)\operatorname{Re}((K_0 - m^{p-1}a_0)u, u|u|^{p-2}) \\
&\leq [1 + k_1(p)]\left\{|(f, m^{p-1}u|u|^{p-2})| + (p-1)\left|\left(m^{p-2}\sum_{i,j=1}^{n} a_{i,j}D_{x_i}mD_{x_j}u, u|u|^{p-2}\right)\right|\right\},
\end{aligned} \tag{3.16}$$

where we have made use of the elementary inequality

$$m(x)^p \leq \|m\|_\infty m(x)^{p-1}, \qquad x \in \overline{\Omega}.$$

We now estimate the last term in (3.16) with the aid of (1.9). Using twice Hölder's inequality, we get

$$\begin{aligned}
&\left|\left(m^{p-2}\sum_{i,j=1}^{n} a_{i,j}D_{x_i}mD_{x_j}u, u|u|^{p-2}\right)\right| \leq \int_\Omega m^{p-2}|u|^{p-1}\left|\sum_{i,j=1}^{n} a_{i,j}D_{x_i}mD_{x_j}u\right| \mathrm{d}x \\
&\leq \int_\Omega m^{p-2}|u|^{p-1}\left|\sum_{i,j=1}^{n} a_{i,j}D_{x_i}mD_{x_j}m\right|^{1/2}\left|\sum_{i,j=1}^{n} a_{i,j}D_{x_i}uD_{x_j}u\right|^{1/2} \mathrm{d}x \\
&\leq C_7\int_\Omega m^{p-2+\rho}|u|^{p-1}|\nabla u|\,\mathrm{d}x = C_7\int_\Omega m^{p\rho/2}|u|^{p/2}m^{(p-2)(2-\rho)/2}|u|^{-1+p/2}|\nabla u|\,\mathrm{d}x \\
&\leq C_7\left(\int_\Omega m^{p\rho}|u|^{p\rho}|u|^{p(1-\rho)}\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega m^{(p-2)(2-\rho)}|u|^{p-2}|\nabla u|^2\,\mathrm{d}x\right)^{1/2} \\
&\leq C_7\|Mu\|_p^{p\rho/2}\|u\|_p^{(1-\rho)p/2}\|m\|_\infty^{(p-2)(2-\rho)/2}\left(\int_\Omega |u|^{p-2}|\nabla u|^2\,\mathrm{d}x\right)^{1/2}.
\end{aligned} \tag{3.17}$$

On account of (2.3), (2.14) and (2.15), we easily observe the estimate

$$\int_\Omega |u|^{p-2}|\nabla u|^2\,\mathrm{d}x \leq C_8(p)\|f\|_p^p. \tag{3.18}$$

From (2.15), (3.17) and (3.18) we finally deduce the estimates

$$\left|\left(m^{p-2}\sum_{i,j=1}^{n} a_{i,j}D_{x_i}mD_{x_j}u, u|u|^{p-2}\right)\right| \le C_9(p)\|f\|_p^{p(2-\rho)/2}\|Mu\|_p^{p\rho/2}. \tag{3.19}$$

Moreover, we have

$$|(f, m^{p-1}u|u|^{p-2})| \le \|f\|_p\|Mu\|_p^{p-1}. \tag{3.20}$$

Finally, from (3.16), (3.19), (3.20) and Lemma 3.2 with $K = K_0 - m^{p-1}a_0$ (which makes use of the assumption $p \in [2, +\infty)$) we deduce the inequality

$$\begin{aligned}
&[\operatorname{Re}\lambda + k_1(p)|\operatorname{Im}\lambda| + \gamma\|m\|_\infty^{-1}]\|Mu\|_p^p \\
&\quad + \left(1 - k_1(p)\frac{|p-2|}{2\sqrt{p-1}}\right)\operatorname{Re}((K_0 - m^{p-1}a_0)u, u|u|^{p-2}) \\
&\le C_{10}(p)[\|f\|_p\|Mu\|_p^{p-1} + \|f\|_p^{p(2-\rho)/2}\|Mu\|_p^{p\rho/2}], \qquad \lambda \in \Sigma_1.
\end{aligned} \tag{3.21}$$

We now introduce the sector

$$\Sigma_2 = \left\{\lambda \in \mathbf{C} : \operatorname{Re}\lambda + \frac{k_1(p)}{2}|\operatorname{Im}\lambda| + \frac{\gamma}{2\|m\|_\infty} \ge 0\right\}.$$

Since $h_1(p) \in (0, 1)$, (cf. (2.11)), we immediately deduce the inclusion $\Sigma_2 \subset \Sigma_1$ (see the definition of Σ_2).

Then, recalling that $\operatorname{Re}((K_0 - m^{p-1}a_0)u, u|u|^{p-2})$ is non-negative (cf. Lemma 3.2) and applying Proposition 2.1, we obtain

$$\begin{aligned}
&(|\lambda| + 1)\|Mu\|_p^p \le \gamma\int_\Omega m^{p-1}|u|^p\,\mathrm{d}x + \operatorname{Re}((K_0 - m^{p-1}a_0)u, u|u|^{p-2}) \\
&\le C_{11}(p)[\|f\|_p\|Mu\|_p^{p-1} + \|f\|_p^{p(2-\rho)/2}\|Mu\|_p^{p\rho/2}], \qquad \lambda \in \Sigma_2.
\end{aligned} \tag{3.22}$$

Consequently, since $\|u\|_p \le C_{12}(p)\|f\|_p$ (cf. (2.15)), (3.15) and (3.22) imply

$$\begin{aligned}
&(|\lambda| + 1)\|Mu\|_p^{p(2-\rho)/2} \\
&\le C_{13}(p)[\|f\|_p\|Mu\|_p^{p-1-p\rho/2} + \|f\|_p^{p(2-\rho)/2}], \qquad \lambda \in \Sigma_2.
\end{aligned} \tag{3.23}$$

By Proposition 2.2, it is verified that $\lambda M + L$ is surjective on $L^p(\Omega)$. Hence, estimate (1.5) holds with $\alpha = 1$ and $\beta = 2[p(2-\rho)]^{-1}$.

We can summarize the results in this section in Theorem 3.1.

Theorem 3.1. *Let L and M be the linear operators defined by* (1.7) *and* (1.8), *the coefficients $a_{i,j}$ $i, j = 1, \dots, n$, a_0 enjoying properties* (2.1) *and* (2.2) *and m being* *a non-negative function satisfying* (1.9). *Then the spectral equation* $\lambda Mu + Lu = f$, *with* $f \in L^p(\Omega)$, *admits, for any* $\lambda \in \Sigma_2 = \{\mu \in \mathbf{C} : \operatorname{Re}\mu + (k_1(p)/2)|\operatorname{Im}\mu| + (\gamma/2\|m\|_\infty) \geq 0\}$ *and* $p \in [2, +\infty)$, *a unique solution* $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ *satisfying the estimates*

$$\|u\|_p \leq C_{14}(p)\|f\|_p, \qquad \|Mu\|_p \leq C_{15}(p)|\lambda|^{-2/[p(2-\rho)]}\|f\|_p, \qquad \lambda \in \Sigma_2,$$
$$\|Lu\|_p \leq C_{16}(p)(1 + |\lambda|^{[p(2-\rho)-2]/[p(2-\rho)]})\|f\|_p, \qquad \lambda \in \Sigma_2.$$

EXAMPLE 3.1. Let Ω be a bounded domain and let x_0 be a fixed point in $\partial\Omega$. Define then $r = \max_{x\in\overline{\Omega}} |x - x_0|$ and choose

$$m(x) = [(|x - x_0|(r - |x - x_0| - r_1)]^q, \qquad q \in (1, +\infty).$$

An elementary computation shows that

$$|\nabla m(x)| = q[|x - x_0|(r - |x - x_0|)]^{q-1}|2|x - x_0| - r| \leq qrm(x)^{(q-1)/q}, \quad x \in \Omega.$$

Consequently, function m satisfies condition (1.9).

We notice that for *any* open interval $\Omega \subset \mathbf{R}$ we have $r = \text{length}(\Omega)$.

4. The case when $p \in (1, 2)$

In this section we are going to considering the case $p \in (1, 2)$. From (2.4) we immediately deduce that the estimate

$$\begin{aligned} |\operatorname{Im}(Lu, u|u|^{p-2})| + \gamma\|u\|_p^p &\leq |\operatorname{Im}(Lu, u|u|^{p-2})| + \int_\Omega a_0(x)|u(x)|^p\,\mathrm{d}x \\ &\leq \frac{|p-2|}{2\sqrt{p-1}}\operatorname{Re}(Lu, u|u|^{p-2}), \qquad u \in \mathcal{D}(L) \end{aligned} \tag{4.1}$$

holds true for any $p \in (1, +\infty)$.

Consider again the spectral problem

$$u \in \mathcal{D}(L), \qquad \lambda Mu + Lu = f \in L^p(\Omega). \tag{4.2}$$

Multiplying both sides in (4.2) by $u|u|^{p-2}$ and integrating over Ω, we get

$$\lambda\|m^{1/p}u\|_p^p + (Lu, u|u|^{p-2}) = (f, u|u|^{p-2}). \tag{4.3}$$

Taking the real and imaginary parts, from (4.3) we deduce

$$\operatorname{Re}\lambda\|m^{1/p}u\|_p^p + \operatorname{Re}(L_0u, u|u|^{p-2}) + (a_0u, u|u|^{p-2}) = \operatorname{Re}(f, u|u|^{p-2}), \tag{4.4}$$

$$\operatorname{Im}\lambda\|m^{1/p}u\|_p^p + \operatorname{Im}(L_0u, u|u|^{p-2}) = \operatorname{Im}(f, u|u|^{p-2}), \tag{4.5}$$

where we have set

$$L_0 = L - a_0. \tag{4.6}$$

Then from Okazawa [4, p.703] we get

$$(L_0 u, u|u|^{p-2}) = \lim_{\delta \to 0+} I_p(u, \delta), \tag{4.7}$$

where $\delta > 0$ and

$$I_p(u, \delta) = -\int_\Omega \big(|u(x)|^2 + \delta\big)^{(p-2)/2} \overline{u(x)} \sum_{j,k=1}^{n} D_{x_k}\big[a_{j,k}(x) D_{x_j} u(x)\big]\, \mathrm{d}x. \tag{4.8}$$

As mentioned at the beginning of Section 2, we have

$$\operatorname{Re}(L_0 u, u|u|^{p-2}) \geq c_0 \int_\Omega |u(x)|^{p-2} |\nabla u(x)|^2 \, \mathrm{d}x, \quad \text{if } p \in [2, +\infty), \tag{4.9}$$

$$\operatorname{Re}(L_0 u, u|u|^{p-2}) \geq c_0(p-1) \int_\Omega \big(|u(x)|^2 + \delta\big)^{(p-2)/2} |\nabla u(x)|^2 \, \mathrm{d}x, \quad \text{if } p \in (1, 2). \tag{4.10}$$

From (4.1) and (4.5) we deduce the inequalities

$$\begin{aligned} |\operatorname{Im}\lambda| \|m^{1/p} u\|_p^p &\leq |\operatorname{Im}(Lu, u|u|^{p-2})| + \|f\|_p \|u\|_p^{p-1} \\ &\leq \frac{|p-2|}{2\sqrt{p-1}} \operatorname{Re}(Lu, u|u|^{p-2}) + \|f\|_p \|u\|_p^{p-1}. \end{aligned} \tag{4.11}$$

Multiply then both sides in (4.11) by a positive constant ν and add the obtained inequality to equation (4.4) to get (cf. (2.2))

$$\begin{aligned} &(\operatorname{Re}\lambda + \nu|\operatorname{Im}\lambda|)\|m^{1/p}u\|_p^p + \left(1 - \nu\frac{|p-2|}{2\sqrt{p-1}}\right)\operatorname{Re}(L_0 u, u|u|^{p-2}) + \frac{\gamma}{2}\|u\|_p^p + \frac{\gamma}{2}\|u\|_p^p \\ &\leq (\operatorname{Re}\lambda + \nu|\operatorname{Im}\lambda|)\|m^{1/p}u\|_p^p + \left(1 - \nu\frac{|p-2|}{2\sqrt{p-1}}\right)\operatorname{Re}(L_0 u, u|u|^{p-2}) + (a_0 u, u|u|^{p-2}) \\ &\leq \operatorname{Re}(f, u|u|^{p-2}) + \nu\|f\|_p\|u\|_p^{p-1} \leq (1+\nu)\|f\|_p\|u\|_p^{p-1}. \end{aligned} \tag{4.12}$$

Choose now $\nu = \nu(p)$ so small as to satisfy

$$\nu_1(p) =: 1 - \nu(p)\frac{|p-2|}{2\sqrt{p-1}} > 0, \qquad \forall p \in (1, +\infty). \tag{4.13}$$

On the other hand, since $m \in L^\infty(\Omega)$, $\|u\|_p \geq \|m\|_\infty^{-1/p}\|m^{1/p}u\|_p$. Then (4.12) and (4.13) imply

$$\left(\frac{\gamma}{2\|m\|_\infty} + \operatorname{Re}\lambda + \nu|\operatorname{Im}\lambda|\right)\|m^{1/p}u\|_p^p + \nu_1(p)\operatorname{Re}(L_0 u, u|u|^{p-2}) + \frac{\gamma}{2}\|u\|_p^p$$

$$\leq [1+\nu(p)]\|f\|_p\|u\|_p^{p-1}. \tag{4.14}$$

In other words, there exist two positive constants C_{18} and C_{19} such that

$$\begin{aligned}&\left(\frac{\gamma}{2\|m\|_\infty}+\operatorname{Re}\lambda+\nu|\operatorname{Im}\lambda|\right)\|m^{1/p}u\|_p^p+C_{18}\operatorname{Re}(L_0u,u|u|^{p-2})+\frac{\gamma}{2}\|u\|_p^p\\ &\leq C_{19}\|f\|_p\|u\|_p^{p-1},\qquad \lambda\in\Sigma,\end{aligned} \tag{4.15}$$

the sector Σ being defined by

$$\Sigma=\left\{\lambda\in\mathbf{C}:\operatorname{Re}\lambda+\nu|\operatorname{Im}\lambda|+\frac{\gamma}{2\|m\|_\infty}\geq\varepsilon_0>0\right\}.$$

Notice that (4.7), (4.8), (4.10), (4.15) yield, in particular, the basic bounds

$$\|u\|_p\leq\frac{2}{\gamma}C_{19}\|f\|_p,\qquad \operatorname{Re}(L_0u,u|u|^{p-2})\leq C_{20}\|f\|_p^p, \tag{4.16}$$

and

$$\begin{aligned}&(p-1)c_0\lim_{\delta\to0+}\int_\Omega\left(|u(x)|^2+\delta\right)^{(p-2)/2}|\nabla u(x)|^2\,\mathrm{d}x\\ &\leq\lim_{\delta\to0+}\operatorname{Re}I_p(u,\delta)\leq C_{20}\|f\|_p\|u\|_p^{p-1},\quad \lambda\in\Sigma.\end{aligned} \tag{4.17}$$

From (4.3) we deduce the estimates

$$\begin{aligned}|\lambda|\|m^{1/p}u\|_p^p&\leq|(Lu,u|u|^{p-2})|+\|f\|_p\|u\|_p^{p-1}\\ &\leq\left(1+\frac{|p-2|}{2\sqrt{p-1}}\right)\operatorname{Re}(Lu,u|u|^{p-2})+\|f\|_p\|u\|_p^{p-1}\\ &\leq C_{21}\|f\|_p\|u\|_p^{p-1}\leq C_{22}^p\|f\|_p^p.\end{aligned} \tag{4.18}$$

Consequently, (4.18) immediately yields

$$|\lambda|^{1/p}\|Mu\|_p\leq C_{23}\|f\|_p. \tag{4.19}$$

This, in turn, implies that (1.5) holds with $\alpha=1$ and $\beta=1/p$ and provides a different proof to (1.5).

Now we focus our attention to the case when $m\in C^1(\overline{\Omega})$ satisfies inequality (1.9) with

$$\rho\in(2-p,1]. \tag{4.20}$$

Multiplying both sides in (4.2) by $m(x)^{p-1}\overline{u(x)}|u(x)|^{p-2}$ and integrating over Ω, we easily get

$$
\begin{aligned}
&\lambda\|Mu\|_p^p - \lim_{\delta\to 0+}\int_\Omega m(x)^{p-1}\overline{u(x)}(|u(x)|^2+\delta)^{(p-2)/2}\sum_{j,k=1}^n D_{x_j}[a_{j,k}(x)D_{x_k}u(x)]\,\mathrm{d}x\\
&\quad + \int_\Omega a_0(x)m(x)^{p-1}|u(x)|^p\,\mathrm{d}x = \int_\Omega f(x)m(x)^{p-1}\overline{u(x)}|u(x)|^{p-2}\,\mathrm{d}x.
\end{aligned}
\tag{4.21}
$$

An integration by parts in the integral appearing in the limit, which takes into account (4.20) and (4.21), easily yields

$$
\begin{aligned}
&-\int_\Omega m(x)^{p-1}\overline{u(x)}(|u(x)|^2+\delta)^{(p-2)/2}\sum_{j,k=1}^n D_{x_j}[a_{j,k}(x)D_{x_k}u(x)]\,\mathrm{d}x\\
&= \int_\Omega (|u(x)|^2+\delta)^{(p-2)/2}m(x)^{p-1}\sum_{j,k=1}^n a_{j,k}(x)D_{x_j}\overline{u(x)}D_{x_k}u(x)\,\mathrm{d}x\\
&\quad + (p-1)\int_\Omega \overline{u(x)}(|u(x)|^2+\delta)^{(p-2)/2}\sum_{j,k=1}^n m(x)^{p-2}D_{x_j}m(x)a_{j,k}(x)D_{x_k}u(x)\,\mathrm{d}x\\
&\quad + (p-2)\int_\Omega m(x)^{p-1}(|u(x)|^2+\delta)^{(p-4)/2}\sum_{j,k=1}^n a_{j,k}(x)\,\mathrm{Re}\left(\overline{u(x)}D_{x_j}u(x)\right)\overline{u(x)}D_{x_k}u(x)\,\mathrm{d}x\\
&=: I_1(\delta)+(p-1)I_2(\delta)-(2-p)I_3(\delta).
\end{aligned}
\tag{4.22}
$$

We have made use here of the following Proposition 4.1 whose proof is postponed to Section 6.

Proposition 4.1. *Let m satisfy property* (1.9). *Then for any $\beta\in(1-\rho,1)$, the function $m(\cdot)^\beta$ belongs to $C^1(\overline{\Omega})$ and $\nabla[m(\cdot)^\beta](x)=m_1(x)$ for any $x\in\overline{\Omega}$, where*

$$
m_1(x)=\begin{cases}0, & x\in Z(m),\\ \beta m(x)^{\beta-1}\nabla m(x), & x\notin Z(m),\end{cases}
\tag{4.23}
$$

and $Z(m)$ denotes the zero-set of m. Moreover,

$$
|\nabla[m(\cdot)^\beta](x)|\le Cm(x)^{\beta-1+\rho},\qquad x\in\overline{\Omega}.
$$

Since the matrix $(a_{j,k}(x))_{j,k=1,\dots,n}$ is real-valued and positive definite, from (4.22) we immediately deduce that

$$
I_1(\delta)\ \text{ and }\ \mathrm{Re}\,I_3(\delta)\ \text{ are positive for any }\ \delta\in\mathbf{R}_+.
\tag{4.24}
$$

Then we observe that $I_2(\delta)$ has a limit as $\delta \to 0+$ and

$$\lim_{\delta\to 0+} I_2(\delta) = \int_\Omega \overline{u(x)}|u(x)|^{p-2} \sum_{j,k=1}^{n} m(x)^{p-2} D_{x_j} m(x) a_{j,k}(x) D_{x_k} u(x)\,\mathrm{d}x. \tag{4.25}$$

Note that the integral in the right-hand side is well-defined on the whole of $W^{1,p}(\Omega)$ since $\overline{u}|u|^{p-2} \in L^{p'}(\Omega)$, $m^{p-2}D_{x_j}m \in L^\infty(\Omega)$ and $D_{x_j}u \in L^p(\Omega)$.

Further, (4.25) implies that there exists also $\lim_{\delta\to 0+}[I_1(\delta) - (2-p)I_3(\delta)]$. From (4.24) we deduce that there exist the limits

$$\lim_{\delta\to 0+} \operatorname{Im} I_3(\delta) \quad \text{and} \quad \lim_{\delta\to 0+} [I_1(\delta) - (2-p)\operatorname{Re} I_3(\delta)].$$

We can now prove the following Lemma 4.1.

Lemma 4.1. *The following estimates hold for any* $\delta \in \mathbf{R}_+$, $p \in (1,2)$ *and* $\eta \in (0, 2(p-1)(2-p)^{-1})$:

$$I_1(\delta) - (2-p)\operatorname{Re} I_3(\delta) - \eta(2-p)|\operatorname{Im} I_3(\delta)| \geq 0, \tag{4.26}$$

$$\begin{aligned} &I_1(\delta) + (p-1)\operatorname{Re} I_2(\delta) - (2-p)\operatorname{Re} I_3(\delta) \\ &\qquad - \eta|(p-1)\operatorname{Im} I_2(\delta) - (2-p)\operatorname{Im} I_3(\delta)| \geq -(p-1)(1+\eta^2)^{1/2}|I_2(\delta)|, \end{aligned} \tag{4.27}$$

$$\begin{aligned} &\lim_{\delta\to 0+} \left[I_1(\delta) + (p-1)\operatorname{Re} I_2(\delta) - (2-p)\operatorname{Re} I_3(\delta)\right] \\ &\qquad - \eta \lim_{\delta\to 0+} |(p-1)\operatorname{Im} I_2(\delta) - (2-p)\operatorname{Im} I_3(\delta)| \\ &\geq -C_{24}\|f\|_p^{p/2}\|Mu\|_p^{p-2+\rho}\|u\|_p^{2-\rho-p/2}, \end{aligned} \tag{4.28}$$

C_{24} *being a suitable positive constant.*

Proof. Since the matrix $(a_{j,k}(x))_{j,k=1,\dots,n}$ is real-valued and positive definite, we immediately deduce the equality

$$\sum_{j,k=1}^{n} a_{j,k}(x)\zeta_j\overline{\zeta}_k = \sum_{j,k=1}^{n} a_{j,k}(x)[\operatorname{Re}(\zeta_j)\operatorname{Re}(\zeta_k) + \operatorname{Im}(\zeta_j)\operatorname{Im}(\zeta_k)], \qquad \forall \zeta \in \mathbf{C}^n.$$

Consider now the formulae

$$\begin{aligned} I_1(\delta) &= \int_\Omega (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \sum_{j,k=1}^{n} a_{j,k}(x)\overline{u(x)}D_{x_j}u(x)\overline{\overline{u(x)}D_{x_k}u(x)}\,\mathrm{d}x \\ &\quad + \delta \int_\Omega (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \sum_{j,k=1}^{n} a_{j,k}(x) D_{x_j}u(x)\overline{D_{x_k}u(x)}\,\mathrm{d}x \\ &= \int_\Omega (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \Bigg\{ \sum_{j,k=1}^{n} a_{j,k}(x)\operatorname{Re}\left[\overline{u(x)}D_{x_j}u(x)\right]\operatorname{Re}\left[\overline{u(x)}D_{x_k}u(x)\right] \end{aligned}$$

$$+\sum_{j,k=1}^{n} a_{j,k}(x)\,\mathrm{Im}\left[\overline{u(x)}D_{x_j}u(x)\right]\mathrm{Im}\left[\overline{u(x)}D_{x_k}u(x)\right]\Bigg\}\,\mathrm{d}x$$

$$+\,\delta\int_\Omega(|u(x)|^2+\delta)^{(p-4)/2}m(x)^{p-1}\sum_{j,k=1}^{n} a_{j,k}(x)D_{x_j}u(x)\overline{D_{x_k}u(x)}\,\mathrm{d}x,\quad \forall\delta\in\mathbf{R}_+, \tag{4.29}$$

$$I_1(\delta)-(2-p)\,\mathrm{Re}\,I_3(\delta)=\int_\Omega(|u(x)|^2+\delta)^{(p-4)/2}m(x)^{p-1}$$

$$\times\Bigg\{(p-1)\sum_{j,k=1}^{n} a_{j,k}(x)\,\mathrm{Re}\left[\overline{u(x)}D_{x_j}u(x)\right]\mathrm{Re}\left[\overline{u(x)}D_{x_k}u(x)\right]$$

$$+\sum_{j,k=1}^{n} a_{j,k}(x)\,\mathrm{Im}\left[\overline{u(x)}D_{x_j}u(x)\right]\mathrm{Im}\left[\overline{u(x)}D_{x_k}u(x)\right]\Bigg\}\,\mathrm{d}x$$

$$+\,\delta\int_\Omega(|u(x)|^2+\delta)^{(p-4)/2}m(x)^{p-1}\sum_{j,k=1}^{n} a_{j,k}(x)D_{x_j}u(x)\overline{D_{x_k}u(x)}\,\mathrm{d}x,\quad \forall\delta\in\mathbf{R}_+, \tag{4.30}$$

$$\mathrm{Im}\,I_3(\delta)$$

$$=\left|\int_\Omega(|u(x)|^2+\delta)^{(p-4)/2}m(x)^{p-1}\sum_{j,k=1}^{n} a_{j,k}(x)\,\mathrm{Re}\left[\overline{u(x)}D_{x_j}u(x)\right]\mathrm{Im}\left[\overline{u(x)}D_{x_k}u(x)\right]\mathrm{d}x\right|$$

$$\le\frac{1}{2}\int_\Omega(|u(x)|^2+\delta)^{(p-4)/2}m(x)^{p-1}\sum_{j,k=1}^{n} a_{j,k}(x)\Big\{\mathrm{Re}\left[\overline{u(x)}D_{x_j}u(x)\right]\mathrm{Re}\left[\overline{u(x)}D_{x_k}u(x)\right]$$

$$+\,\mathrm{Im}\left[\overline{u(x)}D_{x_j}u(x)\right]\mathrm{Im}\left[\overline{u(x)}D_{x_k}u(x)\right]\Big\}\,\mathrm{d}x,\quad\forall\delta\in\mathbf{R}_+. \tag{4.31}$$

We have here used the Cauchy–Schwarz inequality and the geometric-arithmetic mean, i.e.

$$\left|\sum_{j,k=1}^{n} a_{j,k}(x)\xi_j\eta_k\right|\le\left(\sum_{j,k=1}^{n} a_{j,k}(x)\xi_j\xi_k\right)^{1/2}\left(\sum_{j,k=1}^{n} a_{j,k}(x)\eta_j\eta_k\right)^{1/2}$$

$$\le\frac{1}{2}\left(\sum_{j,k=1}^{n} a_{j,k}(x)\xi_j\xi_k+\sum_{j,k=1}^{n} a_{j,k}(x)\eta_j\eta_k\right)=\frac{1}{2}\sum_{j,k=1}^{n} a_{j,k}(x)\big[\xi_j\xi_k+\eta_j\eta_k\big],\quad\forall\xi,\eta\in\mathbf{R}^n.$$

From (4.24) and (4.31) we deduce the following inequality, where we take advantage of the membership $\eta\in(0,2(p-1)(2-p)^{-1})$:

$$I_1(\delta)-(2-p)\,\mathrm{Re}\,I_3(\delta)-\eta(2-p)|\,\mathrm{Im}\,I_3(\delta)|=\int_\Omega(|u(x)|^2+\delta)^{(p-4)/2}m(x)^{p-1}$$

$$\times\Bigg\{\left[p-1-\frac{1}{2}\eta(2-p)\right]\sum_{j,k=1}^{n} a_{j,k}(x)\,\mathrm{Re}\left[\overline{u(x)}D_{x_j}u(x)\right]\mathrm{Re}\left[\overline{u(x)}D_{x_k}u(x)\right]$$

$$+\left[1-\frac{1}{2}\eta(2-p)\right]\sum_{j,k=1}^{n} a_{j,k}(x)\,\mathrm{Im}\left[\overline{u(x)}D_{x_j}u(x)\right]\mathrm{Im}\left[\overline{u(x)}D_{x_k}u(x)\right]\Bigg\}\,\mathrm{d}x$$

$$+\,\delta\int_\Omega(|u(x)|^2+\delta)^{(p-4)/2}m(x)^{p-1}\sum_{j,k=1}^{n} a_{j,k}(x)D_{x_j}u(x)\overline{D_{x_k}u(x)}\,\mathrm{d}x \geq 0,\quad \forall\delta\in\mathbf{R}_+. \tag{4.32}$$

We have thus proved (4.26).

Then we note that (4.27) is a consequence of (4.26):

$$\begin{aligned}
&I_1(\delta)+(p-1)\,\mathrm{Re}\,I_2(\delta)-(2-p)\,\mathrm{Re}\,I_3(\delta)-\eta|(p-1)\,\mathrm{Im}\,I_2(\delta)-(2-p)\,\mathrm{Im}\,I_3(\delta)|\\
\geq\ &I_1(\delta)-(2-p)\,\mathrm{Re}\,I_3(\delta)-\eta(2-p)|\,\mathrm{Im}\,I_3(\delta)|+(p-1)[\mathrm{Re}\,I_2(\delta)-\eta|\,\mathrm{Im}\,I_2(\delta)|]\\
\geq\ &-(p-1)(1+\eta^2)^{1/2}|I_2(\delta)|,\quad\forall\delta\in\mathbf{R}_+.
\end{aligned}\tag{4.33}$$

To conclude the proof of the lemma we take into account the relations

$$\begin{aligned}
&\lim_{\delta\to0+}\left[I_1(\delta)+(p-1)\,\mathrm{Re}\,I_2(\delta)-(2-p)\,\mathrm{Re}\,I_3(\delta)\right]\\
&\quad-\eta\lim_{\delta\to0+}|(p-1)\,\mathrm{Im}\,I_2(\delta)-(2-p)\,\mathrm{Im}\,I_3(\delta)|\\
\geq\ &\lim_{\delta\to0+}\Big\{\mathrm{Re}\,I_1(\delta)+(p-1)\,\mathrm{Re}\,I_2(\delta)-(2-p)\,\mathrm{Re}\,I_3(\delta)\\
&\quad-\eta|(p-1)\,\mathrm{Im}\,I_2(\delta)-(2-p)\,\mathrm{Im}\,I_3(\delta)|\Big\}\\
\geq\ &-(p-1)(1+\eta^2)^{1/2}\lim_{\delta\to0+}|I_2(\delta)|,\qquad\forall\delta\in\mathbf{R}_+.
\end{aligned}\tag{4.34}$$

Next, consider the following chain of inequalities, which holds for any $\delta\in\mathbf{R}_+$:

$$\begin{aligned}
&\lim_{\delta\to0+}|I_2(\delta)|\\
\leq\ &\limsup_{\delta\to0+}\int_\Omega(|u(x)|^2+\delta)^{(p-1)/2}\sum_{j,k=1}^{n} m(x)^{p-2}|D_{x_j}m(x)|\,|a_{j,k}(x)D_{x_k}u(x)|\,\mathrm{d}x\\
\leq\ &\limsup_{\delta\to0+}\int_\Omega(|u(x)|^2+\delta)^{p/4}\\
&\quad\times(|u(x)|^2+\delta)^{(p-2)/4}\sum_{j,k=1}^{n} m(x)^{p-2}|D_{x_j}m(x)|\,|a_{j,k}(x)|\,|D_{x_k}u(x)|\,\mathrm{d}x\\
\leq\ &C_1\limsup_{\delta\to0+}\left[\int_\Omega m(x)^{2(p-2+\rho)}(|u(x)|^2+\delta)^{p/2}\,\mathrm{d}x\right]^{1/2}\\
&\quad\times\limsup_{\delta\to0+}\left[\int_\Omega\sum_{j,k=1}^{n}|a_{j,k}(x)D_{x_k}u(x)|^2(|u(x)|^2+\delta)^{(p-2)/2}\,\mathrm{d}x\right]^{1/2}\\
&\qquad\text{(cf. (4.16), (4.17))}
\end{aligned}$$

$$
\begin{aligned}
&\le C_{25} \lim_{\delta\to 0+} \left\{ \left[\int_\Omega m(x)^{2(p-2+\rho)} (|u(x)|^2+\delta)^{p/2}\,\mathrm{d}x \right]^{1/2} \right. \\
&\qquad\qquad \left. \times \left[\int_\Omega (|u(x)|^2+\delta)^{(p-2)/2} |\nabla u(x)|^2\,\mathrm{d}x \right]^{1/2} \right\} \\
&\le C_{26} \left[\int_\Omega m(x)^{2(p-2+\rho)} |u(x)|^p\,\mathrm{d}x \right]^{1/2} \|f\|_p^{p/2} \\
&\le C_{27} \|f\|_p^{p/2} \|Mu\|_p^{p-2+\rho} \|u\|_p^{2-\rho-p/2}. \qquad (4.35)
\end{aligned}
$$

To derive the last inequality we have applied Hölder's inequality with index $q = p[2(p-2+\rho)]^{-1}$ to the integral

$$
\int_\Omega [m(x)|u(x)|]^{2(p-2+\rho)} |u(x)|^{-p+4-2\rho}\,\mathrm{d}x.
$$

From (4.34) and (4.35) we immediately conclude (4.28). □

Taking now the real part and the modulus of the imaginary part in (4.21) and using (4.22), we easily derive the relations

$$
\begin{aligned}
&\operatorname{Re}\lambda \|Mu\|_p^p + \lim_{\delta\to 0+} [I_1(\delta) + (p-1)\operatorname{Re} I_2(\delta) - (2-p)\operatorname{Re} I_3(\delta)] \\
&\quad + \int_\Omega a_0(x) m(x)^{p-1} |u(x)|^p\,\mathrm{d}x = \operatorname{Re} \int_\Omega m(x)^{p-1} f(x)\overline{u(x)} |u(x)|^{p-2}\,\mathrm{d}x, \qquad (4.36) \\
&|\operatorname{Im}\lambda| \|Mu\|_p^p \le \lim_{\delta\to 0+} |[(p-1)\operatorname{Im} I_2(\delta) - (2-p)\operatorname{Im} I_3(\delta)]| \\
&\quad + \left| \operatorname{Im} \int_\Omega m(x)^{p-1} f(x)\overline{u(x)} |u(x)|^{p-2}\,\mathrm{d}x \right|, \qquad \forall \lambda \in \mathbf{C}. \qquad (4.37)
\end{aligned}
$$

Add now member by member (4.36) and (4.37) multiplied by $\eta \in (0, 2\sqrt{p-1}(2-p)^{-1})$ and use (4.28) and (2.2). We easily deduce the following estimate for any $\lambda \in \Sigma =: \{\mu \in \mathbf{C} : \operatorname{Re}\mu + \eta|\operatorname{Im}\mu| \ge 0\}$:

$$
\begin{aligned}
&\left[\operatorname{Re}\lambda + \eta|\operatorname{Im}\lambda| + \frac{\gamma}{\|m\|_\infty} \right] \|Mu\|_p^p \\
&\le -\Big[\lim_{\delta\to 0+} [I_1(\delta) + (p-1)\operatorname{Re} I_2(\delta) - (2-p)\operatorname{Re} I_3(\delta)] \\
&\qquad - \eta \lim_{\delta\to 0+} |[(p-1)\operatorname{Im} I_2(\delta) - (2-p)\operatorname{Im} I_3(\delta)]| \Big] \\
&\quad + \operatorname{Re} \int_\Omega f(x) m(x)^{p-1} \overline{u(x)} |u(x)|^{p-2}\,\mathrm{d}x + \eta \left| \operatorname{Im} \int_\Omega f(x) m(x)^{p-1} \overline{u(x)} |u(x)|^{p-2}\,\mathrm{d}x \right| \\
&\le -\lim_{\delta\to 0+} \big[[I_1(\delta) + (p-1)\operatorname{Re} I_2(\delta) - (2-p)\operatorname{Re} I_3(\delta)] \\
&\qquad - \eta |[(p-1)\operatorname{Im} I_2(\delta) - (2-p)\operatorname{Im} I_3(\delta)]| \big]
\end{aligned}
$$

$$+ \operatorname{Re} \int_\Omega f(x)m(x)^{p-1}\overline{u(x)}|u(x)|^{p-2}\,dx + \eta \left| \operatorname{Im} \int_\Omega f(x)m(x)^{p-1}\overline{u(x)}|u(x)|^{p-2}\,dx \right|$$
$$\le C_{28}\|f\|_p^{p/2}\|Mu\|_p^{p-2+\rho}\|u\|_p^{2-\rho-p/2} + (1+\eta^2)^{1/2}\|f\|_p\|Mu\|_p^{p-1}. \tag{4.38}$$

Take λ in the sector

$$\Sigma_3 = \left\{ \mu \in \mathbf{C} : \operatorname{Re}\mu + \frac{\eta}{2}|\operatorname{Im}\mu| + \frac{\gamma}{2\|m\|_\infty} \ge 0 \right\}. \tag{4.39}$$

Then, since $\|u\|_p \le C_{19}\|f\|_p$ (cf. (2.11), (2.12) and our definition of η) and $2-\rho-p/2 > 0$ (cf. (4.20)), by Proposition 2.1 we immediately derive the inequality

$$(|\lambda|+1)\|Mu\|_p^{2-\rho} \le C_{24}\big[\|f\|_p^{2-\rho} + \|f\|_p\|Mu\|_p^{1-\rho}\big], \qquad \text{if} \quad \lambda \in \Sigma_3. \tag{4.40}$$

Finally, $\|Mu\|_p \le \|m\|_\infty\|u\|_p \le C_{19}\|m\|_\infty\|f\|_p$ implies

$$(|\lambda|+1)\|Mu\|_p^{2-\rho} \le C_{30}\|f\|_p^{2-\rho}, \qquad \text{if} \quad \lambda \in \Sigma_3. \tag{4.41}$$

We can now collect the result in this section in the following Theorem 4.1.

Theorem 4.1. *Let L and M be the linear operators defined by* (1.7) *and* (1.8), *the coefficients $a_{i,j}$ $i,j = 1,\dots,n$, a_0 enjoying properties* (2.1) *and* (2.2) *and m being a non-negative function satisfying* (1.9). *Then the spectral equation $\lambda Mu + Lu = f$, with $f \in L^p(\Omega)$, admits, for any $\lambda \in \Sigma_3$ and $p \in (1,2)$, $\rho \in [2-p, 1]$, a unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ satisfying the estimates*

$$\|u\|_p \le C_{30}\|f\|_p, \qquad \|Mu\|_p \le C_{31}(p)|\lambda|^{-(2-\rho)^{-1}}\|f\|_p, \qquad \lambda \in \Sigma_3,$$
$$\|Lu\|_p \le C_{32}(1+|\lambda|^{(1-\rho)(2-\rho)^{-1}})\|f\|_p, \qquad \lambda \in \Sigma_3. \tag{4.42}$$

EXAMPLE 4.1. Let $n = 1$, $m(x) = x^q(1-x)^q$, $q \in (1,+\infty)$, $\Omega = (0,1)$. Then

$$m'(x) = q(1-2x)m(x)^{(q-1)/q}, \qquad x \in (0,1).$$

Hence (4.25) holds true for any $q \in (1,+\infty)$. If we have to deal with $L^p(0,1)$ with $p \in (1,2)$, to satisfy (4.20) we are forced to assume $q > (p-1)^{-1}$.

5. Solving problem (1.1)–(1.3)

Taking the spectral Theorems 2.1, 3.1, 4.1 into account, from Theorem 3.26 in [3] we can easily derive our existence and uniqueness result. For this purpose we need to introduce the following interpolation space

$$L^p_{\theta,\infty} = \left\{ g \in L^p(\Omega) : \sup_{t\ge 1} t^\theta \|L(tM+L)^{-1}\|_{L^p(\Omega)} < +\infty \right\}. \tag{5.1}$$

In particular, any $g = mh$ belongs to $L^p_{\theta,\infty}$, whenever $m \in L^\infty(\Omega)$ and $h \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$. Notice that $L^p_{\theta,\infty} \subset (X; D(LM^{-1}))_{\theta,\infty}$.

Theorem 5.1. *Let $p \in (1, +\infty)$, let $m \in L^\infty(\Omega)$ be a non-negative function and let the coefficients $a_{i,j}$ $i, j = 1, \dots, n$, a_0 enjoy properties* (2.1) *and* (2.2). *Then for any*

$$u_0 \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega), \quad f \in C^\theta([0,T]; L^p(\Omega)), \quad \theta \in (1-\beta, 1), \tag{5.2}$$

with $\beta = 1/p$ and

$$-A(x, D_x)u_0 + f(0,\cdot) = g_0, \qquad g_0 \in L^p_{\theta,\infty}, \tag{5.3}$$

problem (1.1)–(1.3) *admits a unique solution*

$$mu \in C^{\theta+\beta}([0,T]; L^p(\Omega)), \qquad u \in C^{\theta+\beta-1}([0,T]; W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)). \tag{5.4}$$

Moreover, if m is a non-negative function satisfying (1.9) *and β is defined by* (1.10), *the same result holds under assumptions* (5.1) *and* (5.2) *on* (u_0, f).

6. Proofs of the propositions

Proof of Proposition 2.1. Let $\lambda \in \Sigma_{k,\varepsilon}$ and $\operatorname{Re}\lambda \geq 0$. Then it is clear that $|\operatorname{Re}\lambda| \leq \operatorname{Re}\lambda + k|\operatorname{Im}\lambda| + \varepsilon$. On the other hand, if $\lambda \in \Sigma_{k,\varepsilon}$ and $\operatorname{Re}\lambda < 0$, then $|\operatorname{Re}\lambda| = -\operatorname{Re}\lambda \leq (k/2)|\operatorname{Im}\lambda| + (\varepsilon/2) \leq \operatorname{Re}\lambda + k|\operatorname{Im}\lambda| + \varepsilon$. Therefore, $|\operatorname{Re}\lambda| \leq \operatorname{Re}\lambda + k|\operatorname{Im}\lambda| + \varepsilon$ for any $\lambda \in \Sigma_{k,\varepsilon}$. In the meantime it is obvious that $|\operatorname{Im}\lambda| + 1 \leq 2\{(1/k)+(1/\varepsilon)\}\{(k/2)|\operatorname{Im}\lambda|+(\varepsilon/2)\} \leq 2(1/k+(1/\varepsilon))(\operatorname{Re}\lambda+k|\operatorname{Im}\lambda|+\varepsilon)$ for any $\lambda \in \Sigma_{k,\varepsilon}$. Hence we conclude that $|\lambda|+1 \leq |\operatorname{Re}\lambda|+|\operatorname{Im}\lambda|+1 \leq \{2/k+(2/\varepsilon)+1\}(\operatorname{Re}\lambda+k|\operatorname{Im}\lambda|+\varepsilon)$, $\lambda \in \Sigma_{k,\varepsilon}$. □

Proof of Proposition 2.2. We consider the set $J = \{\theta \in [0,1]; \mathcal{R}(A(\theta)) = X\}$ and shall prove that this set is an open and closed subset of the interval $[0,1]$ under (2.17) and (2.18). In fact, let $\theta \in J$; then, it follows from (2.17) that $A(\theta)^{-1} \in \mathcal{L}(X)$ with $\|A(\theta)^{-1}\| \leq \delta^{-1}$. Moreover, for any $\theta' \in [0,1]$, we have

$$A(\theta') = [1 + \{A(\theta') - A(\theta)\}A(\theta)^{-1}]A(\theta).$$

Since $\|\{A(\theta')-A(\theta)\}A(\theta)^{-1}\| \leq N\delta^{-1}|\theta'-\theta|$, the operator $1+\{A(\theta')-A(\theta)\}A(\theta)^{-1}$ is a linear isomorphism of X provided $|\theta'-\theta| < N^{-1}\delta$. This then shows that $\theta' \in J$ for any θ' such that $|\theta'-\theta| < N^{-1}\delta$; hence, J is an open set. Consider now a sequence $\theta_n \in J$ and assume that $\theta_n \to \overline{\theta}$ as $n \to +\infty$. Let $f \in X$ be any vector; then, there exists a sequence $u_n \in \mathcal{D}$ such that $A(\theta_n)u_n = f$. From (2.17) it follows that $\|u_n\| \leq \delta^{-1}\|f\|$. Furthermore we observe that $\|A(\overline{\theta})u_n - f\| \leq \|\{A(\overline{\theta}) - A(\theta_n)\}u_n\| \leq N\delta^{-1}|\overline{\theta} - \theta_n|\|f\|$;

therefore, $A(\overline{\theta})u_n \to f$ as $n \to +\infty$. In the meantime, $\delta\|u_m - u_n\| \le \|A(\overline{\theta})(u_m - u_n)\| \le \|A(\overline{\theta})u_m - f\| + \|f - A(\overline{\theta})u_n\| \to 0$ as $m, n \to +\infty$. So, u_n has a limit $u \in X$ as $n \to +\infty$. Since $A(\overline{\theta})$ is a closed operator, $u \in \mathcal{D}$ and $A(\overline{\theta})u = f$; hence, $\overline{\theta} \in J$. That is, J is a closed set. As $1 \in J \neq \emptyset$, we conclude that $J = [0, 1]$. □

Proof of Proposition 4.1. According to (1.9), we have the inclusion $Z(m) \subset Z(\nabla m)$. Moreover, formula (4.23) is trivial if $x \notin Z(m)$. This therefore shows that we have to deal with the case $x \in Z(m)$ only.

First we will consider the one-dimensional case ($n = 1$). For this purpose assume $x_0 \in Z(m)$. Our starting point is the following formula:

$$\begin{aligned}\lim_{x\to x_0}\left|\frac{m(x)^\beta - m(x_0)^\beta}{x - x_0}\right| &= \lim_{x\to x_0}\left|\lim_{\varepsilon\to 0+}\frac{[m(x)+\varepsilon]^\beta - \varepsilon^\beta}{x - x_0}\right| \\ &= \lim_{x\to x_0}\left|\lim_{\varepsilon\to 0+}\frac{\beta}{x - x_0}\int_{x_0}^{x}[m(t)+\varepsilon]^{\beta-1}m'(t)\,\mathrm{d}t\right|. \end{aligned}\tag{6.1}$$

We next notice that $\lim_{\varepsilon\to 0+}[m(t)+\varepsilon]^{\beta-1}m'(t) = m_1(t)$ for any $t \in \Omega$ and that

$$\begin{aligned}|[m(t)+\varepsilon]^{\beta-1}m'(t)| &\le C[m(t)+\varepsilon]^{\beta-1}m(t)^\rho \\ &= C\left[\frac{m(t)}{m(t)+\varepsilon}\right]^{1-\beta} m(t)^{\beta-1+\rho} \le Cm(t)^{\beta-1+\rho}, \qquad \forall t \in \Omega.\end{aligned}$$

By virtue of the dominated convergence theorem and by the bound $|m_1(t)| \le Cm(t)^{\beta-1+\rho}$ for any $t \in \overline{\Omega}$, we deduce the following relations:

$$\begin{aligned}&\lim_{x\to x_0+}\left|\frac{m(x)^\beta - m(x_0)^\beta}{x - x_0}\right| = \lim_{x\to x_0+}\left|\frac{1}{x - x_0}\int_{x_0}^{x} m_1(t)\,\mathrm{d}t\right| \\ &\le \lim_{x\to x_0+}\frac{1}{x - x_0}\int_{x_0}^{x}|m_1(t)|\,\mathrm{d}t \le \lim_{x\to x_0+}\frac{C}{x - x_0}\int_{x_0}^{x} m(t)^{\beta-1+\rho}\,\mathrm{d}t = 0.\end{aligned}\tag{6.2}$$

Note here that $m(\cdot)^{\beta-1+\rho}$ is continuous in Ω and $x_0 \in Z(m)$. An analogous argument holds for $\lim_{x\to x_0-}|\{m(x)^\beta - m(x_0)^\beta\}/(x - x_0)|$ also.

We have thus shown that there exists $D_x[m(\cdot)^\beta](x_0)$ and coincides with $0 = m_1(x_0)$. Therefore the formula $D_x[m(\cdot)^\beta](x) = m_1(x)$ holds for any $x \in \Omega$. Since $\beta \in (1-\rho, 1)$, bound (1.9) and (4.23) immediately imply that $m_1 \in C(\Omega)$. Consequently, $m(\cdot)^\beta \in C(\Omega)$.

Finally, the multi-dimensional case is an immediate consequence of the case $n = 1$. □

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