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SELF-COINCIDENCE OF FIBRE MAPS

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Abstract

We study coincidence points for maps $f_1, f_2\colon E \to B$ into manifolds such that f_1 is homotopic to f_2. We analyze the first and higher obstructions to deform f_1 away to f_2. The main results consist in solving this one problem for the (generalized) Hopf bundles, which are G-principal bundles $p_nG\colon E_nG \to B_nG$ (the n-th stage of Milnor's construction), with $G = S^1, S^3$. We also consider the question for general maps $f\colon E_nG \to B_nG$ with $G = S^1, S^3$.

1. Introduction

Given two maps $f_1, f_2\colon E \to B$ we study the problem of making them coincidence-free, i.e. deforming them, $f_1 \approx g_1$, $f_2 \approx g_2$, such that $g_1(x) \neq g_2(x)$, $\forall x \in E$. For the most and main results we assume B to be a manifold. This has the advantage that we need only deform one of the maps, say $f_2 \approx g_2'$, and obtain $f_1(x) \neq g_2'(x)$ whenever $g_1(x) \neq g_2(x)$ (see [1]). We use the following notation: $f_1 \parallel f_2$ if $f_1(x) \neq f_2(x)$, $\forall x \in E$ and $f_1 \mathrel{|\wr} f_2$ if f_2 is homotopic to some g such that $f_1 \parallel g$.

For instance, if $\dim E < \dim B$ (both paracompact) then $f_1 \mathrel{|\wr} f_2$ for all $f_1, f_2\colon E \to B$. If $\dim E = \dim B$ and B is connected, non-compact, then again $f_1 \mathrel{|\wr} f_2$, $\forall f_1, f_2$. If B is compact connected and $\dim E = \dim B = n$ then there is one potential obstruction which can be regard as an element of $H^n(E, \underline{\mathbb{Z}})$ (with some local coefficient system $\underline{\mathbb{Z}}$, see Proposition 2.11) for $f_1 \mathrel{|\wr} f_2$.

New viewpoints arise when $\dim E > \dim B$, and this is the subject of the present paper. We consider simple cases, however, e.g. fibre bundles $p\colon E \to B$ and ask whether $p \mathrel{|\wr} p$. For the classical Hopf bundles $p_n\mathbb{C}\colon S^{2n+1} \to \mathbb{C}P_n$, $p_n\mathbb{H}\colon S^{4n+3} \to \mathbb{H}P_n$, we obtain

Theorem 1.1 (see Theorem 3.5). *We have* $p_n\mathbb{C} \mathrel{|\wr} p_n\mathbb{C} \Longleftrightarrow 2|n+1$.

Theorem 1.2 (see Theorem 3.9). $p_n\mathbb{H} \mathrel{|\wr} p_n\mathbb{H} \Longleftrightarrow 24|n+1$.

General principal S^1- resp. S^3-bundles $p_n^k\colon E_n^k \to \mathbb{C}P_n$ resp. $p_n^\gamma\colon E_n^\gamma \to \mathbb{H}P_n$ are

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classified by $k \in Map[\mathbb{C}P_n, \mathbb{C}P_\infty] = H^2\mathbb{C}P_\infty = \mathbb{Z}$ resp. $\gamma \in Map[\mathbb{H}P_n, \mathbb{H}P_\infty] \xrightarrow{k} H^4(\mathbb{H}P_n) = \mathbb{Z}$, where in the latter case the map k is defined right before Proposition 3.3.

Theorem 1.3 (see Theorem 3.2). $p_n^k \wr p_n^k \Longrightarrow k|n+1$.

Theorem 1.4 (see Theorem 3.4). $p_n^\gamma \wr p_n^\gamma \Longrightarrow k(\gamma)|n+1$.

Examples of non-bundles which we can handle are the integral multiples of $[p_n\mathbb{C}] \in \pi_{2n+1}(\mathbb{C}P_n)$ and $[p_n\mathbb{H}] \in \pi_{4n+3}(\mathbb{H}P_n)$, as follows

Proposition 1.5 (see Proposition 4.3). *If n is even then $lp_n\mathbb{C} \wr lp_n\mathbb{C} \Longleftrightarrow 2|l$.*

Proposition 1.6 (see Proposition 4.4). *If $n+1$ is not divisible by 24 then*:
(1) $lp_1\mathbb{H} \wr lp_1\mathbb{H}$ *if* $12|l$
(2) *For* $n > 1$, $lp_n\mathbb{H} \wr lp_n\mathbb{H}$ *if* $24|l$.

For the case where $G = S^0$ (the cyclic group $\mathbb{Z}_2$) then we have the maps $p_n : S^n \to \mathbb{R}P_n$. So we have maps between manifolds of the same dimension. We leave to the reader to verify that $p_n \wr p_n \Longleftrightarrow 2|n+1$.

The paper is divided into three sections. In Section 2 we discuss generalities about the obstruction to make (f, f) coincidence free, when $f : E \to B$ is continuous or differentiable. Then we compute the primary obstruction assuming that B is a manifold (Proposition 2.11). Also, we express the problem of removing $coin(p, p)$ in two different ways: one in terms of the existence of a lifting of the map $f : E \to B$ into the sphere bundle of the tangent bundle of B, (Proposition 2.13) and the other in terms of nowhere-zero cross section of the horizontal tangent bundle of f (Proposition 2.16).

In Section 3, using the explicit calculation of the cohomology of G-principal bundles over B_nG, for G either equal to S^1 or S^3, and the primary obstruction, we show that (p_n^k, p_n^k) and (p_n^γ, p_n^γ) can not be made coincidence free for certain pairs of values of k, n and $k(\gamma)$, n, respectively (Theorems 3.2 and 3.4). Then, by using results of Section 2, we prove the cases where $k = 1$ and γ induces the fibre map p_nS^3, which are Theorems 3.5 and 3.9. Also some other G-bundles over B_nG ($G = S^1$, S^3) are considered.

In Section 4 we analyze for which multiples of a map $f : S^n \to B$, can $(l \cdot f, l \cdot f)$ be made coincidence free. In the appendix we show that for Hopf fibration $H : S^{15} \to S^8$, (H, H) can not be made coincidence free. We were not able to prove this last result, using the techniques developed in Sections 3 and 4.

2. Generalities

Let $p\colon E \to B$ denote a fibration, and $f\colon E \to B$ an arbitrary map. In this section we study the problem of making $(f, f)\colon E \to B$ coincidence free. We consider several cases, assuming certain hypotheses on the spaces and on the map f. The case where f is a fibration, introduces some simplifications and will be treated in the next section. A variation of the above problem is to study the problem of making $(f,\ f)\colon E \to B$ coincidence free by small deformation. The precise formulation of this question is given after Proposition 2.12 in terms of the normal bundle of the graph of f. We use the following notations and definitions: *A pair of maps $f, g\colon E \to B$ is disjoint* (*coincidence free*), *in symbols $f \parallel g$*, if $f(x) \neq g(x)$, $\forall x \in E$. *They are homotopy disjoint, in symbol $f \mathrel{|\wr} g$*, if deformations $f_t\colon E \to B$, $g_t\colon E \to B$, $0 \le t \le 1$ exist such that $f_0 = f$, $g_0 = g$, and $f_1 \parallel g_1$. Otherwise *they are not disjoint, in symbols $f \nparallel g$*, or *they are not homotopy disjoint, in symbols $f \mathrel{\not{|\wr}} g$*, respectively. We have chosen the unsymetric $|\wr$ notation because in many cases one of the deformations is redundant. For example, if B is a manifold Proposition 2.4 shows the problem of making $(f, g)\colon E \to B$ coincidence free by deforming both maps is equivalent to the problem of making them coincidence free by either deforming the first map or the second map.

For $f\colon E \to B$ an arbitrary map we have:

Proposition 2.1. *If the identity of B can be deformed to a fixed point free map, then for all $f\colon E \to B$, $f \mathrel{|\wr} f$ by composing f with the given deformation.*

Corollary 2.2. *If the base B is a non-compact connected manifold, then for all $f\colon E \to B$ $f \mathrel{|\wr} f$. If B is a compact connected manifold and $\chi(B) = 0$, then $f \mathrel{|\wr} f$ for all $f\colon E \to B$.*

Proof. In both cases B admits a deformation without fixed points. □

Proposition 2.3. *Let G be a topological group which acts freely on a space E, and $K \subset G$ a subgroup. If the connected path component of G which contains the identity is different from the connected component of K which contains the identity* (*i.e. $K_e \neq G_e$*), *then we have $q \mathrel{|\wr} q$, where $q\colon E \to E/K$ is the projection.*

Proof. Take any element $\gamma \in G_e - K$. As $\gamma \notin K$ and the G-action is free we have that $q(\gamma x) \neq q(x)$, for all $x \in E$. On the other hand, as $\gamma \in G_e$, there is path $\omega\colon [0, 1] \to G$ with $\omega(0) = \gamma$, $\omega(1) = e$ = identity. Therefore $\omega(t) \cdot x$ provides a deformation of $\gamma\cdot$ into id, hence $q\gamma \approx q$ □

EXAMPLE. If $G = S^3$ acts freely on E, and $K = S^1 \subset S^3$ then Proposition 2.3 applies to the projection $q\colon E \to E/K$, i.e. $q \mathrel{|\wr} q$. If $E = E_nG = S^{4n+3} = E_{2n+1}K$ this gives $q \wr\wr q$ for the Hopf map $q = p_{2n+1}K\colon S^{4n+3} \to \mathbb{C}P_{2n+1}$. This proves a half of Theorem 3.5.

For $(f_1, f_2)\colon E \to B$ an arbitrary pair of maps, where B is assumed to be a manifold, we have (see [1, Theorem 1]):

Proposition 2.4. *If f_1' and f_2' are homotopic to f_1 and f_2, respectively, then there exists $\overline{f}_2$ homotopic to f_2 such that*

$$coin(f_1', f_2') \supset coin(f_1, \overline{f}_2).$$

Proof. The following is a simple proof. Let $(B \times B, B \times B - \Delta) \xrightarrow{p_1} B$, where p_1 is the projection in the first coordinate. Since B is a manifold, by [3], this is a fibre pair. Let H be a homotopy from f_1' to f_1. This homotopy at the level $t = 0$ has a lifting, namely $(f_1', f_2')\colon E \to B \times B$, $(f_1', f_2')(E - coin(f_1', f_2')) \subset B \times B - \Delta$. By the lifting property of fibre pair, the homotopy H has a lifting H' such that $H'(\cdot, t)(E - coin(f_1', f_2')) \subset B \times B - \Delta$, for all $t \in [0, 1]$. In particular, $H'(\cdot, 1)(E - coin(f_1', f_2')) \subset B \times B - \Delta$. Hence $H'(\cdot, 1)$ is of the form $(f_1, \overline{f}_2)$. $\square$

Proposition 2.5. *Let $p\colon E \to B$ be a fibration and $\varphi\colon E \to E$ a map then $p \wr\wr p \circ \varphi \iff p \parallel p \circ \varphi_1$ for some $\varphi_1 \simeq \varphi$.*

Proof. If $p \wr\wr p \circ \varphi$ then there is a homotopy $f_t\colon E \to B$, $0 \le t \le 1$, with $f_0 = p \circ \varphi$ and $f_1(x) \neq p(x)$, $\forall x \in E$. Since p is a fibration there is a covering homotopy $\varphi_t\colon E \to E$ with $\varphi_0 = \varphi$, and $p \circ \varphi_1 = f_1$. This proves one direction, the other being clear. $\square$

REMARK. If $i\colon A \to B$ is a cofibration and $\psi\colon B \to B$ a map then $i \wr\wr \psi \circ i \iff i \parallel \psi_1 \circ i$ for some $\psi_1 \simeq \psi$.

Corollary 2.6. *Let $p_nS^1\colon S^{2n+1} \to \mathbb{C}P_n$ be the Hopf bundle. If $p_nS^1 \wr\wr f$, then f is homotopic to p_nS^1.*

Proof. The map $f\colon S^{2n+1} \to \mathbb{C}P_n$ lifts to a map $\varphi\colon S^{2n+1} \to S^{2n+1}$ because $p_nS^1\colon S^{2n+1} \to \mathbb{C}P_n$ generates $\pi_{2n+1}(\mathbb{C}P_n)$. By Proposition 2.5, the map φ is homotopic to a fixed point free map. Hence it has degree one. $\square$

Proposition 2.7. *If $p\colon E \to B$ is a fibration then every map $\overline{p}$ homotopic to p is equal to $p \circ \varphi$, where $\varphi\colon E \to E$ is (a deformation) homotopic to the identity.*

Proof. This follows by the covering homotopy property. $\square$

Corollary 2.8. *We have $p \wr p$ if and only if there is a deformation $\varphi\colon E \to E$ of the identity such that $p(x) \neq p \circ \varphi(x)$ for all $x \in E$.*

REMARK. In Theorem 3.5 we will show that $p_nS^1 \not\wr p_nS^1$ for n even. This shows that p_nS^1 has the property that the pair (p_nS^1, f) has always a coincidence point for any map f i.e. p_nS^1 *has the coincidence point property* (in the notation of [2], p_nS^1 is a coincidence producing map). Note that $\mathbb{C}P_n$ (n even) has the fixed point property which is the same as the identity has the coincidence point property.

Let $[A_n] \in [\mathbb{C}P_n \to \mathbb{C}P_n]$ be the unique homotopy class of maps which corresponds to the maps inducing in $\pi_2(\mathbb{C}P_n)$ multiplication by -1. We have that $\mathbb{C}P_n$ is a simply connected manifold and, if n is odd, the Lefschetz number of a map in the homotopy class is zero. By the converse of the Lefschetz fixed point theorem there is A_n in the homotopy class which is fixed point free. In fact we can define explicitly one such map by the formula $A_{2m+1}[\xi_0, \xi_1, \ldots, \xi_{2m}, \xi_{2m+1}] = [-\bar{\xi}_1, \bar{\xi}_0, -\bar{\xi}_3, \bar{\xi}_2, \ldots, -\bar{\xi}_{2m+1}, \bar{\xi}_{2m}]$. Thus,

Corollary 2.9. *$A_{2m+1} \circ p_{2m+1}$ is homotopic to p_{2m+1}, $\forall m$.*

REMARK. Corollary 2.9 shows one of the implications of Theorem 3.5, i.e. if n is odd $\Longrightarrow p_nS^1 \wr p_nS^1$.

Proposition 2.10. *If $B = S^n$, then $f_1 \wr f_2$ implies that $A \circ f_2$ is homotopic to f_1, where A is the antipodal map on S^n.*

Proof. Let f_1' be a map homotopic to f_1 and $f_1' \parallel f_2$. Then f_1' and $A \circ f_2$ have distance less than π so they are homotopic and we have $f_1 \approx A \circ f_2$. $\square$

E. Fadell and S. Hussein in [4] developed the fixed point theory in terms of classical obstruction theory. Their framework applies also for coincidence theory. In most of cases, the problem of deforming a pair of maps to coincidence free is equivalent to the problem of showing that certain higher obstructions vanish. As a first stage in this direction we compute primary obstructions to deform away self-coincidences, i.e. to deform (f, f) to coincidence free pairs, where $f\colon E \to B$ is an arbitrary continuous map and B is a compact. In most cases, however, we'll have to deal with higher obstructions. Also under the hypothesis that the domain is a co-H-space, we compare the obstruction to deform the pair $(f+g, f+g)$ to coincidence free with the obstructions to deform the pairs (f, f) and (g, g) to coincidence free. The background in obstruction theory can be found in [8] or [11].

Proposition 2.11. *If B is a connected n-dimensional manifold (compact, in view of* Corollary 2.2) *and $f: E \to B$, then the primary obstruction to lift (f, f) in*

$$\begin{array}{ccc} & & B \times B - \Delta \\ & & \downarrow \\ E & \xrightarrow{f,f} & B \times B \end{array}$$

by deformation, is the f^*-image of the primary obstruction to lift (id, id) in

$$\begin{array}{ccc} & & B \times B - \Delta \\ & & \downarrow \\ B & \xrightarrow{id,id} & B \times B \end{array}$$

by deformation. The latter is the twisted Euler class of B, i.e. $= \chi(B)\cdot\mu_B$, where $\chi(B)$ is the Euler characteristic of B and $\mu_B = \pi_1 B$-twisted fundamental class.

Proof. That the primary obstruction to lift (id, id) is the Euler twisted class follows from [4, Remark 4.8]. The rest is by naturality. □

Proposition 2.12. *Let E be a co-H-space and $f, g: E \to B$ two maps. If for given i an obstruction Θ^i is defined both for f and for g then also for $f + g$, and for these Θ^i, $\Theta^i(f + g) = \Theta^i(f) + \Theta^i(g)$.*

Proof. Consider the diagram:

$$\begin{array}{ccccccccc} & & & & & & & & B \times B - \Delta \\ & & & & & & & & \downarrow \\ E & \xrightarrow{m} & E \vee E & \xrightarrow{f \vee g} & B \vee B & \xrightarrow{\nabla} & B & \xrightarrow{\Delta} & B \times B \end{array}$$

The obstruction to lift $\Delta \circ \nabla \circ (f \vee g)$ is the cohomology class defined by the sum of two cocycles c_1, c_2, where each one c_1, c_2 represents the obstruction to lift $\Delta \circ \nabla \circ (f \vee g)$ restricted to each one of the two copies of E in $E \vee E$. Hence c_1, c_2 represent $\Theta^i(f)$, $\Theta^i(g)$, respectively. So the composite with m represents $\Theta^i(f + g)$. □

A variation of our original question arises naturally in our discussion. Given $p: E \to B$ denote by $\Gamma_p \subset E \times B$ the graph of p. We define: *A pair of maps* $(f, f): E \to B$ *is homotopy disjoint by small deformation* if $N\Gamma_p$, the normal bundle of Γ_p, admits a nowhere-zero cross section. It follows from the definition that if a pair (f, f) is homotopy disjoint by small deformation then it is homotopy disjoint. The converse is likely not to be true.

In the next result we give a homotopy condition which is equivalent (f, f) be homotopy disjoint by small deformation. Also, under certain hypotheses, we show that being homotopy disjoint by small deformation is equivalent to being homotopy disjoint. Let $f\colon E \to B$ be a continuous map (E arbitrary), and B a compact differentiable manifold. Let τ_B be the tangent bundle of the differentiable manifold B, $S(\tau_B)$ the sphere bundle and $q\colon S(\tau_B) \to B$ the projection map. We have:

Proposition 2.13. *The map $f\colon E \to B$ admits a lift to $S(\tau_B)$ if and only if (f, f) is homotopy disjoint by small deformation.*

Proof. The lift provides a nowhere-zero vector field transverse to Γ_p. So we obtain a nowhere-zero cross section of the normal bundle. The converse is similar. $\square$

Now we will show some results that if a pair is homotopy disjoint then it is also homotopy disjoint by a small deformation and will be used in our applications. Let E be a CW complex, $n = \dim B$ and $i\colon (D^n, S^{n-1}) \to (B, B - y)$ be the inclusion of a small closed n-dimensional disk around the point y.

Lemma 2.14. *Let $f\colon E \to B$ be a map and $H^k(E, \pi_k(B, B - y)) = H^k(E, \pi_k(B)) = 0$ for $k < m_0$. If $m_0 < 2\dim B - 2$ or $i_\#\colon \pi_{m_0}(D^n, S^{n-1}) \to \pi_{m_0}(B, B - y)$ is a split monomorphism, then the $m - th$ obstruction to have $f \wr f$ vanishes if and only if the $m - th$ obstruction for $f\colon E \to B$ admits a lift to $S(\tau_B)$ vanishes.*

Proof. Because of the hypotheses the m_0-th obstruction is well defined in both situation. Suppose that the m-th obstruction for $f\colon E \to B$ admits a lift to $S(\tau_B)$ is zero. Then the map $f\colon E \to B$ restricted to the m_0-skeleton, denoted by f_1, admits a lift to $S(\tau_B)$, which we denote by $\overline{f}_1$. Then by Proposition 2.13 the pair (f_1, f_1) is homotopy disjoint by small deformation; hence $f_1 \wr f_1$ and the m-th obstruction to have $f \wr f$ vanishes.

Conversely suppose the m-th obstruction to have $f \wr f$ vanishes. Let $i\colon (D^n, S^{n-1}) \to (B, B-y)$ be the inclusion of a closed disk and $j\colon (B, B-int(D^n)) \to (S^n, y_0)$ the map of degree one which takes the subspace $B-int(D^n)$ into the point y_0. The composite induces a map $\pi_k(D^n, S^{n-1}) = \pi_{k-1}(S^{n-1}) \to \pi_k(S^n, y_0)$. It is not hard to see that this homomorphism is the suspension homomorphism. Now we argue by obstruction theory. Consider the map of pairs $exp\colon (D(\tau_B), S(\tau_B)) \to (B\times B, B\times B-\Delta)$ given by the exponential map where $D(\tau_B)$ is the disk bundle of the tangent bundle of B. The inclusions $i_1\colon (D^n, S^{n-1}) \to (D(\tau_B), S(\tau_B))$ and $i_2\colon (B, B - y) \to (B \times B, B \times B - \Delta)$ induce isomorphisms in the homotopy groups. So we can look at the obstructions to lift to the sphere bundle and to deform to coincidence free having coefficients in $\pi_k(D^n, S^{n-1})$ and $\pi_k(B, B-y)$, respectively. Let $\beta \in H^{m_0}(E, \pi_{m_0-1}(S^{n-1}))$ be the m_0-th obstruction to find a lifting to $S(\tau_B)$. Since $j_*(\beta)$ is the obstruction at dimension m_0 to have $p \wr\!\wr p$, we have $j_*(\beta) = 0$. Since the suspension homomorphism is an isomorphism for $m_0 < 2n-2$, we conclude that $\beta = 0$. Finally if $i_\#: \pi_{m_0}(D^n, S^{n-1}) \to \pi_{m_0}(B, B-y)$ is a split monomorphism, then $H^{m_0}(E; \pi_{m_0}(D^n, S^{n-1}))$ is a summand of $H^{m_0}(E; \pi_{m_0}(B, B-y))$. □

Corollary 2.15. *Let E be the m-sphere and B either a sphere of dimension* 2, 4 *or* 8, *or a* $K(\pi, 1)$. *Then the following conditions are equivalent*:

(1) $f \wr\!\wr f$
(2) *the map* $f: E \to B$ *has a lift to* $S(\tau_B)$
(3) (f, f) *is homotopy disjoint by small deformation.*

Proof. Since (2) and (3) are equivalent and (3) implies (1), it is enough to show that (1) implies (2). Let dimension of B be either 2,4 or 8. Since the suspension map $\pi_k(D^n, S^{n-1}) = \pi_{k-1}(S^{n-1}) \to \pi_k(S^n, y_0)$ has a left inverse for $n = 2$, 4 and 8, see [9, Introduction] or [8, 21.2], we have in particular that $i_\#: \pi_m(D^n, S^{n-1}) \to \pi_m(B, B-y)$ is a split monomorphism and the result follows by Lemma 2.14. Now let B be a $K(\pi, 1)$. Call $\tilde{B}$ the universal cover of B. Then $\tilde{B}$ is a contractible space and the homotopy groups $\pi_j(B, B-y)$ are isomorphic to $\pi_j(\tilde{B}, \tilde{B} - \Gamma)$, where Γ is a discrete subset which has the cardinality of the group $\pi_1(B)$. But $\tilde{B} - \Gamma$ has the homotopy type of a bouquet of spheres. The inclusion of one of these spheres corresponds to the inclusion $i: (D^n, S^{n-1}) \to (B, B-y)$ and again we obtain that $i_\#: \pi_m(D^n, S^{n-1}) \to \pi_m(B, B-y)$ is a split monomorphism. □

REMARK. Suppose E is the sphere S^m and $f: E \to B$ is a fibration which has the property that every great circle is contained in some fibre. Then $f \wr\!\wr f$ if and only if the map $f: E \to B$ admits a lift to $S(\tau_B)$ without the hypothesis $m < 2\dim B - 2$. This follows easily from Proposition 2.5 and the uniqueness of a geodesic connecting two non antipodal points in the sphere.

Suppose that $f: E \to B$ is a map (B smooth manifolds). Then we have the horizontal bundle with respect to the map f, denoted by $f^*(\tau_B)$, which is the pullback of the tangent bundle of B by f. Then we have:

Proposition 2.16. *If $f: E \to B$ is a map where B is a smooth manifold then the map $f: E \to B$ admits a lift to $S(\tau_B)$ if and only if the horizontal tangent bundle $f^*(\tau_B)$ over E has a nowhere-zero cross section. Therefore we have that (f, f) is homotopy disjoint by small deformation if and only if the horizontal tangent bundle $f^*(\tau_B)$ over E has a nowhere-zero cross section.*

Proof. The second part follows from the first part together with Proposition 2.13. To show the first part let us first assume that $f: E \to B$ has a lift to $S(\tau_B)$. The universal property of a pullback together with the lifting of f provides a nowhere-zero cross section of $f^*(\tau_B)$. Conversely let the horizontal tangent bundle $f^*(\tau_B)$ over E has a nowhere-zero cross section. Then let f' be the composite of this section with the projection $f^*(\tau_B) \to \tau_B$. Since the cross section is nowhere-zero, by dividing it by its norm, if necessary, we get the desired lift of f. □

3. Principal S^1 and S^3-bundles over $\mathbb{C}P_n$ and $\mathbb{H}P_n$, respectively

In this section we discuss self-coincidences for principal G-bundle maps over projective spaces where G is either S^1 or S^3. In the particular case of the generalized Hopf bundles we denote the complex ones by $p_n\mathbb{C}: E_n\mathbb{C} \to \mathbb{C}P_n$ and the quaternionic ones by $p_n\mathbb{H}: E_n\mathbb{H} \to \mathbb{H}P_n$. We divide this section into two parts. In Part I we treat those cases where the problem is solved by using the primary obstruction and the main results are Theorems 3.2 and 3.4. In Part II we analyze higher obstructions. Then we obtain the main results which are Theorems 3.5, 3.9 and Proposition 3.10.

Part I: The primary obstruction

Let us start with the principal S^1-bundles over B_nS^1 which is $\mathbb{C}P_n$. Over $\mathbb{C}P_n$, for each integer k, we have the S^1-principal bundle $p_n^k: E_n^k \to \mathbb{C}P_n$ where the total space is obtained by taking the quotient of S^{2n+1} by the cyclic subgroup $\mathbb{Z}_k \in S^1$ of order k. The quotients E_n^k, are the Lens spaces $L(k; 1, \dots, 1)$ which fibre over $\mathbb{C}P_n$. This bundle $p_n^k: E_n^k \to \mathbb{C}P_n$ is the principal S^1-bundle classified by the map $\mathbb{C}P_n \to \mathbb{C}P_\infty$ which represents $k \in H^2(\mathbb{C}P_n, \mathbb{Z}) \cong \mathbb{Z}$, and which we also denote by k. Unless stated, the coefficients will be the integers.

Proposition 3.1. *The principal S^1-bundles over $\mathbb{C}P_n$ are classified by integers k, say $E_n^k \to \mathbb{C}P_n$, and*

$$H^i(E_n^k) = \begin{cases} \mathbb{Z} & i = 0, 2n+1 \\ 0 & i = 1, 3, \dots, 2n-1 \\ \mathbb{Z}/k\mathbb{Z} & i = 2, 4, \dots, 2n \end{cases}.$$

Proof. This follows easily from the Gysin sequence of p_n^k. □

Theorem 3.2. $p_n^k \wr\!\wr\, p_n^k \Longrightarrow k|n+1$.

Proof. The primary obstruction to make (p_n^k, p_n^k) coincidence free is $p_n^{k*}(\chi(\mathbb{C}P_n).\mu_{\mathbb{C}P_n}) = (n+1).p_n^{k*}(\mu_{\mathbb{C}P_n})$. From the Gysin sequence (see Proposition 3.1) $p_n^{k*}(\mu_{\mathbb{C}P_n})$ generates $H^{2n}(E_n^k) \cong \mathbb{Z}_k$. Therefore if $p_n^k \wr\!\wr\, p_n^k$ the primary obstruction vanishes and $k|n+1$. □

Now we look at the symplectic case. Even though the classification of the S^3-principal bundles is more complicated than in the complex case, we obtain similar results. The S^3-principal bundles over $\mathbb{H}P_n$ are classified by homotopy classes of maps into BS^3 or into $\mathbb{H}P_n$. The homomorphism induced by the classifying map of the S^3-principal bundles on cohomology at dimension 4, is multiplication by an integer. For a given classifying map γ denote this integer by $k(\gamma)$ and by E_n^γ the S^3-principal bundle over $\mathbb{H}P_n$ which is classified by the map γ.

Proposition 3.3. *As in* Proposition 3.1, *the total space of the S^3-principal bundles $p_n^\gamma\colon E_n^\gamma \to \mathbb{H}P_n$ classified by the map γ has*

$$H^i(E_n^\gamma) = \begin{cases} \mathbb{Z} & i = 0, 4n+3 \\ 0 & i = 1,2,3,5,6,7,9,\ldots,4n-1,4n+1,4n+2 \\ \mathbb{Z}/k(\gamma)\mathbb{Z} & i = 4,8,\ldots,4n \end{cases} .$$

Here $k(\gamma)$ is given by the induced homomorphism in H^4 of the classifying map γ of the principal bundle.

Proof. Similar to the proof of 3.1. □

Theorem 3.4. $p_n^\gamma \wr p_n^\gamma \Longrightarrow k(\gamma)|n+1$.

Proof. Similar to the proof of 3.2. □

Part II: Higher obstructions

Now we will consider the remaining principal G-bundles which were not analyzed in Part I, which correspond to the values of k and $k(\gamma)$ which divide $n+1$. See Theorem 3.2 and Theorem 3.4. We first consider the generalized Hopf bundles, in which case $k = 1$, $p_n^1 = p_n\mathbb{C}$, and $k(\gamma) = 1$, $p_n^\gamma = p_n\mathbb{H}$.

We consider first the complex case.

Theorem 3.5. *We have* $p_n\mathbb{C} \wr p_n\mathbb{C} \Longleftrightarrow 2|n+1$.

Proof. Let n be odd. By Corollary 2.9 follows that $A_n \circ p_n$ is homotopic to p_n. Since they are disjoint follows $p_n\mathbb{C} \wr p_n\mathbb{C}$. For the converse assume that $p_n\mathbb{C} \wr p_n\mathbb{C}$. By Proposition 2.16, Lemma 2.14 for $n > 1$ and Proposition 2.16, Corollary 2.15 for $n = 1$ there is a nowhere-zero section of the horizontal bundle. The fact that the vertical bundle is trivial, implies the existence of at least two linearly independent vector fields over the sphere S^{2n+1}. By the formula which gives the number of vector field on the spheres (see [5] Theorem 8.2 page 156) we must have $n+1$ even. □

For the quaternionic case we give a description of the horizontal bundle of the fibre map.

Proposition 3.6. *The horizontal tangent bundle of the projection* $p_n S^3 \colon E_n S^3 \to B_n S^3$ *is isomorphic to* $(Sp(n+1) \times \mathbb{H}^n)/Sp(n) \to S^{4n+3}$. $Sp(n)$ *acts on the left of* $\mathbb{H}^n$, *in the obvious way, and on the right of* $Sp(n+1)$ *by multiplication by the inverse.*

Proof. Consider the evaluation map from $Sp(n+1) \times \mathbb{H}^n$ to the horizontal bundle. Namely, given $\alpha \in Sp(n+1)$ and $v \in \mathbb{H}^n$ then $\alpha(v)$ is a vector which is perpendicular to the symplectic subspace generated by $\alpha(e_{n+1})$. So it is an element of the horizontal bundle, and the result follows by routine argument. □

For the sake of clarity we will start by studying the Hopf fibration, $p_1\mathbb{H} \colon S^7 \to S^4$.

Proposition 3.7. *We have* $p_1\mathbb{H} \not{|\wr}\, p_1\mathbb{H}$.

Proof. Since the base of the bundle is the sphere S^4 by Corollary 2.15 and Proposition 2.16 the claim is equivalent to the non existence of a nowhere-zero horizontal vector field. We consider the clutching function of the principal fibration $Sp(1) \to Sp(2) \to S^7$ which is a map from S^6 into $Sp(1)$. It is not hard to see that when we compose the inclusion of $Sp(1)$ into $SO(4)$ followed by the projection onto S^3, we get a map homotopic to the identity, after identify $Sp(1)$ with S^3. So the composition of the clutching function with the composition above is homotopic to the clutching function, which we know is not trivial. In fact it is a generator of $\pi_6(S^3)$. So the bundle does not reduce. □

REMARK. The argument used in the proof above shows that if we replace $Sp(2)$ by any non trivial S^3-principal bundle over S^7, then the associated 4-dimensional vector bundle R^4 over S^7 does not reduce.

Now we will prove the remaining cases in the symplectic situation, namely when the fibre maps are $p_n(S^3)n > 1$.

Proposition 3.8. $p_{n-1}\mathbb{H} \;|\wr\; p_{n-1}\mathbb{H} \iff$ *the homomorphism* $\pi_{4n-2}(Sp(n-2)) \to \pi_{4n-2}(Sp(n-1))$, *induced by the inclusion* $Sp(n-2) \to Sp(n-1)$ *is a bijection.*

Proof. The bundles in question satisfy the hypothesis of Lemma 2.14. So by Proposition 2.16 the first condition is also equivalent to the existence of a nowhere-zero horizontal vector field. We have already seen in Proposition 3.6 that the horizontal bundle is isomorphic to $(Sp(n) \times \mathbb{H}^{n-1})/Sp(n-1) \to S^{4n-1}$. The clutching function for this vector bundle is the composition $S^{4n-2} \xrightarrow{\phi_n} Sp(n-1) \to SO(4n-4)$ where the first map called ϕ_n is the clutching function for the principal bundle $Sp(n-1) \to Sp(n) \to S^{4n-1}$. Let $\overline{\phi}_n$ denote the composite map. We want to decide whether the map $\overline{\phi}_n$

$$\begin{tikzcd} & SO(4n-5) \arrow[d] \\ S^{4n-2} \arrow[r, "\overline{\phi}_n"] & SO(4n-4) \arrow[d] \\ & S^{4n-5} \end{tikzcd}$$

factors through $SO(4n-5)$. From the commutative diagram of fibrations

$$\begin{tikzcd} Sp(n-2) \arrow[r] \arrow[d] & SO(4n-5) \arrow[d] \\ Sp(n-1) \arrow[r] \arrow[d] & SO(4n-4) \arrow[d] \\ S^{4n-5} \arrow[r, "="] & S^{4n-5} \end{tikzcd}$$

by taking the long exact sequences in homotopy, we get the commutative diagram

$$\begin{tikzcd} \pi_{4n-2}(Sp(n-2)) \arrow[r] \arrow[d] & \pi_{4n-2}(Sp(n-1)) \arrow[r, "p_*"] \arrow[d] & \pi_{4n-2}(S^{4n-5}) \arrow[r] \arrow[d, equal] & {} \\ \pi_{4n-2}(SO(4n-5)) \arrow[r] & \pi_{4n-2}(SO(4n-4)) \arrow[r, "p_*"] & \pi_{4n-2}(S^{4n-5}) \arrow[r] & {} \end{tikzcd}$$

and $\overline{\phi}_n$ factors through $SO(4n-5)$ if and only if $p_*\overline{\phi}_n$ is zero. The first row above, which is the exact sequence homotopy associated with the fibration $Sp(n-2) \to Sp(n-1) \to S^{4n-5}$, becomes:

$$\longrightarrow \pi_{4n-1}(S^{4n-5}) \longrightarrow \pi_{4n-2}(Sp(n-2)) \longrightarrow \pi_{4n-2}(Sp(n-1)) \longrightarrow \pi_{4n-2}(S^{4n-5}) \longrightarrow$$

where the first group is the 4-stem homotopy group of the sphere which is zero, see [9]. So follows that the induced homomorphism $i_{4n-2}\colon \pi_{4n-2}(Sp(n-2)) \to \pi_{4n-2}(Sp(n-1))$ is injective. The clutching function ϕ_n of the principal bundle $Sp(n-1) \to Sp(n) \to S^{4n-1}$ is a generator of the cyclic group $\pi_{4n-2}(Sp(n-1))$ by [10, Theorem 2.2]. Therefore this element is in the image of the homomorphism if $i_{4n-2}\colon \pi_{4n-2}(Sp(n-2)) \to \pi_{4n-2}(Sp(n-1))$ is surjective. $\square$

By using the knowledge of some metastable homotopy groups of $Sp(n)$ we have.

Theorem 3.9. $p_{n-1}\mathbb{H} \wr p_{n-1}\mathbb{H} \Longleftrightarrow 24|n$.

Proof. We divide into two cases. Suppose first that n is even. By the previous Proposition we have that it suffices to see when the two groups $\pi_{4n-2}(Sp(n-2))$ and $\pi_{4n-2}(Sp(n-1))$ have the same cardinality. By [7] and [6] we have that $\pi_{4n-2}(Sp(n-1)) = \mathbb{Z}_{2.(2n-1)!}$ and $\pi_{4n-2}(Sp(n-2)) = \mathbb{Z}_{((2n-1)!(n,24))/12}$, respectively, where $(n, 24)$ is the g.c.d. of n and 24. Therefore the two groups have the same cardinality if and only if $(n, 24)/12 = 2$, that is, $24|n$.

Therefore the first group has cardinality strictly bigger and we have that $p_{n-1}\mathbb{H} \not\wr p_{n-1}\mathbb{H}$. If $24|n$ then the groups have the same cardinality and the result follows.

Now suppose that n is odd. By [7] and [6] we have that $\pi_{4n-2}(Sp(n-1)) = \mathbb{Z}_{(2n-1)!}$ and $\pi_{4n-2}(Sp(n-2)) = \mathbb{Z}_{((2n-1)!(n,24))/24}$, respectively. So the first group has cardinality greater than the second. □

From the proof above we can see that the image of the generator of $\pi_{4n-2}(Sp(n-2))$ in $\pi_{4n-2}(Sp(n-1))$ is equal to $\alpha_n \cdot 24/(n, 24)$. This might have some relation with the higher obstruction which is an element in the group $\pi_{4n+2}(S^{4n-1}) \cong Z_{24}$ for $n > 1$.

REMARK. The result above shows that, at least in the case n is not divisible by 24, no non-vanishing tangent vector-field of S^{4n-1} is transverse to the vertical bundle. E.g. no nontrivial linear combination of the $\varrho(4n) - 1$ fields is transversal to the vertical bundle, where $\varrho(m)$ means the number of linearly independent vector fields on the sphere S^{m-1}. Note that $\varrho(m)$ is always greater than 4 if m is divisible by 8. □

Finally, using the results above we analyze some cases where $k \neq 1$ is a divisor of $n + 1$ and the principal bundle is over $\mathbb{C}P_n$.

Proposition 3.10. *Suppose that one of the conditions below holds*:

(1) *n is odd, greater than* 1, *and k is odd*

(2) *n is even*

(3) $n = 1$.

Then we have $p_n^1 \wr p_n^1 \Longleftrightarrow$ *we have* $p_n^k \wr p_n^k$.

Proof. The universal cover of the space E_n^k is the sphere S^{2n+1}; let $p\colon S^{2n+1} \to E_n^k$ be the covering map. This is a map of degree k and the composite of p with p_n^k is p_n^1. By standard obstruction theory, we have that $p^*(\Theta^{2n+1}(p_n^k))$ is the obstruction to have $p_n^1 \wr p_n^1$. The map p^* is multiplication by k and the obstructions lie in the groups $H^{2n+1}(E_n^k, R)$ and $H^{2n+1}(S^{2n+1}, R)$, respectively, where all these groups are isomorphic to the coefficient R. The coefficient R, by [4], is $\pi_{2n+1}(\mathbb{C}P_n, \mathbb{C}P_n - y)$. In order to compute this group, first observe that we can replace $\mathbb{C}P_n - y$ by $\mathbb{C}P_{n-1}$, since the inclusion $\mathbb{C}P_{n-1} \to \mathbb{C}P_n - y$ is a homotopy equivalence. Because the fibrations $S^{2n-1} \to \mathbb{C}P_{n-1}$ and $S^{2n+1} \to \mathbb{C}P_n$ have the same fibre, follows that the group in question is isomorphic to $\pi_{2n+1}(S^{2n+1}, S^{2n-1})$. Since the inclusion $S^{2n-1} \to S^{2n+1}$ is homotopic to the constant map $c\colon S^{2n-1} \to S^{2n+1}$, the homotopy fibre of the constant map is $S^{2n-1} \times \Omega S^{2n+1}$. Therefore it follows that $\pi_{i+1}(\mathbb{C}P_n, \mathbb{C}P_n - y) = \pi_i(S^{2n-1} \times \Omega S^{2n+1})$ for all $i \geq 0$ and in particular $\pi_{2n+1}(\mathbb{C}P_n, \mathbb{C}P_n - y)$ is equal to $\mathbb{Z}_2 + \mathbb{Z}$ for $n > 1$ and $\mathbb{Z}$ for $n = 1$. To finish the argument, first take $n = 1$. Since the coefficient R is isomorphic to $\mathbb{Z}$, multiplication by k in R is injective. So we have $p_n^k \mathrel{|\wr} p_n^k$ if and only if $p_n^1 \mathrel{|\wr} p_n^1$. In the remaining cases k is always odd and multiplication by k is injective. □

As a corollary of the Theorem 3.5 and Proposition 3.10 we have:

Corollary 3.11. *If n is even $p_n^k \mathrel{\not{|\wr}} p_n^k$ for arbitrary k. If either $n = 1$ and $k = 2$, or n and k are odd, then $p_n^k \mathrel{|\wr} p_n^k$.*

REMARK. 1) It is not clear how to extend Proposition 3.10 to the quaternionic case.
2) In the next section Corollary 3.11 will be slightly improved.

4. The multiple of certain bundle maps

Consider the generalized Hopf bundles $p_n\mathbb{C}\colon S^{2n+1} \to \mathbb{C}P_n$ and $p_n\mathbb{H}\colon S^{4n+3} \to \mathbb{H}P_n$. In this section we study the following question: determine the integers $\alpha(n)$ such that $\alpha(n).p_n \mathrel{|\wr} \alpha(n).p_n$, where p_n denote the fibre map. Finally, as result of the techniques used, we make some improvement on Corollary 3.11.

We start with a general result.

Proposition 4.1. *Let $f\colon S^n \to B$ be a map into a manifold B where n is different from $2\dim(B) - 2$ if the dimension of B is even. Then, there is a number l such that we have $(\alpha \cdot f, \alpha \cdot f)$ homotopy disjoint by small deformation if and only if $\alpha \in l.\mathbb{Z}$.*

Proof. By Proposition 2.13, the pairs $(\alpha \cdot f, \alpha \cdot f)$ which are homotopy disjoint by small deformation are precisely those belonging to the kernel of the boundary homomorphism $\partial\colon \pi_n(B) \to \pi_{n-1}(S^{m-1})$ of the long exact sequence in homotopy of the sphere bundle of the tangent bundle of B. But by hypotheses $n - 1 \neq 2(m-1) - 1$ which implies that $\pi_{n-1}(S^{m-1})$ is finite. □

Corollary 4.2. *The obstructions to deform (f, f) and (f', f') to coincidence free lie in the torsion part of $H^{2n+1}(S^{2n+1}, \mathbb{Z} + \mathbb{Z}_2)$ and $H^{4n+3}(S^{4n+3}, \mathbb{Z} + \mathbb{Z}_{24})$, respectively, for $n > 1$ and $f\colon S^{2n+1} \to \mathbb{C}P_n$ and $f'\colon S^{4n+3} \to \mathbb{H}P_n$ arbitrary maps. For $n = 1$ in* *the quaternionic case the group referred above is* $H^7(S^7, \mathbb{Z} + \mathbb{Z}_{12})$.

Proof. The obstruction to deform the pair $(l.f, l.f)$ coincidence free is l times the obstruction to deform the pair (f, f) coincidence free by Proposition 2.12. Since by Proposition 4.1, the obstruction to deform the pair $(l.f, l.f)$ vanishes for some integer l, the result follows. For the symplectic case let us first calculate the group where the higher obstruction lies. Following [4], this group is $H^{4n+3}(S^{4n+3}, R) \cong R \cong \pi_{4n+2}(S^{4n-1} \times \Omega S^{4n+3})$ (in fact $\pi_{i+1}(\mathbb{H}P_n, \mathbb{H}P_n - y) = \pi_i(S^{4n-1} \times \Omega S^{4n+3})$ for all $i \geq 0$) which is equal to $\mathbb{Z}_{24} + \mathbb{Z}$ for $n > 1$ and $\mathbb{Z}_{12} + \mathbb{Z}$ for $n = 1$. From now on this is similar to the complex case. $\square$

Since by the previous section $p_n S^1 \wr p_n S^1$ for n odd we are only interested in the case n even.

Proposition 4.3. *If n is even then $lp_n\mathbb{C} \wr lp_n\mathbb{C} \Longleftrightarrow 2|l$.*

Proof. We know that $p_n\mathbb{C} \not\wr p_n\mathbb{C}$. This together with Corollary 4.2 above tell us that the obstruction to deform the pair to coincidence free lies in $\mathbb{Z}_2$. The obstruction of $l.f$ will be zero if and only if l is even. $\square$

For the quaternionic case we have only a weak version of Proposition 4.3. As result of Theorem 3.9 we assume that 24 does not divide $n + 1$.

Proposition 4.4. *If $n + 1$ is not divisible by 24 then*:
(1) $lp_1\mathbb{H} \wr lp_1\mathbb{H}$ *if* $12|l$
(2) *For* $n > 1$, $lp_n\mathbb{H} \wr lp_n\mathbb{H}$ *if* $24|l$.

Proof. Corollary 4.2 tells us that the obstruction to $p_n\mathbb{H} \wr p_n\mathbb{H}$ lies either in the group $\mathbb{Z}_{24}$ or $\mathbb{Z}_{12}$. Then we apply Proposition 2.12. $\square$

Now we make one last application of Corollary 4.2 which improves Corollary 3.11.

Proposition 4.5. *Let k be an integer such that $2k$ divides $n + 1$. Then we have $p_n^k \wr p_n^k$.*

Proof. First observe that the obstruction to deform the pair (p_n^k, p_n^k) also lies in the torsion part of the correspondent cohomology group. To see this, as in the proof of Proposition 3.10, we have that the covering map $S^{2n+1} \to E_n^k$ induces a map in cohomology at dimension $2n+1$ which is multiplication by k, and carries the obstruction to deform (p_n^k, p_n^k) coincidence free into the obstruction to deform (p_n^1, p_n^1) coincidence free. Since the last one belongs to the torsion part by Corollary 4.2, the first one also does. Now consider the two fold cover $E_n^k \to E_n^{2k}$. This map in cohomology takes the obstruction for p_n^{2k} into the obstruction for p_n^k and the map is multiplication by 2. But the obstruction for p_n^{2k} is torsion annulated by multiplication by 2. $\square$

REMARK. Let 2^l be the highest power of 2 which divides $n + 1$. The Proposition 4.5 together with the other information from Section 2 tells us that if one knows the answer for $p_n^{2^l}$ then one has the answer for all cases.

Appendix

Let $H\colon S^{15} \to S^8$ be the Hopf fibration defined using the Cayley multiplication. We show that $H \not\parallel\!\wr\, H$. The same proof works for the quaternionic Hopf fibration $H\colon S^7 \to S^4$. It is not clear how the method used in the proof here, could be extended to the S^3-principal fibrations $S^{4n+3} \to \mathbb{H}P^n$. Also we do not see how the method used in the Theorem 3.9 could be applied to analyze the fibration $H\colon S^{15} \to S^8$.

Proposition. *We have $H \not\parallel\!\wr\, H$ for the Hopf fibration $H\colon S^{15} \to S^8$.*

Proof. If we have $H \parallel\!\wr\, H$, let H' be homotopic to H and $H \parallel H'$. Then $A \circ H'\colon S^{15} \to S^8$ is homotopic to H by Proposition 2.10, where A is the antipodal map. By the formula given in [11, section XI Theorem 8.5], and that the Whitehead product $[\iota_8, \iota_8] = 2H - \nu$, see [9, Propositions 2.5 and 5.15], we have that $A \circ H = H - \nu$. So we cannot have $H \parallel\!\wr\, H$. $\square$

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