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SLICE AND GORDIAN NUMBERS OF TRACK KNOTS

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Abstract

We present a class of knots associated with labelled generic immersions of intervals into the plane and compute their Gordian numbers and 4-dimensional invariants. At least 10 % of the knots in Rolfsen's table belong to this class of knots. We call them track knots. They are contained in the class of quasipositive knots. In this connection, we classify quasipositive knots and strongly quasipositive knots up to ten crossings.

1. Introduction

Several classes of knots are closely related to generic immersions of compact 1-manifolds into the plane. The class of track knots we shall present subsequently is a partial generalization of the class of divide knots. A divide is the intersection of a plane curve with the unit disk in $\mathbb{R}^2$, provided the plane curve is transverse to the unit circle. The concept of knots associated with divides is due to Norbert A'Campo and emerged from the study of isolated singularities of complex plane curves (see [1]). In [2] and [3], A'Campo specified some properties of divide knots, including fiberedness and a Gordian number result. Mikami Hirasawa gave an algorithm for drawing diagrams of divide links and extended the Gordian number result to certain arborescent links (see [9]). A large extension of the class of divide links was introduced by William Gibson and Masaharu Ishikawa [8]. They kept the Gordian number result, too. Tomomi Kawamura [11] and Ishikawa independently proved the quasipositivity of these links of free divides.

Borrowing from all these, we propose a new construction of knots associated with labelled generic immersions of intervals into the plane.

Let C be the image of a generic immersion of the interval $[0, 1]$ into the plane. In particular, C has no multiple points apart from a finite number of transversal double points, none of which is the image of 0 or 1. Further we enrich C, as follows (see Fig. 1 for an illustration).

(i) A small disk around each double point of C is cut into four regions by C. Label each of these regions by a sign, such that the sum of the four signs is non-negative. There are four types of patterns of signs around a double point, called a, b, c and d. They are shown in Fig. 2. If the tangent space T_pC at a double point p of C is the set $\{(x, y) \in \mathbb{R}^2 \mid (y - y(p))^2 = (x - x(p))^2\}$, then we may represent patterns

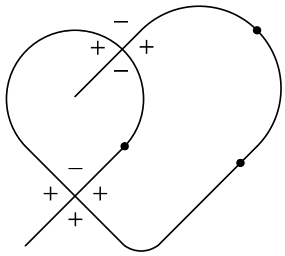


Fig. 1.

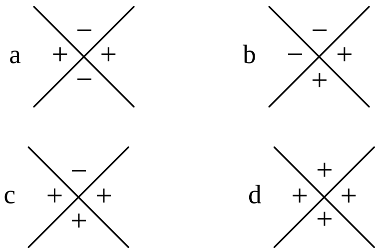


Fig. 2. Patterns of signs

of four signs at p by one of the following symbols:

$$a, a_1, b, b_1, b_2, b_3, c, c_1, c_2, c_3, d.$$

An index i at a symbol means that the corresponding pattern has to be turned counter-clockwise by the angle $i(\pi/2)$.

For example, b_1  stands for the pattern .

Henceforth we shall use these symbols.

(ii) Specify a finite number of different points $p_1, p_2, \dots, p_r$ on the edges of C (i.e. on the connected components of $C - \{\text{double points}\}$, such that $C - \{p_1, p_2, \dots, p_r\}$ is simply connected, but not necessarily connected. r is greater than or equal to the number of double points of C.

A labelled generically immersed interval in the plane will always be denoted by C_λ.

The following algorithm associates a knot diagram, hence a knot in the 3-space, to a labelled generically immersed interval C_λ.

(1) Draw a parallel companion of C_λ. In other words, replace C_λ by the boundary of a small band following C_λ. Join the two strands with an arc at both end points of C_λ and orient the resulting plane curve clockwise, in regard of the small band (see Fig. 3).

(2) At each double point of C_λ, place over- and under-crossings according to the signs of the four regions, as shown in Fig. 4.

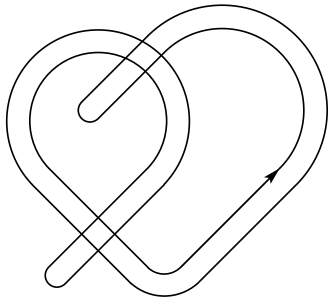


Fig. 3.

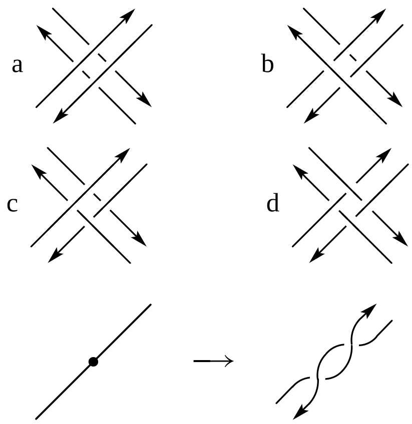


Fig. 4.

The characters a, b, c and d stand for 'above,' 'between,' 'conventional' and 'double,' respectively. 'Conventional' crossings appear in the visualization of links of divides, see Hirasawa [9].

(3) Add a full twist to the band at each specified point of C_λ, in a manner that gives rise to two positive crossings (see Fig. 4).

The knot diagram arising from C_λ by these three steps will be denoted by $D(C_\lambda)$, the corresponding knot by $K(C_\lambda)$.

DEFINITION. A *track knot* is a knot which can be realized as a knot associated with a labelled generically immersed interval C_λ. If it can be realized without any double point of type *b*, then we call it a *special track knot*.

REMARK. We observe that the classes of track knots and special track knots are closed under connected sum. The connected sum operation corresponds to the gluing of two labelled immersed intervals along end points. This is not true for knots of free divides; the connected sum of the free divide knot 5_2 (in Rolfsen's numbering [16]) with itself is not a free divide knot.

2. Slice and Gordian numbers

Let L be an oriented link with n components in $S^3 = \partial B^4$. The slice number $\chi_s(L)$ of L is the maximal Euler characteristic among all smooth, oriented surfaces in B^4 which are bounded by L and have no closed components. The surfaces in consideration need not be connected. If K is a knot, the 4-genus $g^*(K)$ is defined as $(1/2)(1-\chi_s(K))$. The clasp number $c_s(L)$ of a link L is the minimal number of transversal double points of n generically immersed disks in B^4 with boundary L. We will also be concerned with the Gordian number $u(L)$, which is the minimal number of crossing changes needed to transform L into the trivial link with n components. The following two inequalities relate these numbers:

$$u(L) \geq c_s(L) \geq \frac{1}{2}(1-\chi_s(L)). \tag{1}$$

They can be shown by purely geometrical arguments, see Kawamura [10].

Gordian numbers and 4-dimensional invariants of track knots are easy to determine. Let K be a track knot associated with a labelled generically immersed interval C_λ. Further let A, B, C and D be the numbers of double points of C_λ with patterns of signs of type a, b, c and d, respectively.

Theorem 1. *The clasp number and the* 4*-genus of K equal $C+2D$. If B is zero, then the Gordian number and the ordinary genus of K equal $C+2D$, too.*

Corollary. *Both the clasp number and the* 4*-genus are additive under connected sum of track knots. Moreover, the Gordian number is additive under connected sum of special track knots.*

REMARK. The connected sum of a knot with its mirror image always bounds an embedded disk in B^4, thus the clasp number and the 4-genus are not additive under connected sum of knots in general. It is still a conjecture that the Gordian number is additve under connected sum of knots (see M. Boileau and C. Weber [5]).

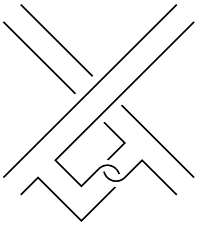
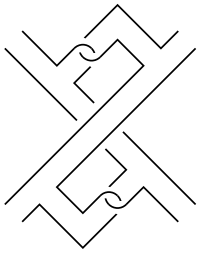
Proof of Theorem 1. We first show that the 4-genus of K does not exceed $C+2D$. If $C = D = 0$, then K is clearly slice, i.e. K bounds a disk in B^4. Indeed, the band following C_λ provides an immersed disk in S^3 with boundary K. At each double point of type b we may push a part of the band into $\mathring{B}^4$ to get an embedded disk. But then, at each double point of type c, we add one handle to the band, as Fig. 5 (c) suggests. Similarly, we add two handles to the band at each double point of type d, see Fig. 5 (d). This creates an embedded surface in B^4 of genus $C+2D$ with c  d 

Fig. 5.

boundary K. If $B = 0$, it is an embedded surface in S^3.

We remark that the spots where we add handles to the band can be interpreted as clasp singularities of the immersed band. Therefore the clasp number of K does not exceed $C + 2D$, either. Next, we show that the Gordian number of K does not exceed $C + 2D$, provided B is zero. If $C = D = 0$, then K is the unknot since it bounds an embedded disk in S^3. On a knot diagram level, double points of type c differ from double points of type a only by one crossing change, see Fig. 4. Similarly, double points of type d differ from double points of type a by two crossing changes. Hence we conclude $u(K) \leq C + 2D$.

We still have to prove that $C+2D$ is a lower bound for the four numbers in question. If we prove $g_*(K) \geq C + 2D$, then we are done, thanks to (1). For this purpose we need the slice-Bennequin inequality. Let D_L be the diagram of an oriented link L. The writhe $w(D_L)$ is the number of positive minus the number of negative crossings of the diagram D_L. Smoothing D_L at all crossings produces a union of Seifert circles. Let $s(D_L)$ be their number.

Theorem (Slice-Bennequin Inequality).

$$\chi_s(L) \leq s(D_L) - w(D_L).$$

The slice-Bennequin inequality was first established for closed braid diagrams by Lee Rudolph [17]; the proof of the general case can be found in Rudolph [18] and Kawamura [10]. A '3-dimensional' version of the inequality (concerning Seifert surfaces) was proved by Daniel Bennequin [4].

Now let us compute $w(D(C_\lambda))$ and $s(D(C_\lambda))$ for the knot diagram of the labelled generically immersed interval C_λ.

(1) $w(D(C_\lambda)) = 2C + 4D + 2r$, where r is the number of specified points on C_λ.

(2) Each double point and each specified point of C_λ gives rise to a small Seifert circle, see Fig. 6. Moreover, each connected component of $C_\lambda - \{p_1, p_2, \ldots, p_r\}$ gives one Seifert circle. The number of connected components of $C_\lambda - \{p_1, p_2, \ldots, p_r\}$ be-

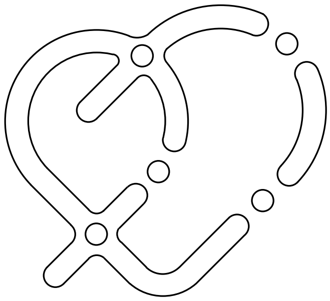


Fig. 6.

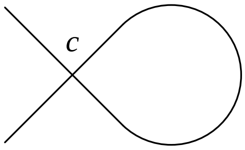


Fig. 7.

ing $1 + r - (A + B + C + D)$, we conclude

$$s(D(C_\lambda)) = A + B + C + D + r + 1 + r - (A + B + C + D) = 2r + 1.$$

Thus the slice-Bennequin inequality yields $\chi_s(K) \leq 1 - 2C - 4D$ and $g^*(K) = (1/2)(1 - \chi_s(K)) \geq C + 2D$. □

REMARKS. (i) If we renounce twisting the band at some specified points, then the statements of Theorem 1 are no longer true. The labelled immersed interval (without specified points) of Fig. 7 has one double point of type c and gives the unknot.
(ii) The statement of Theorem 1 about the Gordian number can be extended for track knots with $B = 1$. However, if $B \geq 2$, then the Gordian number may be greater than $C + 2D$. E.g. the knots 9_{46} and 10_{140} are slice track knots (see Table 2) and their Gordian numbers are certainly not zero.

3. The knots 10_{131} and 10_{148}

The knot 10_{131} is the track knot corresponding to the labelled immersed interval of Fig. 8. Its 4-genus and Gordian number equal 1. The latter is declared unknown in Akio Kawauchi's table of knots [12]. It is a curious fact that we can see the unknotting operation on its minimal diagram both in Rolfsen's and Kawauchi's table.

The knot 10_{148} is not a genuine track knot; it corresponds to a labelled immersed interval with too little specified points, see Fig. 9. Copying the first part of the proof

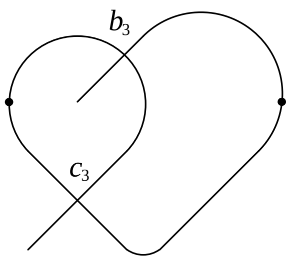


Fig. 8.

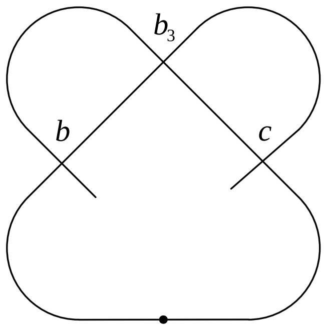


Fig. 9.

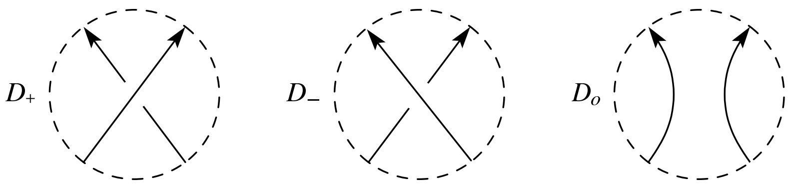


Fig. 10.

of Theorem 1, we see that its 4-genus is 1 at most. Since the knot 10_{148} is already known not to be slice, we conclude that its 4-genus is 1. This entry in Kawauchi's table of knots has been corrected a few years ago, see [13]. However, we cannot decide whether its Gordian number is 1 or 2.

4. The HOMFLY polynomial and quasipositivity

The HOMFLY polynomial $P_L(v, z) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ of an oriented link L is defined by the following two requirements (see [7]):

1. Normalization: $P_O(v, z) = 1$, where O stands for the regular diagram consisting of one trivial circle.

2. Relation: $(1/v)P_{D_+}(v,z) - vP_{D_-}(v,z) = zP_{D_o}(v,z)$. Here D_+, D_- and D_o denote regular diagrams which coincide outside a standard disk and differ, as in Fig. 10, inside this disk.

Writing $P_L(v,z) = \sum_{k=e(L)}^{E(L)} a_k(z)v^k$, with $a_{e(L)}(z)$, $a_{E(L)}(z) \neq 0$, as a Laurent polynomial in one variable v, we define its range in v as $[e(L), E(L)]$. H.R. Morton gave some bounds for $e(L)$ and $E(L)$ in terms of the writhe and the number of Seifert circles of a diagram of L.

Theorem (Morton [14]). *For any diagram D_L of an oriented link L*

$$w(D_L) - (s(D_L) - 1) \leq e(L) \leq E(L) \leq w(D_L) + (s(D_L) - 1).$$

The first inequality is tailor-made for track knots.

Theorem 2. $2g^*(K) \leq e(K)$ *for any track knot K.*

Proof. Choose a track knot diagram D of K. The proof of Theorem 1 tells us that $g^*(K) = (1/2)(1-s(D)+w(D))$, which is exactly half the lower bound in Morton's theorem. □

Theorem 2 draws our attention to quasipositive knots. A quasipositive knot is a knot which can be realized as the closure of a quasipositive braid. A quasipositive braid is a product of conjugates of a positive standard generator of the braid group. The slice-Bennequin inequality being an equality for closed quasipositive braid diagrams, we see that Theorem 2 is true both for track knots and for quasipositive knots.

Theorem 3. *Track knots are quasipositive.*

We adopt the pattern of Takuji Nakamura's proof of strong quasipositivity of positive links (see [15]). Any planar knot diagram gives rise to a system of Seifert circles with signed arcs, where each arc stands for a crossing joining two Seifert circles, as shown in Fig. 11. The sign of an arc tells us whether the crossing is positive or negative.

DEFINITION. A knot diagram is *quasipositive* if its set of crossings can be partitioned into single crossings and pairs of crossings, such that the following three conditions are satisfied.
(1) Each single crossing is positive.
(2) Each pair of crossings consists of one positive and one negative crossing joining the same two Seifert circles.
(3) A pair of crossings does not separate other pairs of crossings. More precisely, going from one crossing of a pair to its opposite counterpart along a Seifert circle, one

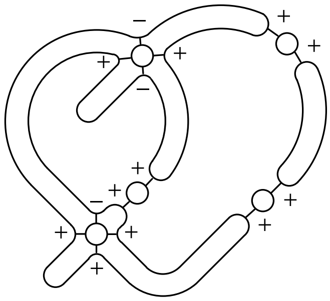


Fig. 11. A system of Seifert circles

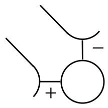


Fig. 12. A pair of crossings

cannot meet only one crossing of a pair.

EXAMPLES.

- Positive knot diagrams are obviously quasipositive.
- Track knot diagrams are quasipositive: negative arcs are incident with a small Seifert circle corresponding to a double point of type a, b or c. They can be paired with neighbouring positive crossings of the same small Seifert circle (see Fig. 12). At this point, it is essential that $C_\lambda - \{p_1, p_2, \ldots, p_r\}$ is simply connected. This guarantees that pairs of crossings do not get entangled (see Fig. 11).
- Quasipositive braid diagrams are quasipositive.

Lemma. *A quasipositive knot diagram represents a quasipositive knot.*

Proof. Any link diagram can be deformed into a braid representation, i.e. a system of concentric Seifert circles, by a finite sequence of bunching operations or concentric deformations of two types, without changing the writhe and the number of Seifert circles of the link diagram. This algorithm is due to Shuji Yamada, see [20]. We shall explain these two deformations and their effect on quasipositive knot diagrams.

First of all, we may consider only knot diagrams which have an outermost Seifert circle S_1, i.e. one that contains all the other Seifert circles. This corresponds to choosing a point on the sphere S^2 appropriately.

If S_1 contains a maximal Seifert circle S_2 with the opposite orientation of S_1, then

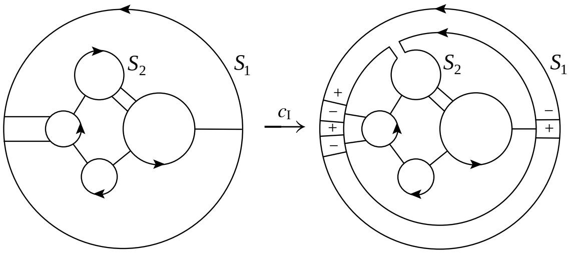


Fig. 13. A concentric deformation of type I

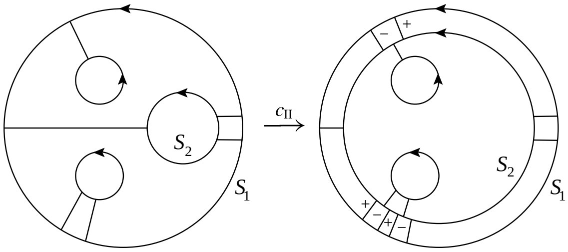


Fig. 14. A concentric deformation of type II

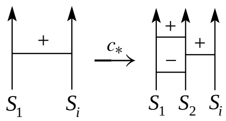


Fig. 15.

we apply a concentric deformation of type I to S_2, as shown in Fig. 13.

If S_1 contains maximal Seifert circles with the same orientation as S_1 only, then we apply a concentric deformation of type II to any of these maximal Seifert circles, say to S_2, as shown in Fig. 14.

In the next step, we consider maximal Seifert circles inside S_2, and so on. This algorithm clearly ends in a braid representation. Now we observe that concentric deformations of both types preserve the quasipositivity of knot diagrams in the above sense. They merely introduce new pairs of crossings, which do not get entangled. Fig. 15 and 16 show how a positive crossing (or a pair of crossings, respectively) gets more 'conjugated' by new pairs of crossings after a concentric deformation.

Thus, starting with a quasipositive knot diagram, we end up with a quasipositive braid diagram, which clearly represents a quasipositive knot. □

Theorems 2 and 3 reduce the number of potential track knots. In the following, we consider prime knots up to 10 crossings. Looking at Kawauchi's table of knots, we see that 60 of 249 prime knots up to 10 crossings satisfy the inequality $2g^*(K) \leq$

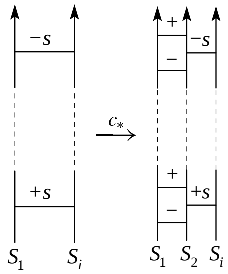


Fig. 16.

$e(K)$. Among these 60 knots, 42 have positive diagrams:

$$3_1, 5_1, 5_2, 7_1, 7_2, 7_3, 7_4, 7_5, 8_{15}, 8_{19},$$
$$9_1, 9_2, 9_3, 9_4, 9_5, 9_6, 9_7, 9_9, 9_{10}, 9_{13}, 9_{16}, 9_{18}, 9_{23}, 9_{35}, 9_{38}, 9_{49},$$
$$10_{49}, 10_{53}, 10_{55}, 10_{63}, 10_{66}, 10_{80}, 10_{101}, 10_{120},$$
$$10_{124}, 10_{128}, 10_{134}, 10_{139}, 10_{142}, 10_{152}, 10_{154}, 10_{161}.$$

REMARKS. (i) Since knots are always listed up to mirror image, we must be more precise: 'a knot K satisfies the inequality…' means 'either K or its mirror image $!K$ satisfies the inequality…'
(ii) The 4-genus of the knot 10_{51} is not known. However, it is known not to be slice, hence the inequality $2g^*(10_{51}) \leq e(10_{51}) = 0$ is not satisfied.

According to the result of Nakamura [15] and Rudolph [18], positive knots are strongly quasipositive, i.e. they can be realized as the closure of a braid which is a product of positive embedded bands of the form

$$\sigma_{i,j} = (\sigma_i \cdots \sigma_{j-2})\sigma_{j-1}(\sigma_i \cdots \sigma_{j-2})^{-1},$$

where σ_i is the i-th positive standard generator of the braid group. The remaining 18 knots are listed in Table 1, except for the knot 10_{132}, which is not quasipositive. Alexander Stoimenow already pointed out that the quasipositivity of the knot 10_{132} would imply the quasipositivity of its untwisted 2-cable link, together with a violation of Morton's inequality, which is a contradiction (see [19]). Table 1 contains one strongly quasipositive, non-positive knot: 10_{145}. It is non-positive since it is non-homogeneous (see P.R. Cromwell [6]). The other 16 knots are not strongly quasipositive since their 4-genus is smaller than their genus. In particular, they are non-positive. So in Table 1 we list all quasipositive, non-positive prime knots up to 10 crossings in Rolfsen's numbering, together with a quasipositive braid representation, the 4-genus g^* and the ordinary genus g. In the second column a, b, … and A, B, …

Table 1. Quasipositive, non-positive prime knots up to 10 crossings

Knot	quasipositive braid representation	g^*	g
8_{20}	(abAbaBA)(baB)	0	1
8_{21}	(abA)b(Abba)	1	2
9_{45}	a(Bcb)b(bacB)	1	2
9_{46}	(abbcBBA)(bacB)	0	1
10_{126}	aa(aaabAAA)b	1	3
10_{127}	abbb(bAbbaB)	2	3
10_{131}	a(aaBCbdBcbAA)(BcbdcBCb)d(Bcb)	1	2
10_{133}	aab(bDCbcdB)(bCBcACbcdCBcaCbcB)(bCBcaCbcB)	1	2
10_{140}	(abbbcBBBA)b(Cbc)	0	2
10_{143}	a(BBBaaabbb)	1	3
10_{145}	(abA)cd(abA)(bcB)(bcdCB)(cdC)b	2	2
10_{148}	ab(bbacBB)(cbC)	1	3
10_{149}	a(bbCbccBB)a(bcccB)	2	3
10_{155}	(abA)(ABcbCba)(bcB)	0	3
10_{157}	a(Baab)b(baaB)	2	3
10_{159}	a(BBaabb)(baB)	1	3
10_{166}	(abcBA)(acbA)(Bcb)(Aba)	1	2

stand for $\sigma_1, \sigma_2, \dots$ and $\sigma_1^{-1}, \sigma_2^{-1}, \dots$ and have nothing to do with symbols of labelled immersed intervals. Parentheses should help to recognize positive bands. The braid of the knot 10_{145} is strongly quasipositive.

This classification of quasipositive and strongly quasipositive knots gives us an interesting criterion for detecting strongly quasipositive knots.

Proposition. *A knot with* 10 *crossings at most is strongly quasipositive, if and only if it is quasipositive and its* 4*-genus equals its ordinary genus.*

We conclude this section with some questions and problems arising from the study of track knots and quasipositive knots.

(1) Does there exist a quasipositive knot which is not a track knot? (The free divide knots 9_{16}, 10_{124}, 10_{152} and 10_{154} might be good candidates.)

(2) Classify track knots up to 10 crossings. For this purpose, find new criterions for detecting track knots.

(3) Is it true that a knot is strongly quasipositive, if and only if it is quasipositive and its 4-genus equals its ordinary genus? In particular, is it true that special track knots are strongly quasipositive?

(4) Do alternating quasipositive knots have positive diagrams? (Up to ten crossings, this is true.)

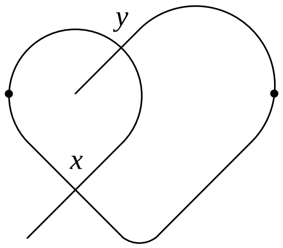


Fig. 17.

(5) Generalize the class of track knots in order to get some new Gordian number results.
(6) Prove the additivity of the Gordian number under connected sum of knots.

5. Examples of track knots

In this section we look at the labelled immersed interval shown in Fig. 17. It has two double points and two specified points.

There are 11^2 patterns of signs, represented by a symbol at each double point. Knots associated with different patterns of signs need not be different. It is still remarkable that we obtain 24 different prime knots in this way. They are listed in Table 2. The second and third column of Table 2 show the Dowker-Thistlethwaite numbering and the Rolfsen numbering, respectively. The fourth column tells us whether the knot is a free divide knot or not. In [8], Gibson and Ishikawa have listed knots of free divides. Up to 10 crossings, their list is complete. We add the 4-genus in the fifth column. It equals the clasp number and, except for the knots 9_{46}, 10_{140} and $11n139$, also the Gordian number.

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Table 2. Knots associated with a special immersed interval

(x, y)	DT numbering	Rolfsen numbering	free divide	g^*
(b, c)	$7a4$	7_2	No	1
(b, c_1)	$5a1$	5_2	Yes	1
(b, d)	$7a5$	7_3	Yes	2
(b_1, b_1)	$9n5$	9_{46}	No	0
(b_1, b_3)	$10n29$	10_{140}	No	0
(b_1, c)	$12n121$	–	No	1
(b_1, c_1)	$3a1$	3_1	Yes	1
(b_1, d)	$10n14$	10_{145}	Yes	2
(b_3, b_3)	$11n139$	–	No	0
(b_3, c_3)	$10n4$	10_{133}	No	1
(c, c_3)	$8a2$	8_{15}	No	2
(c, d)	$10n30$	10_{142}	No	3
(c_1, b_1)	$8n2$	8_{21}	No	1
(c_1, b_3)	$9n2$	9_{45}	No	1
(c_1, c_1)	$5a2$	5_1	Yes	2
(c_1, c_3)	$7a3$	7_5	Yes	2
(c_1, d)	$10n31$	10_{161}	Yes	3
(c_3, b_3)	$10n19$	10_{131}	No	1
(c_3, d)	$10n22$	10_{128}	No	3
(d, b_1)	$11n118$	–	No	2
(d, b_3)	$12n407$	–	No	2
(d, c_1)	$7a7$	7_1	Yes	3
(d, c_3)	$10n6$	10_{134}	No	3
(d, d)	$12n591$	–	?	4

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