Sunagawa, H.
Osaka J. Math.
42 (2005), 65–83

LARGE TIME ASYMPTOTICS OF SOLUTIONS TO NONLINEAR KLEIN-GORDON SYSTEMS

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(Received July 8, 2003)

Abstract

Consider a nonlinear system of two Klein-Gordon equations with masses m and μ. We construct a solution whose amplitude is modulated by the nonlinear interaction when $\mu = m$ or $\mu = 3m$, whereas, when $\mu \neq m$ and $\mu \neq 3m$, the influence of the nonlinearity is negligible and the solution behaves like a free solution as $t \to \infty$.

1. Introduction and the main result

We are concerned with the Cauchy problem for

$$\begin{cases} (\Box + m^2)u = F(v), \\ (\Box + \mu^2)v = G(u), \end{cases} \qquad t > 0, \ \ x \in \mathbb{R} \tag{1.1}$$

with sufficiently small, smooth, compactly-supported initial data. Here $\Box = \partial_t^2 - \partial_x^2$, m, μ are positive constnats, F, G are smooth functions of unknowns and they are cubic nonlinear terms in the sense that

$$|F(w)| + |G(w)| \leq C|w|^3 \quad \text{if} \quad |w| \leq \varsigma$$

for some constants C and $\varsigma > 0$. Though it is possible to consider much more general situations (including derivative nonlinear or quasi-linear cases), we do not go into such directions for the sake of simplicity.

Recently, much efforts are made for study of the large time behavior of solutions to the Cauchy problem for the systems of critical nonlinear Klein-Gordon equations with possibly different masses ([8], [6], [7], [2], [3] etc). According to [6], the Cauchy problem (1.1) admits a unique global classical solution which tends to a free solution as $t \to \infty$ if $(m - \mu)(m - 3\mu)(3m - \mu) \neq 0$. On the other hand, it turns out that the case $(m - \mu)(m - 3\mu)(3m - \mu) = 0$ is much more delicate and the previous works leave the problem open except a few partial results. In [3], some structual condition on the nonlinear terms is studied in one-dimensional cubic quasi-linear case under which the solution exists globally and it has a free profile even if $(m - \mu)(m - 3\mu)(3m - \mu) = 0$ (see also [2] for the corresponding result in the case of two-dimensional quadratic nonlinearity). We do not state their condition precisely but only point out that their condition is not satisfied if the nonlinear terms do not contain the derivatives of unknowns. In particular, the system of type (1.1) does not satisfy their condition if $(m-\mu)(m-3\mu)(3m-\mu)=0$.

In the present paper, we concentrate our attention on the following example:

$$\begin{cases} (\Box + m^2)u = \alpha v^4, & t>0,\ x\in\mathbb{R}, \\ (\Box + \mu^2)v = \beta u^3, & \\ (u, \partial_t u, v, \partial_t v)\big|_{t=0} = (\varepsilon u_0, \varepsilon u_1, \varepsilon v_0, \varepsilon v_1), & x\in\mathbb{R}, \end{cases} \tag{1.2}$$

where $\alpha,\beta\in\mathbb{R}$, $\varepsilon>0$ is a small parameter, and $u_0,\ u_1,\ v_0,\ v_1\in C_0^\infty(\mathbb{R})$. We will find the large time asymptotics for the solution of (1.2) to show that, as $t\to\infty$, the amplitude of v is modulated by the long range interaction when $\mu=m$ or $\mu=3m$, whereas, when $\mu\neq m$, $\mu\neq 3m$, the influence of nonlinearity disappears eventually and v behaves like a free solution in the large time. More precisely, we will prove the following:

Theorem 1. *For any $u_0,\ u_1,\ v_0,\ v_1\in C_0^\infty(\mathbb{R})$, there exists $\varepsilon_0>0$ such that* (1.2) *admits a unique global classical solution if $\varepsilon\in\,]0,\varepsilon_0]$. Moreover, the following asymptotics is valid as $t\to\infty$, uniformly with respect to $x\in\mathbb{R}$:*

$$u(t,x) = \frac{1}{m\sqrt{t}}\,\mathrm{Re}\left[a\Big(\frac{x}{t}\Big)e^{im(t^2-|x|^2)_+^{1/2}}\right] + \mathcal{O}(t^{-1+\delta}),$$
$$v(t,x) = \frac{1}{\mu\sqrt{t}}\,\mathrm{Re}\left[\Big\{A\Big(\frac{x}{t}\Big)\log t + b\Big(\frac{x}{t}\Big)\Big\}e^{i\mu(t^2-|x|^2)_+^{1/2}}\right] + \mathcal{O}(t^{-1+\delta}).$$

Here $(\,\cdot\,)_+=\max\{\,\cdot\,,0\}$, $i=\sqrt{-1}$, δ is an arbitrary small positive number, $a(y)$, $b(y)$ are $\mathbb{C}$-valued smooth functions which vanish when $|y|\geq 1$, and $A(y)$ is given by

$$A(y) = \begin{cases} \dfrac{\beta}{i8m^3}(1-|y|^2)_+^{1/2}a(y)^3 & \text{if}\quad \mu=3m, \\[2ex] \dfrac{3\beta}{i8m^3}(1-|y|^2)_+^{1/2}|a(y)|^2a(y) & \text{if}\quad \mu=m, \\[2ex] 0 & \text{if}\quad \mu\neq 3m,\ \mu\neq m. \end{cases}$$

REMARK. It is interesting to compare this result with the corresponding one to the scalar case

$$(\Box+1)w = \beta w^3, \qquad t>0,\ x\in\mathbb{R}. \tag{1.3}$$

This has been extensively studied by Delort [1] in much more general situations including quasi-linear case. According to his result, w has the following asymptotics:

$$w(t,x) = \frac{1}{\sqrt{t}}\operatorname{Re}\left[a\left(\frac{x}{t}\right)\exp\left\{i\left(t^2-|x|^2\right)_+^{1/2} + i\varphi\left(\frac{x}{t}\right)\log t\right\}\right] + \mathcal{O}(t^{-1+\delta}), \quad t\to\infty$$

with

$$\varphi(y) = -\frac{3\beta}{8}\left(1-|y|^2\right)_+^{1/2}|a(y)|^2.$$

Roughly speaking, this shows that the long range character of nonlinearity appears at the level of the *phase* of oscillation of the solution for the scalar equation (1.3), while our main result claims that the long range character appears at the level of the *amplitude* of the solution for the system (1.2).

We can obtain the similar results for a bit more general systems, such as

$$\text{(1.4)} \qquad \begin{cases} (\Box + m_1^2)u_1 = F_1(u,\partial u), \\ (\Box + m_2^2)u_2 = F_2(u,\partial u), \\ (\Box + m_3^2)u_3 = F_3(u,\partial u), \\ (\Box + m_4^2)u_4 = \gamma u_1u_2u_3 + F_4(u,\partial u), \end{cases} \qquad t>0,\ x\in\mathbb{R},$$

with the initial data

$$\text{(1.5)} \qquad (u_j,\partial_t u_j)\big|_{t=0} = (\varepsilon u_{0j},\varepsilon u_{1j}), \quad j=1,2,3,4.$$

Here $u=(u_j)_{1\le j\le 4}$, $\partial=(\partial_t,\partial_x)$, $\gamma\in\mathbb{R}$ and $F_j(u,\partial u) = O(|u|^4+|\partial u|^4)$ near $(u,\partial u)=(0,0)$. When we put

$$\Lambda := \left\{(\lambda_1,\lambda_2,\lambda_3)\in\{\pm1\}^3 : m_4 = \lambda_1 m_1 + \lambda_2 m_2 + \lambda_3 m_3\right\},$$

the corresponding result to Theorem 1 is stated as follows:

Theorem 2. *For any u_{0j}, $u_{1j}\in C_0^\infty(\mathbb{R})$, there exists $\varepsilon_0>0$ such that* (1.4)–(1.5) *admits a unique global C^∞ solution if $\varepsilon\in]0,\varepsilon_0]$. Moreover, the following asymptotics is valid as $t\to\infty$, uniformly with respect to $x\in\mathbb{R}$:*

$$u_j(t,x) = \frac{1}{m_j\sqrt{t}}\operatorname{Re}\left[a_j\left(\frac{x}{t}\right)e^{im_j(t^2-|x|^2)_+^{1/2}}\right] + \mathcal{O}\left(t^{-1+\delta}\right), \quad j=1,2,3,$$

$$u_4(t,x) = \frac{1}{m_4\sqrt{t}}\operatorname{Re}\left[\left\{A\left(\frac{x}{t}\right)\log t + a_4\left(\frac{x}{t}\right)\right\}e^{im_4(t^2-|x|^2)_+^{1/2}}\right] + \mathcal{O}\left(t^{-1+\delta}\right).$$

Here, δ is an arbitrary small positive number, a_j ($j = 1, 2, 3$) are $\mathbb{C}$-valued smooth functions which vanish when $|y| \geq 1$, and $A(y)$ is given by

$$A(y) = \begin{cases} \dfrac{\gamma}{i8m_1m_2m_3}(1-|y|^2)_+^{1/2} \displaystyle\sum_{(\lambda_1,\lambda_2,\lambda_3)\in\Lambda} a_1^{(\lambda_1)}(y)a_2^{(\lambda_2)}(y)a_3^{(\lambda_3)}(y) & \text{if} \quad \Lambda \neq \emptyset, \\ 0 & \text{if} \quad \Lambda = \emptyset, \end{cases}$$

where $a_j^{(+1)}(y) = a_j(y)$, $\quad a_j^{(-1)}(y) = \overline{a_j(y)}$.

REMARK. This is an extension of the previous result [7, Theorem 2.1], where only the simplest case ($F_1 = F_2 = F_3 = F_4 = 0$) is treated by using the explicit representation of the free solution.

2. Reduction of the problem

In this section, we perform some reduction following the idea developed by [1], [2], [3]. In what follows, we fix $B > 0$ so that

$$\operatorname{supp}(u_0, u_1, v_0, v_1) \subset \big\{x \in \mathbb{R} : |x| \leq B\big\}.$$

Also we fix $\rho_0 > \max\{1, 2B\}$. We begin with the fact that without loss of generality we may treat the problem as if the Cauchy data is given on the upper branch of the hyperbola

$$\big\{(t,x) \in \mathbb{R}^{1+1} : (t+2B)^2 - |x|^2 = \rho_0^2,\ \ t > 0\big\}$$

and it is sufficiently small, smooth, and compactly supported. (This is a consequence of the classical local existence theorem and the property of finite propagation speed. See [1, Proposition 1.4], [2], [3] and [5], [4, Chapter 7] for detail.) Next, as in [1], [5], we introduce the hyperbolic coordinates (ρ, θ) in the interior of the light cone, i.e.,

$$t + 2B = \rho\cosh\theta, \ \ x = \rho\sinh\theta, \quad \text{for} \quad |x| < t + 2B$$

so that

$$\begin{cases} \partial_t = (\cosh\theta)\partial_\rho - \dfrac{1}{\rho}(\sinh\theta)\partial_\theta, \\ \partial_x = -(\sinh\theta)\partial_\rho + \dfrac{1}{\rho}(\cosh\theta)\partial_\theta, \end{cases}$$

$$\Box + m^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + m^2 - \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}$$

and

$$\rho = \sqrt{(t+2B)^2 - |x|^2}.$$

We also introduce the new unknowns $(\tilde{u}, \tilde{v})$ as follows:

$$u(t,x) = \frac{\tilde{u}(\rho,\theta)}{\rho^{1/2}\cosh\kappa\theta}, \quad v(t,x) = \frac{\tilde{v}(\rho,\theta)}{\rho^{1/2}\cosh\kappa\theta},$$

where $\kappa > 0$ is a parameter which is determined later. Roughly speaking, κ measures the decay of the solution outside the light cone because $(\cosh\kappa\theta)^{-1} \approx \big((1-|x/t|)_+ + 1/t\big)^{\kappa/2}$. Now, let us derive the equations which $(\tilde{u}, \tilde{v})$ satisfies when (u, v) is a solution of (1.2). Since

$$v^4 = \rho^{-2}(\cosh\kappa\theta)^{-4}\tilde{v}^4$$

and

$$(\Box + m^2)u = \rho^{-1/2}(\cosh\kappa\theta)^{-1}(\tilde{\Box}_\kappa + m^2)\tilde{u},$$

we have

$$(\tilde{\Box}_\kappa + m^2)\tilde{u} = \frac{\alpha}{\rho^{3/2}(\cosh\kappa\theta)^3}\tilde{v}^4,$$

where

$$\tilde{\Box}_\kappa = \frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} + \frac{2\kappa\tanh\kappa\theta}{\rho^2}\frac{\partial}{\partial\theta} + \frac{1}{\rho^2}\left(\frac{1}{4} + \kappa^2\big(1 - 2(\tanh\kappa\theta)^2\big)\right).$$

In the same way, we see that $\tilde{v}$ satisfies

$$(\tilde{\Box}_\kappa + \mu^2)\tilde{v} = \frac{\beta}{\rho(\cosh\kappa\theta)^2}\tilde{u}^3.$$

Summing up, the original problem (1.2) is reduced to the following Cauchy problem:

$$\begin{cases} (\tilde{\Box}_\kappa + m^2)\tilde{u} = \dfrac{\alpha}{\rho^{3/2}(\cosh\kappa\theta)^3}\tilde{v}^4, \\ (\tilde{\Box}_\kappa + \mu^2)\tilde{v} = \dfrac{\beta}{\rho(\cosh\kappa\theta)^2}\tilde{u}^3, \end{cases} \quad \rho > \rho_0, \ \theta \in \mathbb{R} \tag{2.1}$$

with the initial data

$$\begin{cases} (\tilde{u}, \partial_\rho\tilde{u})|_{\rho=\rho_0} = (\varepsilon\tilde{u}_0, \varepsilon\tilde{u}_1) \\ (\tilde{v}, \partial_\rho\tilde{v})|_{\rho=\rho_0} = (\varepsilon\tilde{v}_0, \varepsilon\tilde{v}_1). \end{cases} \tag{2.2}$$

Our strategy is to prove global exitence and uniqueness of the solution to (2.1)–(2.2) (see Proposition 3 below), find the asymptotics for $(\tilde{u}, \tilde{v})$ as $\rho \to \infty$ (see (4.2), (4.3) in §4), and finally return to the solution of the original problem.

In the next section we shall prove the following proposition. In what follows, we denote by $H^s(\mathbb{R})$ the standard Sobolev space for $s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Proposition 3. *Let $\kappa \geq 0$ and let σ be an integer larger than $1 + 4\kappa$. For any $(\tilde{u}_0, \tilde{u}_1)$, $(\tilde{v}_0, \tilde{v}_1) \in H^{2\sigma}(\mathbb{R}_\theta) \times H^{2\sigma-1}(\mathbb{R}_\theta)$, there exists $\varepsilon_0 > 0$ such that the Cauchy problem* (2.1)–(2.2) *admits a unique global solution*

$$(\tilde{u}, \tilde{v}) \in \bigcap_{j=0}^{1} C^j\big(\rho_0, \infty; H^{2\sigma-j}(\mathbb{R}_\theta)\big) \times \bigcap_{j=0}^{1} C^j\big(\rho_0, \infty; H^{2\sigma-j}(\mathbb{R}_\theta)\big)$$

if $\varepsilon \in]0, \varepsilon_0]$. Moreover, we have

$$\|\partial_\rho^{l_1}\partial_\theta^{j+l_2}\tilde{u}(\rho,\cdot)\|_{H^\sigma(\mathbb{R}_\theta)} \leq C\rho^{(j/4)+l_2},$$
$$\|\partial_\rho^{l_1}\partial_\theta^{j+l_2}\tilde{v}(\rho,\cdot)\|_{H^\sigma(\mathbb{R}_\theta)} \leq C\rho^{\delta+(j/4)+l_2}$$

for each $0 \leq j \leq \sigma - 1$, $0 \leq l_1 + l_2 \leq 1$, and for arbitaray small $\delta > 0$. Here C is a positive constant independent of ρ.

REMARK. Consequently we have

$$\|\tilde{u}(\rho,\cdot)\|_{H^\sigma(\mathbb{R}_\theta)} \leq C, \tag{2.3}$$
$$\|\tilde{v}(\rho,\cdot)\|_{H^\sigma(\mathbb{R}_\theta)} \leq C\rho^\delta \tag{2.4}$$

for any $\rho \geq \rho_0$.

3. Proof of Proposition 3

This section is devoted to the proof of the Proposition 3. The proof is done by means of the contraction mapping principle. For this purpose, we prepare a version of the energy estimate for $(\tilde{\Box}_\kappa + M^2)$, which is essentially due to Delort–Fang–Xue [2] (see also [3]). We state and prove it here with minor modifications.

Let us define

$$\|\phi(\rho)\|_{E(M)} := \left(\int_{\theta\in\mathbb{R}} |\partial_\rho\phi(\rho,\theta)|^2 + \frac{1}{\rho^2}|\partial_\theta\phi(\rho,\theta)|^2 + M^2|\phi(\rho,\theta)|^2\, d\theta\right)^{1/2}$$

for smooth function ϕ and positive constant M. We start by the following basic estimates.

Lemma 4. *For $\kappa \geq 0$, $M > 0$, $s \in \mathbb{N}_0$ and for smooth function $\phi(\rho, \theta)$, we have*

$$\frac{d}{d\rho}\|\partial_\theta^s \phi(\rho)\|_{E(M)}^2 \leq \frac{2\kappa}{\rho}\|\partial_\theta^s \phi(\rho)\|_{E(M)}^2 + \frac{C}{\rho^2}\sum_{j=0}^{s}\|\partial_\theta^j \phi(\rho)\|_{E(M)}^2 + C\|\partial_\theta^s \phi(\rho)\|_{E(M)}\left\|\partial_\theta^s(\tilde{\Box}_\kappa + M^2)\phi(\rho)\right\|_{L^2(\mathbb{R}_\theta)} \tag{3.1}$$

and

$$\frac{d}{d\rho}\|\partial_\theta^s \phi(\rho)\|_{E(M)}^2 \leq \frac{C}{\rho^2}\sum_{j=0}^{s+1}\|\partial_\theta^j \phi(\rho)\|_{E(M)}^2 + C\|\partial_\theta^s \phi(\rho)\|_{E(M)}\left\|\partial_\theta^s(\tilde{\Box}_\kappa + M^2)\phi(\rho)\right\|_{L^2(\mathbb{R}_\theta)}, \tag{3.2}$$

provided that the right hand side is finite. Here C is a positive constant depending only on κ, M, s.

Proof. In what follows, we denote by C various positive constants which might be different line by line. We first show the case where $s = 0$. As in the standard energy integral method, we start from the following calculations

$$\begin{aligned}
\frac{d}{d\rho}\|\phi(\rho)\|_{E(M)}^2 &= 2\int_{\mathbb{R}} (\partial_\rho \phi)(\partial_\rho^2 \phi) + \frac{1}{\rho^2}(\partial_\theta \phi)(\partial_\rho \partial_\theta \phi) + M^2 \phi(\partial_\rho \phi) - \frac{1}{\rho^3}|\partial_\theta \phi|^2 \, d\theta \\
&\leq 2\int_{\mathbb{R}} (\partial_\rho \phi)\left(\partial_\rho^2 \phi - \frac{1}{\rho^2}\partial_\theta^2 \phi + M^2 \phi\right) d\theta \\
&= 2\int_{\mathbb{R}} \frac{-2\kappa \tanh \kappa\theta}{\rho^2}(\partial_\rho \phi)(\partial_\theta \phi) \\
&\quad - \frac{1}{\rho^2}\left\{\frac{1}{4} + \kappa^2 - 2\kappa^2(\tanh \kappa\theta)^2\right\}\phi(\partial_\rho \phi) + (\partial_\rho \phi)(\tilde{\Box}_\kappa + M^2)\phi \, d\theta \\
&\leq \frac{4\kappa}{\rho^2}\int_{\mathbb{R}} |\partial_\rho \phi||\partial_\theta \phi| \, d\theta + \frac{C}{\rho^2}\|\phi(\rho)\|_{E(M)}^2 + C\|\phi(\rho)\|_{E(M)}\left\|(\tilde{\Box}_\kappa + M^2)\phi\right\|_{L^2}.
\end{aligned}$$

The inequalities $(3.1)_{s=0}$ and $(3.2)_{s=0}$ follow from the fact that the first term in the right hand side is dominated by

$$\frac{2\kappa}{\rho}\|\phi(\rho)\|_{E(M)}^2 \quad \text{and} \quad \frac{2\kappa}{\rho^2}\left(\|\phi(\rho)\|_{E(M)}^2 + \frac{1}{M^2}\|\partial_\theta \phi(\rho)\|_{E(M)}^2\right),$$

respectively. Next, we consider the case where $s \geq 1$. Using $(3.1)_{s=0}$ with ϕ replaced by $\partial_\theta^s \phi$, we have

$$\begin{aligned}
\frac{d}{d\rho}\|\partial_\theta^s \phi(\rho)\|_{E(M)}^2 \leq{}& \frac{2\kappa}{\rho}\|\partial_\theta^s \phi(\rho)\|_{E(M)}^2 + \frac{C}{\rho^2}\|\partial_\theta^s \phi(\rho)\|_{E(M)}^2 \\
&+ C\|\partial_\theta^s \phi(\rho)\|_{E(M)}\left\|\partial_\theta^s(\tilde{\Box}_\kappa + M^2)\phi(\rho)\right\|_{L^2}
\end{aligned}$$

$$+ C\|\partial_\theta^s \phi(\rho)\|_{E(M)} \left\| \left[(\tilde{\Box}_\kappa + M^2), \partial_\theta^s \right] \phi(\rho) \right\|_{L^2}.$$

On the other hand, we have the following commutation relation:

$$\left[\tilde{\Box}_\kappa + M^2, \partial_\theta^s \right] = \frac{1}{\rho^2} \sum_{j=0}^{s} \gamma_{j,s}(\theta) \partial_\theta^j \tag{3.3}$$

with appropriate coefficients $\gamma_{j,s}(\theta)$ satisfying $\|\gamma_{j,s}\|_{L^\infty} < \infty$, from which it follows that

$$\left\| \left[(\tilde{\Box}_\kappa + M^2), \partial_\theta^s \right] \phi(\rho) \right\|_{L^2} \le \frac{1}{\rho^2} \sum_{j=0}^{s} \|\gamma_{j,s}\|_{L^\infty} \|\partial_\theta^j \phi(\rho)\|_{L^2} \le \frac{C}{\rho^2} \sum_{j=0}^{s} \|\partial_\theta^j \phi(\rho)\|_{E(M)}.$$

Summing up, we obtain (3.1). In the same way (3.2) follows. □

Next, we show the following energy inequality, which is the main tool for the proof of Proposition 3.

Proposition 5. *Let ϕ be a smooth function of $(\rho, \theta) \in [\rho_0, \infty[\times \mathbb{R}$, and let $\kappa \ge 0$, $M > 0$, $\nu \ge 0$, s_1, $s_2 \in \mathbb{N}_0$. If $s_1 \ge 4\kappa$, we have*

$$\begin{aligned} &\sup_{\rho \ge \rho_0} \left(\sum_{j_1=0}^{s_1} \sum_{j_2=0}^{s_2} \rho^{-(\nu + j_1/4)} \left\| \partial_\theta^{j_1+j_2} \phi(\rho) \right\|_{E(M)} \right) \\ &\qquad \le C\Big(\|\phi(\rho_0)\|_{H^{s_1+s_2+1}} + \|\partial_\rho \phi(\rho_0)\|_{H^{s_1+s_2}} \Big) \\ &\qquad\qquad + C \sum_{j_1=0}^{s_1} \sum_{j_2=0}^{s_2} \int_{\rho_0}^{\infty} \rho^{-(\nu + j_1/4)} \|\partial_\theta^{j_1+j_2} (\tilde{\Box}_\kappa + M^2) \phi(\rho)\|_{L^2(\mathbb{R}_\theta)} \, d\rho, \end{aligned}$$

provided that the right hand side is finite. Here C is a positive constant independent of ν, ρ_0.

Proof. We first note that we can choose some constant $C_s \ge 1$ so that

$$C_s^{-1} \mathcal{E}_s(\rho) \le \sum_{j=0}^{s} \rho^{-(\nu + j/4)} \left\| \partial_\theta^j \phi(\rho) \right\|_{E(M)} \le C_s \mathcal{E}_s(\rho)$$

holds, where

$$\mathcal{E}_s(\rho) := \left(\sum_{j=0}^{s} \rho^{-(2\nu + j/2)} \left\| \partial_\theta^j \phi(\rho) \right\|_{E(M)}^2 \right)^{1/2}.$$

Straightforward calculation yields

$$\begin{aligned}
\frac{d}{d\rho}\mathcal{E}_{s_1}(\rho)^2 &= \sum_{j=0}^{s_1}\left\{\rho^{-(2\nu+j/2)}\frac{d}{d\rho}\left\|\partial_\theta^j\phi(\rho)\right\|_{E(M)}^2 - \left(2\nu+\frac{j}{2}\right)\rho^{-(2\nu+j/2)-1}\left\|\partial_\theta^j\phi(\rho)\right\|_{E(M)}^2\right\}\\
&\le \sum_{j=0}^{s_1-1}\rho^{-(2\nu+j/2)}\frac{d}{d\rho}\left\|\partial_\theta^j\phi(\rho)\right\|_{E(M)}^2 + \rho^{-(2\nu+s_1/2)}\frac{d}{d\rho}\left\|\partial_\theta^{s_1}\phi(\rho)\right\|_{E(M)}^2\\
&\quad - \left(2\nu+\frac{s_1}{2}\right)\rho^{-(2\nu+s_1/2)-1}\left\|\partial_\theta^{s_1}\phi(\rho)\right\|_{E(M)}^2.
\end{aligned}$$

Using Lemma 4 and the relation $-(2\nu+s_1/2)\le -2\kappa$, we have

$$\begin{aligned}
&\frac{d}{d\rho}\mathcal{E}_{s_1}(\rho)^2\\
&\le \sum_{j=0}^{s_1-1}\rho^{-(2\nu+j/2)}\left\{\frac{C}{\rho^2}\sum_{l=0}^{j+1}\|\partial_\theta^l\phi(\rho)\|_{E(M)}^2 + C\|\partial_\theta^j\phi(\rho)\|_{E(M)}\left\|\partial_\theta^j(\tilde{\Box}_\kappa+M^2)\phi(\rho)\right\|_{L^2}\right\}\\
&\quad + \rho^{-(2\nu+s_1/2)}\left\{\frac{2\kappa}{\rho}\|\partial_\theta^{s_1}\phi(\rho)\|_{E(M)}^2 + \frac{C}{\rho^2}\sum_{l=0}^{s_1}\|\partial_\theta^l\phi(\rho)\|_{E(M)}^2\right.\\
&\qquad\qquad \left. + C\|\partial_\theta^{s_1}\phi(\rho)\|_{E(M)}\left\|\partial_\theta^{s_1}(\tilde{\Box}_\kappa+M^2)\phi(\rho)\right\|_{L^2}\right\}\\
&\quad - 2\kappa\rho^{-(2\nu+s_1/2)-1}\|\partial_\theta^{s_1}\phi(\rho)\|_{E(M)}^2\\
&= \frac{C}{\rho^2}\left\{\sum_{j=0}^{s_1-1}\sum_{l=0}^{j+1}\rho^{-(2\nu+j/2)}\|\partial_\theta^l\phi(\rho)\|_{E(M)}^2 + \sum_{l=0}^{s_1}\rho^{-(2\nu+s_1/2)}\|\partial_\theta^l\phi(\rho)\|_{E(M)}^2\right\}\\
&\quad + C\sum_{j=0}^{s_1}\rho^{-(2\nu+j/2)}\|\partial_\theta^j\phi(\rho)\|_{E(M)}\left\|\partial_\theta^j(\tilde{\Box}_\kappa+M^2)\phi(\rho)\right\|_{L^2}\\
&=: I_1 + I_2.
\end{aligned}$$

To estimate I_1, we note the following relation:

$$\begin{aligned}
\sum_{j=0}^{s_1-1}\sum_{l=0}^{j+1}\rho^{-(2\nu+j/2)}\|\partial_\theta^l\phi(\rho)\|_{E(M)}^2 &= \rho^{1/2}\sum_{j=0}^{s_1-1}\sum_{l=0}^{j+1}\rho^{-(j+1-l)/2}\rho^{-(2\nu+l/2)}\|\partial_\theta^l\phi(\rho)\|_{E(M)}^2\\
&\le \rho^{1/2}s_1\sum_{l=0}^{s_1}\rho^{-(2\nu+l/2)}\|\partial_\theta^l\phi(\rho)\|_{E(M)}^2\\
&= s_1\rho^{1/2}\mathcal{E}_{s_1}(\rho)^2.
\end{aligned}$$

This relation gives us

$$I_1 \le \frac{C}{\rho^2}\left\{s_1\rho^{1/2}\mathcal{E}_{s_1}(\rho)^2 + \mathcal{E}_{s_1}(\rho)^2\right\} \le \frac{C}{\rho^{3/2}}\mathcal{E}_{s_1}(\rho)^2.$$

As for I_2, the Cauchy-Schwarz inequality implies

$$I_2 \le C \left\{ \sum_{j=0}^{s_1} \rho^{-(2\nu+j/2)} \|\partial_\theta^j \phi(\rho)\|_{E(M)}^2 \right\}^{1/2} \left\{ \sum_{j=0}^{s_1} \rho^{-(2\nu+j/2)} \left\| \partial_\theta^j (\tilde{\Box}_\kappa + M^2)\phi(\rho) \right\|_{L^2}^2 \right\}^{1/2}$$
$$\le C\mathcal{E}_{s_1}(\rho) \sum_{j=0}^{s_1} \rho^{-(\nu+j/4)} \left\| \partial_\theta^j (\tilde{\Box}_\kappa + M^2)\phi(\rho) \right\|_{L^2}.$$

Summing up, we obtain

$$\frac{d\mathcal{E}_{s_1}}{d\rho}(\rho) \le \frac{C}{\rho^{3/2}}\mathcal{E}_{s_1}(\rho) + C \sum_{j=0}^{s_1} \rho^{-(\nu+j/4)} \left\| \partial_\theta^j (\tilde{\Box}_\kappa + M^2)\phi(\rho) \right\|_{L^2},$$

which implies

$$\mathcal{E}_{s_1}(\rho) \le e^{-C/\rho^{1/2}+C/\rho_0^{1/2}} \mathcal{E}_{s_1}(\rho_0)$$
$$+ C \sum_{j=0}^{s_1} \int_{\rho_0}^{\rho} e^{-C/\rho^{1/2}+C/\tau^{1/2}} \tau^{-(\nu+j/4)} \left\| \partial_\theta^j (\tilde{\Box}_\kappa + M^2)\phi(\tau) \right\|_{L^2} d\tau.$$

Using the fact that $1 \le e^{C/\rho^{1/2}} \le e^{C/\rho_0^{1/2}} \le e^C$ for any $\rho \ge \rho_0 \ge 1$, we obtain the desired inequality with $s_2 = 0$. Concerning the case of $s_2 \ge 1$, we have only to use the commutation relation (3.3) and the Gronwall inequality. □

Now, let us introduce the function space

$$Y^{\sigma,\delta} := \Big\{ \boldsymbol{\phi} = (\phi_1, \phi_2) \in C^0\big(\rho_0, \infty; H^{2\sigma}(\mathbb{R}; \mathbb{R}^2)\big) \cap C^1\big(\rho_0, \infty; H^{2\sigma-1}(\mathbb{R}; \mathbb{R}^2)\big) :$$
$$0 \le^{\forall} j \le 2\sigma - 1,\ {}^{\exists}C_j > 0 \text{ s.t.}$$
$$\|\partial_\theta^j \phi_1(\rho)\|_{E(m)} \le C_j \rho^{(1/4)(j-\sigma)_+},\ \ \|\partial_\theta^j \phi_2(\rho)\|_{E(\mu)} \le C_j \rho^{\delta+(1/4)(j-\sigma)_+} \Big\}$$

equipped with the norm

$$\|\boldsymbol{\phi}\|_{Y^{\sigma,\delta}} = \sup_{\rho \ge \rho_0} \sum_{j_1=0}^{\sigma-1} \sum_{j_2=0}^{\sigma} \left(\rho^{-j_1/4} \left\| \partial_\theta^{j_1+j_2} \phi_1(\rho) \right\|_{E(m)} + \rho^{-(\delta+j_1/4)} \left\| \partial_\theta^{j_1+j_2} \phi_2(\rho) \right\|_{E(\mu)} \right).$$

Here, $\delta \in]0, 1/10]$ and $\sigma \ge 1 + 4\kappa$. We denote by $Y^{\sigma,\delta}(r)$ the closed ball in $Y^{\sigma,\delta}$ of radius r centered at the origin, i.e.,

$$Y^{\sigma,\delta}(r) := \left\{ \boldsymbol{\phi} \in Y^{\sigma,\delta} : \|\boldsymbol{\phi}\|_{Y^{\sigma,\delta}} \le r \right\}.$$

For $\boldsymbol{\phi} = (\phi_1, \phi_2) \in Y^{\sigma,\delta}$, let $S(\boldsymbol{\phi})$ be the solution $\boldsymbol{\psi} = (\psi_1, \psi_2)$ to the Cauchy problem

$$\begin{cases} (\tilde{\Box}_\kappa + m^2)\psi_1 = \dfrac{\alpha}{\rho^{3/2}(\cosh\kappa\theta)^3}\phi_2^4, & \\ & \rho > \rho_0, \ \ \theta \in \mathbb{R}, \\ (\tilde{\Box}_\kappa + \mu^2)\psi_2 = \dfrac{\beta}{\rho(\cosh\kappa\theta)^2}\phi_1^3, & \\ (\psi_1, \psi_2, \partial_\rho\psi_1, \partial_\rho\psi_2)\,\Big|_{\rho=\rho_0} = (\varepsilon\tilde{u_0}, \varepsilon\tilde{v_0}, \varepsilon\tilde{u_1}, \varepsilon\tilde{v_1}), & \theta \in \mathbb{R}. \end{cases}$$

We shall show that S becomes a contraction mapping on $Y^{\sigma,\delta}(r)$ when we choose ε_0, r appropriately. Then we can apply the fixed point theorem to obtain Proposition 3.

Let $\boldsymbol{\phi} = (\phi_1, \phi_2) \in Y^{\sigma,\delta}(r)$. It follows from Proposition 5 that

$$\|S(\boldsymbol{\phi})\|_{Y^{\sigma,\delta}} \le C\varepsilon + C\int_{\rho_0}^{\infty} G(\rho)\,d\rho,$$

where

$$G(\rho) = \sum_{j_1=0}^{\sigma-1}\sum_{j_2=0}^{\sigma}\Big(\rho^{-(\delta+1+j_1/4)}\left\|\partial_\theta^{j_1+j_2}\{\phi_1(\rho)^3\}\right\|_{L^2(\mathbb{R}_\theta)} + \rho^{-(3/2+j_1/4)}\left\|\partial_\theta^{j_1+j_2}\{\phi_2(\rho)^4\}\right\|_{L^2(\mathbb{R}_\theta)}\Big).$$

Since $[(j_1 + j_2)/2] + 1 \le [\sigma - 1/2] + 1 = \sigma$, the Leibniz formula and the Sobolev imbedding yield

$$\begin{aligned} \left\|\partial_\theta^{j_1+j_2}\{\phi_1(\rho)^3\}\right\|_{L^2} &\le C\|\phi_1(\rho)\|^2_{W^{[(j_1+j_2)/2],\infty}} \sum_{l=0}^{j_1+j_2} \|\partial_\theta^l \phi_1(\rho)\|_{L^2} \\ &\le C\|\phi_1(\rho)\|^2_{H^\sigma} \sum_{l=0}^{j_1+j_2} r\rho^{(1/4)(l-j_2)_+} \\ &\le Cr^3\rho^{j_1/4}. \end{aligned}$$

Here $[\,\cdot\,]$ stands for the integer part. Similarly we have

$$\begin{aligned} \left\|\partial_\theta^{j_1+j_2}\{\phi_2(\rho)^4\}\right\|_{L^2} &\le C\|\phi_2(\rho)\|^3_{H^\sigma} \sum_{l=0}^{j_1+j_2} r\rho^{\delta+(1/4)(l-j_2)_+} \\ &\le Cr^4\rho^{4\delta+j_1/4}. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \|S(\boldsymbol{\phi})\|_{Y^{\sigma,\delta}} &\le C\varepsilon + C\int_{\rho_0}^{\infty} r^3\rho^{-(1+\delta)} + r^4\rho^{-(3/2-4\delta)}\,d\rho \\ &\le C\varepsilon + C(1+r)r^3\int_{\rho_0}^{\infty} \rho^{-(1+\delta)}\,d\rho \end{aligned}$$

$$\leq C\varepsilon + \frac{C(1+r)r^3}{\delta}.$$

Note that $1+\delta \leq 3/2 - 4\delta$ since $\delta \leq 1/10$. When we take $\varepsilon_0 := r/2C$ and choose $r > 0$ so small that $C(1+r)r^2 \leq \delta/2$, we have

$$\|S(\boldsymbol{\phi})\|_{Y^{\sigma,\delta}} \leq r,$$

provided that $\varepsilon \in]0, \varepsilon_0]$.

Next, we put $\tilde{\boldsymbol{\psi}} = S(\boldsymbol{\phi}) - S(\boldsymbol{\phi}')$ for $\boldsymbol{\phi} = (\phi_1, \phi_2), \boldsymbol{\phi}' = (\phi_1', \phi_2') \in Y^{\sigma,\delta}(r)$. Then $\tilde{\boldsymbol{\psi}} = (\tilde{\psi}_1, \tilde{\psi}_2)$ satisfies

$$\begin{cases} (\tilde{\Box}_\kappa + m^2)\tilde{\psi}_1 = \dfrac{\alpha}{\rho^{3/2}(\cosh\kappa\theta)^3}(\phi_2^4 - {\phi'}_2^4), \\ (\tilde{\Box}_\kappa + \mu^2)\tilde{\psi}_2 = \dfrac{\beta}{\rho(\cosh\kappa\theta)^2}(\phi_1^3 - {\phi'}_1^3) \end{cases}$$

with the initial data

$$\tilde{\psi}_1 = \tilde{\psi}_2 = \partial_\rho\tilde{\psi}_1 = \partial_\rho\tilde{\psi}_2 = 0$$

at $\rho = \rho_0$. Using Proposition 5 again, we have

$$\|\tilde{\boldsymbol{\psi}}\|_{Y^{\sigma,\delta}} \leq C\sum_{j_1=0}^{\sigma-1}\sum_{j_2=0}^{\sigma}\int_{\rho_0}^{\infty} G_{j_1j_2}(\rho)\,d\rho,$$

where

$$\begin{aligned} G_{j_1j_2}(\rho) = \rho^{-(\delta+1+j_1/4)}&\left\|\partial_\theta^{j_1+j_2}\left\{\phi_1(\rho)^3 - {\phi'}_1(\rho)^3\right\}\right\|_{L^2(\mathbb{R}_\theta)} \\ &+ \rho^{-(3/2+j_1/4)}\left\|\partial_\theta^{j_1+j_2}\left\{\phi_2(\rho)^4 - \phi_2'(\rho)^4\right\}\right\|_{L^2(\mathbb{R}_\theta)}. \end{aligned}$$

In the same way as before, we have

$$\begin{aligned} G_{j_1j_2}(\rho) &\leq Cr^2\rho^{-(1+\delta+j_1/4)}\sum_{l=0}^{j_1+j_2}\left\|\partial_\theta^l\left\{\phi_1(\rho) - {\phi'}_1(\rho)\right\}\right\|_{L^2} \\ &\quad + Cr^3\rho^{-(3/2-3\delta+j_1/4)}\sum_{l=0}^{j_1+j_2}\left\|\partial_\theta^l\left\{\phi_2(\rho) - {\phi'}_2(\rho)\right\}\right\|_{L^2} \\ &\leq Cr^2\rho^{-(1+\delta)}\sum_{l=0}^{j_1+j_2}\rho^{(1/4)\{-j_1+(l-j_2)_+\}}\|\boldsymbol{\phi} - \boldsymbol{\phi}'\|_{Y^{\sigma,\delta}} \\ &\quad + Cr^3\rho^{-(3/2-4\delta)}\sum_{l=0}^{j_1+j_2}\rho^{(1/4)\{-j_1+(l-j_2)_+\}}\|\boldsymbol{\phi} - \boldsymbol{\phi}'\|_{Y^{s,\delta}} \end{aligned}$$

$$\leq C(1+r)r^2\rho^{-(1+\delta)}\|\boldsymbol{\phi}-\boldsymbol{\phi}'\|_{Y^{\sigma,\delta}}.$$

Therefore, if r is chosen so small that $\sigma(\sigma+1)C(1+r)r^2 \leq \delta/2$ holds, then we have

$$\begin{aligned}\|S(\boldsymbol{\phi})-S(\boldsymbol{\phi}')\|_{Y^{\sigma,\delta}} &\leq \frac{\delta}{2}\left(\int_{\rho_0}^{\infty}\rho^{-(1+\delta)}\,d\rho\right)\|\boldsymbol{\phi}-\boldsymbol{\phi}'\|_{Y^{\sigma,\delta}}\\ &\leq \frac{1}{2}\|\boldsymbol{\phi}-\boldsymbol{\phi}'\|_{Y^{\sigma,\delta}}.\end{aligned}$$

This completes the proof of Proposition 3. □

4. Proof of the Main Theorem

In what follows, we only treat the case where $\mu = 3m$. The other cases can be treated in the same manner.

First, we rewrite (2.1) as

$$\begin{cases}(\partial_\rho^2+m^2)\tilde{u} = \dfrac{1}{\rho^{3/2-4\delta}}R_1,\\ (\partial_\rho^2+\mu^2)\tilde{v} = \dfrac{\beta}{\rho(\cosh\kappa\theta)^2}\tilde{u}^3+\dfrac{1}{\rho^{2-\delta}}R_2,\end{cases}$$

where

$$R_1 = \frac{\alpha}{(\cosh\kappa\theta)^3}\left(\rho^{-\delta}\tilde{v}\right)^4+\frac{1}{\rho^{1/2+4\delta}}\mathcal{L}_\kappa\tilde{u},\quad R_2=\rho^{-\delta}\mathcal{L}_\kappa\tilde{v}$$

with

$$\mathcal{L}_\kappa = \rho^2(\partial_\rho^2-\tilde{\Box}_\kappa) = \partial_\theta^2-2\kappa(\tanh\kappa\theta)\partial_\theta-\frac{1}{4}-\kappa^2+2\kappa^2(\tanh\kappa\theta)^2.$$

It follows from (2.3), (2.4) and the Sobolev imbedding that

$$\sup_{\rho\geq\rho_0}\Big(\|R_1(\rho,\cdot)\|_{L^\infty(\mathbb{R}_\theta)}+\|R_2(\rho,\cdot)\|_{L^\infty(\mathbb{R}_\theta)}\Big)<\infty.$$

Next, we put

$$\begin{aligned}\tilde{a}_\pm &= e^{\mp im\rho}(m\mp i\partial_\rho)\tilde{u},\\ \tilde{b}_\pm &= e^{\mp i\mu\rho}(\mu\mp i\partial_\rho)\tilde{v}.\end{aligned}$$

Note that $\tilde{a}_\pm$ and $\tilde{b}_\pm$ satisfy

$$\partial_\rho\tilde{a}_\pm = \mp ie^{\mp im\rho}(\partial_\rho^2+m^2)\tilde{u} = \frac{\pm e^{\mp im\rho}}{i\rho^{3/2-4\delta}}R_1$$

and

$$
\begin{aligned}
\partial_\rho \tilde{b}_\pm &= \frac{\pm e^{\mp i\mu\rho}\beta}{i\rho(\cosh\kappa\theta)^2}\left(\frac{e^{+im\rho}\tilde{a}_+ + e^{-im\rho}\tilde{a}_-}{2m}\right)^3 + \frac{\pm e^{\mp i\mu\rho}}{i\rho^{2-\delta}}R_2 \\
&= \frac{\pm\beta}{i8m^3(\cosh\kappa\theta)^2}\sum_{l=0}^{3}\binom{3}{l}\frac{e^{i\{(3-2l)m\mp\mu\}\rho}}{\rho}(\tilde{a}_+)^{3-l}(\tilde{a}_-)^l + \frac{\pm e^{\mp i\mu\rho}}{i\rho^{2-\delta}}R_2. \qquad (4.1)
\end{aligned}
$$

We are going to find the asymptotics of $\tilde{a}_\pm$, $\tilde{b}_\pm$ as $\rho \to \infty$. It is easy to do it for $\tilde{a}_\pm$. Indeed, since

$$\int_\rho^\infty \left|\partial_\rho \tilde{a}_\pm(\tau,\theta)\right| d\tau \le C\int_\rho^\infty \tau^{-3/2+4\delta}\,d\tau \le C\rho^{-1/2+4\delta},$$

we have

$$\tilde{a}_\pm(\rho,\theta) = \tilde{a}_\pm^\infty(\theta) + \mathcal{O}(\rho^{-1/2+4\delta}), \qquad (4.2)$$

where

$$\tilde{a}_\pm^\infty(\theta) := e^{\mp im\rho_0}\left\{m\tilde{u}_0(\theta) \mp i\tilde{u}_1(\theta)\right\} + \int_{\rho_0}^\infty \frac{\pm e^{\mp im\tau}}{i\tau^{3/2-4\delta}}R_1(\tau,\theta)\,d\tau.$$

To get the asymptotics of $\tilde{b}_\pm$, we use the following lemma.

Lemma 6. *Let $\nu \in \mathbb{R}$ and let $\psi_j(\rho,\theta)$ $(j = 1, 2, \dots, N)$ be smooth functions which satisfy*

$$|\psi_j(\rho,\theta)| \le C_0, \quad |\partial_\rho\psi_j(\rho,\theta)| \le C_0\rho^{-\nu}$$

for some constant $C_0 \ge 0$. Then we have

$$\frac{e^{i\omega\rho}}{\rho}\prod_{j=1}^N \psi_j(\rho,\theta) = \frac{\partial}{\partial\rho}\left\{\frac{e^{i\omega\rho}}{i\omega\rho}\prod_{j=1}^N \psi_j(\rho,\theta)\right\} + \mathcal{O}\left(\rho^{-\min\{2,1+\nu\}}\right)$$

for $\omega \in \mathbb{R}\backslash\{0\}$, while

$$\frac{1}{\rho}\prod_{j=1}^N \psi_j(\rho,\theta) = \frac{\partial}{\partial\rho}\left\{(\log\rho)\prod_{j=1}^N \psi_j(\rho,\theta)\right\} + \mathcal{O}\left(\rho^{-\nu}\log\rho\right).$$

Proof is quite simple. Indeed, using the relation

$$\frac{e^{i\omega\rho}}{\rho}\prod_{j=1}^N \psi_j = \frac{\partial}{\partial\rho}\left\{\frac{e^{i\omega\rho}}{i\omega\rho}\prod_{j=1}^N \psi_j\right\} - \frac{e^{i\omega\rho}}{i\omega}\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\prod_{j=1}^N \psi_j\right),$$

we have

$$\left|\frac{e^{i\omega\rho}}{\rho}\prod_{j=1}^{N}\psi_j-\frac{\partial}{\partial\rho}\left\{\frac{e^{i\omega\rho}}{i\omega\rho}\prod_{j=1}^{N}\psi_j\right\}\right|\leq\frac{1}{|\omega|\rho^2}\prod_{j=1}^{N}|\psi_j|+\frac{1}{|\omega|\rho}\sum_{k=1}^{N}|\partial_\rho\psi_k|\prod_{j\neq k}|\psi_j|$$
$$\leq\frac{C_0^N}{|\omega|\rho^2}+\frac{NC_0^N}{|\omega|\rho^{1+\nu}}.$$

The other one follows similarly from the relation

$$\frac{1}{\rho}\prod_{j=1}^{N}\psi_j=\frac{\partial}{\partial\rho}\left\{(\log\rho)\prod_{j=1}^{N}\psi_j\right\}-(\log\rho)\frac{\partial}{\partial\rho}\Big(\prod_{j=1}^{N}\psi_j\Big).$$ □

Applying the above lemma to (4.1), we obtain

$$\partial_\rho(\tilde{b}_\pm-\Phi_\pm)=\Psi_\pm+\frac{\pm e^{\mp i\mu\rho}}{i\rho^{2-\delta}}R_2,$$

where

$$\Phi_+(\rho,\theta)=(\log\rho)\frac{\beta\left\{\tilde{a}_+(\rho,\theta)\right\}^3}{i8m^3(\cosh\kappa\theta)^2}+\mathcal{O}(\rho^{-1}),\quad \Phi_-(\rho,\theta)=\overline{\Phi_+(\rho,\theta)},$$

and

$$\Psi_+(\rho,\theta)=\mathcal{O}(\rho^{-3/2+4\delta}\log\rho),\quad \Psi_-(\rho,\theta)=\overline{\Psi_+(\rho,\theta)}.$$

From this it follows that

$$\left|\tilde{b}_\pm(\rho,\theta)-\Phi_\pm(\rho,\theta)-\tilde{b}_\pm^\infty(\theta)\right|\leq\int_\rho^\infty|\Psi_\pm(\tau,\theta)|+\left|\frac{R_2(\tau,\theta)}{\tau^{2-\delta}}\right|d\tau\leq C\rho^{-1/2+5\delta}$$

where

$$\tilde{b}_\pm^\infty(\theta):=e^{\mp i\mu\rho_0}\left\{\mu\tilde{v}_0(\theta)\mp i\tilde{v}_1(\theta)\right\}-\Phi_\pm(\rho_0,\theta)+\int_{\rho_0}^\infty\Psi_\pm(\tau,\theta)+\frac{\pm e^{\mp i\mu\tau}}{i\tau^{2-\delta}}R_2(\tau,\theta)\,d\tau.$$

Therefore we obtain

$$\tilde{b}_\pm(\rho,\theta)=\pm(\log\rho)\frac{\beta\left(\tilde{a}_\pm^\infty(\theta)\right)^3}{i8m^3(\cosh\kappa\theta)^2}+\tilde{b}_\pm^\infty(\theta)+\mathcal{O}(\rho^{-1/2+5\delta})\tag{4.3}$$

as $\rho\to\infty$, uniformly with respect to $\theta\in\mathbb{R}$.

Now, we are going back to the original variables. Remember that

$$u(t,x)=\frac{e^{im\rho}\tilde{a}_+(\rho,\theta)+e^{-im\rho}\tilde{a}_-(\rho,\theta)}{2m\rho^{1/2}\cosh\kappa\theta},$$
$$v(t,x)=\frac{e^{i\mu\rho}\tilde{b}_+(\rho,\theta)+e^{-i\mu\rho}\tilde{b}_-(\rho,\theta)}{2\mu\rho^{1/2}\cosh\kappa\theta},$$
$$\rho=\sqrt{(t+2B)^2-|x|^2},\quad \theta=\frac{1}{2}\log\left(\frac{2B+t+x}{2B+t-x}\right)$$

and $t\gg 1$, $|x|<t+2B$. Using the relations $\tilde{a}_-^\infty=\overline{\tilde{a}_+^\infty}$, $\tilde{b}_-^\infty=\overline{\tilde{b}_+^\infty}$ and

$$\begin{aligned}\frac{\rho^{-\nu}}{\cosh\kappa\theta}&\le\frac{Ct^{-\nu}}{(\cosh\theta)^\kappa}\left(\frac{t+2B}{\rho}\right)^\nu\\&=Ct^{-\nu}\left(1-\left|\frac{x}{t+2B}\right|^2\right)^{\kappa-\nu}\\&\le Ct^{-\nu}\end{aligned}$$

for $\kappa\ge\nu\ge 0$, we have

$$u(t,x)=\mathrm{Re}\left[\frac{e^{im\rho(t,x)}}{m\sqrt{t+2B}}\tilde{\tilde{a}}\big(\theta(t,x)\big)\right]+\mathcal{O}\left(t^{-1+4\delta}\right),\tag{4.4}$$

$$v(t,x)=\mathrm{Re}\left[\frac{e^{i\mu\rho(t,x)}}{\mu\sqrt{t+2B}}\left\{\frac{\beta}{i8m^3}\left\{\tilde{\tilde{a}}\big(\theta(t,x)\big)\right\}^3\frac{\log\rho(t,x)}{\cosh\theta(t,x)}+\tilde{\tilde{b}}\big(\theta(t,x)\big)\right\}\right]+\mathcal{O}\left(t^{-1+5\delta}\right),\tag{4.5}$$

where

$$\tilde{\tilde{a}}(\theta)=\frac{(\cosh\theta)^{1/2}\tilde{a}_+^\infty(\theta)}{\cosh\kappa\theta},\quad \tilde{\tilde{b}}(\theta)=\frac{(\cosh\theta)^{1/2}\tilde{b}_+^\infty(\theta)}{\cosh\kappa\theta}.$$

Next, we put

$$a(y):=\begin{cases}e^{i2Bm\sqrt{1-|y|^2}}\tilde{\tilde{a}}\big(\theta_0(y)\big) & \text{if } |y|<1,\\ 0 & \text{if } |y|\ge 1,\end{cases}$$
$$A(y):=\frac{\beta}{i8m^3}(1-|y|^2)_+^{1/2}a(y)^3,$$
$$b(y):=\begin{cases}e^{i2B\mu\sqrt{1-|y|^2}}\tilde{\tilde{b}}\big(\theta_0(y)\big)+A(y)\log\sqrt{1-|y|^2} & \text{if } |y|<1,\\ 0 & \text{if } |y|\ge 1\end{cases}$$

with

$$\theta_0(y)=\frac{1}{2}\log\left(\frac{1+y}{1-y}\right).$$

Note that the following estimates are valid (cf. [1, p.58–59]):

$$|\partial_y^j a(y)| \le C(1-|y|)_+^{\kappa/2-1/4-j},$$
$$\left|a\Big(\frac{x}{t+2B}\Big)-a\Big(\frac{x}{t}\Big)\right| \le Ct^{-1}\left\{\Big(1-\Big|\frac{x}{t}\Big|\Big)_+ + \frac{1}{t}\right\}^{\kappa/2-5/4},$$
$$1_{|x|<t}\left|e^{imt\sqrt{1-|x/(t+2B)|^2}}-e^{imt\sqrt{1-|x/t|^2}}\right| \le Ct^{-1}\left\{\Big(1-\Big|\frac{x}{t}\Big|\Big)_+ + \frac{1}{t}\right\}_+^{-1/2}.$$

These relations give us

$$\begin{aligned}
&\left|\frac{e^{imt\sqrt{1-|x/(t+2B)|^2}}}{\sqrt{t+2B}}a\Big(\frac{x}{t+2B}\Big)-\frac{e^{im\sqrt{t^2-|x|^2}}}{\sqrt{t}}a\Big(\frac{x}{t}\Big)\right|\\
&\le \frac{1}{\sqrt{t+2B}}\left|a\Big(\frac{x}{t+2B}\Big)-a\Big(\frac{x}{t}\Big)\right| + \left|\frac{1}{\sqrt{t+2B}}-\frac{1}{\sqrt{t}}\right|\left|a\Big(\frac{x}{t}\Big)\right|\\
&\quad + \frac{1}{\sqrt{t}}\left|e^{imt\sqrt{1-|x/(t+2B)|^2}}-e^{imt\sqrt{1-|x/t|^2}}\right|\left|a\Big(\frac{x}{t}\Big)\right|\\
&\le Ct^{-3/2}\left\{\Big(1-\Big|\frac{x}{t}\Big|\Big)_+ + \frac{1}{t}\right\}^{\kappa/2-5/4} + Ct^{-3/2}\Big(1-\Big|\frac{x}{t}\Big|\Big)_+^{\kappa/2-1/4}\\
&\quad + Ct^{-3/2}\left\{\Big(1-\Big|\frac{x}{t}\Big|\Big)_+ + \frac{1}{t}\right\}^{\kappa/2-1/4-1/2}\\
&\le Ct^{-3/2},
\end{aligned}$$

provided that $\kappa > 5/2$. Summing up, we have

$$\begin{aligned}
\frac{e^{im\rho(t,x)}}{m\sqrt{t+2B}}\tilde{\tilde{a}}\big(\theta(t,x)\big) &= \frac{e^{imt\sqrt{1-|x/(t+2B)|^2}}}{m\sqrt{t+2B}}a\Big(\frac{x}{t+2B}\Big)\\
&= \frac{e^{im\sqrt{t^2-|x|^2}}}{m\sqrt{t}}a\Big(\frac{x}{t}\Big) + \mathcal{O}(t^{-3/2}).
\end{aligned}$$

Substituting it for the first term of the right hand side of (4.4), we obtain the asymptotics of u. In the same way, the first term of the right hand side of (4.5) can be written as

$$\operatorname{Re}\left[\frac{e^{i\mu\sqrt{t^2-|x|^2}}}{\mu\sqrt{t}}\left\{A\Big(\frac{x}{t}\Big)\log t + b\Big(\frac{x}{t}\Big)\right\}\right] + \mathcal{O}(t^{-3/2}\log t),$$

which yields the asymptotics of v. This completes the proof of Theorem 1. ☐

5. Concluding remarks

(1) We can prove the analogous result for two-dimensional case, such as

$$(5.1)\qquad \begin{cases} (\Box + m^2)v_1 = \alpha v_2^3, \\ (\Box + \mu^2)v_2 = \beta v_1^2, \end{cases} \quad t > 0, \ x \in \mathbb{R}^2,$$

where $\alpha, \beta \in \mathbb{R}$, or

$$(5.2)\qquad \begin{cases} (\Box + m_1^2)u_1 = F_1(u, \partial u), \\ (\Box + m_2^2)u_2 = F_2(u, \partial u), \\ (\Box + m_3^2)u_3 = \gamma u_1 u_2 + F_3(u, \partial u), \end{cases} \quad t > 0, \ x \in \mathbb{R}^2,$$

where $u = (u_j)_{1\le j\le 3}$, $\partial = (\partial_t, \partial_{x_1}, \partial_{x_2})$, $\gamma \in \mathbb{R}$ and $F_j(u, \partial u) = O(|u|^3 + |\partial u|^3)$ near $(u, \partial u) = (0, 0)$ ($j = 1, 2, 3$). For the solution v_2 of (5.1) (resp. u_3 of (5.2)), the long range effect as in Theorem 1 (resp. Theorem 2) is observed if and only if $\mu = 2m$ (resp. $m_3 = \lambda_1 m_1 + \lambda_2 m_2$ for some $\lambda_1, \lambda_2 \in \{\pm 1\}$).
(2) One might expect a result similar to Theorem 1 for one-dimensional cubic homogeneous case, such as

$$(5.3)\qquad \begin{cases} (\Box + m^2)u = \alpha v^3, \\ (\Box + \mu^2)v = \beta u^3. \end{cases}$$

However, it seems still open whether this holds true or not. The main reason is that Lemma 6 is not effective for (5.3). Indeed, we can not regard $\mathcal{O}(\rho^{-\nu}\log\rho)$ as the remainder term in this case since it is impossible to take ν greater than 1.

ACKNOWLEDGEMENT. The author would like to express his sincere gratitude to Professors Nakao Hayashi, Soichiro Katayama, Hideo Kubo and Daoyuan Fang for their valuable advices and fruitful discussions. The author is also grateful to Professor Akitaka Matsumura for his encouragement. Finally, the author thanks the referee for reading the original manuscript carefully and pointing out the mistakes in it.

ADDITIONAL REMARK. After submitting this paper, the author was informed of the following paper:
D. Fang and R. Xue: *Global existence and asymptotics behavior of solutions for a resonant Klein-Gordon systems in two space dimensions*, preprint (2003),
where analogous problems in two space dimensions are discussed.

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