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ON FINITELY DEGENERATE HYPERBOLIC OPERATORS OF SECOND ORDER

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1. Introduction

We are interested in the Cauchy problem

$$\text{(CP)} \qquad \begin{cases} P(t,x,\partial_t,\partial_x)u(t,x) = f(t,x) \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x) \end{cases}$$

on $[0,T] \times \mathbf{R}^n$ where

$$P(t,x,\partial_t,\partial_x) = P_2(t,\partial_t,\partial_x) + P_1(t,x,\partial_t,\partial_x) + c(t,x),$$
$$P_2(t,\partial_t,\partial_x) = \partial_t^2 - \sum_{i,j=1}^{n} a_{ij}(t)\partial_{x_i}\partial_{x_j},$$
$$P_1(t,x,\partial_t,\partial_x) = \sum_{j=1}^{n} b_j(t,x)\partial_{x_j}.$$

We assume that $a_{ij} \in C^\infty([0,T])$, b_j and $c \in C([0,T];C^\infty(\mathbf{R}^n))$; moreover

$$a(t,\xi) = \sum_{i,j=1}^{n} a_{ij}(t)\xi_i\xi_j \geq 0, \quad \forall \xi \in \mathbf{R}^n, \quad t \in [0,T].$$

It is well known that the question of the C^∞ well posedness of the Cauchy problem for general linear weakly hyperbolic equations is not settled. Restricting our attention to the second order equations, there are two main difficulties in studying C^∞ well posedness of the Cauchy problem:
1) For the Cauchy problem to be C^∞ well posed, the lower order term must be dominated in a suitable sense by the principal part of the operator (the so called Levi conditions). For instance

$$u_{tt} - u_x = 0$$

is not solvable in C^∞ but only in Gevrey class of order 2.

2) Oscillations of the coefficients of the principal symbol with respect to the time variable can destroy the solvability in C^∞. For instance, in [5] they show by an example that the Cauchy problem for

$$u_{tt} - a(t)u_{xx} = 0$$

where the function $a(t)$ is C^∞ verifying $a(0) = 0$, $a(t) > 0$ for $t > 0$ and has an infinite number of oscillations as $t \downarrow 0$, may be not locally solvable in C^∞.

Thus in order to obtain positive results concerning the C^∞ well posedness, some additional assumptions both on the principal symbol and on the lower order terms are needed.

It is well known that the Cauchy problem is C^∞ well posed for any lower order term if and only if it is effectively hyperbolic (see [9] and its bibliography) . We recall that the effective hyperbolicity on our operator is equivalent to

$$\partial_t^2 a(t,\xi) > 0$$

whenever $a(t,\xi) = 0$, or we can express the condition as

$$\sum_{j=0}^{2} |\partial_t^j a(t,\xi)| \neq 0, \quad \forall |\xi| = 1, \quad t \in [0,T].$$

In this note we assume that there is $k \in \mathbf{N}$, $k \geq 2$ such that

$$\sum_{j=0}^{k} |\partial_t^j a(t,\xi)| \neq 0, \quad \forall |\xi| = 1, \quad t \in [0,T]. \tag{1.1}$$

If $k > 2$ and $\partial_t^j a(\bar{t},\bar{\xi}) = 0$, $0 \leq j < k$, $\partial_t^k a(\bar{t},\bar{\xi}) \neq 0$ then as noted above (see [7]) the lower order term $b(t,x,\xi)$ must verify some conditions for the Cauchy problem to be C^∞ well posed.

Let us denote

$$b(t,x,\xi) = \sum_{j=1}^{n} b_j(t,x)\xi_j$$

and introduce the following assumption on $b(t,x,\xi)$:

$$|\partial_x^\alpha b(t,x,\xi)| \leq C_\alpha a(t,\xi)^\gamma |\xi|^{1-2\gamma} \tag{1.2}$$

for any α with

$$\gamma = \frac{k-2}{2(k-1)}. \tag{1.3}$$

For simplicity we also assume that $b(t, x, \xi) = 0$ for $|x| \geq R$ with some $R > 0$. Then we have

Theorem 1.1. *Assume* (1.1), (1.2) *and* (1.3). *Then the Cauchy problem* (CP) *is C^∞ well posed.*

Note that a positive result in this direction is obtained in [2] where the authors studied the lower order terms $b(t, \xi)$ and $c(t)$ which are independent of x. They showed the C^∞ well posedness of the Cauchy problem under the conditions (1.1) and (1.2) with

$$\gamma = \frac{1}{2} - \frac{1}{k}.$$

Moreover in [1] the authors considered the case of first order term $b(t, \xi)$ independent of x, but zero order term $c(t, x)$ depending on all the variables, obtaining C^∞ well-posedness under the conditions (1.1) and (1.2) with

$$\gamma > \frac{1}{2} - \frac{1}{k}.$$

In the special case that $n = 1$, $a(t, \xi_1) = t^{2l}\xi_1^2$ and $b(t, \xi_1) = t^\nu \xi_1$, $c(t) = 0$, a necessary and sufficient condition for the C^∞ well posedness is (see [7], [10]) that

$$|b(t, \xi_1)| \leq C t^{l-1} |\xi_1|$$

with some $C > 0$. This shows that $\gamma = 1/2 - (1/k)$ is optimal.

We note that when $k = 2$ any lower order term verifies (1.2). This is a special case of effectively hyperbolic case as we remarked before. On the other hand we get $\gamma = 1/2$ formally when $k = +\infty$. The condition (1.2) with $\gamma = 1/2$ is sufficient for the C^∞ well posedness for any $a(t, x, \xi) \geq 0$ and $b(t, x, \xi)$ analytic with respect to t and x, if the space dimension n is equal to 1 (see [8]), or, for every $n \geq 1$, if $a(t, \xi) \geq 0$ and $b(t, \xi)$ depend, analytically, only on t (see [3]).

2. Preliminaries

Assume (1.1) at $t = 0$:

$$\sum_{j=0}^{k} |\partial_t^j a(0, \xi)| \neq 0, \quad \forall |\xi| = 1. \tag{2.1}$$

Let us set

$$\tilde{a}(t, \xi) = \frac{a(t, \xi)}{|\xi|^2}$$

so that $\widetilde{a}(t,\xi)$ is homogeneous of degree 0 in ξ and start with

Lemma 2.1. *There exist $c > 0$, $\delta > 0$ such that for any $|\xi| = 1$ one can find $0 \le r(\xi) \le k$ so that we have*

- $|\partial_t^{r(\xi)}\widetilde{a}(t,\xi)| \ge c, \quad |t| \le \delta$
- $\partial_t^j\widetilde{a}(t,\xi) = 0$, $0 \le j < r(\xi)$, $|t| \le \delta$ *has at most $r(\xi) - j$ roots with respect to t.*

We first prove

Lemma 2.2. *Let $|\bar{\xi}| = 1$ be fixed. Then there exist $0 \le r \le k$, $c > 0$, $\delta > 0$ and a neighborhood V of $\bar{\xi}$ such that*

- $|\partial_t^r\widetilde{a}(t,\xi)| \ge c, \quad |t| \le \delta, \quad \xi \in V$
- $\partial_t^j\widetilde{a}(t,\xi) = 0$, $0 \le j < r$, $\xi \in V$, $|t| \le \delta$ *has at most $r - j$ roots with respect to t.*

Proof. If $\widetilde{a}(0,\bar{\xi}) \neq 0$ the assertion is clear with $r = 0$. Assume $\widetilde{a}(0,\bar{\xi}) = 0$. From (2.1) there is $1 \le r \le k$ such that

$$\partial_t^\mu\widetilde{a}(0,\bar{\xi}) = 0, \quad 0 \le \mu < r, \quad \partial_t^r\widetilde{a}(0,\bar{\xi}) \neq 0.$$

Hence one can choose $c > 0$, $\delta^{(r)} > 0$ and a neighborhood $V^{(r)}$ of $\bar{\xi}$ so that

$$|\partial_t^r\widetilde{a}(t,\xi)| \ge c, \quad |t| \le \delta^{(r)}, \quad \xi \in V^{(r)}.$$

Consider $\partial_t^j\widetilde{a}(t,\xi)$ for $0 \le j < r$. Note that

$$\partial_t^i(\partial_t^j\widetilde{a})(0,\bar{\xi}) = 0, \quad 0 \le i < r - j, \quad \partial_t^{r-j}(\partial_t^j\widetilde{a})(0,\bar{\xi}) \neq 0.$$

By the Malgrange preparation theorem, one can find $\delta^{(j)}$ and a neighborhood $V^{(j)}$ of $\bar{\xi}$ such that one can write

$$\partial_t^j\widetilde{a}(t,\xi) = e^{(j)}(t,\xi)\left[t^{r-j} + \widetilde{a}_1^{(j)}(\xi)t^{r-j-1} + \cdots + \widetilde{a}_{r-j}^{(j)}(\xi)\right]$$

for $|t| \le \delta^{(j)}$, $\xi \in V^{(j)}$ where $\widetilde{a}_\mu^{(j)}(\bar{\xi}) = 0$ and $e^{(j)}(t,\xi) \neq 0$ (for $|t| \le \delta^{(j)}$, $\xi \in V^{(j)}$). Thus we conclude that if $\xi \in V^{(j)}$, $|t| \le \delta^{(j)}$ then $\partial_t^j\widetilde{a}(t,\xi) = 0$ has at most $r - j$ roots with respect to t. Now taking

$$\delta = \min_{0 \le j \le r} \delta^{(j)}, \qquad V = \bigcap_{j=0}^{r} V^{(j)}$$

we get the desired assertion. $\square$

Proof of Lemma 2.1. From Lemma 2.2, for any $|\xi| = 1$, there exist $0 \le r(\xi) \le k$, $c(\xi) > 0$, $\delta(\xi) > 0$ and a neighborhood $V(\xi)$ of ξ such that

$$\begin{gathered}|\partial_t^{r(\xi)}\widetilde{a}(t,\eta)| \ge c(\xi), \quad \text{for} \quad |t| \le \delta(\xi), \quad \eta \in V(\xi),\\ \partial_t^j\widetilde{a}(t,\eta) = 0, \quad 0 \le j < r(\xi), \quad \eta \in V(\xi) \text{ has at most}\\ r(\xi) - j \text{ roots with respect to } t \text{ in } |t| \le \delta(\xi).\end{gathered}$$

Since $\{|\xi| = 1\}$ is compact one can find $\xi_1, \dots, \xi_M$ so that

$$\{|\xi| = 1\} \subset \bigcup_{i=1}^{M} V(\xi_i).$$

Let us set

$$0 < \delta = \min_{1\le j\le M} \delta(\xi_j), \quad c = \min_{1\le j\le M} c(\xi_j) > 0.$$

Then for any $|\xi| = 1$ there is i such that $\xi \in V(\xi_i)$. Taking $r(\xi) = r(\xi_i)$ we get the desired assertion. $\square$

For $s < t$ we set

$$F^{(j)}(s,t;\xi) = \max\left(\frac{|\widetilde{a}^{(j)}(t,\xi)| + |\xi|^{-1}}{|\widetilde{a}^{(j)}(s,\xi)| + |\xi|^{-1}}, \frac{|\widetilde{a}^{(j)}(s,\xi)| + |\xi|^{-1}}{|\widetilde{a}^{(j)}(t,\xi)| + |\xi|^{-1}}\right) \tag{2.2}$$

where $\widetilde{a}^{(j)}(t,\xi) = \partial_t^j\widetilde{a}(t,\xi)$. It is obvious that

$$B|\xi| \ge F^{(j)}(s,t;\xi) \ge 1$$

with some $B > 0$. We define $W^{(j)}(t,\xi)$ as follows: let N be fixed (which will be determined later). We set

$$W^{(j)}(t,\xi) = \sup \sum_{i=0}^{N-1} \log F^{(j)}(t_i, t_{i+1};\xi), \tag{2.3}$$

where supremum is taken over all sequences $\{t_i\}_{i=0}^{N}$ such that

$$0 \le t_0 \le t_1 \le \cdots \le t_N \le t. \tag{2.4}$$

Note that $W^{(j)}(t,\xi)$ is an increasing function in t by definition so that $W^{(j)}(t,\xi)$ is differentiable almost everywhere and

$$\frac{d}{dt}W^{(j)}(t,\xi) \ge 0 \quad \text{a.e.}$$

Let us put

$$W(t,\xi)=\sum_{j=0}^{k} W^{(j)}(t,\xi).$$

Then we have

$$W(t,\xi)\le C\log(2+|\xi|),$$

with some $C>0$. We now recall that $\widetilde{a}(t,\xi)$ is non negative:

$$\widetilde{a}(t,\xi)\ge 0.$$

For $s<t$ we put

$$F^*(s,t;\xi)=\max\left(\frac{\widetilde{a}(t,\xi)+|\xi|^{-2}}{\widetilde{a}(s,\xi)+|\xi|^{-2}},\frac{\widetilde{a}(s,\xi)+|\xi|^{-2}}{\widetilde{a}(t,\xi)+|\xi|^{-2}}\right).$$

Define $W^*(t,\xi)$ by the same formula (2.3) where $F^{(j)}(t_i,t_{i+1};\xi)$ is replaced by $F^*(t_i,t_{i+1};\xi)$.

Lemma 2.3. *$e^{W^*(t,\xi)}$ and $e^{W^{(j)}(t,\xi)}$ are temperate, that is we have*

$$W^*(t,\xi)\le C\log(2+|\xi-\eta|)+W^*(t,\eta),$$
$$W^{(j)}(t,\xi)\le C\log(2+|\xi-\eta|)+W^{(j)}(t,\eta)$$

with some $C>0$, for $|\xi|$, $|\eta|\ge 1$.

Proof. We prove the first assertion. We fix a small $0<\epsilon\ll 1$. When $|\xi-\eta|\ge\epsilon|\xi|$ we proceed as follows. Note that

$$\frac{\widetilde{a}(t,\xi)+|\xi|^{-2}}{\widetilde{a}(s,\xi)+|\xi|^{-2}}\le C|\xi|^2\le\epsilon^{-2}C|\xi-\eta|^2.$$

This shows that

$$F^*(s,t;\xi)\le C\epsilon^{-2}|\xi-\eta|^2\le C'(2+|\xi-\eta|)^2.$$

Since $F^*(s,t;\eta)\ge 1$ one gets

$$F^*(s,t;\xi)\le C'(2+|\xi-\eta|)^2F^*(s,t;\eta). \tag{2.5}$$

Let $\{t_i\}_{i=0}^N$ be any sequence verifying (2.4). Then we have

$$\sum_{i=0}^{N-1}\log F^*(t_i,t_{i+1};\xi)\le NC''\log(2+|\xi-\eta|)+\sum_{i=0}^{N-1}\log F^*(t_i,t_{i+1};\eta)$$

by (2.5). Since the right-hand side is bounded by

$$NC'' \log (2 + |\xi - \eta|) + W^*(t, \eta)$$

and $\{t_i\}_{i=0}^N$ is arbitrary we get the desired assertion.

We turn to the case $|\xi - \eta| \leq \epsilon|\xi|$ and hence $C^{-1}|\xi| \leq |\eta| \leq C|\xi|$ with some $C > 0$. It is enough to show that

$$\tilde{a}(t, \xi) + |\xi|^{-2} \leq C(2 + |\xi - \eta|)^3[\tilde{a}(t, \eta) + |\eta|^{-2}]. \tag{2.6}$$

Assume that (2.6) is proved. Then exchanging ξ and η and taking $t = s$ one gets

$$[\tilde{a}(s, \xi) + |\xi|^{-2}]^{-1} \leq C(2 + |\xi - \eta|)^3[\tilde{a}(s, \eta) + |\eta|^{-2}]^{-1}. \tag{2.7}$$

Thus from (2.6) and (2.7) we have

$$F^*(s, t; \xi) \leq C^2(2 + |\xi - \eta|)^6 F^*(s, t; \eta). \tag{2.8}$$

The rest of the proof is just a repetition of the case $|\xi - \eta| \geq \epsilon|\xi|$. We now prove (2.6). Let us recall that $a(t, \xi)$ is homogeneous of degree 2 with respect to ξ. By the Glaeser inequality one has

$$|\partial_{\xi_i} a(t, \xi)| \leq C\sqrt{a(t, \xi)}.$$

Hence we have

$$a(t, \xi) \leq a(t, \eta) + C|\xi - \eta|\sqrt{a(t, \eta)} + C|\xi - \eta|^2$$

from the Taylor expansion. Since $2\sqrt{a(t, \eta)} \leq a(t, \eta) + 1$ it follows that

$$a(t, \xi) + 1 \leq C[a(t, \eta) + 1](2 + |\xi - \eta|)^2. \tag{2.9}$$

Noting $C^{-1}|\xi| \leq |\eta| \leq C|\xi|$ and multiplying (2.9) by $|\xi|^{-2}$ we get (2.6). This completes the proof of the first assertion.

To prove the second assertion we use the following inequality in place of (2.6):

$$|a^{(j)}(t, \xi)| + |\xi|^{-1} \leq C(2 + |\xi - \eta|)(|a^{(j)}(t, \eta)| + |\eta|^{-1}). \tag{2.10}$$

To see (2.10) let us put $\phi(t, \xi) = \tilde{a}^{(j)}(t, \xi)|\xi|^2$. Since

$$\phi(t, \xi) = \phi(t, \eta) + (\xi - \eta) \cdot \nabla_\xi \phi(t, \eta + \theta(\xi - \eta))$$

we have

$$|\phi(t, \xi)| \leq |\phi(t, \eta)| + C|\xi - \eta||\xi|$$

because $C^{-1}|\xi| \le |\eta + \theta(\xi - \eta)| \le C|\xi|$. Thus we have

$$|\phi(t,\xi)| + |\xi| \le |\phi(t,\eta)| + |\eta| + \tilde{C}|\xi||\xi - \eta| \le \tilde{C}(|\phi(t,\eta)| + |\eta|)(2 + |\xi - \eta|)$$

recalling $C^{-1}|\xi| \le |\eta| \le C|\xi|$. Multiplying $|\xi|^{-2}$ to the above inequality we get the desired result. □

In what follows we take $N = 2k + 1$.

Lemma 2.4. *There is $D > 0$ and $c > 0$ such that we have for any ξ*

$$\frac{d}{dt}\left[W^*(t,\xi) + Dt\right] \ge \frac{|a'(t,\xi)|}{a(t,\xi)+1}$$
$$\frac{d}{dt}\left[W(t,\xi) + Dt\right] \ge \frac{c|\xi|^{2/k}}{(a(t,\xi)+|\xi|)^{1/k}}$$

in $|t| \le \delta$.

Proof. We prove the second assertion. From Lemma 2.1 for any ξ there is $r(\xi)$ such that the assertion of Lemma 2.1 holds. Let $r(\xi) = 0$ then one has

$$\widetilde{a}(t,\xi) \ge c, \quad |t| \le \delta.$$

In this case the assertion holds obviously if we take $D > 0$ large because

$$\frac{d}{dt}W(t,\xi) \ge 0.$$

We show the assertion when $r(\xi) \ge 1$. From Lemma 2.1 it follows that $\widetilde{a}^{(j)}(t,\xi)$ and $\widetilde{a}^{(j+1)}(t,\xi)$, $0 \le j \le r(\xi) - 1$ have at most k zeros in $|t| \le \delta$. Choosing $t_0 = 0$, $t_N = t$ and $t_1 \le t_2 \le \cdots \le t_{N-1}$ to be the zeros of $\widetilde{a}^{(j)}(s,\xi)$ and $\widetilde{a}^{(j+1)}(s,\xi)$ in $(0,t)$ we get

$$\int_0^t \frac{|\widetilde{a}^{(j+1)}(s,\xi)|}{|\widetilde{a}^{(j)}(s,\xi)| + |\xi|^{-1}}\,ds = \sum_{i=0}^{N-1} \log F^{(j)}(t_i, t_{i+1}; \xi) \le W^{(j)}(t,\xi).$$

On the other hand we have

$$\log F^{(j)}(s,t;\xi) \le \int_s^t \frac{|\widetilde{a}^{(j+1)}(\tau,\xi)|}{|\widetilde{a}^{(j)}(\tau,\xi)| + |\xi|^{-1}}\,d\tau \tag{2.11}$$

for any $s < t$ (see [4], proof of Lemma 2.2). This shows that

$$W^{(j)}(t,\xi) \le \int_0^t \frac{|\widetilde{a}^{(j+1)}(s,\xi)|}{|\widetilde{a}^{(j)}(s,\xi)| + |\xi|^{-1}}\,ds.$$

Hence one gets

$$W^{(j)}(t,\xi) = \int_0^t \frac{|\widetilde{a}^{(j+1)}(s,\xi)|}{|\widetilde{a}^{(j)}(s,\xi)| + |\xi|^{-1}}\,ds \tag{2.12}$$

for $0 \le j \le r(\xi) - 1$.

Now we have

$$\begin{aligned} \frac{d}{dt} W(t,\xi) &= \frac{d}{dt}\sum_{j=r(\xi)}^{k} W^{(j)}(t,\xi) + \frac{d}{dt}\sum_{j=0}^{r(\xi)-1} W^{(j)}(t,\xi) \\ &\ge \sum_{j=0}^{r(\xi)-1} \frac{|\widetilde{a}^{(j+1)}(t,\xi)|}{|\widetilde{a}^{(j)}(t,\xi)| + |\xi|^{-1}}. \end{aligned} \tag{2.13}$$

We note that

$$\begin{aligned} \frac{|\widetilde{a}^{(r(\xi))}(t,\xi)|}{\widetilde{a}(t,\xi) + |\xi|^{-1}} &= \frac{|\widetilde{a}^{(r(\xi))}(t,\xi)|}{|\widetilde{a}^{(r(\xi)-1)}(t,\xi)| + |\xi|^{-1}} \cdot \frac{|\widetilde{a}^{(r(\xi)-1)}(t,\xi)| + |\xi|^{-1}}{|\widetilde{a}^{(r(\xi)-2)}(t,\xi)| + |\xi|^{-1}} \cdots \frac{|\widetilde{a}^{(1)}(t,\xi)| + |\xi|^{-1}}{\widetilde{a}(t,\xi) + |\xi|^{-1}} \\ &\le \left(\sum_{j=0}^{r(\xi)-1} \frac{|\widetilde{a}^{(j+1)}(t,\xi)| + |\xi|^{-1}}{|\widetilde{a}^{(j)}(t,\xi)| + |\xi|^{-1}} \right)^{r(\xi)} \\ &\le \left(r(\xi) + \sum_{j=0}^{r(\xi)-1} \frac{|\widetilde{a}^{(j+1)}(t,\xi)|}{|\widetilde{a}^{(j)}(t,\xi)| + |\xi|^{-1}} \right)^{r(\xi)} \\ &\le \left(k + \sum_{j=0}^{r(\xi)-1} \frac{|\widetilde{a}^{(j+1)}(t,\xi)|}{|\widetilde{a}^{(j)}(t,\xi)| + |\xi|^{-1}} \right)^{k}. \end{aligned}$$

Since $|\widetilde{a}^{(r(\xi))}(t,\xi)| \ge c$ in $|t| \le \delta$ by Lemma 2.1 it follows that

$$\frac{c}{(\widetilde{a}(t,\xi) + |\xi|^{-1})^{1/k}} \le \left(k + \sum_{j=0}^{r(\xi)-1} \frac{|\widetilde{a}^{(j+1)}(t,\xi)|}{|\widetilde{a}^{(j)}(t,\xi)| + |\xi|^{-1}} \right).$$

Thanks to (2.13), the right-hand side is estimated by

$$\frac{d}{dt}\,(kt + W(t,\xi))$$

and this proves the assertion. □

Lemma 2.5. *We have*

$$|\xi|^{2/k} \le C(a(t,\xi) + |\xi|^{2/k})^{1/2} \frac{|\xi|^{2/k}}{(a(t,\xi) + |\xi|)^{1/k}}.$$

Let $\gamma = (k-2)/2(k-1)$. *Then we have*

$$a(t,\xi)^{\gamma}|\xi|^{1-2\gamma} \le C(a(t,\xi)+|\xi|^{2/k})^{1/2}\frac{|\xi|^{2/k}}{(a(t,\xi)+|\xi|)^{1/k}}.$$

Proof. The first assertion is obvious because

$$\frac{(a(t,\xi)+|\xi|)^{1/k}}{(a(t,\xi)+|\xi|^{2/k})^{1/2}}$$

is bounded since $k \ge 2$.

We prove the second assertion. Noticing $1-2\gamma-2/k = -2\gamma/k$ it suffices to show

$$\Big(\frac{a(t,\xi)^k}{|\xi|^2}\Big)^{\gamma} \le C\frac{(a(t,\xi)+|\xi|^{2/k})^{k/2}}{a(t,\xi)+|\xi|}$$

or rather

$$a(t,\xi)^{k\gamma}|\xi|^{-2\gamma}(a(t,\xi)+|\xi|) \le C(a(t,\xi)^{k/2}+|\xi|).$$

Since $k\gamma+1-k/2 = (k-2)/(2(k-1)) = \gamma$ it is clear that

$$a(t,\xi)^{k\gamma+1}|\xi|^{-2\gamma} = \Big(\frac{a(t,\xi)}{|\xi|^2}\Big)^{\gamma}a(t,\xi)^{k/2} \le C(a(t,\xi)^{k/2}+|\xi|).$$

On the other hand, remarking that $p = 1/(2\gamma) > 1$, $q = 1/(1-2\gamma) > 1$ because $0 \le \gamma < 1/2$ we have from the Young's inequality that

$$a(t,\xi)^{k\gamma}|\xi|^{1-2\gamma} \le 2\gamma a(t,\xi)^{k\gamma p} + (1-2\gamma)|\xi|^{(1-2\gamma)q} \le C(a(t,\xi)^{k/2}+|\xi|).$$

This proves the assertion. □

3. Energy estimate

In this section we prove Theorem 1.1. We apply the Fourier transform with respect to the space variable to the equation, thus we obtain the following ordinary differential equation in t, depending on the parameter ξ

$$v'' + a(t,\xi)v + i\widehat{bu} + \widehat{cu} = \widehat{f}, \tag{3.1}$$

where v denotes the Fourier transform of u with respect to x and the symbol $\widehat{\ }$ denotes the Fourier transform with respect to x.

We consider the following energy function

$$\mathcal{E}(t) = \int_{\mathbf{R}^n}\widetilde{E}(t,\xi)\,d\xi = \int_{\mathbf{R}^n}E(t,\xi)K(t,\xi)\,d\xi \tag{3.2}$$

with

$$E(t,\xi) = |v'(t,\xi)|^2 + (a(t,\xi) + |\xi|^{2/k} + 1)|v(t,\xi)|^2$$

and

$$K(t,\xi) = e^{A\Lambda(t,\xi)}$$

where

$$\Lambda(t,\xi) = -Dt - W^*(t,\xi) - W(t,\xi).$$

Differentiating $\mathcal{E}(t)$ with respect to the time we have

$$\mathcal{E}'(t) = \int_{\mathbf{R}^n} \left(E'(t,\xi) + A\Lambda'(t,\xi)E(t,\xi)\right) K(t,\xi)\, d\xi.$$

Note that

$$E' = 2\,\mathrm{Re}(v'', v') + 2\,\mathrm{Re}(v', v)(a + |\xi|^{2/k} + 1) + a'|v|^2$$

and using (3.1) we have

$$E'(t,\xi) \le 2|\widehat{bu}||v'| + 2|\widehat{cu}||v'| + 2|\widehat{f}||v'| + 2|\xi|^{2/k}|v||v'| + 2|v||v'| + |a'||v|^2.$$

Since

$$-\Lambda' \ge \frac{|a'(t,\xi)|}{a(t,\xi)+1} + \frac{c|\xi|^{2/k}}{(a(t,\xi)+|\xi|)^{1/k}} + 1$$

from Lemma 2.5 it follows that

$$2|\xi|^{2/k}|v||v'| \le C\frac{|\xi|^{2/k}}{(a+|\xi|)^{1/k}}((a+|\xi|^{2/k})|v|^2 + |v'|^2) \le -C\Lambda'(t,\xi)E(t,\xi).$$

Now we use the following estimates

$$2|\widehat{cu}||v'| \le |\widehat{cu}|^2 + E, \quad 2|\widehat{f}||v'| \le |\widehat{f}|^2 + E, \quad 2|v||v'| \le E.$$

Thus we obtain

$$E' \le 2|\widehat{bu}||v'| + |\widehat{cu}|^2 - C\Lambda' E + |\widehat{f}|^2. \tag{3.3}$$

We now estimate

$$2\int (|\widehat{bu}(t,\xi)||v'(t,\xi)| + |\widehat{cu}(t,\xi)|^2)K(t,\xi)\, d\xi.$$

Let us recall that

$$|\partial_x^\alpha b(t,x,\xi)| \le C_\alpha a(t,\xi)^\gamma |\xi|^{1-2\gamma} \tag{3.4}$$

for every α. We denote

$$\widehat{b}(t,\eta,\xi) = \int e^{-i\eta x} b(t,x,\xi)\,dx.$$

Note that, by integration by parts, for any l we have

$$|\widehat{b}(t,\eta,\xi)| \le C_l (2+|\eta|)^{-l} a(t,\xi)^\gamma |\xi|^{1-2\gamma}. \tag{3.5}$$

Lemma 3.1. *Assume* (3.4). *Then we have*

$$\int_{|\xi|\ge 1} |\widehat{bu}(t,\xi)||v'(t,\xi)|K(t,\xi)\,d\xi \le -C\int \Lambda'(t,\xi)E(t,\xi)K(t,\xi)\,d\xi$$

with some $C>0$.

Proof. Note that

$$\begin{aligned}2\int_{|\xi|\ge 1} &|\widehat{bu}||v'|K(t,\xi)\,d\xi\\ &\le \int_{|\xi|\ge 1} \frac{(a+|\xi|)^{1/k}}{|\xi|^{2/k}}|\widehat{bu}|^2 K(t,\xi)\,d\xi + \int \frac{|\xi|^{2/k}}{(a+|\xi|)^{1/k}}|v'|^2 K(t,\xi)\,d\xi.\end{aligned}$$

From Lemma 2.4 the second term of the right-hand side is bounded by

$$-C\int \Lambda'(t,\xi)E(t,\xi)K(t,\xi)\,d\xi.$$

Thus it is enough to show that

$$\int_{|\xi|\ge 1} \frac{(a(t,\xi)+|\xi|)^{1/k}}{|\xi|^{2/k}}|\widehat{bu}(t,\xi)|^2 K(t,\xi)\,d\xi \le -C\int \Lambda'(t,\xi)E(t,\xi)K(t,\xi)\,d\xi.$$

Note that

$$\begin{aligned}|\widehat{bu}(t,\xi)|^2 &= \left|\int \widehat{b}(t,\xi-\eta,\eta)\widehat{u}(t,\eta)\,d\eta\right|^2\\ &\le \int (2+|\xi-\eta|)^{-N_1}\,d\eta \int (2+|\xi-\eta|)^{N_1}|\widehat{b}(t,\xi-\eta,\eta)|^2|\widehat{u}(t,\eta)|^2\,d\eta\end{aligned}$$

$$\leq C \int (2 + |\xi - \eta|)^{N_1} |\widehat{b}(t, \xi - \eta, \eta)|^2 |\widehat{u}(t, \eta)|^2 \, d\eta.$$

From Lemma 2.3 it follows that

$$K(t, \xi) \leq (2 + |\xi - \eta|)^{N_2} K(t, \eta).$$

It is easy to check that

$$\frac{(a(t, \xi) + |\xi|)^{1/k}}{|\xi|^{2/k}} \leq (2 + |\xi - \eta|)^{N_3} \frac{(a(t, \eta) + |\eta|)^{1/k}}{|\eta|^{2/k}}$$

for $|\xi|$, $|\eta| \geq 1$. Then using these estimates one gets

$$\begin{aligned}
&\int_{|\xi| \geq 1} \frac{(a(t, \xi) + |\xi|)^{1/k}}{|\xi|^{2/k}} |\widehat{bu}(t, \xi)|^2 K(t, \xi) \, d\xi \\
&\quad \leq C \int (2 + |\xi - \eta|)^{N_1 + N_2 + N_3} \frac{(a(t, \eta) + |\eta|)^{1/k}}{|\eta|^{2/k}} |\widehat{b}(t, \xi - \eta, \eta)|^2 |\widehat{u}(t, \eta)|^2 K(t, \eta) \, d\xi \, d\eta.
\end{aligned}$$

From (3.5) and Lemma 2.5 it follows that

$$|\widehat{b}(t, \xi - \eta, \eta)|^2 \leq C(2 + |\xi - \eta|)^{-2l} \frac{(a(t, \eta) + |\eta|^{2/k})}{(a(t, \eta) + |\eta|)^{2/k}} |\eta|^{4/k}. \tag{3.6}$$

We plug the estimate (3.6) into the above estimate to get

$$\begin{aligned}
&C \int (2 + |\xi - \eta|)^{N - 2l} \frac{(a(t, \eta) + |\eta|^{2/k})}{(a(t, \eta) + |\eta|)^{1/k}} |\eta|^{2/k} |\widehat{u}(t, \eta)|^2 K(t, \eta) \, d\xi \, d\eta \\
&\quad \leq C' \int \frac{(a(t, \eta) + |\eta|^{2/k})}{(a(t, \eta) + |\eta|)^{1/k}} |\eta|^{2/k} |\widehat{u}(t, \eta)|^2 K(t, \eta) \, d\eta
\end{aligned}$$

where $N = N_1 + N_2 + N_3$ and we have taken l so that $N - 2l < -n$. This proves the assertion because

$$-\Lambda'(t, \eta) \geq \frac{c|\eta|^{2/k}}{(a(t, \eta) + |\eta|)^{1/k}}. \qquad \square$$

Lemma 3.2. *We have*

$$\int |\widehat{cu}(t, \xi)|^2 K(t, \xi) \, d\xi \leq C \int |\widehat{u}(t, \xi)|^2 K(t, \xi) \, d\xi$$

with some $C > 0$.

Proof. Since $K(t, \xi) \leq C_1 (2 + |\xi - \eta|)^{N_2} K(t, \eta)$ we see

$$\int |\widehat{cu}(t, \xi)|^2 K(t, \xi) \, d\xi$$

$$
\begin{aligned}
&\le \int K(t,\xi)\int |\widehat{c}(t,\xi-\eta)||\widehat{u}(t,\eta)|^2\,d\eta\,d\xi \int |\widehat{c}(t,\eta_1)|\,d\eta_1\\
&\le C_1 \iint K(t,\eta)|\widehat{u}(t,\eta)|^2|\widehat{c}(t,\xi-\eta)|(2+|\xi-\eta|)^{N_2}\,d\eta\,d\xi\\
&\le C_2 \int K(t,\eta)|\widehat{u}(t,\eta)|^2\,d\eta \int |\widehat{c}(t,\xi)|(2+|\xi|)^{N_2}\,d\xi\\
&\le C_3 \int K(t,\eta)|\widehat{u}(t,\eta)|^2\,d\eta. \qquad \square
\end{aligned}
$$

Multiply the inequality (3.3) by $K(t,\xi)$ and integrate with respect to ξ. In view of Lemma 3.1 and Lemma 3.2 one has

$$\int E'K\,d\xi \le -C'' \int \Lambda' EK\,d\xi + \int |\widehat{f}|^2 K\,d\xi.$$

Taking $A \ge C''$ in the definition of energy we conclude that

$$\mathcal{E}'(t) \le \int |\widehat{f}(t,\xi)|^2 K(t,\xi)\,d\xi.$$

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