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INTEGRABLE IRREDUCIBLE HIGHEST WEIGHT MODULES FOR $sl_2(\mathbf{C}_p[x^{\pm 1}, y^{\pm 1}])$

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1. Introduction

Modules for loop algebras $g \otimes \mathbf{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ and their universal central extensions have been extensively investigated where g is a finite dimensional simple Lie algebra. The universal central extensions are affine Lie algebras in the case $d = 1$ and called toroidal Lie algebras in the case $d > 1$ [1]. Affine Lie algebras and their q-analogues have been well studied and found applications in several areas. Compared with these algebras, the investigations of the representation theory of toroidal Lie algebras are still under way, though much progress was made in references [1], [2], [3], [4], to name a few, and their q-analogues were applied in [5], [6], [7] and [8].

Among modules for these Lie algebras we are interested in integrable modules. In the case $d = 1$ (with the scaling element), in [9] and [10], irreducible integrable modules with finite dimensional weight spaces were classified and, when the central element acts trivially, they were shown to be isomorphic to loop modules of the tensor product of some modules depending on continuous parameters or their irreducible submodules. The q-analogue of this problem was investigated in [11]. In the case $d = 2$, integrable modules were studied in [4] where some of the central elements act nontrivially. For $d \geq 2$ these modules were considered in [3] and references therein.

In this paper we consider a loop algebra with the algebra of Laurent polynomials replaced by a quantum torus [12]. For generic $p \in \mathbf{C}^\times$ let $\mathbf{C}_p[x^{\pm 1}, y^{\pm 1}]$ be the $\mathbf{C}$ algebra of the Laurent polynomials in the two variables x, y satisfying $yx = pxy$. We shall denote this algebra by $\mathcal{C}_p$. Let $gl_2(\mathcal{C}_p)$ be the Lie algebra of 2×2 matrices with entries in $\mathcal{C}_p$ with the usual commutator. We consider the Lie algebra $sl_2(\mathcal{C}_p) := [gl_2(\mathcal{C}_p), gl_2(\mathcal{C}_p)]$. Lie algebras of this kind appeared in the study of some extended affin Lie algebras in [13] and when taking the $q = 1$ limit of the quantum toroidal algebras in [5]. Representations of these Lie algebras were considered in [14] and those of their central extensions were studied in [15], [16] and [17] in terms of vertex operators. The main result of this paper is the classification of integrable irreducible highest weight modules for $sl_2(\mathcal{C}_p)$. Our line of thought and result are similar to those in [9] and [10] but more complex.

This paper is organized as follows. In Section 2, after giving the necessary definitions, we state our result for the classification of integrable irreducible highest weight modules for $sl_2(\mathcal{C}_p)$. The outline of the proof of this is given in Section 3 and the technical details are proven in Sections 4 and 5.

2. Main result

2.1. The Lie algebra $sl_2(\mathcal{C}_p)$. Fixing a nonzero complex number p which is not a root of unity, let $\mathcal{C}_p$ be the $\mathbf{C}$ algebra defined by generators $x^{\pm 1}$, $y^{\pm 1}$ and relations

$$x^{\pm 1}x^{\mp 1} = 1, \quad y^{\pm 1}y^{\mp 1} = 1, \quad yx = pxy.$$

We consider the Lie algebra $\mathcal{L} = sl_2(\mathcal{C}_p) := [gl_2(\mathcal{C}_p), gl_2(\mathcal{C}_p)]$. Set $\mathbf{0} = (0,0)$. For $\mathbf{k} = (k,l) \in \mathbf{Z}^2$, $\mathbf{m} = (m,n) \in \mathbf{Z}^2 \setminus \{\mathbf{0}\}$ and $i = 1, 2$, define the following elements of $\mathcal{L}$:

$$e_{\mathbf{k}} = E_{12}x^k y^l, \quad f_{\mathbf{k}} = E_{21}x^k y^l, \quad h = E_{11} - E_{22}, \quad \epsilon_i(\mathbf{m}) = E_{ii}x^m y^n.$$

Then these elements form a basis of $\mathcal{L}$ and satisfy the relations

$$\begin{aligned}
&[e_{\mathbf{k}}, f_{\mathbf{m}}] = \begin{cases} p^{lm}\epsilon_1(\mathbf{k}+\mathbf{m}) - p^{nk}\epsilon_2(\mathbf{k}+\mathbf{m}) & \text{if } \mathbf{k}+\mathbf{m} \neq \mathbf{0}, \\ p^{lm}h & \text{if } \mathbf{k}+\mathbf{m} = \mathbf{0}, \end{cases}\\
&[e_{\mathbf{k}}, e_{\mathbf{m}}] = 0 = [f_{\mathbf{k}}, f_{\mathbf{m}}],\\
&[h, e_{\mathbf{k}}] = 2e_{\mathbf{k}}, \quad [h, f_{\mathbf{k}}] = -2f_{\mathbf{k}}, \quad [h, \epsilon_i(\mathbf{m})] = 0,\\
&[\epsilon_1(\mathbf{k}), e_{\mathbf{m}}] = p^{lm}e_{\mathbf{k}+\mathbf{m}}, \quad [\epsilon_2(\mathbf{k}), e_{\mathbf{m}}] = -p^{nk}e_{\mathbf{k}+\mathbf{m}},\\
&[\epsilon_1(\mathbf{k}), f_{\mathbf{m}}] = -p^{nk}f_{\mathbf{k}+\mathbf{m}}, \quad [\epsilon_2(\mathbf{k}), f_{\mathbf{m}}] = p^{lm}f_{\mathbf{k}+\mathbf{m}},\\
&[\epsilon_i(\mathbf{k}), \epsilon_j(\mathbf{m})] = \begin{cases} \delta_{ij}(p^{lm} - p^{nk})\epsilon_i(\mathbf{k}+\mathbf{m}) & \text{if } \mathbf{k}+\mathbf{m} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{k}+\mathbf{m} = \mathbf{0} \end{cases}
\end{aligned}$$

where $\mathbf{k} = (k,l)$ and $\mathbf{m} = (m,n)$.

Set

$$\mathcal{N}_- = \bigoplus \mathbf{C} f_{\mathbf{k}}, \quad \mathcal{N}_+ = \bigoplus \mathbf{C} e_{\mathbf{k}},$$
$$\mathcal{H} = \bigoplus_{i=0}^{2} \mathcal{H}_i, \quad \mathcal{H}_0 = \mathbf{C}h, \quad \mathcal{H}_i = \bigoplus_{\mathbf{m}\neq\mathbf{0}} \mathbf{C}\epsilon_i(\mathbf{m}) \quad (i = 1, 2).$$

Then $\mathcal{N}_\pm$, $\mathcal{H}_i$ and $\mathcal{H}$ are subalgebras and the Lie algebra $\mathcal{L}$ is the direct sum of these subalgebras:

$$\mathcal{L} = \mathcal{N}_- \oplus \mathcal{H} \oplus \mathcal{N}_+.$$

Note that $[\mathcal{H}_i, \mathcal{H}_j] = 0$ $(i \neq j)$ but $\mathcal{H}$ is not commutative. Set

$$\mathcal{H}' = \mathbf{C}h \oplus \bigoplus_{\substack{l\neq 0\\ i=1,2}} \mathbf{C}\epsilon_i(0,l).$$

Then this is a maximal commutative subalgebra.

2.2. Main result. In this paper we call an $\mathcal{L}$ module V *highest weight* if there exists $0 \neq v \in V$ such that

$$V = U(\mathcal{L})v, \quad \mathcal{N}_+ v = 0, \quad uv \in \mathbf{C}v \quad (\forall u \in \mathcal{H}').$$

We call such a vector v a highest weight vector. An $\mathcal{L}$ module V is said to be *integrable* if it admits a weight space decomposition with respect to h

$$V = \bigoplus_{\alpha \in \mathbf{C}} V_\alpha, \quad V_\alpha = \{v \in V \mid hv = \alpha v\}$$

and

$$(e_{\mathbf{m}})^M u = 0 \quad \text{and} \quad (f_{\mathbf{m}})^M u = 0 \quad (M \gg 1)$$

for any $u \in V$ and any $\mathbf{m} \in \mathbf{Z}^2$.

The main result of this paper is the classification of integrable irreducible highest weight $\mathcal{L}$ modules. To describe the result, we introduce some notation.

Let D be a linear map on $\mathbf{C}[z^{\pm 1}]$ such that $Dz^m = p^m z^m$ $(m \in \mathbf{Z})$ and let the E_{ij} denote the matrix units acting on $\mathbf{C}^2$. For $a \in \mathbf{C}^\times$, let $V^1(a)$ denote the $\mathcal{L}$ module $\mathbf{C}^2[z^{\pm 1}](= \mathbf{C}^2 \otimes \mathbf{C}[z^{\pm 1}])$ on which $\mathcal{L}$ acts as

$$\begin{aligned} &e_{\mathbf{k}} = a^l E_{12} z^k D^l, \quad f_{\mathbf{k}} = a^l E_{21} z^k D^l, \quad h = E_{11} - E_{22}, \\ &\epsilon_i(\mathbf{m}) = a^n E_{ii} z^m D^n \quad (i = 1,\ 2) \end{aligned}$$

and let $V^2(a)$ signify the $\mathcal{L}$ module $\mathbf{C}^2[z^{\pm 1}]$ on which $\mathcal{L}$ acts as

$$\begin{aligned} &e_{\mathbf{k}} = a^l E_{12} D^l z^{-k}, \quad f_{\mathbf{k}} = a^l E_{21} D^l z^{-k}, \quad h = E_{11} - E_{22}, \\ &\epsilon_1(\mathbf{m}) = -a^n E_{22} D^n z^{-m}, \quad \epsilon_2(\mathbf{m}) = -a^n E_{11} D^n z^{-m}. \end{aligned}$$

For each Young tableau T let c_T denote the corresponding Young symmetrizer, i.e.,

$$c_T = \sum_{\sigma \in C(T)} \sum_{\tau \in R(T)} \operatorname{sign}(\sigma) \sigma \tau \in \mathbf{C}[\mathcal{S}_r]$$

where r is $|T|$, the number of boxes of T, and $R(T)$ and $C(T)$ are the subgroups of $\mathcal{S}_r$ preserving the entries in each row and column of T, respectively.

In this paper we shall neither consider a partition of 0 nor the empty tableau. For each partition λ fix a tableau T_λ of shape λ. For a partition λ, a nonzero complex number a and $i = 1,\ 2$, set

$$U^i_\lambda(a) = c_{T_\lambda} V^i(a)^{\otimes r}$$

where $r = |\lambda|$ and the symmetric group $\mathcal{S}_r$ acts on the tensor product $V^i(a)^{\otimes r}$ by permuting the factors. Since the action of $\mathcal{L}$ on $V^i(a)^{\otimes r}$ commutes with that of $\mathcal{S}_r$, $U^i_\lambda(a)$ is an $\mathcal{L}$ module. Let $p^{\mathbf{Z}} = \{p^m \mid m \in \mathbf{Z}\}$. For a nonnegative integer n, partitions $\lambda_1, \ldots, \lambda_n$ and nonzero complex numbers $a_1, \ldots, a_n$ such that $a_i/a_j \notin p^{\mathbf{Z}}$ $(i \neq j)$, set

$$U^i_{\lambda_1,\ldots,\lambda_n}(a_1, \ldots, a_n) = U^i_{\lambda_1}(a_1) \otimes \cdots \otimes U^i_{\lambda_n}(a_n) \quad (i = 1,\ 2)$$

where in the case $n = 0$ the right hand side should be understood as the trivial $\mathcal{L}$ module $\mathbf{C}$.

Theorem 1. *Set $U = U^1_{\lambda_1,\ldots,\lambda_n}(a_1, \ldots, a_n) \otimes U^2_{\mu_1,\ldots,\mu_m}(b_1, \ldots, b_m)$. The submodule of U generated by the weight space U_{n+m} has a unique irreducible quotient, which we denote by $V_{\lambda_1,\ldots,\lambda_n;\mu_1,\ldots,\mu_m}(a_1, \ldots, a_n; b_1, \ldots, b_m)$. Then*
(1) *An $\mathcal{L}$ module is integrable, irreducible and highest weight if and only if it is isomorphic to one of the $V_{\lambda_1,\ldots,\lambda_n;\mu_1,\ldots,\mu_m}(a_1, \ldots, a_n; b_1, \ldots, b_m)$.*
(2) *The $\mathcal{L}$ modules*

$$V_{\lambda_1,\ldots,\lambda_n;\mu_1,\ldots,\mu_m}(a_1, \ldots, a_n; b_1, \ldots, b_m) \text{ and } V_{\lambda'_1,\ldots,\lambda'_{n'};\mu'_1,\ldots,\mu'_{m'}}(a'_1, \ldots, a'_{n'}; b'_1, \ldots, b'_{m'})$$

are isomorphic if and only if $n' = n$, $m' = m$ and there exist $\sigma \in \mathcal{S}_n$ and $\tau \in \mathcal{S}_m$ such that

$$\lambda'_{\sigma(i)} = \lambda_i, \quad a'_{\sigma(i)}/a_i \in p^{\mathbf{Z}}, \quad \mu'_{\tau(j)} = \mu_j, \quad b'_{\tau(j)}/b_j \in p^{\mathbf{Z}}$$

for $1 \leq i \leq n$ and $1 \leq j \leq m$.

REMARK 1. Since U in the theorem admits a weight space decomposition with respect to the commutative subalgebra $\mathcal{H}'$, so does
$V_{\lambda_1,\ldots,\lambda_n;\mu_1,\ldots,\mu_m}(a_1, \ldots, a_n; b_1, \ldots, b_m)$.

3. Proof of Theorem 1

In this section we shall prove Theorem 1, leaving the technical details to later sections. First we introduce some notation. For a subalgebra K of $\mathcal{L}$ and a complex number α, set

$$U(K)_\alpha = \{u \in U(K) \mid [h, u] = \alpha u\}.$$

Then

$$U(\mathcal{H}) = U(\mathcal{H})_0 \quad \text{and} \quad U(\mathcal{N}_\pm) = \bigoplus_{n=0}^{\infty} U(\mathcal{N}_\pm)_{\pm 2n}.$$

If V is a highest weight $\mathcal{L}$ module with highest weight vector v and $hv = \alpha v$ ($\alpha \in \mathbf{C}$), then

$$V = \bigoplus_{n\geq 0} V_{\alpha-2n}, \quad V_{\alpha-2n} = U(\mathcal{N}_-)_{-2n}U(\mathcal{H})v.$$

Let W be an irreducible $\mathcal{H}$ module on which h acts as a scalar α and set

$$M(W) = U(\mathcal{L}) \otimes_{U(B_+)} W$$

where W is regarded as a $B_+ := \mathcal{H} + \mathcal{N}_+$ module by letting $\mathcal{N}_+ W = 0$. Since $M(W) \simeq U(\mathcal{N}_-) \otimes W$ as vector spaces, we get

$$M(W) = \bigoplus_{n\geq 0} M(W)_{\alpha-2n}, \quad M(W)_{\alpha-2n} \simeq U(\mathcal{N}_-)_{-2n} \otimes W.$$

So, noting that W is irreducible, we find that the $\mathcal{L}$ module $M(W)$ has a unique maximal submodule N, which intersects $M(W)_\alpha \simeq W$ trivially. We set $V(W) = M(W)/N$. Let w be a nonzero vector in W and v the image of $1 \otimes w$ in $V(W)$. Then we can identify W with the $\mathcal{H}$ submodule $V(W)_\alpha = U(\mathcal{H})v$.

Define $\mathcal{W}$ to be the set of irreducible $\mathcal{H}$ modules generated by a nonzero vector w such that

$$uw \in \mathbf{C}w \quad (\forall u \in \mathcal{H}'). \tag{3.1}$$

Note that h acts on an $\mathcal{H}$ module in $\mathcal{W}$ as a scalar. Suppose that $W \in \mathcal{W}$ and that $w \in W$ is a nonzero vector satisfying the above condition. Let v denote the image of $1 \otimes w$ in $V(W)$. Then $V(W)$ is an irreducible highest weight $\mathcal{L}$ module with highest weight vector v. Note that any highest weight vector of $V(W)$ is obtained in this way.

Proposition 1. (1) *An $\mathcal{L}$ module V is irreducible and highest weight if and only if there exists $W \in \mathcal{W}$ such that $V \simeq V(W)$.*
(2) *For $W, W' \in \mathcal{W}$, $V(W) \simeq V(W')$ if and only if $W \simeq W'$.*

Proof. (1) The 'if part' has just been proved before the proposition. We show the 'only if' part. Suppose that V is an irreducible highest weight module with highest weight vector v. Set $W = U(\mathcal{H})v$. Then we can easily see that $W \in \mathcal{W}$ and $V \simeq V(W)$.
(2) Follows from the definition and the fact $V(W)_\alpha \simeq W$ as $\mathcal{H}$ modules. □

The argument used above for $V(W)$ is quite standard. See [9] and [3] for similar constructions in the case of loop algebras.

By the standard theory of $sl_2(\mathbf{C})$, we can see that for an integrable highest weight $\mathcal{L}$ module V with highest weight vector v there exists a nonnegative integer N such that

$$V = \bigoplus_{i=0}^{N} V_{N-2i} \quad \text{and} \quad v \in V_N.$$

For a nonnegative integer N let $\mathcal{IH}(N)$ denote the set of pairs of an integrable highest weight $\mathcal{L}$ module V and its highest weight vector v satisfying the above condition.

The following lemma was proved for one variable loop algebras in [9, Lemma 4.4]. The proof in our case is essentially the same.

Lemma 1. *Let V be a highest weight $\mathcal{L}$ module with highest weight vector v such that $hv = Nv$ for some nonnegative integer N. Then*
(1) *V is integrable if and only if $f_{\mathbf{m}_{N+1}} \cdots f_{\mathbf{m}_1} v = 0$ for any $\mathbf{m}_i \in \mathbf{Z}^2$ $(\forall i)$.*
(2) *If V is irreducible, then it is integrable if and only if*

$$e_{\mathbf{k}_1} \cdots e_{\mathbf{k}_{N+1}} f_{\mathbf{m}_{N+1}} \cdots f_{\mathbf{m}_1} v = 0$$

for any $\mathbf{k}_i$, $\mathbf{m}_i \in \mathbf{Z}^2$ $(\forall i)$.

Set $\mathcal{A} = U(\mathcal{H}_1)$ and $\mathcal{B} = U(\mathcal{H}_2)$. Note that $\mathcal{A}$ and $\mathcal{B}$ commute with each other. Let $a(\mathbf{k}) = \epsilon_1(\mathbf{k})$ and $b(\mathbf{k}) = -\epsilon_2(\mathbf{k})$ for $\mathbf{k} \neq \mathbf{0}$. Define $\Lambda^{\pm}_{\mathcal{A}}(u) = \sum_{n\geq 0} \Lambda_{\mathcal{A},\pm n} u^n \in \mathcal{A}[[u]]$ and $\Lambda^{\pm}_{\mathcal{B}}(u) = \sum_{n\geq 0} \Lambda_{\mathcal{B},\pm n} u^n \in \mathcal{B}[[u]]$ by

$$\Lambda^{\pm}_{\mathcal{A}}(u) = \exp\left(-\sum_{i>0} \frac{a(0,\pm i)}{i} u^i\right) \quad \text{and} \quad \Lambda^{\pm}_{\mathcal{B}}(u) = \exp\left(-\sum_{i>0} \frac{b(0,\pm i)}{i} u^i\right),$$

respectively.

Lemma 2. *For $(V, v) \in \mathcal{IH}(N)$ define formal power series $\lambda^{\pm}_{\mathcal{A}}(u)$ by*

$$\Lambda^{\pm}_{\mathcal{A}}(u) v = \lambda^{\pm}_{\mathcal{A}}(u) v$$

and $\lambda^{\pm}_{\mathcal{B}}(u)$ similarly. Then there exist nonnegative integers r and s and nonzero complex numbers $a_1, \ldots, a_r$, $b_1, \ldots, b_s$ such that $r + s = N$,

$$\lambda^{\pm}_{\mathcal{A}}(u) = \prod_{i=1}^{r} \left(1 - a_i^{\pm 1} u\right) \quad \text{and} \quad \lambda^{\pm}_{\mathcal{B}}(u) = \prod_{i=1}^{s} \left(1 - b_i^{\pm 1} u\right). \tag{3.2}$$

To prove this lemma, we need the following result [18]. Let L be the loop algebra of type sl_2, $sl_2(\mathbf{C}) \otimes \mathbf{C}[t, t^{-1}]$, and set for $k \in \mathbf{Z}$

$$e_k = E_{12} \otimes t^k, \quad f_k = E_{21} \otimes t^k, \quad h_k = (E_{11} - E_{22}) \otimes t^k.$$

Define $\Lambda^{\pm}(u) \in U(L)[[u]]$ by

$$\Lambda^{\pm}(u) = \sum_{n\geq 0} \Lambda_{\pm n} u^n = \exp\left(- \sum_{k>0} \frac{h_{\pm k}}{k} u^k \right).$$

An L module is said to be integrable if it admits a weight space decompositoin with respect to h_0 and the e_k and the f_k act on it locally nilpotently.

Lemma 3 (Proposition 1.1, [18]). *Let V be an integrable L module and v a vector in V such that $e_k v = 0$ for any k and $hv = Nv$ for a nonnegative integer N. Then*

$$\Lambda_N \Lambda^-(u)v = u^N \Lambda^+\left(\frac{1}{u}\right) v.$$

In particular, this implies

$$\Lambda_n v = 0 \quad (|n| > N) \quad \textit{and} \quad \Lambda_N \Lambda_{-N} v = v.$$

Proof of Lemma 2. For an integer m, set

$$\Lambda^{\pm}(u; m) = \exp\left(- \sum_{i>0} \frac{p^{\pm im} a(0, \pm i) + b(0, \pm i)}{i} u^i \right)$$

so that

$$\Lambda^{\pm}(u; m) = \Lambda^{\pm}_{\mathcal{A}}(p^{\pm m} u) \Lambda^{\pm}_{\mathcal{B}}(u).$$

For each m the subalgebra of $\mathcal{L}$ generated by the elements $e_{-m,l}$ and $f_{m,l}$ ($l \in \mathbf{Z}$) is isomorphic to L and for this subalgebra

$$h_k = p^{km} a(0, k) + b(0, k) \quad \text{for } k \neq 0$$

and $h_0 = h$. Applying Lemma 3 to these subalgebras, we find that for any integer m

$$\lambda(m) \lambda^-_{\mathcal{A}}\left(\frac{u}{p^m}\right) \lambda^-_{\mathcal{B}}(u) = u^N \lambda^+_{\mathcal{A}}\left(\frac{p^m}{u}\right) \lambda^+_{\mathcal{B}}\left(\frac{1}{u}\right)$$

where $\lambda(m)$ is a nonzero complex number. From this we get the assertion. $\square$

For nonnegative integers r and s we let $\mathcal{IH}(r, s)$ denote the subset of $\mathcal{IH}(r + s)$ consisting of the elements for which (3.2) holds for some nonzero complex numbers $a_1, \ldots, a_r, b_1, \ldots, b_s$.

For a nonnegative integer r, $M \geq 1$ and $\mathbf{k}_i = (k_i, l_i) \in \mathbf{Z}^2$ $(1 \leq i \leq M)$, define $A^r_M(\mathbf{k}_1, \dots, \mathbf{k}_M) \in \mathcal{A}$ and $B^r_M(\mathbf{k}_1, \dots, \mathbf{k}_M) \in \mathcal{B}$ by the recurrence relations

$$A^r_1(\mathbf{k}) = \begin{cases} a(\mathbf{k}) & \text{if } \mathbf{k} \neq \mathbf{0}, \\ r & \text{if } \mathbf{k} = \mathbf{0}, \end{cases} \tag{3.3}$$

$$\begin{aligned} A^r_M(\mathbf{k}_1, \dots, \mathbf{k}_M) &= A^r_1(\mathbf{k}_M) A^r_{M-1}(\mathbf{k}_1, \dots, \mathbf{k}_{M-1}) \\ &- \sum_{i=1}^{M-1} p^{l_M k_i} A^r_{M-1}(\mathbf{k}_1, \dots, \mathbf{k}_i + \mathbf{k}_M, \dots, \mathbf{k}_{M-1}) \quad (M \geq 2) \end{aligned} \tag{3.4}$$

and

$$B^r_1(\mathbf{k}) = \begin{cases} b(\mathbf{k}) & \text{if } \mathbf{k} \neq \mathbf{0}, \\ r & \text{if } \mathbf{k} = \mathbf{0}, \end{cases} \tag{3.5}$$

$$\begin{aligned} B^r_M(\mathbf{k}_1, \dots, \mathbf{k}_M) &= B^r_1(\mathbf{k}_M) B^r_{M-1}(\mathbf{k}_1, \dots, \mathbf{k}_{M-1}) \\ &- \sum_{i=1}^{M-1} p^{k_M l_i} B^r_{M-1}(\mathbf{k}_1, \dots, \mathbf{k}_i + \mathbf{k}_M, \dots, \mathbf{k}_{M-1}) \quad (M \geq 2), \end{aligned} \tag{3.6}$$

respectively. In the case $M = 0$ we set $A^r_0 = B^r_0 = 1$.

The following lemma easily follows from Lemmas 7 and 15 in Section 4.

Lemma 4. *The elements $A^r_M(\mathbf{k}_1, \dots, \mathbf{k}_M)$ and $B^r_M(\mathbf{k}_1, \dots, \mathbf{k}_M)$ are symmetric in the variables $\mathbf{k}_1, \dots, \mathbf{k}_M$.*

For the study of $\mathcal{IH}(r, s)$ we need the following two lemmas, the proofs of which will be given in Section 5. Let $<>$ denote the projection from

$$U(\mathcal{L})_0 = U(\mathcal{H}) \oplus \bigoplus_{n \geq 1} U(\mathcal{N}_-)_{-2n} U(\mathcal{H}) U(\mathcal{N}_+)_{2n}$$

to $U(\mathcal{H})$. For $M \geq 1$ and $\mathbf{k}_i = (k_i, l_i)$, $\mathbf{m}_i = (m_i, n_i) \in \mathbf{Z}^2$ $(1 \leq i \leq M)$, set

$$F_M(\mathbf{k}_1, \dots, \mathbf{k}_M; \mathbf{m}_1, \dots, \mathbf{m}_M) = \langle e_{\mathbf{k}_1} \cdots e_{\mathbf{k}_M} f_{\mathbf{m}_M} \cdots f_{\mathbf{m}_1} \rangle.$$

For a nonnegative integer r let J_r denote the left ideal of $U(\mathcal{H})$ genereted by the element $h - r$. Since $h - r$ commutes with $U(\mathcal{H})$, J_r is an ideal.

Lemma 5. *For nonnegative integers r and s, the following hold in the quotient algebra $U(\mathcal{H})/J_{r+s}$:*

$$\begin{aligned} &F_M(\mathbf{k}_1, \dots, \mathbf{k}_M; \mathbf{m}_1, \dots, \mathbf{m}_M) \\ &\quad = \sum_{\sigma \in \mathcal{S}_M} \sum_{I \sqcup J = \{1, \dots, M\}} p^{\sum_{\alpha=1}^{a} l_{\sigma(i_\alpha)} m_{i_\alpha} + \sum_{\beta=1}^{b} k_{\sigma(j_\beta)} n_{j_\beta}} \end{aligned}$$

$$\times A^r_a(\mathbf{k}_{\sigma(i_1)} + \mathbf{m}_{i_1}, \ldots, \mathbf{k}_{\sigma(i_a)} + \mathbf{m}_{i_a}) B^s_b(\mathbf{k}_{\sigma(j_1)} + \mathbf{m}_{j_1}, \ldots, \mathbf{k}_{\sigma(j_b)} + \mathbf{m}_{j_b}).$$

Here $I = \{i_1, \ldots, i_a\}$, $i_1 < \cdots < i_a$ *and* $J = \{j_1, \ldots, j_b\}$, $j_1 < \cdots < j_b$ *in the summand.*

Lemma 6. *For* $(V, v) \in \mathcal{IH}(r, s)$,

$$A^r_{r+1}(\mathbf{k}_1, \ldots, \mathbf{k}_{r+1})v = 0 \quad \textit{and} \quad B^s_{s+1}(\mathbf{k}_1, \ldots, \mathbf{k}_{s+1})v = 0$$

for any $\mathbf{k}_i \in \mathbf{Z}^2$ $(\forall i)$.

Admitting the above two lemmas for a while, we can prove the following proposition.

Proposition 2. *An irreducible highest weight* $\mathcal{L}$ *module* V *with highest weight vector* v *is integrable if and only if there exist nonnegative integers* r *and* s *such that* $hv = (r + s)v$ *and*

$$A^r_{r+1}(\mathbf{k}_1, \ldots, \mathbf{k}_{r+1})v = 0 \quad \textit{and} \quad B^s_{s+1}(\mathbf{k}_1, \ldots, \mathbf{k}_{s+1})v = 0$$

for any $\mathbf{k}_i \in \mathbf{Z}^2$ $(\forall i)$.

Proof. If V is integrable, then $(V, v) \in \mathcal{IH}(r, s)$ for some nonnegative integers r and s by Lemma 2. Hence the 'only if' part follows from Lemma 6. Conversely if the conditions in the proposition hold, then

$$A^r_M(\mathbf{k}_1, \ldots, \mathbf{k}_M)v = 0 \quad \text{and} \quad B^s_{M'}(\mathbf{k}_1, \ldots, \mathbf{k}_{M'})v = 0$$

for $M \geq r+1$, $M' \geq s+1$ and $\mathbf{k}_i \in \mathbf{Z}^2$ $(\forall i)$ by the recurrence relations (3.4) and (3.6). Hence, by Lemma 5,

$$F_{r+s+1}(\mathbf{k}_1, \ldots, \mathbf{k}_{r+s+1}; \mathbf{m}_1, \ldots, \mathbf{m}_{r+s+1})v = 0$$

for any $\mathbf{k}_i$, $\mathbf{m}_i \in \mathbf{Z}^2$ $(\forall i)$. Therefore the 'if' part follows from part (2) of Lemma 1. □

For nonnegative integers r and s, let $I^r_{\mathcal{A}}$ and $I^s_{\mathcal{B}}$ be the left ideals of $\mathcal{A}$ and $\mathcal{B}$ generated by the elements $A^r_{r+1}(\mathbf{k}_1, \ldots, \mathbf{k}_{r+1})$ $(\mathbf{k}_i \in \mathbf{Z}^2, \forall i)$ and the elements $B^s_{s+1}(\mathbf{k}_1, \ldots, \mathbf{k}_{s+1})$ $(\mathbf{k}_i \in \mathbf{Z}^2, \forall i)$, respectively. Let further $I^{r,s}$ be the left ideal of $U(\mathcal{H})$ generated by $I^r_{\mathcal{A}}$, $I^s_{\mathcal{B}}$ and J_{r+s}.

Proposition 3. (1) *The left ideals* $I^r_{\mathcal{A}}$, $I^s_{\mathcal{B}}$ *and* $I^{r,s}$ *are ideals of* $\mathcal{A}$, $\mathcal{B}$ *and* $U(\mathcal{H})$, *respectively.*

(2) *Set* $\mathcal{A}^r = \mathcal{A}/I^r_{\mathcal{A}}$, $\mathcal{B}^s = \mathcal{B}/I^s_{\mathcal{B}}$ *and* $U(\mathcal{H})^{r,s} = U(\mathcal{H})/I^{r,s}$. *Then the map* $\mathcal{A}^r \otimes \mathcal{B}^s \to U(\mathcal{H})^{r,s}$ ($\overline{a} \otimes \overline{b} \mapsto \overline{ab}$) *is an isomorphism of* $\mathbf{C}$ *algebras where* $\overline{a}$ *denotes the image of* a *in* $\mathcal{A}^r$ *for* $a \in \mathcal{A}$, *and* $\overline{b}$ *and* $\overline{ab}$ *are defined similarly.*

The proof of this proposition will be given in Section 4.4.

Proposition 4. *Suppose that* $W \in \mathcal{W}$. *Then the irreducible highest weight* $\mathcal{L}$ *module* $V(W)$ *is integrable if and only if there exist nonnegative integers* r *and* s *such that the* $U(\mathcal{H})$ *module structure on* W *induces a* $U(\mathcal{H})^{r,s}$ *module structure on* W.

Proof. Let v be a highest weight vector of $V(W)$. Then, by Proposition 3 (1), $I^{r,s}$ annihilates $W \simeq U(\mathcal{H})v$ if and only if $I^{r,s}v = 0$. Hence we get the claim by Proposition 2. $\square$

By Propositions 1, 3 and 4, the classification of integrable irreducible highest weight $\mathcal{L}$ modules is reduced to that of $\mathcal{H}$ modules in $\mathcal{W}$, the $U(\mathcal{H})$ module structures on which induce $U(\mathcal{H})^{r,s} (\simeq \mathcal{A}^r \otimes \mathcal{B}^s)$ module structures for some nonnegative integers r and s. The answer for the latter problem is given in the following theorem.

For $a \in \mathbf{C}^\times$ let $N^1(a)$ denote the $\mathcal{H}$ module $\mathbf{C}[z^{\pm 1}]$ on which $\mathcal{H}$ acts as

$$a(k,l) = a^l z^k D^l, \quad b(k,l) = 0, \quad h = 1$$

and let $N^2(a)$ signify the $\mathcal{H}$ module $\mathbf{C}[z^{\pm 1}]$ on which $\mathcal{H}$ acts as

$$a(k,l) = 0, \quad b(k,l) = a^l D^l z^{-k}, \quad h = 1.$$

For $a \in \mathbf{C}^\times$ and a partition λ set

$$N^i_\lambda(a) = c_{T_\lambda} N^i(a)^{\otimes r} \quad (i = 1,\ 2)$$

where $r = |\lambda|$. For a nonnegative integer n, partitions $\lambda_1, \ldots, \lambda_n$ and nonzero complex numbers $a_1, \ldots, a_n$ such that $a_i/a_j \notin p^{\mathbf{Z}}$ $(i \neq j)$, set

$$N^i_{\lambda_1,\ldots,\lambda_n}(a_1,\ldots,a_n) = N^i_{\lambda_1}(a_1) \otimes \cdots \otimes N^i_{\lambda_n}(a_n) \quad (i = 1,\ 2)$$

where in the case $n = 0$ the right hand side should be understood as the trivial $\mathcal{H}$ module $\mathbf{C}$.

Theorem 2. (1) *Let* r *and* s *be nonnegative integers. An* $\mathcal{H}$ *module* W *is in* $\mathcal{W}$ *and the* $U(\mathcal{H})$ *module structure on it induces a* $U(\mathcal{H})^{r,s}$ *module structure if and only if it is isomorphic to one of the* $N^1_{\lambda_1,\ldots,\lambda_n}(a_1,\ldots,a_n) \otimes N^2_{\mu_1,\ldots,\mu_m}(b_1,\ldots,b_m)$ *with* $\sum|\lambda_i| = r$ *and* $\sum|\mu_i| = s$.

(2) *The* $\mathcal{H}$ *modules*

$$N^1_{\lambda_1,\dots,\lambda_n}(a_1,\dots,a_n) \otimes N^2_{\mu_1,\dots,\mu_m}(b_1,\dots,b_m) \quad \text{and}$$
$$N^1_{\lambda'_1,\dots,\lambda'_n}(a'_1,\dots,a'_n) \otimes N^2_{\mu'_1,\dots,\mu'_m}(b'_1,\dots,b'_m)$$

are isomorphic if and only if $n' = n$, $m' = m$ *and there exist* $\sigma \in \mathcal{S}_n$ *and* $\tau \in \mathcal{S}_m$ *such that*

$$\lambda'_{\sigma(i)} = \lambda_i, \quad a'_{\sigma(i)}/a_i \in p^{\mathbf{Z}}, \quad \mu'_{\tau(j)} = \mu_j, \quad b'_{\tau(j)}/b_j \in p^{\mathbf{Z}}$$

for $1 \leq i \leq n$ *and* $1 \leq j \leq m$.

The proof of this theorem will be given in Section 4.5.

Now we can give the

Proof of Theorem 1. Set $W = N^1_{\lambda_1,\dots,\lambda_n}(a_1,\dots,a_n) \otimes N^2_{\mu_1,\dots,\mu_m}(b_1,\dots,b_m)$. Then we can and do identify W with U_{n+m} as $\mathcal{H}$ modules and $W \in \mathcal{W}$ by Theorem 2. Let us denote the submodule of U generated by U_{n+m} by U'. By considering the $\mathcal{L}$ module homomorphism $M(W) = U(\mathcal{L}) \otimes_{U(B_+)} W \to U$ $(u \otimes w \mapsto uw)$, we find that U' is isomorphic to a quotient of $M(W)$. Therefore U' has a unique maximal submodule N and $U'/N \simeq V(W)$. So the claim follows from Propositions 1 and 4 and Theorem 2. □

4. Proofs of Proposition 3 and Theorem 2

In this section we shall prove Proposition 3 and Theorem 2.

4.1. The algebra $\mathcal{A}^r$.

Lemma 7. (1) *The* $A^r_M(\mathbf{k}_1,\dots,\mathbf{k}_M)$ *are symmetric in the variables* $\mathbf{k}_1,\dots \mathbf{k}_M$.
(2) *The* $A^r_M(\mathbf{k}_1,\dots,\mathbf{k}_M)$ *satisfy the following*:

(i)
$$A^r_M(\mathbf{k}_1,\dots,\mathbf{k}_M) = A^r_{M-1}(\mathbf{k}_1,\dots,\mathbf{k}_{M-1})A^r_1(\mathbf{k}_M) - \sum_{i=1}^{M-1} p^{l_i k_M} A^r_{M-1}(\mathbf{k}_1,\dots,\mathbf{k}_i+\mathbf{k}_M,\dots,\mathbf{k}_{M-1}).$$

(ii)
$$A^r_M(\mathbf{k}_1,\dots,\mathbf{k}_{M-1},\mathbf{0}) = (r+1-M)A^r_{M-1}(\mathbf{k}_1,\dots,\mathbf{k}_{M-1}).$$

Proof. (1) We show the claim by induction on M. Suppose that the $A^r_M(\mathbf{k}_1,\dots,\mathbf{k}_M)$ are symmetric for $M < n$. Then by the definition $A^r_n(\mathbf{k}_1,\dots,\mathbf{k}_n)$ is symmetric in the variables $\mathbf{k}_1,\dots,\mathbf{k}_{n-1}$. So it is sufficient to show that it is also symmetic in the variables $\mathbf{k}_{n-1}$ and $\mathbf{k}_n$. Substituting (3.4) for $M = n-1$ into the right hand side of (3.4) for $M = n$, we get

$$\begin{aligned}
&A^r_n(\mathbf{k}_1, \dots, \mathbf{k}_{n-2}, \mathbf{k}_{n-1}, \mathbf{k}_n) - A^r_n(\mathbf{k}_1, \dots, \mathbf{k}_{n-2}, \mathbf{k}_n, \mathbf{k}_{n-1})\\
&\quad = [A^r_1(\mathbf{k}_n), A^r_1(\mathbf{k}_{n-1})]A^r_{n-2}(\mathbf{k}_1, \dots, \mathbf{k}_{n-2})\\
&\qquad - \left(p^{l_n k_{n-1}} - p^{l_{n-1} k_n}\right)\left(\sum_{i=1}^{n-2} p^{(l_{n-1}+l_n)k_i} A^r_{n-2}(\mathbf{k}_1, \dots, \mathbf{k}_i + \mathbf{k}_{n-1} + \mathbf{k}_n, \dots, \mathbf{k}_{n-2})\right.\\
&\qquad \left. + A^r_{n-1}(\mathbf{k}_1, \dots, \mathbf{k}_{n-2}, \mathbf{k}_{n-1} + \mathbf{k}_n)\right)\\
&\quad = 0.
\end{aligned}$$

This completes the proof.
(2) Part (ii) is immediate from (3.4) We shall show (i). Let us denote $A^r_M(\mathbf{k}_1, \dots, \mathbf{k}_M)$ defined by (3.3) and the recurrence relation in part (i) (instead of (3.4)) by $A'^r_M(\mathbf{k}_1, \dots, \mathbf{k}_M)$. Suppose that $A'^r_M(\mathbf{k}_1, \dots, \mathbf{k}_M) = A^r_M(\mathbf{k}_1, \dots, \mathbf{k}_M)$ for $M < n$. Since $A^r_n(\mathbf{k}_1, \dots, \mathbf{k}_n)$ is symmetric in the variables $\mathbf{k}_1, \dots, \mathbf{k}_n$ by part (1), the following holds:

$$\begin{aligned}
A^r_n(\mathbf{k}_1, \dots, \mathbf{k}_n) &= A^r_1(\mathbf{k}_1)A^r_{n-1}(\mathbf{k}_2, \dots, \mathbf{k}_n) - p^{l_1 k_n} A^r_{n-1}(\mathbf{k}_2, \dots, \mathbf{k}_{n-1}, \mathbf{k}_1 + \mathbf{k}_n)\\
&\quad - \sum_{i=2}^{n-1} p^{l_1 k_i} A^r_{n-1}(\mathbf{k}_2, \dots, \mathbf{k}_i + \mathbf{k}_1, \dots, \mathbf{k}_n).
\end{aligned}$$

We rewrite the right hand side of the above equality by substituting the relation in (i) for $M = n - 1$ into the first term and the summand of the last sum. By rewriting the right hand side of (i) for $M = n$ by making use of (3.4) similarly, we find that $A'^r_n(\mathbf{k}_1, \dots, \mathbf{k}_n) = A^r_n(\mathbf{k}_1, \dots, \mathbf{k}_n)$. □

Recall that for a nonnegative integer r $I^r_{\mathcal{A}}$ is the left ideal of $\mathcal{A}$ generated by the elements $A^r_{r+1}(\mathbf{k}_1, \dots, \mathbf{k}_{r+1})$ ($\mathbf{k}_i \in \mathbf{Z}^2$, $\forall i$). Fix a total order $\preceq$ on $\mathbf{Z}^2$.

Proposition 5. (1) *The left ideal $I^r_{\mathcal{A}}$ is an ideal of $\mathcal{A}$.*
(2) *Set $\mathcal{A}^r = \mathcal{A}/I^r_{\mathcal{A}}$ and let $^-: \mathcal{A} \to \mathcal{A}^r$ be the canonical map. For $r \geq 1$ the* **C** *algebra homomorphism $\phi^r: \mathcal{A}^r \to \mathcal{C}_p^{\otimes r}$ defined by*

$$\phi^r(\bar{a}(k, l)) = \sum_{i=1}^{r} 1^{\otimes i-1} \otimes x^k y^l \otimes 1^{\otimes r-i}$$

is injective.
(3) $\mathcal{A}^0 \simeq \mathbf{C}$ $(\bar{a}(k, l) \leftrightarrow 0)$ *and* $\mathcal{A}^r = \bigoplus_{\mathbf{k}_1 \preceq \cdots \preceq \mathbf{k}_r} \mathbf{C}\bar{A}^r_r(\mathbf{k}_1, \dots, \mathbf{k}_r)$ *for* $r \geq 1$.

Proof. From (3.4) and part (i) of Lemma 7 (2) we get

$$A^r_{M-1}(\mathbf{k}_1, \dots, \mathbf{k}_{M-1})A^r_1(\mathbf{k}_M)$$

$$= A_1^r(\mathbf{k}_M)A_{M-1}^r(\mathbf{k}_1, \dots, \mathbf{k}_{M-1}) + \sum_{i=1}^{M-1}(p^{l_i k_M} - p^{l_M k_i})A_{M-1}^r(\mathbf{k}_1, \dots, \mathbf{k}_i + \mathbf{k}_M, \dots, \mathbf{k}_{M-1}).$$

Since $\mathcal{A}$ is generated by the $A_1^r(\mathbf{k})$, part (1) follows from the above equality for $M = r + 2$.

Next we show the case $r \geq 1$ of parts (2) and (3), the proof of the case $r = 0$ being similar. Let

$$X = \sum_{\mathbf{k}_1 \preceq \cdots \preceq \mathbf{k}_r} \mathbf{C}\overline{A}_r^r(\mathbf{k}_1, \dots, \mathbf{k}_r) \subset \mathcal{A}^r$$

By part (ii) of Lemma 7 (2), $A_r^r(\mathbf{0}, \dots, \mathbf{0}) = r!$. So $\overline{1} \in X$. Hence, to prove that $\mathcal{A}^r = X$, it is sufficient to show that X is preserved by left multiplication by the $\overline{a}(\mathbf{k})$. Since $A_r^r(\mathbf{k}_1, \dots, \mathbf{k}_r)$ is symmetric in the variables $\mathbf{k}_1, \dots, \mathbf{k}_r$ by Lemma 7 (1), this can be checked, using (3.4) and the fact $\overline{A}_{r+1}^r(\mathbf{k}_1, \dots, \mathbf{k}_{r+1}) = 0$.

Let $\phi \colon \mathcal{A} \to \mathcal{C}_p^{\otimes r}$ be the $\mathbf{C}$ algebra homomorphism determined by $\phi(a(k,l)) = \sum_{i=1}^r x_i^k y_i^l$ where $x_i = 1^{\otimes i-1} \otimes x \otimes 1^{\otimes r-i}$ and y_i is defined similarly. Then since

$$\phi(A_M^r(\mathbf{k}_1, \dots, \mathbf{k}_M)) = \begin{cases} \sum_{\substack{1 \leq i_1, \dots, i_M \leq r \\ i_1, \dots, i_M \text{ distinct}}} \prod_{j=1}^M x_{i_j}^{k_j} y_{i_j}^{l_j} & \text{if } M \leq r \\ 0 & \text{if } M > r, \end{cases}$$

this induces a $\mathbf{C}$ algebra homomorphism $\phi^r \colon \mathcal{A}^r \to \mathcal{C}_p^{\otimes r}$. Let $v_{\mathbf{k}} = x^k y^l$ for $\mathbf{k} = (k, l) \in \mathbf{Z}^2$. Then

$$\phi^r\left(\overline{A}_r^r(\mathbf{k}_1, \dots, \mathbf{k}_r)\right) = \sum_{\sigma \in \mathcal{S}_r} v_{\mathbf{k}_{\sigma(1)}} \otimes \cdots \otimes v_{\mathbf{k}_{\sigma(r)}}.$$

Since the vectors $v_{\mathbf{k}}$ form a basis of $\mathcal{C}_p$, we can see that the vectors $\phi^r(\overline{A}_r^r(\mathbf{k}_1, \dots, \mathbf{k}_r))$ $(\mathbf{k}_1 \preceq \cdots \preceq \mathbf{k}_r)$ are linearly independent. This proves that the sum on the right hand side of X is direct and that ϕ^r is injective. □

Corollary 1. *In $\mathcal{A}^r$ the following hold*:

$$\overline{\Lambda}_{\mathcal{A},n} = 0 \quad \textit{for } |n| > r \quad \textit{and} \quad \overline{\Lambda}_{\mathcal{A},r}\overline{\Lambda}_{\mathcal{A},-n} = \overline{\Lambda}_{\mathcal{A},r-n} \quad \textit{for } 0 \leq n \leq r.$$

Proof. The claim follows from the following equality.

$$\phi^r\left(\overline{\Lambda}_{\mathcal{A}}^{\pm}(u)\right) = \prod_{i=1}^r \left(1 - y_i^{\pm 1}u\right). \qquad \square$$

Compare this corollary with Lemma 3.

REMARK 2. An $\mathcal{A}^r$ module is an $\mathcal{A}$ (or $\mathcal{H}_1$) module which the elements $A^r_{r+1}(\mathbf{k}_1, \ldots, \mathbf{k}_r)$ annihilate.

Letting $\mathbf{1} = (0, 1)$, set

$$\Pi_{\pm n} = \begin{cases} (-1)^n \frac{A^r_n(\pm\mathbf{1}, \ldots, \pm\mathbf{1})}{n!} & \text{if } n > 0, \\ 1 & \text{if } n = 0. \end{cases}$$

Note that Π_n is independent of r.

Lemma 8. *In $\mathcal{A}$ the following hold.*

(1) $$(-1)^{n-1}\frac{A^r_n(\pm\mathbf{1}, \ldots, \pm\mathbf{1}, \mathbf{k})}{(n-1)!} = \sum_{i=0}^{n-1} A^r_1(\mathbf{k} \pm i\mathbf{1})\Pi_{\pm(n-1-i)} \quad \text{for } r \geq 0 \text{ and } n \geq 1.$$

(2) $\Pi_n = \Lambda_{\mathcal{A},n}$ *for any* n.

Proof. (1) The assertion can be easily proven by induction on n, using (3.4) and Lemma 7 (1).
(2) Letting $\mathbf{k} = \pm\mathbf{1}$ in (1), we get

$$n\Pi_{\pm n} = -\sum_{i=1}^{n} a(0, \pm i)\Pi_{\pm(n-i)}$$

for any $n \geq 1$. This can be rewritten as

$$\frac{d}{du}\sum_{n\geq 0}\Pi_{\pm n}u^n = -\frac{d}{du}\left(\sum_{i\geq 1}\frac{a(0, \pm i)}{i}u^i\right)\left(\sum_{n\geq 0}\Pi_{\pm n}u^n\right).$$

Noting that the Π_n and the $a(0, i)$ are elements of the commutative algebra $U(\mathcal{H}')$, we find that

$$\sum_{n\geq 0}\Pi_{\pm n}u^n = \exp\left(-\sum_{i\geq 1}\frac{a(0, \pm i)}{i}u^i\right). \qquad \square$$

REMARK 3. Noting Lemma 5, we can see that Lemma 8 was essentially proved for one variable loop algebras in [9] and [18].

Lemma 9. *In $\mathcal{A}^r$ the following hold.*

(1) $$\bar{A}^r_M(\mathbf{k}_1, \ldots, \mathbf{k}_M) = 0 \quad \text{for } M \geq r + 1.$$

(2) $$\sum_{i=0}^{r} \bar{A}^r_M(\mathbf{k}_1, \ldots, \mathbf{k}_{M-1}, \mathbf{k}_M \pm i\mathbf{1})\bar{\Pi}_{\pm(r-i)} = 0 \quad \text{for } M \geq 1.$$

Proof. (1) Follows from the fact $\overline{A}^r_{r+1}(\mathbf{k}_1, \ldots, \mathbf{k}_{r+1}) = 0$ and (3.4).
(2) From Lemma 8 (1) for $n = r + 1$ and part (1), we get

$$\sum_{i=0}^{r} \overline{A}^r_1(\mathbf{k} \pm i\mathbf{1})\overline{\Pi}_{\pm(r-i)} = 0.$$

Hence the claim is proved by induction on M, using part (i) of Lemma 7 (2). □

Lemma 10. *Let $\Delta\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ be the* $\mathbf{C}$ *algebra homomorphism defined by $a(\mathbf{k}) \mapsto a(\mathbf{k}) \otimes 1 + 1 \otimes a(\mathbf{k})$. Then with the notation of* Lemma 5

$$\Delta\left(A^{r+s}_M(\mathbf{k}_1, \ldots, \mathbf{k}_M)\right) = \sum_{I \sqcup J = \{1,\ldots,M\}} A^r_a(\mathbf{k}_{i_1}, \ldots, \mathbf{k}_{i_a}) \otimes A^s_b(\mathbf{k}_{j_1}, \ldots, \mathbf{k}_{j_b})$$

for any nonnegative integers r and s.

Proof. The claim is proved by induction on M. □

Corollary 2. *Let V and W be an $\mathcal{A}^r$ module and an $\mathcal{A}^s$ module, respectively. Then $V \otimes W$ is endowed with a $\mathcal{A}^{r+s}$ module structure via Δ.*

Proof. By Remark 2 it is sufficient to show that the elements $\Delta(A^{r+s}_{r+s+1}(\mathbf{k}_1, \ldots, \mathbf{k}_{r+s+1}))$ annihilate $V \otimes W$. This can be easily checked, using the lemma and part (1) of Lemma 9. □

REMARK 4. $V \otimes W$ in Corollary 2 is nothing but the tensor product module if regarded as an $\mathcal{H}_1$ module.

4.2. $\mathcal{A}^r$ modules. In this subsection, we study $\mathcal{A}$ modules and $\mathcal{A}^r$ modules. We shall endow the tensor product of $\mathcal{A}$ modules with an $\mathcal{A}$ module structure via Δ and consider the tensor product of $\mathcal{A}^r$ modules ($r \geq 0$) as in Corollary 2. By Proposition 5 (3), $\mathcal{A}^0 \simeq \mathbf{C}$. So an $\mathcal{A}^0$ module is nothing but a $\mathbf{C}$ vector space. Therefore we assume that $r \geq 1$ until just before Theorem 3. In the following, for $x \in \mathcal{A}$ we shall denote the image of x in $\mathcal{A}^r$ simply by x.

Set $\mathcal{H}'_1 = \bigoplus_{l \neq 0} \mathbf{C}a(0, l)$. We call an $\mathcal{A}$ (or $\mathcal{A}^r$) module V a weight module if the following holds:

$$V = \bigoplus_{f \in (\mathcal{H}'_1)^*} V_f, \quad V_f = \{v \in V \mid uv = f(u)v \quad \text{for any } u \in \mathcal{H}'_1\}.$$

We shall call the above decomposition a weight space decomposition and a nonzero vector in each weight space a weight vector.

For a finite sequence $X = (a_1, \ldots, a_n)$ of nonzero complex numbers, define $f_X \in$

$(\mathcal{H}_1')^*$ by

$$f_X(a(0,l)) = \sum_{i=1}^{n} a_i^l \quad (l \neq 0).$$

In the following we need the following simple fact.

Lemma 11. *$f_X = f_{X'}$ if and only if X coincides with X' as sets with repetitions allowed.*

For $\mathbf{a} = (a_1, \ldots, a_r) \in (\mathbf{C}^\times)^r$ let $J^r(\mathbf{a})$ be the left ideal of $\mathcal{A}^r$ generated by the elements $a(0,l) - P_l(\mathbf{a})$ $(l \neq 0)$ where $P_l(\mathbf{a}) = \sum_{i=1}^r a_i^l$. Set $M^r(\mathbf{a}) = \mathcal{A}^r / J^r(\mathbf{a})$.

Proposition 6. *Suppose that W is an $\mathcal{A}^r$ module generated by a weight vector v. Then*
(1) *W is isomorphic to a quotient of $M^r(\mathbf{a})$ for some $\mathbf{a} \in (\mathbf{C}^\times)^r$.*
(2) *If W is further irreducible, then W is isomorphic to a quotient of $M^r(\mathbf{a})$ for some $\mathbf{a} \in (\mathbf{C}^\times)^r$ such that the a_i are distinct.*

To prove this proposition, we need the following lemma.

Lemma 12. *Let $\mathbf{a} = (a_1, \ldots, a_r) \in (\mathbf{C}^\times)^r$ and set*

$$T(\mathbf{a}) = (T(\mathbf{a})_{ij})_{1 \le i,j \le r} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & (-1)^{r-1}E_r(\mathbf{a}) \\ 1 & 0 & \cdots & 0 & 0 & (-1)^{r-2}E_{r-1}(\mathbf{a}) \\ 0 & 1 & \cdots & 0 & 0 & (-1)^{r-3}E_{r-2}(\mathbf{a}) \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\ 0 & 0 & \cdots & 1 & 0 & -E_2(\mathbf{a}) \\ 0 & 0 & \cdots & 0 & 1 & E_1(\mathbf{a}) \end{pmatrix}$$

where $E_i(\mathbf{a})$ is the i-th elementary symmetric polynomial in the variables $a_1, \ldots, a_r$. Suppose that v is a vector in an $\mathcal{A}^r$ module satisfying $a(0,l)v = P_l(\mathbf{a})v$ for any $l \neq 0$. Then the following hold for $1 \le l \le r$, k, $n \in \mathbf{Z}$ and $M \ge 1$:

$$A_M^r(\mathbf{k}_1, \ldots, \mathbf{k}_{M-1}, (k, n+l))v = \sum_{i=1}^r A_M^r(\mathbf{k}_1, \ldots, \mathbf{k}_{M-1}, (k,i))v\,(T(\mathbf{a})^n)_{il}.$$

Proof. Fixing M, $\mathbf{k}_j$ $(1 \le j \le M-1)$ and k, set

$$u_l = A_M^r(\mathbf{k}_1, \ldots, \mathbf{k}_{M-1}, (k,l))v.$$

By Lemma 8 (2)

$$\Pi_n v = (-1)^n E_n(\mathbf{a}) v$$

for $n \geq 0$. Hence Lemma 9 (2) gives

$$u_{l+r} = \sum_{i=1}^{r} u_{l+i-1} T(\mathbf{a})_{ir}$$

for any integer l. From this we obtain

$$(u_{n+1}, \ldots, u_{n+r}) = (u_1, \ldots, u_r) T(\mathbf{a})^n$$

for any intger n. The claim follows from this. □

Proof of Proposition 6. (1) By Corollary 1

$$\sum_{n \geq 0} \Lambda_{\mathcal{A}, \pm n} u^n v = \prod_{i=1}^{r} (1 - a_i^{\pm 1} u) v$$

for some nonzero complex numbers $a_1, \ldots, a_r$. This implies $a(0, l)v = P_l(\mathbf{a})v$ for $l \neq 0$ with $\mathbf{a} = (a_1, \ldots, a_r)$. This proves the claim.

(2) For $\mathbf{a} = (a_1, \ldots, a_r)$ let $n(\mathbf{a})$ denote the number of pairs (i, j) such that $i < j$ and $a_i = a_j$. We shall prove the claim by showing that if W is isomorphic to a quotient of $M^r(\mathbf{a})$ with $n(\mathbf{a}) > 0$, then W is isomorphic to a quotient of $M^r(\mathbf{b})$ with $n(\mathbf{b}) < n(\mathbf{a})$ ($\sharp$).

Noting $M^r(a_{\sigma(1)}, \ldots, a_{\sigma(r)}) = M^r(\mathbf{a})$ for any $\sigma \in \mathcal{S}_r$, assume that

$$\mathbf{a} = (\overbrace{\alpha_1, \ldots, \alpha_1}^{n_1}, \ldots, \overbrace{\alpha_i, \ldots, \alpha_i}^{n_i}, \ldots, \overbrace{\alpha_t, \ldots, \alpha_t}^{n_t})$$

where the α_i are distinct and $n_{i_0} > 1$ for some i_0. Noting that the characteristic polynomial of the matrix $T(\mathbf{a})$ is $\prod_{1 \leq i \leq t} (x - \alpha_i)^{n_i}$, let K denote the generalized eigenspace of $T(\mathbf{a})$ corresponding to the eigenvalue α_{i_0}, i.e.,

$$K = \{ f \in \mathbf{C}^r \mid (T(\mathbf{a}) - \alpha_{i_0})^{n_{i_0}} f = 0 \}. \tag{4.1}$$

Let $v \in W$ be the image of $1 + J^r(\mathbf{a})$ under the canonical homomorphism $M^r(\mathbf{a}) \to W$. Then $v \neq 0$. Setting $u(k, l) = a(k, l)v$, let

$$W(k) = \operatorname{span}_{\mathbf{C}} \left\{ \sum_{l=1}^{r} u(k, l) f_l \;\middle|\; f = {}^t(f_1, \ldots, f_r) \in K \right\}$$

for $k \in \mathbf{Z}$. For a while admitting that there exists a nonzero integer k such that $W(k) \neq 0$ and $p^k \neq \alpha_i / \alpha_j$ for $1 \leq i, j \leq t$, we shall prove ($\sharp$). By the relations among the $a(\mathbf{k})$ and Lemma 12 we get

$$(a(0, n) - P_n(\mathbf{a})) u(k, l) = (p^{kn} - 1) u(k, n + l) = (p^{kn} - 1) \sum_{i=1}^{r} u(k, i) (T(\mathbf{a})^n)_{il}$$

for $1 \le l \le r$ and $n \neq 0$. Hence $a(0,n) - P_n(\mathbf{a})$ preserves $W(k)$. Set $d_n = (a(0,n) - P_n(\mathbf{a}))\,|_{W(k)}\,/(p^{kn}-1)$ for $n \neq 0$. Then $(d_n - \alpha_{i_0}^n)^{n_{i_0}} = 0$ by (4.1). Noting this and the fact that the d_n commute with each other, we can see that there exists $0 \neq u \in W(k)$ such that $d_n u = \alpha_{i_0}^n u$ for any $n \neq 0$. This implies that $a(0,n)u = P_n(\mathbf{b})u$ for $n \neq 0$ where $\mathbf{b}$ is the sequence obtained from $\mathbf{a}$ by replacing one α_{i_0} by $p^k\alpha_{i_0}$. Since W is irreducible, this proves ($\sharp$).

Now we shall show the fact assumed above. Let $\mathbf{a}'$ denote the sequence obtained from $\mathbf{a}$ by deleting all the α_{i_0}'s and set $m = r - n_{i_0}$. Letting

$$g = {}^t(g_l)_{1\le l\le r} = {}^t(\overbrace{0,\ldots,0}^{n_{i_0}-1}, (-1)^m E_m(\mathbf{a}'), \ldots, E_2(\mathbf{a}'), -E_1(\mathbf{a}'), 1) \in \mathbf{C}^r,$$

set

$$\gamma(k) = \sum_{l=1}^{r} g_l a(k,l).$$

Then after a little calculation we find that $g \in K$ and that

$$[\gamma(k), \gamma(-k)]v = \phi(p^k)v$$

where $\phi(x)$ is a nonzero Laurent polynomial in x depending on $\mathbf{a}$. (Explicitly $\phi(x) = n_{i_0}\alpha_{i_0}^{2n_{i_0}} h(1)(h(1/x) - h(x))$ with $h(x) = x^{n_{i_0}} \prod_{i\neq i_0}(x\alpha_{i_0} - \alpha_i)^{n_i}$.) Noting $\gamma(\pm k)v \in W(\pm k)$, we can see from this that $\phi(p^k) = 0$ if $W(\pm k) = 0$. Therefore $W(k) \neq 0$ for an infinite number of k. This completes the proof. $\square$

For $a \in \mathbf{C}^\times$ let $N(a) = \mathbf{C}[z^{\pm 1}]$ be the $\mathcal{A}$ module on which $a(\mathbf{k})$ acts as

$$a(k,l) = a^l z^k D^l.$$

As is easily shown, the $\mathcal{A}$ module structure on $N(a)$ induces an $\mathcal{A}^1$ module structure. For $\mathbf{a} = (a_1, \ldots, a_r) \in (\mathbf{C}^\times)^r$ set $N^r(\mathbf{a}) = N(a_1) \otimes \cdots \otimes N(a_r)$. By Corollary 2 we can regard $N^r(\mathbf{a})$ as an $\mathcal{A}^r$ module. We shall identify $N^r(\mathbf{a})$ with $\mathbf{C}[z_1^{\pm 1}, \ldots, z_r^{\pm 1}]$ via the correspondence $1^{\otimes i-1} \otimes z \otimes 1^{\otimes r-i} \leftrightarrow z_i$.

Proposition 7. *If the nonzero complex numbers a_i are distinct, then $M^r(\mathbf{a}) \simeq N^r(\mathbf{a})$.*

Proof. Let $v = 1 + J^r(\mathbf{a}) \in M^r(\mathbf{a})$. Clearly there exists a homomorphism $\varphi\colon M^r(\mathbf{a}) \to N^r(\mathbf{a})$ determined by $\varphi(v) = 1$. We shall show that this map is an isomorphism.

Set

$$u(\mathbf{k}_1, \ldots, \mathbf{k}_r) = A_r^r(\mathbf{k}_1, \ldots, \mathbf{k}_r)v.$$

Then by Proposition 5 (3) and Lemma 12

$$M^r(\mathbf{a}) = \sum_{\substack{k_1,\ldots,k_r\in\mathbf{Z}\\ 1\le l_1,\ldots,l_r\le r}} \mathbf{C}u(\mathbf{k}_1,\ldots,\mathbf{k}_r) \tag{4.2}$$

where $\mathbf{k}_i = (k_i, l_i)$. Rewriting $A_r^r(\mathbf{k}_1,\ldots,\mathbf{k}_r)a(0,n)v$ by making use of part (i) of Lemma 7 (2), Lemma 9 (1) and Lemma 12, we obtain

$$\begin{aligned} P_n(\mathbf{a})u\,(\mathbf{k}_1,\ldots,\mathbf{k}_r) &= \sum_{i=1}^r u(\mathbf{k}_1,\ldots,\mathbf{k}_i + n\mathbf{1},\ldots,\mathbf{k}_r) \\ &= \sum_{i=1}^r \sum_{l=1}^r u(\mathbf{k}_1,\ldots,(k_i,l),\ldots,\mathbf{k}_r)(T(\mathbf{a})^n)_{ll_i} \end{aligned} \tag{4.3}$$

for $n \neq 0$ and $1 \le l_1,\ldots,l_r \le r$. Let S be the inverse of the matrix $(a_i^j)_{1\le i,j\le r}$ so that

$$S^{-1}T(\mathbf{a})S = \mathrm{diag}(a_1,\ldots,a_r).$$

For $1 \le l_1,\ldots,l_r \le r$ set

$$w(\mathbf{k}_1,\ldots,\mathbf{k}_r) = \sum_{1\le l_1',\ldots,l_r'\le r} u(\mathbf{k}_1',\ldots,\mathbf{k}_r')S_{l_1',l_1}\cdots S_{l_r',l_r}$$

where $\mathbf{k}_i' = (k_i, l_i')$. Then from (4.3) we get

$$\left(P_n(\mathbf{a}) - \sum_{i=1}^r a_{l_i}^n\right) w(\mathbf{k}_1,\ldots,\mathbf{k}_r) = 0 \quad \text{for any } n \neq 0$$

for $1 \le l_1,\ldots,l_r \le r$. Since the a_i are distinct, this implies

$$w(\mathbf{k}_1,\ldots,\mathbf{k}_r) = 0 \quad \text{unless } \{l_1,\ldots,l_r\} = \{1,2,\ldots,r\}.$$

So from (4.2) we obtain

$$M^r(\mathbf{a}) = \sum_{k_1,\ldots,k_r\in\mathbf{Z}} \mathbf{C}w\big((k_1,1),(k_2,2),\ldots,(k_r,r)\big).$$

Now it is sufficient to show that the vectors $\varphi(w((k_1,1),(k_2,2),\ldots,(k_r,r)))$ ($k_i \in \mathbf{Z}$, $\forall i$) form a basis of $N^r(\mathbf{a})$. Using Lemma 9 (1) and Lemma 10, we find that

$$\begin{aligned} \varphi(u(\mathbf{k}_1,\ldots,\mathbf{k}_r)) &= \sum_{\sigma\in\mathcal{S}_r} A_1^1(\mathbf{k}_{\sigma(1)})1 \otimes \cdots \otimes A_1^1(\mathbf{k}_{\sigma(r)})1 \\ &= \sum_{\sigma\in\mathcal{S}_r} \prod_{i=1}^r z_{\sigma(i)}^{k_i} a_{\sigma(i)}^{l_i}. \end{aligned}$$

Hence we get $\varphi(w((k_1,1),(k_2,2),\dots,(k_r,r))) = z_1^{k_1}\cdots z_r^{k_r}$. This completes the proof. $\square$

Now we summarize the well known results on the representations of the general linear group. See, for example, [19].

Lemma 13. *Let $V = \mathbf{C}^n$ for $n \geq 1$ and set $W_T = c_T V^{\otimes |T|}$ for a Young tableau T.*
(1) *If the depth of $T \leq n$, then W_T is an irreducible $GL(V)$ (and $gl(V)$) module and if the depth of $T > n$, then $W_T = 0$.*
(2) *$W_T \simeq W_{T'}$ if and only if T and T' are of the same shape.*
(3) *Let $f = \mathrm{diag}(x_1,\dots,x_n) \in GL(V)$ and T of shape λ and depth $\leq n$. Then*

$$\mathrm{tr}_{W_T} f = S_\lambda(x_1,\dots,x_n)$$

where $S_\lambda(x_1,\dots,x_n)$ is the Schur symmetric polynomial corresponding to the partition λ.
(4) *The $GL(V)$ module $V^{\otimes r}$ admits an irreducible decomposition*

$$V^{\otimes r} = \bigoplus_T W_T$$

where the sum is taken over standard tableaux T with r boxes and depth $\leq n$.

For a positive integer r and a subset X of $\mathbf{Z}$ let $I_r(X)$ be the set of nonincreasing sequences of r elements of X. For

$$\mathbf{m} = (\overbrace{m_1,\dots,m_1}^{r_1},\dots,\overbrace{m_s,\dots,m_s}^{r_s}) \in I_r(X) \quad (m_1 > \cdots > m_s)$$

we let $p(\mathbf{m})$ signify a partition of r obtained by reordering $(r_1,\dots,r_s)$. For a Young tableau T we denote the shape of T by $sh(T)$. Let the $K_{\lambda\mu}$ be the Kostka numbers and let $M_\lambda(x_1,\dots,x_n)$ denote the monomial symmetric polynomial corresponding to the partition λ. Part (3) of Lemma 13 implies the following corollary.

Corollary 3. *Let $v_1,\dots,v_n$ be the canonical basis of $V = \mathbf{C}^n$. For a Young tableau T with r boxes and depth $\leq n$ and $\mathbf{m} \in I_r(\{1,2,\dots,n\})$ set*

$$W_{T,\mathbf{m}} = W_T \cap \left(\sum_{(k_1,\dots,k_r)\in \mathcal{S}_r \mathbf{m}} \mathbf{C} v_{k_1} \otimes \cdots \otimes v_{k_r} \right).$$

Then

$$W_T = \bigoplus_{\mathbf{m}\in I_r(\{1,2,\dots,n\})} W_{T,\mathbf{m}} \quad \text{and} \quad \dim W_{T,\mathbf{m}} = K_{sh(T)\, p(\mathbf{m})}.$$

Proof. The first claim is immediate. For a partition λ such that $l(\lambda) \leq n$

$$S_\lambda(x_1, \ldots, x_n) = \sum_\mu K_{\lambda\mu} M_\mu(x_1, \ldots, x_n)$$

and for a partition μ of r such that $l(\mu) \leq n$

$$M_\mu(x_1, \ldots, x_n) = \sum_{\substack{\mathbf{m}=(m_1,\ldots,m_r)\in I_r(\{1,2,\ldots,n\}) \\ p(\mathbf{m})=\mu}} x_{m_1} \cdots x_{m_r}.$$

Therefore we find by the first claim and part (3) of Lemma 13 that

$$\sum \dim W_{T,\mathbf{m}} x_{m_1} \cdots x_{m_r} = \sum K_{sh(T)\, p(\mathbf{m})} x_{m_1} \cdots x_{m_r}$$

where the sums are taken over $\mathbf{m} = (m_1, \ldots, m_r) \in I_r(\{1, 2, \ldots, n\})$. The second claim follows from this. $\square$

For a Young tableau T and a nonzero complex number a, set $N_T(a) = c_T N^r(a, \ldots, a)$ where $r = |T|$. For a positive integer r, let $\mathcal{ST}_r$ denote the set of standard tableaux with r boxes.

Proposition 8. (1) *$N_T(a)$ is an irreducible $\mathcal{A}$ module.*
(2) *For a tableau T with r boxes and $\mathbf{m} \in I_r(\mathbf{Z})$ set*

$$N_{T,\mathbf{m}}(a) = N_T(a) \cap \left(\sum_{\mathbf{k}\in\mathcal{S}_r\mathbf{m}} \mathbf{C} z^{\mathbf{k}} \right)$$

where $z^{\mathbf{k}} = z_1^{k_1} \cdots z_r^{k_r}$ for $\mathbf{k} = (k_1, \ldots, k_r)$. Then

$$N_T(a) = \bigoplus_{\mathbf{m}\in I_r(\mathbf{Z})} N_{T,\mathbf{m}}(a)$$

is a weight space decomposition and the dimension of each weight space is given by

$$\dim N_{T,\mathbf{m}}(a) = K_{sh(T)\, p(\mathbf{m})}.$$

(3) *$N_T(a) \simeq N_{T'}(b)$ if and only if $sh(T) = sh(T')$ and $b/a \in p^{\mathbf{Z}}$.*
(4) *The $\mathcal{A}$ module $N^r(a, \ldots, a)$ has an irreducible decomposition*

$$N^r(a, \ldots, a) = \bigoplus_{T\in\mathcal{ST}_r} N_T(a).$$

Proof. (1) Set $r = |T|$. Since the action of $\mathcal{A}$ on $N^r(a, \ldots, a)$ commutes with that of $\mathcal{S}_r$, $N_T(a)$ is an $\mathcal{A}$ submodule. Regard the $\mathcal{A}$ module $N(a)$ as an $\mathcal{H}_1$ module naturally and let $\rho\colon \mathcal{H}_1 \to \mathrm{End}(N(a))$ denotes the action of $\mathcal{H}_1$. Set $V_n = \bigoplus_{i=-n}^{n} \mathbf{C}z^i \subset N(a)$ for $n \geq 0$. Then for each n there exists a subspace K_n of $\mathcal{H}_1$ such that $\rho(K_n)V_n \subset V_n$ and $\rho(K_n)|_{V_n} = \mathrm{End}\, V_n$ (*). Let $u(\neq 0)$, $v \in N_T(a)$. Then u, $v \in V_n^{\otimes r}$ for some n. Set $W_{n,T} = c_T V_n^{\otimes r}$. Since c_T^2 is a nonzero scalar multiple of c_T (See, for example, [19]), we find that $u, v \in W_{n,T}$. This implies, by Lemma 13 (1), that $2n+1$ is greater than the depth of T and that $W_{n,T}$ is an irreducible $gl(V_n)$ module. Therefore, noting that $N^r(a,\ldots,a)$ is the tensor product module $N(a)^{\otimes r}$ as an $\mathcal{H}_1$ module, we find by (*) that there exists $x \in \mathcal{A}$ such that $xu = v$. This proves the irreducibility of $N_T(a)$.

(2) By Lemma 11 $N^r(a,\ldots,a) = \bigoplus_{\mathbf{m}\in I_r(\mathbf{Z})}(\sum_{\mathbf{k}\in\mathcal{S}_r\mathbf{m}} \mathbf{C}z^{\mathbf{k}})$ is a weight space decomposition. The first claim follows from this. We shall show the second claim. Suppose that $\mathbf{m} = (m_1,\ldots,m_r)$ and $n \geq m_1 \geq \cdots \geq m_r \geq -n$ for some nonnegative integer n. Then with the notation of part (1) $N_{T,\mathbf{m}}(a) = W_{n,T} \cap \left(\sum_{\mathbf{k}\in\mathcal{S}_r\mathbf{m}} \mathbf{C}z^{\mathbf{k}}\right)$. So we are done by Corollaryd 3.

(3) Let r and s denote the numbers of boxes in T and T', respectively. Suppose that $N_T(a) = \bigoplus N_{T,\mathbf{m}}(a)$ and $N_{T'}(b) = \bigoplus N_{T',\mathbf{n}}(b)$ are isomorphic. Then by considering the action of $a(0,l)$ we find that

$$a^l \sum_{i=1}^{r} p^{lm_i} = b^l \sum_{i=1}^{s} p^{ln_i} \quad \text{for any } l \neq 0$$

for some $\mathbf{m} = (m_1,\ldots,m_r)$ and $\mathbf{n} = (n_1,\ldots,n_s)$. From this we get $r = s$ and $b = ap^k$ for some integer k by Lemma 11. Moreover we can see that any isomorphism $N_T(a) \to N_{T'}(b)$ maps $N_{T,\mathbf{m}}(a)$ onto $N_{T',\mathbf{m}'}(b)$ with $\mathbf{m}' = (m_1-k,\ldots,m_r-k)$. By comparing the dimensions of these subspaces, we find by part (2) that

$$K_{sh(T)p(\mathbf{m})} = K_{sh(T')p(\mathbf{m})}$$

for any $\mathbf{m} \in I_r(\mathbf{Z})$. This implies $S_{sh(T)}(x_1,\ldots,x_n) = S_{sh(T')}(x_1,\ldots,x_n)$ for any $n \geq \max\{\text{depth of } T, \text{depth of } T'\}$ and, hence, $sh(T) = sh(T')$.

Conversely suppose that $sh(T) = sh(T')$ and $b = ap^k$ for some integer k. Then there exists $\sigma \in \mathcal{S}_r$ such that $\sigma(T) = T'$ and the map $u \mapsto \sigma(u)(z_1\cdots z_r)^{-k}$ is an isomorphism $N_T(a) \to N_{T'}(b)$.

(4) Since $N_T(a) \cap V_n^{\otimes r} = c_T V_n^{\otimes r}$ with the notation of part (1), we get by parts (1) and (4) of Lemma 13 for $V = V_n$ that

$$V_n^{\otimes r} = \bigoplus_{T\in\mathcal{ST}_r} N_T(a) \cap V_n^{\otimes r}$$

for any $n \geq 0$. From this, we obtain the claim. □

Proposition 9. *Let $a_1,\ldots,a_n$ be nonzero complex numbers such that $a_i/a_j \notin p^{\mathbf{Z}}$.*

(1) *For tableaux* $T_1, \ldots, T_n$, $N_{T_1}(a_1) \otimes \cdots \otimes N_{T_n}(a_n)$ *is an irreducible* $\mathcal{A}$ *module.*
(2) *Let* $N_{T_1}(a_1) \otimes \cdots \otimes N_{T_n}(a_n)$ *and* $N_{T_1'}(b_1) \otimes \cdots \otimes N_{T_m'}(b_m)$ *be* $\mathcal{A}$ *modules as in* (1). *Then they are isomorphic if and only if* $m = n$ *and there exists* $\sigma \in \mathcal{S}_r$ *such that*

$$sh(T'_{\sigma(i)}) = sh(T_i) \quad \textit{and} \quad b_{\sigma(i)}/a_i \in p^{\mathbf{Z}} \quad \textit{for } 1 \leq i \leq n.$$

(3) *Letting* $r_1, \ldots, r_n$ *be positive integers, set*

$$N = N^{r_1 + \cdots + r_n}(\overbrace{a_1, \ldots, a_1}^{r_1}, \overbrace{a_2, \ldots, a_2}^{r_2}, \ldots, \overbrace{a_n, \ldots, a_n}^{r_n}).$$

The $\mathcal{A}$ *module* N *admits an irreducible decomposition*

$$N = \bigoplus_{(T_1, \ldots, T_n) \in \mathcal{ST}_{r_1} \times \cdots \times \mathcal{ST}_{r_n}} N_{T_1}(a_1) \otimes \cdots \otimes N_{T_n}(a_n).$$

To prove this proposition, we need the following lemma, which is a special case of [20, Chap. 8, Sect. 7, Theorem 2].

Lemma 14. *Let* A *and* B *be* $\mathbf{C}$ *algebras. Suppose that* M *and* N *are an irreducible* A *module and an irreducible* B *module such that* $\operatorname{End}_A M \simeq \mathbf{C}$ *and* $\operatorname{End}_B N \simeq \mathbf{C}$, *respectively. Then* $M \otimes N$ *is an irreducible* $A \otimes B$ *module.*

Proof of Proposition 9. (1) For $1 \leq i \leq n$, letting $N_j = N_{T_j}(a_j)$ and $N_{j,\mathbf{m}} = N_{T_j,\mathbf{m}}(a_j)$, set $L_i = N_1 \otimes \cdots \otimes N_i$ and $L_{i,\mathbf{m}_1,\ldots,\mathbf{m}_i} = N_{1,\mathbf{m}_1} \otimes \cdots \otimes N_{i,\mathbf{m}_i}$. Then by Lemma 11

$$L_i = \bigoplus_{(\mathbf{m}_1, \ldots, \mathbf{m}_i) \in I_{r_1}(\mathbf{Z}) \times \cdots \times I_{r_i}(\mathbf{Z})} L_{i,\mathbf{m}_1,\ldots,\mathbf{m}_i}$$

is a weight space decomposition since $a_k/a_l \notin p^{\mathbf{Z}}$ $(k \neq l)$. Any element of $\operatorname{End}_{\mathcal{A}} L_i$ preserves each weight space $L_{i,\mathbf{m}_1,\ldots,\mathbf{m}_i}$ and one of these subspaces is nonzero and finite dimensional. Therefore if we prove that L_i is irreducible, we get $\operatorname{End}_{\mathcal{A}} L_i \simeq \mathbf{C}$. The same argument shows that $\operatorname{End}_{\mathcal{A}} N_j \simeq \mathbf{C}$ $(1 \leq j \leq n)$.

Now we show the irreducibility of L_i by induction on i. Suppose that the claim is proved up to $i-1$. Then, by the discussion in the previous paragraph and Lemma 14, the $\mathcal{A} \otimes \mathcal{A}$ module $L_{i-1} \otimes N_i$ is irreducible. Therefore, to prove the irreducibility of L_i, we have only to show that any submodule W of the $\mathcal{A}$ module L_i is a submodule even if we regard L_i as an $\mathcal{A} \otimes \mathcal{A}$ module via $L_i = L_{i-1} \otimes N_i$. Since L_i is a weight module, W admits a weight space decomposition

$$W = \bigoplus_{(\mathbf{m}_1, \ldots, \mathbf{m}_i) \in I_{r_1}(\mathbf{Z}) \times \cdots \times I_{r_i}(\mathbf{Z})} W_{\mathbf{m}_1,\ldots,\mathbf{m}_i}$$

where $W_{\mathbf{m}_1,\ldots,\mathbf{m}_i} = W \cap (L_{i-1,\mathbf{m}_1,\ldots,\mathbf{m}_{i-1}} \otimes N_{i,\mathbf{m}_i})$. For a sequence of integers $\mathbf{m}$ let $\overline{\mathbf{m}}$ denote the sequence obtained from $\mathbf{m}$ by ordering the entries in nonincreasing order. Set $\mathbf{m}_i = (m_{i,1}, \ldots, m_{i,r_i})$. Since $(a(k,l) \otimes 1)W_{\mathbf{m}_1,\ldots,\mathbf{m}_i} \subset L_{i-1} \otimes N_{i,\mathbf{m}_i}$ and

$$(1 \otimes a(k,l))W_{\mathbf{m}_1,\ldots,\mathbf{m}_i} \subset \sum_{j=1}^{r_i} L_{i-1} \otimes N_{i,\overline{(m_{i,1},\ldots,m_{i,j}+k,\ldots,m_{i,r_i})}},$$

we can see that both $a(k,l) \otimes 1$ and $1 \otimes a(k,l)$ map $W_{\mathbf{m}_1,\ldots,\mathbf{m}_i}$ to W for $k \neq 0$. This implies $(\mathcal{A} \otimes \mathcal{A})W \subset W$ since $\mathcal{A}$ is generated by the elements $a(k,l)$ $(k \neq 0)$.

(2) The ‘if’ part follows from part (3) of Proposition 8. We shall show the ‘only if’ part. Let r_i and s_i be the numbers of boxes of T_i and T_i', respectively. By considering the action of $a(0,l)$ on both modules, we find as in the proof of Proposition 8 (3) that $m = n$ and there exists $\sigma \in \mathcal{S}_n$ such that

$$s_{\sigma(i)} = r_i \quad \text{and} \quad b_{\sigma(i)}/a_i \in p^{\mathbf{Z}}$$

for $1 \leq i \leq n$. Further we can see that

$$K_{sh(T_1)p(\mathbf{m}_1)} \cdots K_{sh(T_n)p(\mathbf{m}_n)} = K_{sh(T'_{\sigma(1)})p(\mathbf{m}_1)} \cdots K_{sh(T'_{\sigma(n)})p(\mathbf{m}_n)}$$

for any $(\mathbf{m}_1, \ldots, \mathbf{m}_n) \in I_{r_1}(\mathbf{Z}) \times \cdots \times I_{r_n}(\mathbf{Z})$. This implies $sh(T'_{\sigma(i)}) = sh(T_i)$ for $1 \leq i \leq n$.

(3) Since $N = N^{r_1}(a_1, \ldots, a_1) \otimes \cdots \otimes N^{r_n}(a_n, \ldots, a_n)$, we get the direct sum decomposition in the proposition by Proposition 8 (4). So part (1) proves the claim. $\square$

Corollary 4 (of the proof). *If $a_i/a_j \notin p^{\mathbf{Z}}$ for any $i \neq j$, then $N_{T_1}(a_1) \otimes \cdots \otimes N_{T_n}(a_n)$ is a weight module and satisfies*

$$\mathrm{End}_{\mathcal{A}}(N_{T_1}(a_1) \otimes \cdots \otimes N_{T_n}(a_n)) \simeq \mathbf{C}.$$

Recalling that T_λ is a fixed tableau of shape λ, set $N_\lambda(a) = N_{T_\lambda}(a)$.

Theorem 3. (1) *An $\mathcal{A}^0$ module W is irreducible if and only if $W \simeq \mathbf{C}$ on which $a(\mathbf{k})$ acts as* 0.

(2) *Suppose that $r \geq 1$. Then W is an irreducible $\mathcal{A}^r$ module generated by a weight vector if and only if there exist a positive integer n, nonzero complex numbers $a_1, \ldots, a_n$ such that $a_i/a_j \notin p^{\mathbf{Z}}$ for any $i \neq j$ and partitions $\lambda_1, \ldots, \lambda_n$ such that $|\lambda_1| + \cdots + |\lambda_n| = r$ for which*

$$W \simeq N_{\lambda_1}(a_1) \otimes \cdots \otimes N_{\lambda_n}(a_n).$$

(3) *Let $N_{\lambda_1}(a_1) \otimes \cdots \otimes N_{\lambda_n}(a_n)$ and $N_{\mu_1}(b_1) \otimes \cdots \otimes N_{\mu_m}(b_m)$ be $\mathcal{A}^r$ modules as in part* (2). *Then these two modules are isomorphic if and only if $m = n$ and there exists*

$\sigma \in \mathcal{S}_n$ such that

$$\mu_{\sigma(i)} = \lambda_i \quad \textit{and} \quad b_{\sigma(i)}/a_i \in p^{\mathbf{Z}} \quad \textit{for } 1 \le i \le n.$$

Proof. (1) Follows from the discussion at the beginning of this subsection.
(2) Suppose that W is an irreducible $\mathcal{A}^r$ module generated by a weight vector. Then, by Propositions 6 and 7, we find that W is isomorphic to a quotient of $N^r(\mathbf{a})$ for some $\mathbf{a} = (a_1, \ldots, a_r) \in (\mathbf{C}^\times)^r$. For $(m_1, \ldots, m_r) \in \mathbf{Z}^r$ and $\sigma \in \mathcal{S}_r$ the map $N^r(\mathbf{a}) \to N^r(a_{\sigma(1)}p^{m_1}, \ldots, a_{\sigma(r)}p^{m_r})$ $(u \mapsto \sigma^{-1}(u)z_1^{-m_1} \cdots z_r^{-m_r})$ is an isomorphism of $\mathcal{A}^r$ modules. Therefore we can assume that $\mathbf{a} = (\overbrace{\alpha_1, \ldots, \alpha_1}^{r_1}, \ldots, \overbrace{\alpha_n, \ldots, \alpha_n}^{r_n})$ with $\alpha_i/\alpha_j \notin p^{\mathbf{Z}}$ $(i \neq j)$. Hence we find by Propositions 9 that W is isomorphic to one of the $\mathcal{A}^r$ modules in the theorem. Since these modules satisfy the condition for W by Proposition 9 and Corollary 4, we are done.
(3) Follows from Proposition 9 (2). ☐

REMARK 5. Since $K_{\lambda\lambda} = 1$, the above theorem and Proposition 8 (2) imply that in the case $r \geq 1$ any irreducible $\mathcal{A}^r$ module generated by a weight vector is infinite dimensional.

4.3. The algebra $\mathcal{B}^r$ and $\mathcal{B}^r$ modules. The following lemma is immediate.

Lemma 15. (1) *There exist a* $\mathbf{C}$ *algebra isomorphism* $\psi\colon \mathcal{A} \to \mathcal{B}$ *determined by*

$$\psi(a(\mathbf{k})) = p^{-kl} b(\mathbf{k}')$$

where $\mathbf{k} = (k, l) \neq \mathbf{0}$ *and* $\mathbf{k}' = (-k, l)$.
(2) *This isomorphism satisfies*

$$\psi(A^r_M(\mathbf{k}_1, \ldots, \mathbf{k}_M)) = p^{-\sum_i k_i l_i} B^r_M(\mathbf{k}'_1, \ldots, \mathbf{k}'_M)$$

for $r \geq 0$ *and* $M \geq 1$ *and induces an isomorphism* $\psi^r\colon \mathcal{A}^r \to \mathcal{B}^r$.
(3) $\psi(a(0, l)) = b(0, l)$ *for* $l \neq 0$ *and* $\psi(\Lambda_{\mathcal{A},n}) = \Lambda_{\mathcal{B},n}$ *for any* n.

By this lemma we can see that any irreducible $\mathcal{B}^r$ module generated by a nonzero vector v such that $b(0, l)v \in \mathbf{C}v$ $(l \neq 0)$ is obtained by regarding an irreducible $\mathcal{A}^r$ module generated by a weight vector v as an $\mathcal{B}^r$ module via the isomorphism $(\psi^r)^{-1}\colon \mathcal{B}^r \to \mathcal{A}^r$. Therefore we can obtain the necessary results for the algebra $\mathcal{B}^r$ and $\mathcal{B}^r$ modules from the results in the previous two subsections.

4.4. Proof of Proposition 3. Set $\mathcal{C} = U(\mathcal{H}_0)$. Since $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ and $[\mathcal{H}_i, \mathcal{H}_j] = 0$ if $i \neq j$, the map $\mathcal{C} \otimes \mathcal{A} \otimes \mathcal{B} \to U(\mathcal{H})$ $(c \otimes a \otimes b \mapsto cab)$ is an isomorphism of $\mathbf{C}$ algebras. If we identify $U(\mathcal{H})$ with $\mathcal{C}\otimes\mathcal{A}\otimes\mathcal{B}$ via this isomorphism, then

$$I^{r,s} = I_{\mathcal{C}}^{r+s}\otimes\mathcal{A}\otimes\mathcal{B}+\mathcal{C}\otimes I_{\mathcal{A}}^{r}\otimes\mathcal{B}+\mathcal{C}\otimes\mathcal{A}\otimes I_{\mathcal{B}}^{s}$$

where $I_{\mathcal{C}}^{r+s}$ is the ideal of $\mathcal{C}$ generated by the element $h-r-s$. By Proposition 5 and Lemma 15, $I_{\mathcal{A}}^{r}$ and $I_{\mathcal{B}}^{s}$ are ideals of $\mathcal{A}$ and $\mathcal{B}$, respectively. Therefore $I^{r,s}$ is an ideal of $U(\mathcal{H})$ and

$$U(\mathcal{H})^{r,s}\simeq\mathcal{C}/I_{\mathcal{C}}^{r+s}\otimes\mathcal{A}^{r}\otimes\mathcal{B}^{s}$$

as $\mathbf{C}$ algebras. Since $\mathcal{C}/I_{\mathcal{C}}^{r+s}\simeq\mathbf{C}$, we are done. □

4.5. Proof of Theorem 2. In this proof, we write $J_{\mathcal{A}}^{r}(\mathbf{a})$, $M_{\mathcal{A}}^{r}(\mathbf{a})$, $N_{\mathcal{A}}^{r}(\mathbf{a})$ and $N_{T}^{\mathcal{A}}(a)$ for $J^{r}(\mathbf{a})$, $M^{r}(\mathbf{a})$, $N^{r}(\mathbf{a})$ and $N_{T}(a)$, respectively. Set $J_{\mathcal{B}}^{r}(\mathbf{a})=\psi(J_{\mathcal{A}}^{r}(\mathbf{a}))$ and $M_{\mathcal{B}}^{r}(\mathbf{a})=\mathcal{B}^{r}/J_{\mathcal{B}}^{r}(\mathbf{a})$. The $\mathcal{B}^{r}$ module $M_{\mathcal{B}}^{r}(\mathbf{a})$ is isomorphic to $M_{\mathcal{A}}^{r}(\mathbf{a})$ considered as a $\mathcal{B}^{r}$ module via $(\psi^{r})^{-1}$. We denote $N_{\mathcal{A}}^{r}(\mathbf{a})$ and $N_{T}^{\mathcal{A}}(a)$ regarded as $\mathcal{B}$ modules via ψ^{-1} by $N_{\mathcal{B}}^{r}(\mathbf{a})$ and $N_{T}^{\mathcal{B}}(a)$, respectively.

By part (2) of Proposition 3 we can identify $\mathcal{A}^{r}\otimes\mathcal{B}^{s}$ modules with $\mathcal{H}$ modules the $U(\mathcal{H})$ module structures of which induce $U(\mathcal{H})^{r,s}$ module structures. Under this identification an $\mathcal{A}^{r}\otimes\mathcal{B}^{s}$ module is endowed with an $\mathcal{H}$ module structure via the composite map $\mathcal{H}\hookrightarrow U(\mathcal{H})\to U(\mathcal{H})^{r,s}\simeq\mathcal{A}^{r}\otimes\mathcal{B}^{s}$ ($h\mapsto r+s$, $a(\mathbf{k})\mapsto a(\mathbf{k})\otimes 1$, $b(\mathbf{k})\mapsto 1\otimes b(\mathbf{k})$) where the second map is the quotient map. For a nonnegative integer n, tableaux $T_1,\ldots,T_n$, nonzero complex numbers $a_1,\ldots,a_n$ such that $a_i/a_j\notin p^{\mathbf{Z}}$ ($i\neq j$) and $i=\mathcal{A}$, $\mathcal{B}$, set

$$N_{T_1,\ldots,T_n}^{i}(a_1,\ldots,a_n)=N_{T_1}^{i}(a_1)\otimes\cdots\otimes N_{T_n}^{i}(a_n)$$

where the right hand side should be understood as $\mathbf{C}$ on which $a(\mathbf{k})$ or $b(\mathbf{k})$ acts as 0 in the case $n=0$. Suppose that $\sum|\lambda_i|=r$ and $\sum|\mu_i|=s$. Then the $\mathcal{A}^{r}\otimes\mathcal{B}^{s}$ module

$$(\natural)\qquad N_{T_{\lambda_1},\ldots,T_{\lambda_n}}^{\mathcal{A}}(a_1,\ldots,a_n)\otimes N_{T_{\mu_1},\ldots,T_{\mu_m}}^{\mathcal{B}}(b_1,\ldots,b_m)$$

is identified with the $\mathcal{H}$ module $N_{\lambda_1,\ldots,\lambda_n}^{1}(a_1,\ldots,a_n)\otimes N_{\mu_1,\ldots,\mu_m}^{2}(b_1,\ldots,b_m)$ in the theorem. Therefore it is sufficient to show, for the proof of part (1) of the theorem, that W is an irreducible $\mathcal{A}^{r}\otimes\mathcal{B}^{s}$ module generated by a nonzero vector v such that $a(0,l)v\in\mathbf{C}v$ and $b(0,l)v\in\mathbf{C}v$ for any $l\neq 0$ (where we identify $\mathcal{A}^{r}$ and $\mathcal{B}^{s}$ with subalgebras of $\mathcal{A}^{r}\otimes\mathcal{B}^{s}$ naturally) if and only if it is isomorphic to an $\mathcal{A}^{r}\otimes\mathcal{B}^{s}$ module of the form ($\natural$) and, for the proof of part (2), that the $\mathcal{A}^{r}\otimes\mathcal{B}^{s}$ module ($\natural$) and the $\mathcal{A}^{r'}\otimes\mathcal{B}^{s'}$ module ($\natural'$) are isomorphic as $\mathcal{H}$ modules if and only if the conditions in part (2) of the theorem hold. Here ($\natural'$) stands for ($\natural$) with n, the λ_i, etc., replaced by n', the λ_i', etc.

First we show part (1). Suppose that W is an irreducible $\mathcal{A}^{r}\otimes\mathcal{B}^{s}$ module satisfying the above condition. Then as in the proof of Proposition 6 (1), by Corollary 1 and

Lemma 15, we find that there exist $\mathbf{a} = (a_1, \ldots, a_r) \in (\mathbf{C}^\times)^r$ and $\mathbf{b} = (b_1, \ldots, b_s) \in (\mathbf{C}^\times)^s$ such that $a(0, l)v = P_l(\mathbf{a})v$ and $b(0, l)v = P_l(\mathbf{b})v$ for $l \neq 0$. Using the argument in the proof of Proposition 6 (2), we can assume that the a_i and the b_i are distinct, respectively. Since $J^r_{\mathcal{A}}(\mathbf{a}) \otimes \mathcal{B}^s + \mathcal{A}^r \otimes J^s_{\mathcal{B}}(\mathbf{b})$ annihilates v, W is a quotient of $M^r_{\mathcal{A}}(\mathbf{a}) \otimes M^s_{\mathcal{B}}(\mathbf{b})$, which is isomorphic to $N^r_{\mathcal{A}}(\mathbf{a}) \otimes N^s_{\mathcal{B}}(\mathbf{b})$ by Proposition 7 and its counterpart for $\mathcal{B}^s$. So, by using the argument in the proof of Theorem 3, we find that W is a quotient of

$$\bigoplus N^{\mathcal{A}}_{T_1,\ldots,T_n}(\alpha_1, \ldots, \alpha_n) \otimes N^{\mathcal{B}}_{T'_1,\ldots,T'_m}(\beta_1, \ldots, \beta_m)$$

where the α_i and the β_i are some nonzero complex numbers and the sum is taken over the standard tableaux T_i and T'_i satisfying $\sum |T_i| = r$, $\sum |T'_i| = s$ and some other conditions. By Corollary 4, its counterpart for $\mathcal{B}^s$ and Lemma 14, each component of the above direct sum is irreducible. Therefore we find, by Proposition 9 (3) and its counterpart for $\mathcal{B}^s$, that W is isomorphic to an $\mathcal{A}^r \otimes \mathcal{B}^s$ module of the form ($\natural$). Since any $\mathcal{A}^r \otimes \mathcal{B}^s$ module of the form ($\natural$) satisfies the condition for W, we get part (1).

Next we show part (2). The 'if' part follows from Proposition 9 (3) and its counterpart for $\mathcal{B}^s$. We show the 'only if' part. Suppose that the $\mathcal{A}^r \otimes \mathcal{B}^s$ module ($\natural$) and the $\mathcal{A}^{r'} \otimes \mathcal{B}^{s'}$ module ($\natural'$) are isomorphic as $\mathcal{H}$ modules. Then they are isomorphic if regarded both as $\mathcal{A}$ modules and as $\mathcal{B}$ modules via the isomorphism $U(\mathcal{H}) \simeq \mathcal{C} \otimes \mathcal{A} \otimes \mathcal{B}$ in the proof of Proposition 3. From the isomorphism as $\mathcal{A}$ modules we find that $N^{\mathcal{A}}_{T_{\lambda_1},\ldots,T_{\lambda_n}}(a_1, \ldots, a_n) \simeq N^{\mathcal{A}}_{T_{\lambda'_1},\ldots,T_{\lambda'_{n'}}}(a'_1, \ldots, a'_{n'})$ as $\mathcal{A}$ modules. So Proposition 9 (3) and Remark 5 give the conditions for n, the λ_i and the a_i. The conditions for m, the μ_i and the b_i are obtained similarly by considering the isomorphism as $\mathcal{B}$ modules. □

5. Proof of Lemmas 5 and 6

In this section, to complete the proof of Theorem 1, we prove Lemmas 5 and 6.

5.1. Proof of Lemma 5.

Lemma 16. *The elements $F_M(\mathbf{k}_1, \ldots, \mathbf{k}_M; \mathbf{m}_1, \ldots, \mathbf{m}_M)$ are determined by the relations*

$$(5.1) \quad F_1(\mathbf{k}; \mathbf{m}) = \begin{cases} p^{lm} a(\mathbf{k} + \mathbf{m}) + p^{kn} b(\mathbf{k} + \mathbf{m}) & \textit{if} \quad \mathbf{k} + \mathbf{m} \neq \mathbf{0} \\ p^{lm} h & \textit{if} \quad \mathbf{k} + \mathbf{m} = \mathbf{0}, \end{cases}$$

$$(5.2) \quad F_M(\mathbf{k}_1, \ldots, \mathbf{k}_M; \mathbf{m}_1, \ldots, \mathbf{m}_M) = \sum_{i=1}^{M} F_1(\mathbf{k}_i; \mathbf{m}_M) F_{M-1}(\mathbf{k}_1, \ldots, \hat{\mathbf{k}}_i, \ldots, \mathbf{k}_M; \mathbf{m}_1, \ldots, \mathbf{m}_{M-1})$$

$$-\sum_{j<i}(q_{ij}+q_{ji})F_{M-1}(\mathbf{k}_1,\dots,\hat{\mathbf{k}}_j,\dots,\hat{\mathbf{k}}_i,\dots,\mathbf{k}_M,\mathbf{k}_j+\mathbf{k}_i+\mathbf{m}_M;\mathbf{m}_1,\dots,\mathbf{m}_{M-1})$$
$$(M\geq 2)$$

where $q_{ij}=p^{k_il_j+l_jm_M+k_in_M}$ *and* $\hat{}$ *denotes omission of variables.*

Proof. First note that $F_M(\mathbf{k}_1,\dots,\mathbf{k}_M;\mathbf{m}_1,\dots,\mathbf{m}_M)$ is symmetric in the variables $\mathbf{k}_1,\dots,\mathbf{k}_M$ since $[e_{\mathbf{k}},e_{\mathbf{k}'}]=0$. Then the claim follows from multiplying the equality

$$\begin{aligned}[e_{\mathbf{k}_1}\cdots e_{\mathbf{k}_M},f_{\mathbf{m}_M}] &= \sum_{i=1}^M e_{\mathbf{k}_1}\cdots e_{\mathbf{k}_{i-1}}F_1(\mathbf{k}_i;\mathbf{m}_M)e_{\mathbf{k}_{i+1}}\cdots e_{\mathbf{k}_M}\\ &= \sum_{i=1}^M F_1(\mathbf{k}_i;\mathbf{m}_M)e_{\mathbf{k}_1}\cdots\hat{e}_{\mathbf{k}_i}\cdots e_{\mathbf{k}_M}\\ &\quad -\sum_{j<i}(q_{ij}+q_{ji})e_{\mathbf{k}_1}\cdots e_{\mathbf{k}_j+\mathbf{k}_i+\mathbf{m}_M}\cdots\hat{e}_{\mathbf{k}_i}\cdots e_{\mathbf{k}_M}\end{aligned}$$

by $f_{\mathbf{m}_{M-1}}\cdots f_{\mathbf{m}_1}$ from the right. □

Utilizing the above lemma, we can give the

Proof of Lemma 5. With the notation of the lemma set

$$\begin{aligned}&G_M(\mathbf{k}_1,\dots,\mathbf{k}_M;\mathbf{m}_1,\dots,\mathbf{m}_M)\\ &=\sum_{I\sqcup J=\{1,\dots,M\}}\mathcal{A}^r_a(\mathbf{k}_{i_1},\dots,\mathbf{k}_{i_a};\mathbf{m}_{i_1},\dots,\mathbf{m}_{i_a})\mathcal{B}^s_b(\mathbf{k}_{j_1},\dots,\mathbf{k}_{j_b};\mathbf{m}_{j_1},\dots,\mathbf{m}_{j_b})\end{aligned}$$

where

$$\begin{aligned}\mathcal{A}^r_a(\mathbf{k}_1,\dots,\mathbf{k}_a;\mathbf{m}_1,\dots,\mathbf{m}_a) &= p^{\sum_{\alpha=1}^a l_\alpha m_\alpha}A^r_a(\mathbf{k}_1+\mathbf{m}_1,\dots,\mathbf{k}_a+\mathbf{m}_a),\\ \mathcal{B}^s_b(\mathbf{k}_1,\dots,\mathbf{k}_b;\mathbf{m}_1,\dots,\mathbf{m}_b) &= p^{\sum_{\beta=1}^b k_\beta n_\beta}B^s_b(\mathbf{k}_1+\mathbf{m}_1,\dots,\mathbf{k}_b+\mathbf{m}_b).\end{aligned}$$

We shall show that the elements

$$F^{r,s}_M(\mathbf{k}_1,\dots,\mathbf{k}_M;\mathbf{m}_1,\dots,\mathbf{m}_M):=\sum_{\sigma\in\mathcal{S}_M}G_M(\mathbf{k}_{\sigma(1)},\dots,\mathbf{k}_{\sigma(M)};\mathbf{m}_1,\dots,\mathbf{m}_M)$$

satisfy the recurrence relations (5.1) and (5.2) in the quotient algebra $U(\mathcal{H})/J_{r+s}$.

Eq. (5.1) can be easily checked, using the definitions (3.3) and (3.5). Utilizing the recurrence relations (3.4) and (3.6), we find that

$$G_M(\mathbf{k}_1,\dots,\mathbf{k}_M;\mathbf{m}_1,\dots,\mathbf{m}_M)$$

$$= G_1(\mathbf{k}_M; \mathbf{m}_M) G_{M-1}(\mathbf{k}_1, \ldots, \mathbf{k}_{M-1}; \mathbf{m}_1, \ldots, \mathbf{m}_{M-1}) - \sum_{I \sqcup J = \{1, \ldots, M-1\}}$$
$$\times \Big(\sum_{\alpha=1}^{a} q_{i_\alpha M} A_a^r(\mathbf{k}_{i_1}, \ldots, \mathbf{k}'_{i_\alpha}, \ldots, \mathbf{k}_{i_a}; \mathbf{m}_{i_1}, \ldots, \mathbf{m}_{i_a}) B_b^s(\mathbf{k}_{j_1}, \ldots, \mathbf{k}_{j_b}; \mathbf{m}_{j_1}, \ldots, \mathbf{m}_{j_b})$$
$$+ \sum_{\beta=1}^{b} q_{M j_\beta} A_a^r(\mathbf{k}_{i_1}, \ldots, \mathbf{k}_{i_a}; \mathbf{m}_{i_1}, \ldots, \mathbf{m}_{i_a}) B_b^s(\mathbf{k}_{j_1}, \ldots, \mathbf{k}'_{j_\beta}, \ldots, \mathbf{k}_{j_b}; \mathbf{m}_{j_1}, \ldots, \mathbf{m}_{j_b}) \Big)$$

where $\mathbf{k}'_i = \mathbf{k}_i + \mathbf{k}_M + \mathbf{m}_M$. By symmetrizing the above equation with respect to the variables $\mathbf{k}_1, \ldots, \mathbf{k}_M$, we can verify (5.2). □

In the next subsection, we need the following corollary of Lemma 5. For $M \geq 1$, $\mathbf{k}_i = (k_i, l_i) \in \mathbf{Z}^2$ $(1 \leq i \leq M)$ and $\mathbf{m} = (m, n) \in \mathbf{Z}^2$, set

$$H_M(\mathbf{k}_1, \ldots, \mathbf{k}_M; \mathbf{m}) = \frac{p^{Mmn} F_M(\mathbf{k}_1 - \mathbf{m}, \ldots, \mathbf{k}_M - \mathbf{m}; \mathbf{m}, \ldots, \mathbf{m})}{M!}.$$

Corollary 5. *Let r and s be nonnegative integers. With the notation of* Lemma 5, *the following hold in $U(\mathcal{H})/J_{r+s}$:*

$$H_M(\mathbf{k}_1, \ldots, \mathbf{k}_M; \mathbf{m}) = \sum_{I \sqcup J = \{1, \ldots, M\}} p^{m \sum_{\alpha=1}^{a} l_{i_\alpha} + n \sum_{\beta=1}^{b} k_{j_\beta}} A_a^r(\mathbf{k}_{i_1}, \ldots, \mathbf{k}_{i_a}) B_b^s(\mathbf{k}_{j_1}, \ldots, \mathbf{k}_{j_b}).$$

5.2. Proof of Lemma 6.

Lemma 17. *For $(V, v) \in \mathcal{IH}(r, s)$ define $\lambda_{\mathcal{A}}^{\pm}(u)$ and $\lambda_{\mathcal{B}}^{\pm}(u)$ as in* Lemma 2. *Let $\lambda_{\mathcal{A},i}$ and $\lambda_{\mathcal{B},i}$ be the coefficient of u^i in $\lambda_{\mathcal{A}}^{+}(u)$ and $\lambda_{\mathcal{B}}^{+}(u)$, respectively. Then for any $M \geq 1$ and any $\mathbf{k}_j \in \mathbf{Z}^2$ $(1 \leq j \leq M)$, the following hold.*

(1) *$A_r^r(\pm\mathbf{1}, \ldots, \pm\mathbf{1})v$ and $B_s^s(\pm\mathbf{1}, \ldots, \pm\mathbf{1})v$ are nonzero scalar multiples of v.*
(2) $\sum_{i=0}^{r} \lambda_{\mathcal{A}, r-i} A_M^r(\mathbf{k}_1, \ldots, \mathbf{k}_{M-1}, \mathbf{k}_M + i\mathbf{1})v = 0$.
(3) $\sum_{i=0}^{s} p^{k_M i} \lambda_{\mathcal{B}, s-i} B_M^s(\mathbf{k}_1, \ldots, \mathbf{k}_{M-1}, \mathbf{k}_M + i\mathbf{1})v = 0$.

Proof. (1) Suppose that

$$\lambda_{\mathcal{A}}^{\pm}(u) = \prod_{i=1}^{r} (1 - a_i^{\pm 1} u) \tag{5.3}$$

where the a_i are nonzero complex numbers. Then, by Lemma 8 (2),

$$\frac{A_r^r(\pm\mathbf{1}, \ldots, \pm\mathbf{1})}{r!} v = (a_1 \cdots a_r)^{\pm 1} v.$$

This proves the claim for A_r^r. The claim for B_s^s is proved similarly, using Lemma 15.

(2) By part (1) and part (2-i) of Lemma 7 it is sufficient to prove the case $M = 1$, i.e.,

$$\sum_{i=0}^{r} \lambda_{\mathcal{A},r-i} A_1^r(k, l+i)v = 0.$$

First we consider the case $k = 0$. Eq. (5.3) and the definition of $\Lambda_{\mathcal{A}}^{\pm}(u)$ give

$$\lambda_{\mathcal{A},n} = (-1)^n E_n(\mathbf{a}) \quad (0 \leq n \leq r) \quad \text{and} \quad a(0,l)v = P_l(\mathbf{a})v \quad (l \neq 0)$$

where $\mathbf{a} = (a_1, \ldots, a_r)$. So, recalling the definition of $A_1^r(\mathbf{k})$, we find that

$$\sum_{i=0}^{r} \lambda_{\mathcal{A},r-i} A_1^r(0, l+i)v = 0$$

for any integer l.

Next we shall show the case $k \neq 0$. Set $N = r+s$. From Lemma 1 and Corollary 5 we get

$$\begin{aligned} 0 &= H_{N+1}(\mathbf{1}, \ldots, \mathbf{1}, \mathbf{k}; \mathbf{m})v \\ &= \sum_{i=0}^{N} \binom{N}{i} \Big(p^{mi+nk} A_i^r(\mathbf{1}, \ldots, \mathbf{1}) B_{N+1-i}^s(\mathbf{1}, \ldots, \mathbf{1}, \mathbf{k}) \\ &\qquad + p^{m(N-i+l)} A_{N+1-i}^r(\mathbf{1}, \ldots, \mathbf{1}, \mathbf{k}) B_i^s(\mathbf{1}, \ldots, \mathbf{1}) \Big) v \end{aligned}$$

for any $\mathbf{m} = (m, n) \in \mathbf{Z}^2$. Recalling that p is not a root of unity, from the coefficients of $p^{m(s+l)}$ in the above equation and part (1) of this lemma, we get

$$A_{r+1}^r(\mathbf{1}, \ldots, \mathbf{1}, \mathbf{k})v = 0.$$

This result and Lemma 8 give

$$\sum_{i=0}^{r} \lambda_{\mathcal{A},r-i} A_1^r(k, l+i)v = (-1)^r \frac{A_{r+1}^r(\mathbf{1}, \ldots, \mathbf{1}, \mathbf{k})}{r!} v = 0.$$

(3) The proof is similar to that of part (2). □

Lemma 18. *Let* $\mathbf{k}_i = (k_i, l_i) \in \mathbf{Z}^2$ *for any* i. *For* $(V, v) \in \mathcal{IH}(r, s)$ *the following hold.*

(1) *If* $l_i > l_{i+1} + \cdots + l_{r+1} + s$ $(1 \leq i \leq r+1)$, *then* $A_{r+1}^r(\mathbf{k}_1, \ldots, \mathbf{k}_{r+1})v = 0$.

(2) *If* $l_i < 0$ $(1 \leq i \leq s+1)$, *then* $B_{s+1}^s(\mathbf{k}_1, \ldots, \mathbf{k}_{s+1})v = 0$.

Proof. We shall show (1). The proof of (2) is similar. Let $I_0 = \{1, \ldots, r+1\}$ and $N = r+s$. Set $\mathbf{k}_j = (0, 1)$ for $r+2 \leq j \leq N+1$. From Lemma 1 and Corollary 5, we get with the notation of Lemma 5

$$\begin{aligned} 0 &= H_{N+1}(\mathbf{k}_1, \dots, \mathbf{k}_{N+1}; \mathbf{m})v \\ &= p^{m \sum_{i \in I_0} l_i} A^r_{r+1}(\mathbf{k}_1, \dots, \mathbf{k}_{r+1}) B^s_s(\mathbf{1}, \dots, \mathbf{1})v \\ &\quad + \sum_{\substack{I \sqcup J = \{1, \dots, N+1\} \\ I \neq I_0}} p^{m \sum_{i \in I} l_i + n \sum_{j \in J} k_j} A^r_a(\mathbf{k}_{i_1}, \dots, \mathbf{k}_{i_a}) B^s_b(\mathbf{k}_{j_1}, \dots, \mathbf{k}_{j_b})v \end{aligned}$$

for any $\mathbf{m} = (m, n) \in \mathbf{Z}^2$. Hence for the proof of the claim, thanks to Lemma 17 (1), it is sufficient to show that if I is a subset of $\{1, \dots, N+1\}$ and $I \neq I_0$, then $\sum_{i \in I} l_i \neq \sum_{i \in I_0} l_i$. This can be easily checked. □

Now we can give the

Proof of Lemma 6. By applying parts (2) and (3) of Lemma 17 repeatedly, we can see that $A^r_{r+1}(\mathbf{k}_1, \dots, \mathbf{k}_{r+1})v$ and $B^s_{s+1}(\mathbf{k}_1, \dots, \mathbf{k}_{s+1})v$ can be written as a linear combination of the terms in Lemma 18. This proves the claim. □

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Added in proof. After the acceptance of our paper the preprint [21] came to our attention. In the paper imaginary Verma modules for a central extension of $sl_2(\mathbf{C}_q)$ are studied. Their quantum torus $\mathbf{C}_q$ is more general than that in our paper.

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