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KNOTTED KLEIN BOTTLES WITH ONLY DOUBLE POINTS

AKIKO SHIMA

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1. Introduction

If an embedded 2-sphere in 4-space $\mathbf{R}^4$ has the singular set of the projection in 3-space $\mathbf{R}^3$ consisting of double points, then the 2-sphere is ambient isotopic to a ribbon 2-sphere (see [19]). Similarly, if an embedded torus in $\mathbf{R}^4$ has the singular set of the projection in $\mathbf{R}^3$ consisting only of double points, then the torus is ambient isotopic to either a ribbon torus or a torus obtained from a symmetry-spun torus by m-fusion (see [15]). In this paper we will show a similar theorem for an embedded Klein bottle in $\mathbf{R}^4$. The following is the main results in this paper.

Theorem 1.1. *Let F be an embedded Klein bottle in $\mathbf{R}^4$. If the singular set $\Gamma^*(F)$ of the projection of F in $\mathbf{R}^3$ consists only of double points, then F is ambient isotopic to either a ribbon Klein bottle or a Klein bottle obtained from a spun Klein bottle by m-fusion.*

Corollary 1.2. *Let F be an embedded Klein bottle in $\mathbf{R}^4$. Suppose that the singular set $\Gamma^*(F)$ of the projection of F in $\mathbf{R}^3$ consists of double points, and every component of the singular set $\Gamma(F)$ on F is not homotopic to zero. If the fundamental group of the complement of F is isomorphic to $\mathbf{Z}_2$, then F is trivial, i.e., F bounds a solid Klein bottle in $\mathbf{R}^4$.*

Let F be an oriented closed surface in $\mathbf{R}^4$. Is F trivial if the fundamental group of the complement of F is isomorphic to $\mathbf{Z}$? In the topological category, the question is affirmatively soloved when if it is a 2-sphere (see [3]). In the PL or smooth category, this is an open question, it is affirmatively soloved when F is one of the following:
(i) F is a 1-fusion ribbon 2-knot ([8]).
(ii) F is a 2-sphere with four critical points ([11]).
(iii) F is a symmetry-spun torus ([17]).
(iv) F is a torus whose singular set on the torus consists only of disjoint simple closed curves with non-homotopic to zero in the torus ([15]).

All homology groups are taken with coefficients in $\mathbf{Z}$, and all submanifolds are

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assumed to be locally flat, thourghout in this papar. We will work in the PL category, thourghout in this papar. Let $\mathbf{R}^n$ be the n-dimensional Euclidean space. Moreover, we regard 3-space $\mathbf{R}^3$ as the subset $\mathbf{R}^3 \times \{0\}$ of $\mathbf{R}^4$.

The paper is organized as follows. In Section 2, we define a ribbon surface, and a Klein bottle obtained from a spun Klein bottle by m-fusion. In Section 3, we study certain types of 2-complexes in $\mathbf{R}^3$. In Section 4, we define diagrams for embedded surfaces. In Section 5, we consider spun Klein bottles in $\mathbf{R}^4$. In Section 6, we prove the main theorem.

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2. Preliminaries and definitions

In this section, we define an m-fusion, a ribbon surface, and a spun Klein bottle.

Let F be a closed surface. A map f from F to $\mathbf{R}^3$ is a *generic map* if for at every point x of F, there exists a regular neigborhood N of $f(x)$ in $\mathbf{R}^3$ such that $(N, f(F) \cap N)$ is homeomorphic to (B^3, Z_1), $(B^3, Z_1 \cup Z_2)$, $(B^3, Z_1 \cup Z_2 \cup Z_3)$ or $(B^3$, the cone on a figure 8), where B^3 is the unit 3-ball in $\mathbf{R}^3$, Z_i is the intersection of B^3 and $x_j x_k$-plane ($\{1, 2, 3\} = \{i, j, k\}$). If $(N, f(F) \cap N)$ is homeomorphic to $(B^3$, the cone on a figure 8), then the point $f(x)$ is called a *branch point*. The point is also known as "Whitney's umbrella" or "a pinch point". A point $x \in f(F)$ is called a *double point* if $f^{-1}(x)$ consists of two points, and a *triple point* if $f^{-1}(x)$ consists of three points.

Let F be an embedded surface in $\mathbf{R}^4$, and let p be the projection defined by $p(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3)$. If $p|F$ is a generic map, then we associate the subset $F^* = p(F)$, and we denote by $\Gamma^*(F)$ the set of all double points, triple points and branch points. And put $\Gamma(F) = p^{-1}(\Gamma^*(F)) \cap F$. In this paper we assume that $p|F$ is a generic map.

An oriented closed surface in $\mathbf{R}^4$ is said to be *trivial* if it is the boundary of the disjoint union of handlebodies in $\mathbf{R}^4$. Note that the boundary of a handlebody is unique up to ambient isotopies of $\mathbf{R}^4$ (see [5]). An embedded Klein bottle in $\mathbf{R}^4$ is said to be *trivial* if it is the boundary of a solid Klein bottle in $\mathbf{R}^4$. Here the solid Klein bottle is homemorphic to the 3-manifold by attaching $B^2 \times \{0\}$ and $B^2 \times \{1\}$ from $B^2 \times [0, 1]$ via the map $q(x, 0) = (-x, 1)$, where B^2 is the unit 2-ball. In other word, the trivial Klein bottle is ambient isotopic to the surface with projection in $\mathbf{R}^3$ as illustrated in Fig. 1.

Let G be an embedded closed surface in $\mathbf{R}^4$, $I = [0, 1]$, B^2 the unit 2-ball. An embedded surface F in $\mathbf{R}^4$ is a *surface obtained from G by m-fusion* if there exists a collection of embeddings $h_i \colon B^2 \times I \longrightarrow \mathbf{R}^4$, $i = 1, 2, \dots, m$, satisfying the following three conditions:

(i) The images of any two maps h_i, h_j are disjoint for any distinct i, j.

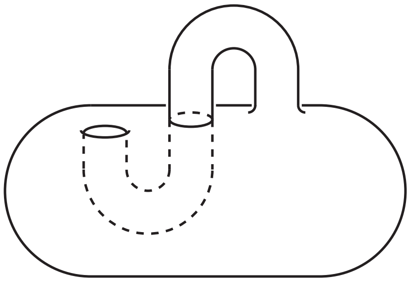


Fig. 1.

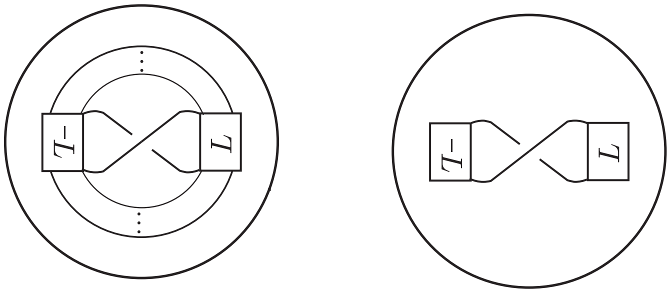
(ii) $h_i(B^2 \times I) \cap G = h_i(B^2 \times \partial I)$ for all i.
(iii) $F = \left(G \setminus \bigcup_{i=1}^m (h_i(B^2 \times \partial I))\right) \cup \left(\bigcup_{i=1}^m h_i(\partial B^2 \times I)\right)$.
An embedded surface in $\mathbf{R}^4$ is a *ribbon surface* if it is obtained from a trivial 2-spheres by m-fusion.

Next, we define a spun Klein bottle in $\mathbf{R}^4$. For $\theta \in [0, 2\pi]$, let $\mathbf{R}^3_\theta = \{(x, y\cos\theta, y\sin\theta, z) \mid y \geq 0\}$, and

$$B_0 = \{(x, y, 0, z) \mid x^2 + (y-2)^2 + z^2 \leq 1\}.$$

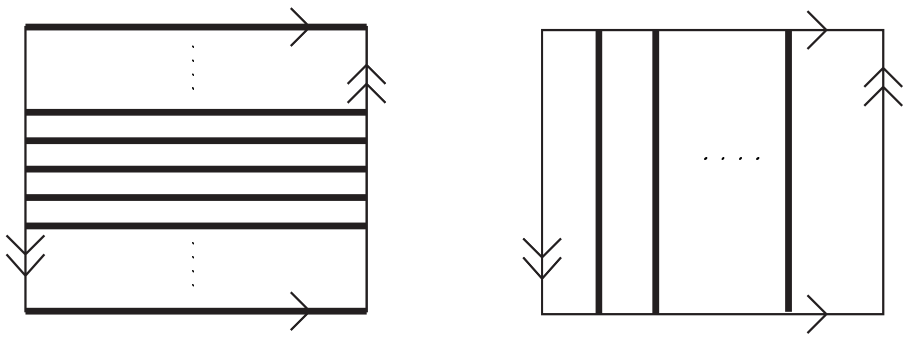
Then B_0 is the 3-ball in $\mathbf{R}^3_0$, and the union of $\mathbf{R}^3_\theta$ for all $\theta \in [0, 2\pi]$ is $\mathbf{R}^4$. Let $r_\theta \colon B_0 \to B_0$ be the θ-rotation map through the axis $(0, 2, 0) \times [-1, 1]$ for $\theta \in [0, 2\pi]$. An embedded Klein bottle F in $\mathbf{R}^4$ is called a *spun Klein bottle* if there exist an integer a and a knot K in the 3-ball B_0 as shown in Fig. 2 (1) such that
(i) K intersects two points to the axis $(0, 2, 0) \times [-1, 1]$,
(ii) $r_\pi(K) = K$, and
(iii) $F = \{(x, y\cos\theta, y\sin\theta, z) \mid (x, y, 0, z) \in r_{(a+(1/2))\theta}(K), \theta \in [0, 2\pi]\}$.
We denote it by $Kl^a(K)$. In particular, if K is a connected sum $L\#(-L)$ of a knot L as shown in Fig. 2 (2), then $Kl^a(K)$ is called a *simple* spun Klein bottle, where $-L$ is the knot with the reverse orientation of L. The symbol L in Fig. 2 (2) is the 1-string tangle so that the tangle sum of L and the trivial tangle is the knot L. In particular, a Klein bottle obtained from a split union of a trivial 2-spheres and a spun Klein bottle by m-fusion is simply called a *Klein bottle obtained from a spun Klein bottle by m-fusion*.

REMARK 2.1. (1) Let $Kl^a(L\#(-L))$ be a simple spun Klein bottle. Then, the fundamental group of the complement of $Kl^a(L\#(-L))$ is isomorphic to $\pi_1(S^3 \setminus L)/\langle m^2 = 1\rangle$ where m is a meridian curve of L (see [18]).
(2) The Klein bottle $Kl^a(K)$ is ambient isotopic to $Kl^{a\pm 2}(K)$ (cf. [17]).



(1) (2)

Fig. 2. The center of each figure is z-axis.



Γ_1 Γ_2

Fig. 3.

3. 2-complexes in $\mathbf{R}^3$

3.1. Embedded Klein bottles in $\mathbf{R}^4$. Let F be an embedded Klein bottle in $\mathbf{R}^4$ such that $p|F$ is a generic map. In this section, we assume that $\Gamma^*(F)$ consists only of double points. First, we consider the singular set $\Gamma(F)$ on F. Let $c_1 = 0 \times I$, $c_2 = (1/2) \times I$, $c_i = i/(2n+1) \times I \cup (2n+1-i)/(2n+1) \times I$, and $d_j = I \times j/(2n)$ where $i = 3, \ldots, 2n$ and $j = 1, 2, \ldots, 2n-1$. Let $\Gamma_1 = c_1 \cup c_2 \cup \cdots \cup c_n/\sim$, $\Gamma_2 = d_1 \cup d_2 \cup \cdots \cup d_{2n-1}/\sim$ where $\sim$ is the relation on $I \times I$ with $(0, t) \sim (1, t)$ and $(t, 0) \sim (1-t, 1)$ for all $t \in I$. Then each of Γ_1 and Γ_2 is a union of disjoint simple closed curves on a Klein bottle (see Fig. 3). Note that Γ_2 consists of an *odd* number of disjoint simple closed curves.

Lemma 3.1 ([16, Lemma 1.4]). *Let F be a Klein bottle in $\mathbf{R}^4$ such that $\Gamma^*(F)$ consists only of double points. Let Γ be the union of the components of $\Gamma(F)$ each of*

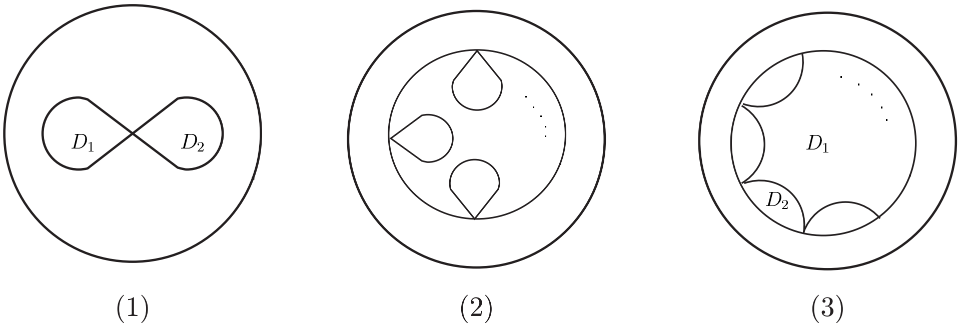


Fig. 4.

which is not homotopic to zero in F. *Then the pair* (Γ, F) *is homeomorphic to* (Γ_1, F) *or* (Γ_2, F).

3.2. Certain types of 2-complexes in $\mathbf{R}^3$. In this subsection, we define certain types of 2-complexes in $\mathbf{R}^3$. For $\theta \in [0, 2\pi]$, let $\mathbf{R}^2_\theta = \{(x, y\cos\theta, y\sin\theta) \mid y \geq 0\}$, and

$$\overline{B}_0 = \{(x, y, 0) \mid x^2 + (y-2)^2 \leq 1\}.$$

Then $\overline{B}_0$ is the 2-ball in $\mathbf{R}^2_0$, and the union of $\mathbf{R}^2_\theta$ for all $\theta \in [0, 2\pi]$ is $\mathbf{R}^3$. Let $\overline{r_\theta}\colon \overline{B}_0 \to \overline{B}_0$ be the θ-rotation map through the point $(0, 2, 0)$ for $\theta \in [0, 2\pi]$. Let α be a 1-complex in $\overline{B}_0$ such that each vertex is a vertex of degree four or three. A 2-complex K in $\mathbf{R}^3$ is called a 2-*complex obtained from* α if there exist integers b, c with $c \neq 0$ such that
(i) If α intersects the point $(0, 2, 0)$, then the point $(0, 2, 0)$ is the vertex of degree four and $c = 2$.
(ii) $\overline{r}_{2\pi/c}(\alpha) = \alpha$, and
(iii) $K = \{(x, y\cos\theta, y\sin\theta) \mid (x, y, 0) \in \overline{r}_{(b/c)\theta}(\alpha), \theta \in [0, 2\pi]\}$,
We denote the 2-complex K by $\alpha(b, c)$, and the above 1-complex α is called a *c-symmetric* 1-*complex*.

EXAMPLE 3.2. (i) Let α_1 be the 2-symmetric 1-complex in $\overline{B}_0$ as shown in Fig. 4 (1) such that the vertex of α_1 is the point $(0, 2, 0)$. Then if b is an odd integer (resp. even integer), then the 2-complex $\alpha_1(b, 2)$ is an immersed Klein bottle (resp. torus) in $\mathbf{R}^3$.
(ii) Let c be an integer with $c \neq 0$, and α_i the c-symmetric 1-complex in $\overline{B}_0$ as shown in Fig. 4 (i) such that α_i does not intersect the point $(0, 2, 0)$ for $i = 2, 3$. Then c is the number of vertices of α_i, and then the 2-complex $\alpha_i(b, c)$ is immersed tori for any integer b.

Lemma 3.3. *Let α be a c-symmetric 1-complex, and $\alpha(b,c)$ a 2-complex in $\mathbf{R}^3$ obtained from α.*
(1) *Let C be a component of $S(\alpha(b,c))$. Then, a regular neighborhood of C in $\alpha(b,c)$ is two immersed annuli, two immersed Möbius bands. Moreover, there is at most one regular neighborhood consisting of two immersed Möbius bands.*
(2) *Removing $S(\alpha(b,c))$, we obtain open annuli.*
Here $S(\alpha(b,c))$ is the set of all point whose neighborhood in $\alpha(b,c)$ is the intersection of two sheets or $Y \times [0,1]$, where Y is the cone on three points.

Proof. (1) If $c = 2$, if b is odd, and if α intersects the point $(0,2,0)$ in $\overline{B}_0$, then we have the component with $(0,2,0)$ in $S(\alpha(b,c))$ whose regular neighborhood in $\alpha(b,c)$ consists of two immersed Möbius bands. Conversely, such a component can be obtained only as above, which yields the result.
(2) From the condition (ii) of the definition of symmetric 1-complexes, we can show (2). □

From Lemma 3.3, we have the following remark:

Remark 3.4. (1) Let b, c be integers with $c \neq 0$, and α a c-symmetric 1-complex in $\overline{B}_0$. If $\alpha(b,c)$ is an immersed Klein bottle, then b is odd, $c = 2$ and there exists a knot K in B_0 with $(Kl^{(b-1)/2}(K))^* = \alpha(b,2)$.
(2) Let K be a knot in B_0 satisfying (i) and (ii) in the definition of spun Klein bottles. Then for any integer a, the projection $(Kl^a(K))^*$ in $\mathbf{R}^3$ is the 2-complex obtained from $p(K)$, i.e., $(Kl^a(K))^* = p(K)(2a+1,2)$.

Definition 3.5. Let α_1 be the 2-symmetric 1-complex as shown in Fig. 4 (1) with $\alpha_1 \subset \overline{B}_0$. Then there exist two 2-balls D_1, D_2 in $\overline{B}_0$ such that $D_1 \cap D_2$ is the point $(0,2,0)$ and $\alpha_1 = \partial D_1 \cup \partial D_2$. For an integer b, the 3-complex X_b is defined by

$$X_b = \{(x, y\cos\theta, y\sin\theta) \mid (x,y,0) \in \overline{r}_{\{(2b+1)/2\}\theta}(D_1 \cup D_2), \theta \in [0,2\pi]\}.$$

Note that the closure of one component of $\mathbf{R}^3 \setminus \{\alpha_1(b,2)\}$ is X_b. Let S^1 be the unit 1-sphere. Then, the 1-sphere S^1 is identified with $[0,2\pi]/0 \sim 2\pi$. We have a natural embedding ψ of the solid torus $\overline{B}_0 \times S^1$ in $\mathbf{R}^3$ defined by $\psi(x,y,\theta) = (x, y\cos\theta, y\sin\theta)$. Let $g\colon \overline{B}_0 \times S^1 \to \mathbf{R}^3$ be an embedding. Then $g(\psi^{-1}(X_b))$ is also a 3-complex in $\mathbf{R}^3$. We call it a *coiled solid torus*. Let α be a c-symmetric 1-complex. Then we also call $g(\psi^{-1}(\alpha(b,c)))$ a 2-complex obtained from α for any integer b.

Let F be an embedded surface in $\mathbf{R}^4$ such that
(K0) F is the disjoint union of one Klein bottle and tori, or the disjoint union of tori,
(K1) $\Gamma^*(F)$ consists only of double points, and
(K2) each component of $\Gamma(F)$ is not homotopic to zero in F, and F^* is connected.

From Lemma 3.1, we have the following lemma.

Lemma 3.6. *Let F be as above. Then we have the following.*
(1) $F^* \setminus \Gamma^*(F)$ *consists of open annuli.*
(2) *Let C be a component of $\Gamma^*(F)$, and $N(C)$ a regular neighborhood of C in $\mathbf{R}^3$. Then $N(C) \cap F^*$ consists of two immersed annuli or two immersed Möbius bands.*

A curve C is an *A-curve* if $N(C) \cap F^*$ is two immersed annuli, and is an *M-curve* if $N(C) \cap F^*$ is two immersed Möbius bands.

In the case of classical knots, any knot diagram in $\mathbf{R}^2$ can be considered in the 2-sphere. Because, by ambient isotopies the bounded region of $\mathbf{R}^2 \setminus \{\text{a knot projection}\}$ can be changed. Similarly, without loss of generality we may consider that the projection of knotted surfaces is in the 3-sphere S^3. Here, we consider the 3-sphere S^3 as a one point compactification of $\mathbf{R}^3$. We discuss about a 2-complex which is the projection into $\mathbf{R}^3$ of an embedded surface in $\mathbf{R}^4$ satisfying (K0), (K1) and (K2). Note that the above 2-complex is called a 2-complex consisting of annuli in [14]. From now on, we assume that such a projection is in the 3-sphere S^3 in this section.

Lemma 3.7 ([16, Lemma 2.1]). *Let F be an embedded Klein bottle in $\mathbf{R}^4$ such that $\Gamma^*(F)$ consists only of one simple closed curve, and each component of $\Gamma(F)$ has a Möbius band neighborhood. Then there exists an odd integer b and an embedding $g\colon \overline{B}_0 \times S^1 \to S^3$ such that F^* can be moved to the* 2*-complex $g(\psi^{-1}(\alpha_1(b, 2)))$ by an ambient isotopy of S^3, where α_1 is the* 2*-symmetric* 1*-complex as shown in* Fig. 4 (1).

3.3. Good solid tori sequences. Let F be an embedded surface in $\mathbf{R}^4$ satisfying the conditions (K0), (K1) and (K2). Then $\Gamma^*(F)$ consists only of A-curves and at most one M-curve. Let $V_1, V_2, \dots, V_k$ be solid tori in S^3, and $\mathfrak{V} = \{V_1, V_2, \dots, V_k\}$. We say that $\mathfrak{V}$ is a *solid tori sequence for F^** if $\mathfrak{V}$ satisfies the following two conditions:
(i) $\partial V_i \subset F^*$ for all i.
(ii) If $i \neq j$, then $V_i \cap V_j = \partial V_i \cap \partial V_j$ is one simple closed curve, an annulus or empty.
Let X be a coiled solid torus, and $\mathfrak{V}$ as above. We say that $\mathfrak{V} \cup \{X\}$ is an *almost solid tori sequence for F^** if $\mathfrak{V} \cup \{X\}$ satisfies the above conditions (i), (ii), and
(iii) the intersection of X and $\overline{S^3 \setminus X}$ is contained in F^*, and
(iv) $X \cap V_i$ is one simple closed curve, an annulus or empty for all i.

EXAMPLE 3.8. Let α_3 be a c-symmetric 1-complex as shown in Fig. 4 (3), and let D_1, D_2 be 2-balls in $\overline{B}_0$ such that $D_1 \subset D_2$ and $\alpha_3 = \partial D_1 \cup \partial D_2$. For an integer b with $(b, c) = 1$, let $W_i = \{(x, y\cos\theta, y\sin\theta) \mid (x, y, 0) \in \overline{r}_{(b/c)\theta}(D_i), \theta \in [0, 2\pi]\}$. Then W_1, W_2 are the solid tori in S^3 with $W_1 \subset W_2$ and $\partial W_1 \cup \partial W_2 = \alpha_3(b, c)$. We see that $\{W_2\}$ is a solid tori sequence for the 2-complex $\alpha_3(b, c)$. Let $V_2 = \overline{S^3 \setminus W_2}$. Then V_2 is a solid torus, $\partial W_1 \cup \partial V_2 = \alpha_3(b, c)$, and $W_1 \cap V_2 = \partial W_1 \cap \partial V_2$ is one simple closed curve, say L. The set $\{W_1, V_2\}$ is a solid tori sequence for $\alpha_3(a, b)$. Let N be a

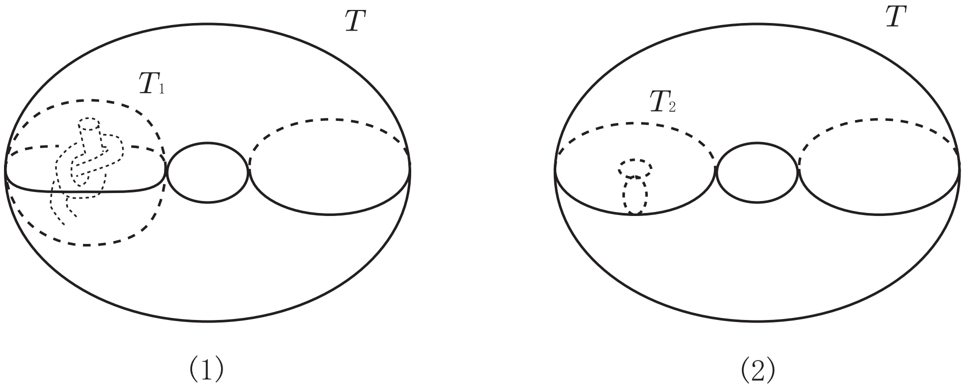


Fig. 5. (1) the 2-complex K_1 (2) the 2-complex K_2.

regular neighborhood of L in S^3. Note that if L is not a trivial knot, then $W_1 \cup V_2 \cup N$ is not a solid torus. Because, $W_1 \cup V_2 \cup N$ is homeomorphic to the complement of an open regular neighborhood of L.

Let F be an embedded surface in $\mathbf{R}^4$ satisfying (K0), (K1) and (K2). Let $\mathfrak{V} = \{V_1, V_2, \ldots, V_k\}$ be a solid tori sequence for the 2-complex F^*. Let c_i be a component of $\Gamma^*(F)$ with $c_i \subset \partial V_i$. Let n be the minimal number of intersection points of c_i and a meridian disk of the solid torus V_i. For the solid torus V_i we define $n(V_i)$ as follows:

$$n(V_i) = \begin{cases} n & \text{if } n \geq 1, \\ 0 & \text{if } n = 0,\ V_i \text{ is non-standard}, \\ \infty & \text{if } n = 0,\ V_i \text{ is standard}. \end{cases}$$

Here, a standard solid torus means a regular neighborhood of a trivial knot in S^3. We would like to distinguish standard and non-standard solid tori. Let T_1, T_2, T be tori in S^3 such that

- T bounds a standard solid torus V,
- T_1, $T_2 \subset V$,
- $T_i \cap T$ is a simple closed curve for $i = 1, 2$,
- T_1 bounds the complement of an open regular neighborhood of a trefoil knot in V, and
- T_2 bounds a solid torus V_2 in V so that V_2 has a 2-ball D in V with $D \cap V_2 = \partial D$.

See Fig. 5. For the torus T_1, there exists a solid torus V_3 with $\partial V_3 = T_1$. Let $K_i = T_i \cup T$ for $i = 1, 2$. Then $\{V\}$ is a solid tori sequence for K_i with $K_i \subset V$ and $n(V) = 0$, and $\{V_3\}$ is a solid torus sequence for K_1 with $n(V_3) = \infty$. However, K_1 is not a 2-complex $\alpha(b, c)$ obtained from any symmetric 1-complex α. If an embedded torus in $\mathbf{R}^4$ has such a projection K_1 into S^3, then by an ambient isotopy of $\mathbf{R}^4$ we can assume that its projection in S^3 is K_2. Let $W = \overline{S^3 \setminus V}$. Note that K_2 has a solid tori sequence

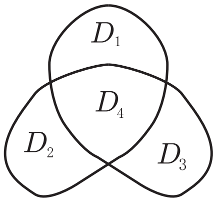


Fig. 6.

$\mathfrak{W} = \{V_2, W\}$ with $K_2 \subset \cup\mathfrak{W}$, $n(V_2) = 1$ and $n(W) = 1$. By Proposition 3.18, we see that K_2 is a 2-complex obtained from some symmetric 1-complex. In this paper we discuss about immersed Klein bottles. It is not important a solid torus V with $n(V) = 0$ or $n(V) = \infty$.

We construct the graph $G(\mathfrak{V})$ obtained by a solid tori sequence $\mathfrak{V}$ as follows. The vertices are in one to one correspondence with the solid tori $\{V_i\}$, and the edges are in one to one correspondence with the set $\{V_i \cap V_j \neq \emptyset\}$. If $V_i \cap V_j \neq \emptyset$, then we connect the vertices $v(V_i)$ and $v(V_j)$ by the edge e_{ij}.

DEFINITION 3.9. Let F be an embedded surface in $\mathbf{R}^4$ satisfying (K0), (K1) and (K2), and $\mathfrak{V} = \{V_1, V_2, \dots, V_k\}$ a solid tori sequence for the 2-complex F^*. A solid tori sequence $\mathfrak{V}$ is *good*, if $\mathfrak{V}$ satisfies the following four conditions:
(i) $G(\mathfrak{V})$ is a connected tree.
(ii) If B is an annulus with $B \subset F^*$ and if $(\cup\mathfrak{V}) \cap B = \partial B$, then $\partial B \subset \partial V_i$ for some i. Namely, for any annulus B in F^* with $\partial B \cap (\cup\mathfrak{V}) = \partial B$, the boundary of B is *not* contained in *different* two solid tori.
(iii) There exists a vertex $v(V_1)$ of $G(\mathfrak{V})$ such that if $V_i \neq V_1$ then $n(V_i) = 1$.
(iv) If $i \neq j$, then $V_i \cap V_j$ is either one simple closed curve or empty.
The vertex $v(V_1)$ is called the *special vertex*.

EXAMPLE 3.10. We give not good solid tori sequences as follows. Let M be the 1-complex in $\overline{B}_0$ as shown in Fig. 6, and let D_1, D_2, D_3, D_4 be the closures of the bounded components of $\overline{B}_0 \backslash M$ as shown in Fig. 6. We naturally embed the 2-complex $M \times S^1 \subset \overline{B}_0 \times S^1$ in S^3 via ψ.
(i) The solid tori sequence $\mathfrak{V}_1 = \{D_1 \times S^1, D_2 \times S^1, D_3 \times S^1\}$ is not a good solid tori for $M \times S^1$, because $G(\mathfrak{V}_1)$ is a circle.
(ii) Let A be the closure of a component of $M \setminus D_1 \cup D_2$. Then A is an arc in ∂D_3. The solid tori sequence $\mathfrak{V}_2 = \{D_1 \times S^1, D_2 \times S^1\}$ is not a good solid tori sequence for $M \times S^1$, because there exists the annulus $A \times S^1$ with $(\partial A \times S^1) \cap (\partial D_i \times S^1) \neq \emptyset$ for $i = 1, 2$.
(iii) Let L, $\alpha_3(b, c)$, W_1, V_2 be as in Example 3.8. Suppose that b, c are integers with $b > 1$ and $c > 1$. Then the knot L wraps b times in the longitudinal direction of W_1, and then L wraps c times in the longitudinal direction of V_2. Moreover, $n(W_1) = b$

and $n(V_2) = c$. Since $b > 1$ and $c > 1$, $\{W_1, V_2\}$ is not a good solid tori sequence for $\alpha_3(b, c)$.
However, there exist good solid tori sequences $\mathfrak{V}$ and $\mathfrak{W}$ for $M \times S^1$ and $\alpha_3(b, c)$, respectively, such that $\alpha_3(b, c) \subset (\cup\mathfrak{V})$ and $M \times S^1 \subset (\cup\mathfrak{W})$. In the case of $M \times S^1$, let $D = D_1 \cup D_2 \cup D_3 \cup D_4$, then $\mathfrak{V} = \{D \times S^1\}$ is a desired solid tori sequence. In the case of $\alpha_3(b, c)$, since V_2 is a standard solid torus, $W = \overline{S^3 \setminus V_2}$ is a solid torus with $W_1 \subset W$. Hence, $\mathfrak{W} = \{W\}$ is a desired solid tori sequence.

For a coiled solid torus X, we define $n(X) = 2$. For an almost solid tori sequence $\mathfrak{V}$, we construct the graph $G(\mathfrak{V})$ in a similar way as above.

DEFINITION 3.11. Let F be an embedded surface in $\mathbf{R}^4$ satisfying (K0), (K1) and (K2). Let X be a coiled solid torus, and $\mathfrak{V} = \{X, V_1, V_2, \dots, V_k\}$ an almost solid tori sequence for F^*. An almost solid tori sequence $\mathfrak{V}$ is *good*, if $\mathfrak{V}$ satisfies the following four conditions:
(i) $G(\mathfrak{V})$ is a connected tree.
(ii) If B is an annulus with $B \subset F^*$ and if $(\cup\mathfrak{V}) \cap B = \partial B$, then $\partial B \subset \partial V_i$ for some i or $\partial B \subset X \cap \overline{S^3 \setminus X}$.
(iii) $n(V_i) = 1$ for all solid tori V_i.
(iv) If $i \neq j$, then $V_i \cap V_j$ and $X \cap V_i$ are one simple closed curve or empty.
The vertex $v(X)$ is called the *special vertex*.

Let $\mathfrak{V} = \{V_1, \dots, V_k\}$ be a (almost) solid tori sequence. If $V_i \cap V_j$ is one simple closed curve, let N_{ij} be a regular neighborhood of $V_i \cap V_j$ in S^3. If $V_i \cap V_j = \emptyset$, let $N_{ij} = \emptyset$. If $V_i \cap V_j$ is an annulus, let $N_{ij} = V_i \cap V_j$. Then we say that $(\cup\mathfrak{V}) \cup (\cup N_{ij})$ is a *shape* of $\mathfrak{V}$.

Lemma 3.12 ([15, Lemma 3.4]). *Let $\{V_1, V_2\}$ be a solid tori sequence. Let V be a shape of $\mathfrak{V}$.*
(1) *If V is a solid torus, then $n(V_1) = 1$ or $n(V_2) = 1$.*
(2) *If V is not a solid torus, then $n(V_1) > 1$, $n(V_2) > 1$, and V_1, V_2 are standard solid tori in S^3.*
Here a standard solid torus means a regular neighborhood of a trivial knot in S^3.

Lemma 3.13. *Let $\{V_1\}$, $\{V_2\}$ be solid tori sequences such that $V_2 \subset V_1$, and $\partial V_1 \cap \partial V_2$ is one simple closed curve or an annulus. If $n(V_2)$ is not equal to* 0, 1, *and* ∞, *then $\partial V_1 \cup \partial V_2$ can be moved a* 2*-complex obtained from one of* Fig. 7 (1), (3) *by an ambient isotopy of S^3. Hence V_1 can be moved to V_2 by an ambient isotopy of S^3.*

Proof. In the case that $\partial V_1 \cap \partial V_2$ is an annulus, by [12, Lemma 2.1] the annulus $B = \overline{\operatorname{Int} V_1 \cap \partial V_2}$ is parallel to a boundary annulus in ∂V_2. The annulus B is decom-

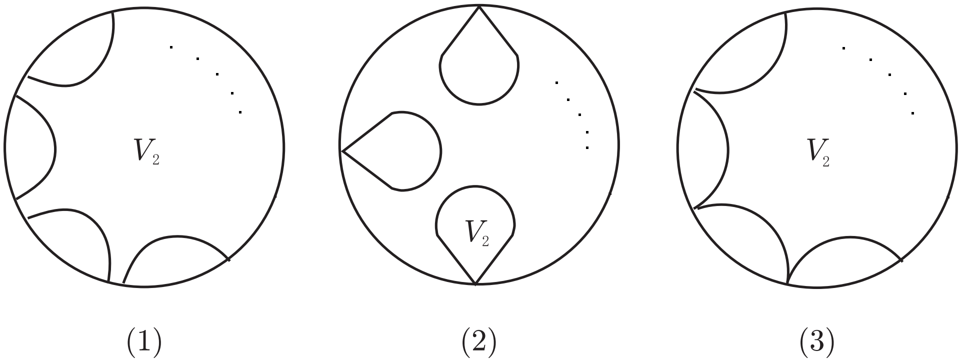


Fig. 7. Cutting a meridian disk.

posed V_1 into two solid tori V_2 and $\overline{V_2 \setminus V_1}$. Note that cutting a meridian disk of V_2, then we have Fig. 7 (1) which is the intersection of the meridian disk and ∂V_2. Since $n(V_2) \neq 0, 1, \infty$, V_1 can be moved to V_2 by an ambient isotopy of S^3.

In the case that $\partial V_1 \cap \partial V_2$ is one simple closed curve C, let N be a regular neighborhood of C in V_1. Let $K = \overline{(\partial V_1 \cup \partial V_2) \setminus N} \cup \overline{\mathrm{Int}\, V_1 \cap \partial N}$. Then, the solid tori sequence $\{\overline{V_1 \setminus N}, \overline{V_2 \setminus N}\}$ for K satisfies the above condition. Cutting a meridian disk of V_1, then we have Fig. 7 (2) or (3) which is the intersection of the meridian disk and ∂V_2. If $\partial V_2 \cap \partial V_1$ is a longitude curve of V_2, i.e., $n(V_2) = 1$, then we see Fig. 7 (2). We have that V_2 can be moved to V_1 by an ambient isotopy of S^3 if and only if we see Fig. 7 (3). Since $n(V_2) \neq 0, 1, \infty$, V_1 can be moved to V_2 by an ambient isotopy of S^3. $\square$

REMARK 3.14. Let F be an embedded Klein bottle in $\mathbf{R}^4$ satisfying (K1) and (K2). Let $\mathfrak{V}$ be a good almost solid tori sequence for F^*, C the M-curve in the coiled solid torus X. Let N be a regular neighborhood of C in S^3, $X' = X \cup N$, $K = (F^* \setminus N) \cup (\partial N \cap \partial X')$. Then X' is a solid torus, $\mathfrak{V}' = \{X'\} \cup (\mathfrak{V} \setminus \{X\})$ is a good solid tori sequence for K with $n(X') = 2$.

Lemma 3.15. *Let F be an embedded surface satisfying* (K0), (K1) *and* (K2). *Let $\mathfrak{V}$ be a good (almost) solid tori sequence for F^* such that $n(V_1) = 2$, where $v(V_1)$ is the special vertex. Let C be an A-curve in $\cup\mathfrak{V}$, and V a shape of $\mathfrak{V}$. Then V is a coiled solid torus if $\mathfrak{V}$ is almost, and V is a solid torus otherwise. Moreover,* $[C] = \pm 2 \in H_1(V)$.

Proof. In the case of a solid tori sequence, we showed in [14, Lemma 7.5]. So, we may assume that $\mathfrak{V}$ is almost. By Remark 3.14, $\mathfrak{V}$ can be changed a solid tori sequence. Given that N is a regular neighborhood of the M-curve in S^3, we have $V \cup N$ is a solid torus, and $[C] = \pm 2 \in H_1(V \cup N) \cong H_1(V)$. This and Lemma 3.13 imply that V can be moved to the coiled solid torus X in $\mathfrak{V}$ by an ambient isotopy of S^3. $\square$

Lemma 3.16. *Let F be an embedded surface satisfying* (K0), (K1) *and* (K2). *Let $\mathfrak{V}$ be a good* (*almost*) *solid tori sequence for F^* with $\cup\mathfrak{V} \not\supset F^*$, and $n(V_1) = 2$, where $v(V_1)$ is the special vertex. Then, there exists a solid torus V such that $\partial V \subset F^*$ and $\partial V \cap (\cup\mathfrak{V})$ is a simple closed curve or an annulus, $n(V) = 2$ if V contains V_1, and $n(V) = 1$ otherwise. Moreover, if the M-curve is a trivial knot in S^3, then there exists a coiled solid torus X with $X \cap (\overline{S^3 \setminus X}) \subset F^*$ such that X can be moved to the* 3*-complex X_b for some integer b of an ambient isotopy of S^3, where X_b is the set in* Definition 3.5. *In paticular, if $b = 0$ or -1, then we can take a solid torus V with* $n(V) = 1$.

Proof. By Remark 3.14, it suffices to prove for a solid tori sequence. Let $\mathfrak{V} = \{V_1, \dots, V_k\}$ be a good solid tori sequence. Since $\cup\mathfrak{V} \not\supset F^*$, by the definition of good, there exists a torus or an annulus, B, in F^* such that

$$B \cap (\cup\mathfrak{V}) = \begin{cases} \text{one simple closed curve}, & \text{if } B \text{ is a torus},\\ B \cap \partial V_i = \partial B, & \text{if } B \text{ is an annulus}. \end{cases}$$

By the solid torus theorem in [10], there exists a solid torus V with $B \subset \partial V \subset F^*$. We see that $\partial V \cap (\cup\mathfrak{V})$ is a simple closed curve or an annulus. Let C be a component of $\Gamma^*(F)$ in $\partial V \cap (\cup\mathfrak{V})$.

CASE 1. V contains V_1.

Let $\mathfrak{V}' = \{V_i \in \mathfrak{V} \mid V_i \subset V\}$. Then $\mathfrak{V}'$ is a good solid tori sequence for F^*. By Lemma 3.15, a shape V' of $\mathfrak{V}'$ is a solid torus and $V_1 \subset V'$. By $[C] = \pm 2 \in H_1(V')$ and Lemma 3.13, we can show that V' can be moved to V by an ambient isotopy of S^3. This implies $n(V) = 2$.

CASE 2. V does not contain V_1.

Let $\mathfrak{V}' = \{V_i \in \mathfrak{V} \mid V_i \not\subset V\}$, then $\mathfrak{V}'$ is a good solid tori sequence for F^*. By Lemma 3.15, a shape V' of $\mathfrak{V}'$ is a solid torus. Since $V' \cap V = \partial V' \cap \partial V$ is a simple closed curve or an annulus, by Lemma 3.12, $n(V) = 1$ or V is standard. If V is standard, then this case can be proved in a similar way to Case 1 by replacing V by $\overline{S^3 \setminus V}$. If $n(V) = 1$, then there is nothing to do.

Moreover, we assume that $\mathfrak{V}$ is a good almost solid tori sequence and the M-curve is a trivial knot in S^3. Then there exists a 2-complex $K \subset F^*$ such that K is a projection of an embedded Klein bottle satisfying (K1) and (K2), K contains only one M-curve and no A-curve. By Lemma 3.7, there exists a coiled solid torus X. Since the M-curve is a trivial knot, we can easily prove that X can be moved to the 3-complex X_b for some b of an ambient isotopy of S^3. Suppose that $b = 0$ or -1. In the case of $n(V) = 1$, there is nothing to do. Suppose $n(V) = 2$. Let $\mathfrak{V}' = \{V_i \in \mathfrak{V} \mid V_i \subset V\}$ and let V' be a shape of $\mathfrak{V}'$. Then $V' \subset V$, $\partial V' \cap \partial V$ is an annulus or a simple closed curve, and V' is the coiled solid torus by Lemma 3.13. So we may assume $V' = X_b$. Since the M-curve is a trivial knot, V is a standard solid torus. Let $W = \overline{S^3 \setminus V}$. Since $b = 0$ or -1, a simple closed curve of $\partial V' \cap \partial V$ is homologous to $\pm 2l \pm m \in H_1(\partial V)$, where m is a meridian curve of V, l is a preferred longitude of V. This implies $n(W) = 1$, and W is a desired solid torus. □

Proposition 3.17. *Let F be an embedded surface satisfying* (K0), (K1) *and* (K2). *Then there exists a good* (*almost*) *solid tori sequence $\mathfrak{V}$ for F^* with $\cup\mathfrak{V} \supset F^*$. Moreover, suppose that the M-curve in F^* is a trivial knot in S^3, and suppose that there exists a good almost solid tori sequence $\{X\}$ for F^* such that X can be moved to the* 3*-complexes X_0 or X_{-1} of an ambient isotopy of S^3. Then we can take that $\mathfrak{V}$ is almost.*

Proof. We only prove for the case that F^* contains an M-curve. There exists a good almost solid tori sequence $\{X\}$ for F^* such that X is maximal, i.e., X is not contained in another coiled solid torus. We prove by induction on the number of the components of $F^* \setminus \Gamma^*(F)$ in a good (almost) solid tori sequence. Let $\mathfrak{V}$ be a good (almost) solid tori sequence for F^*. If $\cup\mathfrak{V} \not\supset F^*$, then by Lemma 3.16 there exists a solid torus V satisfying the condition in Lemma 3.16. By Lemma 3.16, there exists only one solid torus $V_j \in \mathfrak{V}$ such that $(\cup\mathfrak{V}) \cap \partial V = V_j \cap \partial V$ is an annulus or a simple closed curve. Let $\tilde{V} = V \cup V_j$ if $V_j \cap \partial V$ is an annulus, let $\tilde{V} = V$ otherwise. Since $n(V_j) = 1$, $\tilde{V}$ is a solid torus. We have a good solid tori sequence $\mathfrak{W} = \{V_i \in \mathfrak{V} \mid V_i \not\subset \tilde{V}\} \cup \{\tilde{V}\}$ for F^* with $\cup\mathfrak{V} \subset \cup\mathfrak{W}$. In particular, if the M-curve is trivial, and if the coiled solid torus $X \in \mathfrak{V}$ can be moved to the 3-complexs X_0 or X_{-1} of an ambient isotopy of S^3, by Lemma 3.16, then $n(V) = 1$, and $\mathfrak{W}$ contains the coiled solid torus X. Inductively, this completes the proof of Proposition 3.17. □

Proposition 3.18. *Let F be an embedded surface satisfying* (K0), (K1) *and* (K2). *Let $\mathfrak{V}$ be a good* (*almost*) *solid tori sequence for F^* with $\cup\mathfrak{V} \supset F^*$, $n(V_1) \neq 0$ and $n(V_1) \neq \infty$, where $v(V_1)$ is the special vertex. Then F^* can be moved to a* 2*-complex obtained by a c-symmetirc* 1*-complex by an ambient isotopy of S^3, where $b = n(V_1)$,* $(b, c) = 1$. *In particular, if $\mathfrak{V}$ is almost, then $b = 2$.*

Proof. In the case that $\Gamma^*(F)$ consists only A-curves, we showed in [14, Proposition 7.15].

Assume that $\Gamma^*(F)$ contains one M-curve. Let C be the M-curve, N a regular neighborhood of C in S^3, $V = V_1 \cup N$ and $K = (F^* \setminus N) \cup (\partial N \cap \partial V)$. By Remark 3.14, $(\mathfrak{V}\setminus\{V_1\})\cup\{V\}$ is a good solid tori sequence for K. Since it is true for the case of only A-curves, we see that K is a 2-complex obtained from some symmetric 1-complex. Hence, F^* is also a 2-complex obtained from some symmetric 1-complex. □

4. Spun Klein bottles

Proposition 4.1. *Let F be an embedded Klein bottle in $\mathbf{R}^4$ such that $\Gamma^*(F)$ consists only of double points, and each component of $\Gamma(F)$ is not homotopic to zero in $\pi_1(F)$. Then F^* is the projection into $\mathbf{R}^3$ of a spun Klein bottle in $\mathbf{R}^4$. In particular, F is ambient isotopic to a* simple *spun Klein bottle in $\mathbf{R}^4$.*

Proof. By [16, Remark 1.5], the number of components of $\Gamma(F)$ is even. Hence, by Lemma 3.1, $(\Gamma(F), F)$ is homeomorphic to (Γ_1, F). We see that $\Gamma^*(F)$ consists only of A-curves and one M-curve. By Proposition 3.17, there exists a good (almost) solid tori sequence $\mathfrak{V}$ for F^* with $F^* \subset \cup\mathfrak{V}$. By Proposition 3.18 and Remark 3.4, there exists a spun Klein bottle $Kl^a(K)$ in $\mathbf{R}^4$ such that F is ambient isotopic to $Kl^a(K)$.

If $a \neq 0$ and $a \neq -1$, by Remark 2.1 (2), then we may assume that $a = 0$ or -1, and the M-curve of F^* is a trivial knot. Applying Proposition 3.18 again, we obtain a good *almost* solid tori sequence $\mathfrak{V}$ for F^* with $F^* \subset \cup\mathfrak{V}$. Hence F is simple. $\square$

5. Diagrams for embedded surfaces

For an embedded surface, we define a 'diagram' in $\mathbf{R}^3$. In classical knots, it is convenient to represent by a diagram, i.e., an immersed closed curve in the plane that has crossing information indicated at its double points. A 'diagram' for an embedded surface is like a diagram of classical knots.

Let $\varphi : F \to \mathbf{R}^3$ be an immersion of a closed surface F (possibly disconnected, non-orientable) such that the singular set of φ has only transverse double points; each component of its is a circle. Such a circle is called a *crossing circle*. A *diagram D* is an immersion of a union of 2-spheres and a Klein bottle with a mark at each crossing circle satisfying the two conditions:

(i) For any crossing circle C, let N be a regular neighborhood of C in $\mathbf{R}^3$. Then $N \cap \operatorname{Im} D$ consists of two annuli or two Möbius bands, say A_1, A_2.

(ii) One of A_1, A_2 is marked either by 'a' (for 'above') or by 'b' (for 'below').

We define that there is a mark 'a' on A_i if and only if there is a mark 'b' on A_j $(i \neq j)$.

We usually place a mark 'a' or 'b' on only one A_i. A surface A_i with mark 'a' (resp. 'b') is called an *a-tube* (resp. a *b-tube*). We define the associated embedded surface L_D of a diagram D by the following properties.

(i) $p(L_D) = \operatorname{Im} D$, where $p\colon \mathbf{R}^4 = \mathbf{R}^3 \times \mathbf{R} \to \mathbf{R}^3$ is the projection onto $\mathbf{R}^3$.

(ii) $L_D \cap (\mathbf{R}^3 \times \{0\}) = (\operatorname{Im} D \setminus \operatorname{Int}(\text{a-tubes in } D)) \times \{0\}$, and $L_D \subset \mathbf{R}^3 \times [0, \infty)$.

These conditions determine an embedded surface up to ambient isotopy.

The mark 'a' and 'b' are used in [6] and [7]. Yajima [19] uses an arrow. Giller [4, p. 629] uses '+' for our 'a'. Carter and Saito [2, 3] define a broken surface diagram.

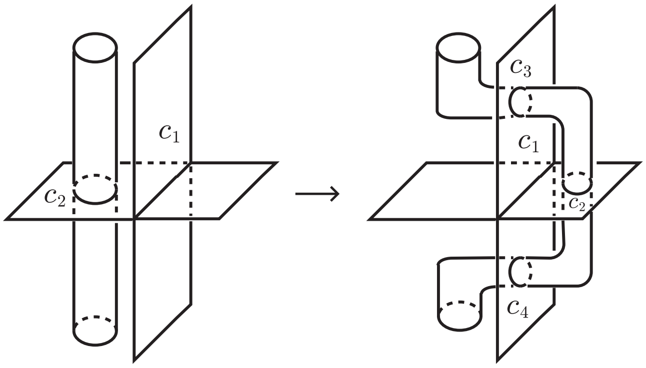


Fig. 8. Type (Ω1) move.

5.1. 1-handles for diagrams. In this subsection, we define a 1-handle for a diagram.

Let D be a diagram. Let $h_i\colon B^2 \times I \to \mathbf{R}^3$, $i = 1, 2, \ldots, m$, be a collection of embeddings with mutually disjoint images such that

$$h_i(B^2 \times I) \cap \operatorname{Im} D = h_i(B^2 \times \{0, t_1, \ldots, t_{i_k}, 1\})$$

for some $t_{i_1}, t_{i_2}, \ldots, t_{i_k}$ with $0 < t_{i_1} < t_{i_2} < \cdots < t_{i_k} < 1$, where B^2 is a 2-ball and $I = [0, 1]$. Define the immersed surface $D + \sum_{i=1}^m h_i$ to be

$$\left(\operatorname{Im} D \setminus \bigcup_{i=1}^{m} h_i(B^2 \times \partial I)\right) \cup \left(\bigcup_{i=1}^{m} h_i(\partial B^2 \times I)\right).$$

We call the embedding h_i *1-handle on the diagram D*, and the diagram with $D + \sum_{i=1}^m h_i$ a *diagram obtained from D by attaching 1-handles*. For a 1-handle h_i, we call the disks $h_i(B^2 \times 0)$ and $h_i(B^2 \times 1)$, *attaching disks*, the disk $h_i(B^2 \times t)$, $0 < t < 1$, a *cocore* of h_i, and the arc $h_i(x \times I)$, $x \in \operatorname{Int} B^2$, a *core*; see [7, Fig. 1].

5.2. Local moves. Local moves between diagrams are defined in [7]. They do not change the ambient isotopy classes of associated embedded surfaces of diagrams. Now, we define three of them.

(Ω1) Moving a 1-handle through a sheet as shown in Fig. 8, where $c_1, c_2 \in \{\mathrm{a}, \mathrm{b}\}$ and

$$c_3 = c_4 = \begin{cases} c_1 & \text{if } c_1 = c_2, \\ \text{either a or b} & \text{if } c_1 \neq c_2. \end{cases}$$

This move adds two crossing circles. (cf. Fig. 4 in [19])

(Ω2) Sliding a 1-handle through a sheet as shown in Fig. 9, where $c_1 = c_2 \in \{a, b\}$. This move adds one crossing circle.

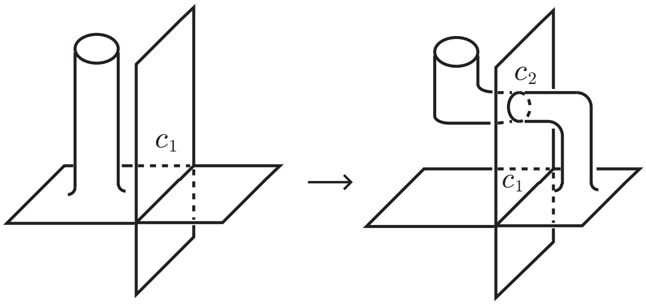


Fig. 9. Type (Ω2) move.

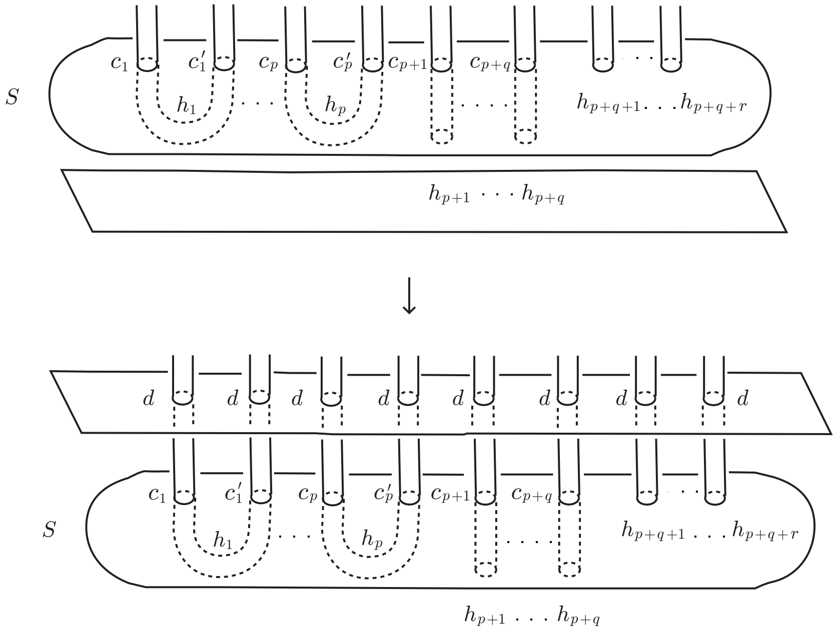


Fig. 10. Type (Ω6) move.

(Ω6) Pulling out a 2-sphere with 1-handles across a sheet as shown in Fig. 10, where S is a 2-sphere bounding a 3-ball B, and h_i, $1 \leq i \leq p + q + r$, are 1-handles such that

(i) $h_1, \ldots, h_p$ are passing through S,

(ii) $h_{p+1}, \ldots, h_{p+q+r}$ are attached on S whose one attaching disks are in S,

(iii) the pair $\left(B, B \cap \left(\bigcup_{i=1}^{p+q} \alpha_i\right)\right)$, where α_i is a core of h_i, is a trivial tangle, meaning that it is homeomorphic to the pair $(D^2, \{x_1, \ldots, x_{p+q}\}) \times [0, 1]$, where x_i are interior points of the 2-ball D^2, and

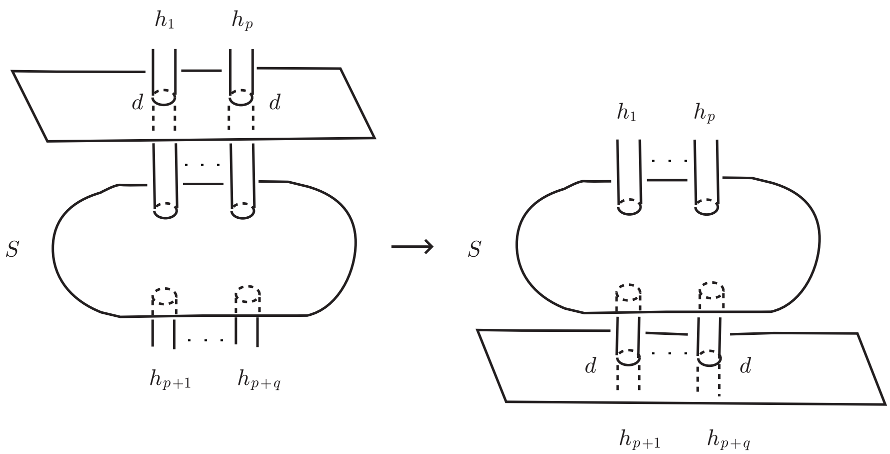


Fig. 11. Type $(\Omega 6)'$ move.

(iv) c_i, c'_j, $d \in \{\mathrm{a}, \mathrm{b}\}$, where $1 \leq i \leq p+q$, $1 \leq j \leq p$.

The following move is a generalization of the move $(\Omega 6)$.

$(\Omega 6)'$ Pulling out a 2-sphere S across a sheet as shown in Fig. 11, where S is bounding a 3-ball B, and h_i, $1 \leq i \leq p+q$, are 1-handles such that
(i) $h_1, \dots, h_{p+q}$ are passing through S or are attached on S, and
(ii) $d \in \{\mathrm{a}, \mathrm{b}\}$.
(cf. Lemma 4.6 in [19])

A diagram D is *with good position*, if it is obtained by attaching 1-handles from 2-spheres $S_1, \dots, S_m$ and an immersed Klein bottle K in $\mathbf{R}^3$ such that
(i) K is the projection of an embedded Klein bottle in $\mathbf{R}^3$ satisfying (K1) and (K2), and
(ii) there exist disjoint 3-balls $B_1, \dots, B_{m+1}$ in $\mathbf{R}^3$ with $S_i \subset \operatorname{Int} B_i$ and $K \subset \operatorname{Int} B_{m+1}$.
Observe that an associated surface as above is a Klein bottle obtained from a spun Klein bottle by m-fusion. Also, a diagram obtained by attaching 1-handles from only 2-spheres $S_1, \dots, S_m$ is called a diagram with good position. Observe that an associated surface of its diagram is a ribbon surface.

Proposition 5.1. *Any diagram can be transformed into a diagram with good position by a sequence of moves* $(\Omega 1)$, $(\Omega 2)$ *and* $(\Omega 6)'$.

Proof. First of all, we show that any diagram can be transformed into a diagram by attaching 1-handles from disjoint 2-spheres in $\mathbf{R}^3$, or a diagram by attaching 1-handles from disjoint 2-spheres and the projection of a spun Klein bottle. Let D be a diagram obtained from a diagram D_0 by attaching 1-handles $h_1, \dots, h_m$, where D_0 is the image of an immersion of a surface F. Let $R(D_0)$ be the components in the singular set of D_0 in $\mathbf{R}^3$ such that one of the preimage bounds a disk in F. We use induction on the number of the components in $R(D_0)$, say n.

In case of $n = 0$, i.e., $R(D_0) = \emptyset$, by Proposition 4.1, D_0 is disjoint 2-spheres in $\mathbf{R}^3$, or disjoint 2-spheres and the projection of a spun Klein bottle in $\mathbf{R}^3$. This implies the desired result.

Assume it is true for less than n, and the number of the components in $R(D_0)$ is n. Choose the disk E in D_0 such that ∂E is a component of $R(D_0)$, and E is a non-singular disk in $\mathbf{R}^3$. If E intersects a cocore of a 1-handle, perform the 1-handle by the move (Ω1) in Fig. 8. See the first move in Fig. 12. By the move (Ω1), two crossing circles appear, but the number of the components in $R(D_0)$ does not change. If E intersects an attaching disk of a 1-handle, then perform the 1-handle by the move (Ω2) in Fig. 9. See the second move in Fig. 12. Similarly, we see that the number of the components in $R(D_0)$ does not change. Hence, we may assume that E does not intersect 1-handles. A regular neighborhood of E in $\mathbf{R}^3$ consists of an annulus A and a disk E' containing E. By replacing the annulus A with two disks, each of which is parallel to E. Then we obtain a diagram D_1 such that D_0 is obtained from D_1 by attaching a 1-handle h such that $h(\partial B^2 \times I) = A$. Thus, D is obtained from D_1 by attaching 1-handles $h_1, \dots, h_m$, h. The number of the components of $R(D_1)$ is less than that of $R(D_0)$, which yields the result.

Next, we consider a diagram obtained by attaching 1-handles $h_1, \dots, h_n$ on 2-spheres $S_1, \dots, S_m$ and immersed Klein bottle K such that K is a 2-complex consisting of annuli. If the 2-spheres and K are contained in the interior of disjoint 3-balls, respectively, then the diagram is a desired diagram. Otherwise, take a 2-sphere, say S_i, such that S_i does not contain any other 2-sphere in $\mathbf{R}^3$. Let B, B_i be 3-balls in $\mathbf{R}^3$ such that the interior of B contains K, $\partial B \cap S_i = \emptyset$ for all i, and $\partial B_i = S_i$. If B_i does not contain K, by a sequence of the move (Ω6)$'$, then we pull out S_i from the 2-sphere that contains S_i. If not, by a sequence of the move (Ω6)$'$, then we pull K, and then we pull out S_i from the 2-sphere that contains S_i. Inductively, we have a diagram with good position. Similarly, we can prove for the case of a diagram obtained by attaching 1-handles on 2-spheres. $\square$

The technique in Proposition 5.1 was used in [7] and [19].

6. Proof of the main theorem

From Proposition 5.1, we have:

Theorem 6.1 (Theorem 1.1). *Let F be an embedded Klein bottle in $\mathbf{R}^4$. If $\Gamma^*(F)$ consists of double points, then F is ambient isotopic to either a ribbon Klein bottle, or a Klein bottle obtained from a spun Klein bottle by m-fusion.*

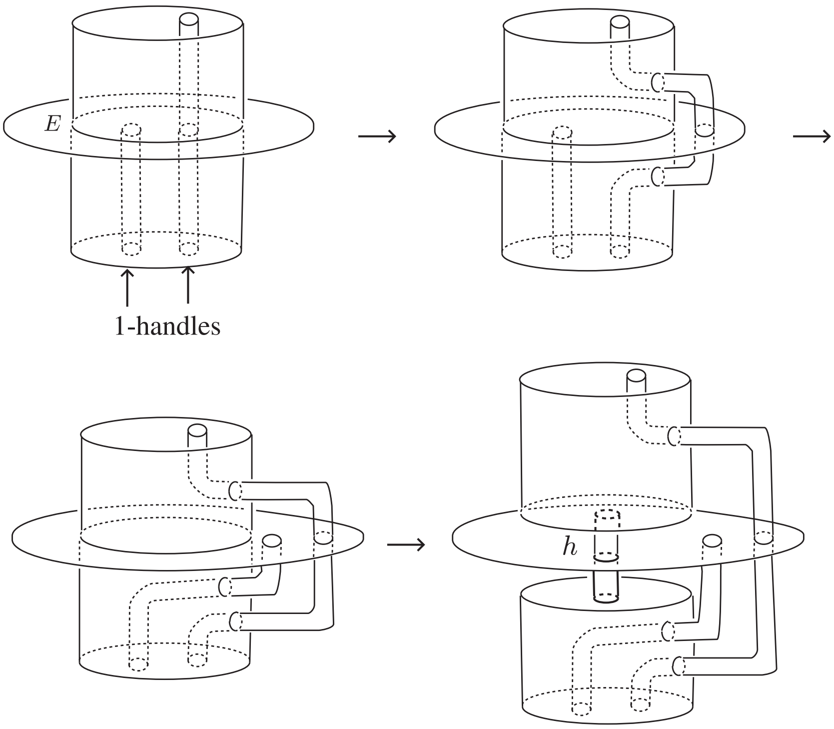


Fig. 12. A transformation for the case that E intersects one cocore and one attaching disk.

Lemma 6.2. *Let L be a knot in S^3. If $\pi_1(S^3 \setminus L)/\langle m^2 = 1\rangle$ is isomorphic to $\mathbf{Z}_2$, then L is trivial.*

Proof. Let N be a regular neighborhood of L in S^3, $E = \overline{S^3 \setminus N}$, E_2 the 2-fold cover, X_2 the 2-fold branch cover. Then we obtain the following exact sequences:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(E_2) \arrow[r, "\tilde{m}=m^2"'] \arrow[d, "\tilde{m}=1"'] & \pi_1(E) \arrow[r] \arrow[d, "m^2=1"'] & \mathbf{Z}_2 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1(X_2) \arrow[r] & \pi_1(E)/\langle m^2 = 1\rangle \arrow[r, "\cong"'] & \mathbf{Z}_2 \arrow[r] & 1
\end{tikzcd}$$

where m is a meridian curve of L. By the above diagram, we have $\pi_1(X_2) \cong 1$. By the Smith Conjecture [9], if $\pi_1(X_2) \cong 1$, then the branch set of X_2 is a trivial knot. And we can show that L is trivial. $\square$

Corollary 6.3 (Corollary 1.2). *Let F be an embedded Klein bottle in $\mathbf{R}^4$. Suppose that $\Gamma^*(F)$ consists only of double points, and all components of the singular set $\Gamma(F)$ are not homotopic to zero in $\pi_1(F)$. If $\pi_1(\mathbf{R}^4 \setminus F)$ is isomorphic to $\mathbf{Z}_2$, then F*

is trivial.

Proof. By assumption, F^* consists only of A-curves and one M-curve. By Proposition 4.1, F is ambient isotopic to a simple spun Klein bottle $Kl^a(L\#(-L))$. By Lemma 6.2 and Remark 2.1 (2), if the fundamental group of the complement of $Kl^a(L\#(-L))$ is isomorphic to $\mathbf{Z}_2$, then the knot L is trivial in S^3. Hence $Kl^a(L\#(-L))$ is ambient isotopic to a Klein bottle F' such that $\Gamma^*(F')$ consists only of one simple closed curve. Hence F' is a boundary of a solid Klein bottle in $\mathbf{R}^4$. Therefore F is trivial. □

6.1. Example of a non-ribbon surface. In [12], [13], and [14], we classified for an embedded torus T whose singular set $\Gamma^*(T)$ consists of at most three disjoint simple closed curves. The twist spun torus of the trefoil knot has the projection into $\mathbf{R}^3$ with the singular set consisting three disjoint simple closed curves. This example is given in [1] or [14].

Proposition 6.4. *The twist spun torus F is not a ribbon surface.*

Proof. Suppose that F is a ribbon surface. Let N be a regular neighborhood of F in $\mathbf{R}^4$. Boyle [1] defined the $\mathbf{Z}_2$-invariant q for a curve c in ∂N which is homologous to zero in $\overline{\mathbf{R}^4 \setminus N}$, this is modulo 2 to the intersection number of a surface with boundary c in $\overline{\mathbf{R}^4 \setminus N}$. Then, there exists a unique simple closed curve C on the boundary of N such that C is homotopic to zero in $\overline{\mathbf{R}^4 \setminus N}$. We see that $q(C) = 1$. However, a ribbon torus has a curve C' on ∂N such that C' is homotopic to zero in $\overline{\mathbf{R}^4 \setminus N}$, and $q(C') = 0$. This is a contradiction. Hence, F is not a ribbon surface. □

Question 6.5. For a trefoil knot L, is the spun Klein bottle $Kl^a(L\#(-L))$ a non-ribbon surface?

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Department of Mathematics
Tokai University
1117 Kitakaname, Hiratuka, Kanagawa 259–1292
Japan
e-mail: shima@keyaki.cc.u-tokai.ac.jp