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CONNECTIVE COVERINGS OF SPACES OF HOLOMORPHIC MAPS

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1. Introduction

For each integer $d \geq 0$, we denote by $\mathrm{Hol}_d(S^2, \mathbb{C}\mathrm{P}^n)$ the space consisting of all holomorphic maps $S^2 \to \mathbb{C}\mathrm{P}^n$ of degree d. The corresponding space of continuous maps is denoted by $\mathrm{Map}_d(S^2, \mathbb{C}\mathrm{P}^n)$. We also denote by $\mathrm{Hol}_d^*(S^2, \mathbb{C}\mathrm{P}^n)$ (resp. $\Omega_d^2\mathbb{C}\mathrm{P}^n$) the subspace of $\mathrm{Hol}_d(S^2, \mathbb{C}\mathrm{P}^n)$ (resp. $\mathrm{Map}_d(S^2, \mathbb{C}\mathrm{P}^n)$) consisting of all maps $f \in \mathrm{Hol}_d(S^2, \mathbb{C}\mathrm{P}^n)$ which preserve the base-points. The space of holomorphic maps are of interest both from a classical and modern point of view (e.g. [1], [3], [6]). It is an elementary and fundamental fact that $\mathrm{Hol}_d(S^2, \mathbb{C}\mathrm{P}^n)$ and $\mathrm{Hol}_d^*(S^2, \mathbb{C}\mathrm{P}^n)$ are connected spaces. If $n = 1$, the fundamental groups of these spaces are $\mathbb{Z}/2d$ and $\mathbb{Z}$, repectively ([7], [12]); if $n \geq 2$, these spaces are simply connected and $2(n-1)$-connected, respectively. The following more general result was obtained by G. Segal:

Theorem 1.1 ([12]). *If*

$$\begin{cases} i_d\colon \mathrm{Hol}_d(S^2, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d(S^2, \mathbb{C}\mathrm{P}^n) \\ \tilde{i}_d\colon \mathrm{Hol}_d^*(S^2, \mathbb{C}\mathrm{P}^n) \to \Omega_d^2\mathbb{C}\mathrm{P}^n \end{cases}$$

are inclusion maps, i_d and $\tilde{i}_d$ are homotopy equivalences up to dimension $D(d,n) = (2n-1)d$.

REMARK. The map $f\colon X \to Y$ is said to be *a homotopy equivalence up to dimension N* if $f_*\colon \pi_k(X) \to \pi_k(Y)$ is bijective when $k < N$ and surjective when $k = N$.

The principal motivation of this paper derives from the work of Segal ([12]), in which he describes the homotopy types of $\mathrm{Hol}_d(S^2, \mathbb{C}\mathrm{P}^n)$ and $\mathrm{Hol}_d^*(S^2, \mathbb{C}\mathrm{P}^n)$ from the point of view of the infinite dimensional Morse theoretical principle by using a technique of scanning maps ([8], [9], [12]). Now the homotopy types of $\mathrm{Hol}_d^*(S^2, \mathbb{C}\mathrm{P}^n)$ were studied well by several authors ([3], [9], [10]). So in this paper we shall study the homotopy types of $\mathrm{Hol}_d(S^2, \mathbb{C}\mathrm{P}^n)$. We identify $S^2 = \mathbb{C} \cup \infty$ and consider the evaluation fibration sequence $\mathrm{Hol}_d^*(S^2, \mathbb{C}\mathrm{P}^n) \xrightarrow{j_d} \mathrm{Hol}_d(S^2, \mathbb{C}\mathrm{P}^n) \xrightarrow{ev} \mathbb{C}\mathrm{P}^n$, where the map ev is given by $ev(f) = f(\infty)$ for $f \in \mathrm{Hol}_d(S^2, \mathbb{C}\mathrm{P}^n)$.

In this situation, we define the space $\tilde{\mathrm{Hol}}_d(S^2, \mathbb{C}\mathrm{P}^n)$ by

$$\tilde{\mathrm{Hol}}_d(S^2, \mathbb{C}\mathrm{P}^n) = \{(f, x) \in \mathrm{Hol}_d(S^2, \mathbb{C}\mathrm{P}^n) \times S^{2n+1} : ev(f) = h_n(x)\},$$

where $h_n \colon S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ denotes the Hopf fibering with fibre S^1.

There is the commutative diagram

$$\begin{array}{ccc} \tilde{\mathrm{Hol}}_d(S^2, \mathbb{C}\mathrm{P}^n) & \xrightarrow{\tilde{ev}} & S^{2n+1} \\ {\scriptstyle p_n}\downarrow & & {\scriptstyle h_n}\downarrow \\ \mathrm{Hol}_d(S^2, \mathbb{C}\mathrm{P}^n) & \xrightarrow{ev} & \mathbb{C}\mathrm{P}^n \end{array}$$

where vertical maps are fibrations. Recall the following result.

Theorem 1.2 ([14]). *Let $d \geq 1$ be an integer and let $\tilde{\mathrm{Hol}}_d^*$ denote the universal covering of $\mathrm{Hol}_d^*(S^2, S^2)$.*

(i) $\tilde{\mathrm{Hol}}_d(S^2, S^2)$ *is a universal covering of* $\mathrm{Hol}_d(S^2, S^2)$.

(ii) *There is a homotopy equivalence* $\tilde{\mathrm{Hol}}_d(S^2, S^2) \simeq \tilde{\mathrm{Hol}}_d^* \times S^3$.

(iii) *So, if $k \geq 2$, there is an isomorphism*

$\pi_k(\mathrm{Hol}_d(S^2, S^2)) \cong \pi_k(\mathrm{Hol}_d^*) \oplus \pi_k(S^3)$.

In particular, if $2 \leq k < d$, there is an isomorphism

$\pi_k(\mathrm{Hol}_d(S^2, S^2)) \cong \pi_{k+2}(S^2) \oplus \pi_k(S^3)$.

We would like to investigate the corresponding results for the case $n \geq 2$. In fact, the main purpose of this paper is to investigate whether a similar result holds or not. Our results are as follows:

Theorem 1.3. *Let $n \geq 2$ and $d \geq 1$ be integers.*

(i) $\tilde{\mathrm{Hol}}_d(S^2, \mathbb{C}\mathrm{P}^n)$ *is the 2-connective covering of* $\mathrm{Hol}_d(S^2, \mathbb{C}\mathrm{P}^n)$.

(ii) *There is a fibration sequence (up to homotopy)*

$$(*)_n \qquad \mathrm{Hol}_d^*(S^2, \mathbb{C}\mathrm{P}^n) \xrightarrow{\tilde{j}_d} \tilde{\mathrm{Hol}}_d(S^2, \mathbb{C}\mathrm{P}^n) \xrightarrow{\tilde{ev}} S^{2n+1}.$$

Moreover, the fibration $(*)_n$ has a section if and only if $n \equiv 1 \pmod 2$ or $n \equiv d \equiv 0 \pmod 2$.

Corollary 1.4. *Let $d \geq 1$ and $n \geq 2$ be integers such that $n \equiv 1 \pmod 2$ or $n \equiv d \equiv 0 \pmod 2$.*

(i) *If $k \geq 3$, there is an isomorphism*

$\pi_k(\mathrm{Hol}_d(S^2, \mathbb{C}\mathrm{P}^n)) \cong \pi_k(\mathrm{Hol}_d^*(S^2, \mathbb{C}\mathrm{P}^n)) \oplus \pi_k(S^{2n+1})$.

(ii) *In particular, if $3 \leq k < d$, there is an isomorphism*

$\pi_k(\mathrm{Hol}_d(S^2, \mathbb{C}\mathrm{P}^n)) \cong \pi_{k+2}(S^{2n+1}) \oplus \pi_k(S^{2n+1})$.

We shall also see that the fibration $(*)_n$ does not have a section if $n \equiv 0 \pmod 2$ and $d \equiv 1 \pmod 2$. However, we can prove the weaker version as follows.

Proposition 1.5. *Let $n \geq 2$ and $d \geq 1$ be integers and let A be an abelian group. Then there are isomorphisms of graded abelian groups and graded rings*:

$$\begin{cases} H_*(\tilde{\text{Hol}}_d(S^2, \mathbb{C}\text{P}^n), A) \cong H_*(\text{Hol}_d^*(S^2, \mathbb{C}\text{P}^n), A) \otimes H_*(S^{2n+1}, A), \\ H^*(\tilde{\text{Hol}}_d(S^2, \mathbb{C}\text{P}^n), A) \cong H^*(\text{Hol}_d^*(S^2, \mathbb{C}\text{P}^n), A) \otimes H^*(S^{2n+1}, A). \end{cases}$$

Finally, we shall study the case $d = 1$ carefully. In this case, we can determine the homotopy types of $\text{Hol}_1(S^2, \mathbb{C}\text{P}^n)$ and $\tilde{\text{Hol}}_1(S^2, \mathbb{C}\text{P}^n)$ explicitly. Let E_k be the $(k \times k)$-identity matrix and $\Delta_k \subset U_k$ be the center of U_k given by $\Delta_k = \{\alpha E_k : \alpha \in \mathbb{C}, |\alpha| = 1\}$.

For each pair of integers (k, m) with $1 \leq k \leq m$, let $W_{m,k}$ denote the complex Stiefel manifold of orthogonal k-frames in $\mathbb{C}^m$ defined by $W_{m,k} = U_m/U_{m-k}$. Similarly, let $X_{m,k}$ be the complex projective Stiefel manifold of orthogonal k-frames in $\mathbb{C}^m$ defined by $X_{m,k} = U_m/(\Delta_k \times U_{m-k}) \cong W_{m,k}/\mathbb{C}^*$.

Theorem 1.6. *If $n \geq 2$, there are homotopy euivalences*

$$\begin{cases} \phi_{1,n} : X_{n+1,2} \xrightarrow{\simeq} \text{Hol}_1(S^2, \mathbb{C}\text{P}^n), \\ \tilde{\phi}_{1,n} : W_{n+1,2} \xrightarrow{\simeq} \tilde{\text{Hol}}_1(S^2, \mathbb{C}\text{P}^n). \end{cases}$$

Corollary 1.7. *There are homotopy equivalences*

$$\begin{cases} \tilde{\text{Hol}}_1(S^2, \mathbb{C}\text{P}^2) \simeq SU_3, \\ \tilde{\text{Hol}}_1(S^2, \mathbb{C}\text{P}^3) \simeq S^5 \times S^7. \end{cases}$$

This paper is organized as follows. In Section 2, we shall show the existence of the fundamental fibration $(*)_n$ and prove Theorem 1.3. In Section 3, we shall compute the (co-)homology of $\tilde{\text{Hol}}_d(S^2, \mathbb{C}\text{P}^n)$ and prove Proposition 1.5. In Section 4, we shall investigate the homogenous space structures of $\tilde{\text{Hol}}_d(S^2, \mathbb{C}\text{P}^n)$ and $\text{Hol}_d(S^2, \mathbb{C}\text{P}^n)$ for the case $d = 1$.

2. The fundamental fibration

First, in this section, we shall prove the following result.

Proposition 2.1. *If $n \geq 2$, there are fibration sequences*

$$\begin{cases} (*)_n \qquad \text{Hol}_d^*(S^2, \mathbb{C}\text{P}^n) \xrightarrow{\tilde{j}_d} \tilde{\text{Hol}}_d(S^2, \mathbb{C}\text{P}^n) \xrightarrow{\tilde{ev}} S^{2n+1}, \\ (\dagger)_n \qquad \tilde{\text{Hol}}_d(S^2, \mathbb{C}\text{P}^n) \xrightarrow{p_n} \text{Hol}_d(S^2, \mathbb{C}\text{P}^n) \xrightarrow{\iota'} K(\mathbb{Z}, 2). \end{cases}$$

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} S^1 & \xrightarrow{=} & S^1 & & \\ \downarrow & & \downarrow & & \\ \tilde{\mathrm{Hol}}_d(S^2,\mathbb{C}\mathrm{P}^n) & \xrightarrow{\tilde{ev}} & S^{2n+1} & & \\ {\scriptstyle p_n}\downarrow & & {\scriptstyle h_n}\downarrow & & \\ \mathrm{Hol}_d(S^2,\mathbb{C}\mathrm{P}^n) & \xrightarrow{ev} & \mathbb{C}\mathrm{P}^n & \xrightarrow{\iota} & K(\mathbb{Z},2)=BS^1 \end{array}$$

where the map $\iota\colon \mathbb{C}\mathrm{P}^n \to BS^1$ represents the generator of the homotopy set $[\mathbb{C}\mathrm{P}^n, K(\mathbb{Z},2)] \cong H^2(\mathbb{C}\mathrm{P}^n,\mathbb{Z}) = \mathbb{Z}$.

If we consider the Serre spectral sequence of the evaluation fibration $\Omega^2 S^{2n+1} \simeq \Omega^2_d\mathbb{C}\mathrm{P}^n \to \mathrm{Map}_d(S^2,\mathbb{C}\mathrm{P}^n) \xrightarrow{ev'} \mathbb{C}\mathrm{P}^n$, it is easy to see that the induced homomorphism $(ev')^*\colon \mathbb{Z} = H^2(\mathbb{C}\mathrm{P}^n,\mathbb{Z}) \xrightarrow{\cong} H^2(\mathrm{Map}_d(S^2,\mathbb{C}\mathrm{P}^n),\mathbb{Z})$ is bijective. Hence there is a map $\iota'\colon \mathrm{Hol}_d(S^2,\mathbb{C}\mathrm{P}^n) \to K(\mathbb{Z},2)$ such that ι' represents the generator of $[\mathrm{Hol}_d(S^2,\mathbb{C}\mathrm{P}^n), K(\mathbb{Z},2)] \cong H^2(\mathrm{Hol}_d(S^2,\mathbb{C}\mathrm{P}^n),\mathbb{Z}) = \mathbb{Z}$ with $\iota\circ ev = \iota'$. Then it follows from [5, (2.1)] that there is a homotopy commutative diagram

$$\begin{array}{ccccc} * & \longrightarrow & S^1 & \xrightarrow{=} & S^1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hol}^*_d(S^2,\mathbb{C}\mathrm{P}^n) & \xrightarrow{\tilde{j}_d} & \tilde{\mathrm{Hol}}_d(S^2,\mathbb{C}\mathrm{P}^n) & \xrightarrow{\tilde{ev}} & S^{2n+1} \\ {\scriptstyle =}\downarrow & & {\scriptstyle p_n}\downarrow & & {\scriptstyle h_n}\downarrow \\ \mathrm{Hol}^*_d(S^2,\mathbb{C}\mathrm{P}^n) & \xrightarrow{j_d} & \mathrm{Hol}_d(S^2,\mathbb{C}\mathrm{P}^n) & \xrightarrow{ev} & \mathbb{C}\mathrm{P}^n \\ \downarrow & & {\scriptstyle \iota'}\downarrow & & {\scriptstyle \iota}\downarrow \\ * & \longrightarrow & K(\mathbb{Z},2) & \xrightarrow{=} & K(\mathbb{Z},2) \end{array}$$

where all horizontal and vertical sequences are fibration sequences. Hence we have the desired fibration sequences $(*)_n$ and $(\dagger)_n$. □

Corollary 2.2. *If $n \geq 2$, the space $\tilde{\mathrm{Hol}}_d(S^2,\mathbb{C}\mathrm{P}^n)$ is a 2-connective covering of* $\mathrm{Hol}_d(S^2,\mathbb{C}\mathrm{P}^n)$.

Proof. This follows from the diagram chasing of the above diagram. □

For a connected space X, let $\mathrm{Map}(S^n,X)$ (resp. $\mathrm{Map}^*(S^n,X)$) denote the space consisting of all (resp. basepoint preserving) continuous maps $f\colon S^n \to X$ with compact-open topology. For $f \in \mathrm{Map}(S^n,X)$, let $\mathrm{Map}_f(S^n,X)$ and $\mathrm{Map}^*_f(S^n,X)$ be the path-component of $\mathrm{Map}(S^n,X)$ or $\mathrm{Map}^*(S^n,X)$ containing the element f. Let us consider the evaluation fibration $ev\colon \mathrm{Map}(S^n, X) \to X$ with fibre $\mathrm{Map}^*_f(S^n, X)$, which is given by $ev(g) = g(s_0)$ for $g \in \mathrm{Map}(S^n, X)$ ($s_0 \in S^n$ is a fixed base point).

First, we recall the following two well-known results.

Lemma 2.3 ([13]). *Let $\partial\colon \pi_k(X) \to \pi_{k-1}(\mathrm{Map}^*_f(S^n, X))$ be the boundary operator of the evaluation fibration. If we identify $\pi_{k-1}(\mathrm{Map}^*_f(S^n, X)) \cong \pi_{k-1+n}(X)$, ∂ is identified with the operator $\partial'\colon \pi_k(X) \to \pi_{k-1+n}(X)$, which is defined by the Whitehead product $\partial'(\alpha) = [\alpha, f]$ for $\alpha \in \pi_k(X)$.*

Proposition 2.4 ([2]). *Let $n \geq 2$ be an integer and let $i_n\colon S^2 \to \mathbb{C}\mathrm{P}^n$ be the inclusion map of the bottom cell e^2 in $\mathbb{C}\mathrm{P}^n$. Then the following equality holds in $\pi_{2n+2}(\mathbb{C}\mathrm{P}^n) = \mathbb{Z}/2 \cdot h_n \circ \eta_{2n+1}$.*

$$[h_n, i_n] = \begin{cases} 0 & \text{if } n \equiv 1 \pmod 2 \\ h_n \circ \eta_{2n+1} \neq 0 & \text{if } n \equiv 0 \pmod 2, \end{cases}$$

where $\eta_2 \in \pi_3(S^2) \cong \mathbb{Z}$ denotes the Hopf map and we take $\eta_k = E^{k-2}\eta_2 \in \pi_{k+1}(S^k) = \mathbb{Z}/2 \cdot \eta_k$ for $k \geq 3$.

Proposition 2.5. *Let $n \geq 2$ and $d \geq 1$ be integers.*
(i) *If $n \equiv 1 \pmod 2$ or $n \equiv d \equiv 0 \pmod 2$, there is a map $s_n\colon S^{2n+1} \to \tilde{\mathrm{Hol}}_d(S^2, \mathbb{C}\mathrm{P}^n)$ which is a section of $(*)_n$ with $\tilde{ev} \circ s_n = \iota_{2n+1}$, where $\iota_k \in \pi_k(S^k)$ denotes the identity map of S^k.*
(ii) *Moreover, if $n \equiv 0 \pmod 2$ and $d \equiv 1 \pmod 2$, there exists no section of $(*)_n$.*

Proof. (i) It is sufficient to show that the induced homomorphism

$$\tilde{ev}_*\colon \pi_{2n+1}(\tilde{\mathrm{Hol}}_d(S^2, \mathbb{C}\mathrm{P}^n)) \to \pi_{2n+1}(S^{2n+1}) = \mathbb{Z} \cdot \iota_{2n+1}$$

is surjective only when $n \equiv 1 \pmod 2$ or $n \equiv d \equiv 0 \pmod 2$. Consider the homotopy exact sequence

$$\begin{array}{ccccc} \pi_{2n+1}(\tilde{\mathrm{Hol}}_d(S^2, \mathbb{C}\mathrm{P}^n)) & \xrightarrow{\tilde{ev}_*} & \pi_{2n+1}(S^{2n+1}) & \xrightarrow{\partial} & \pi_{2n}(\mathrm{Hol}^*_d(S^2, \mathbb{C}\mathrm{P}^n)) \\ & & {\scriptstyle (h_n)_*}\downarrow\cong & & {\scriptstyle \tilde{i}_{d*}}\downarrow\cong \\ & & \pi_{2n+1}(\mathbb{C}\mathrm{P}^n) & \xrightarrow{\partial'} & \pi_{2n+2}(\mathbb{C}\mathrm{P}^n) \cong \pi_{2n}(\Omega^2_d\mathbb{C}\mathrm{P}^n) \end{array}$$

First, assume $n \equiv 1 \pmod 2$. It follows from Lemma 2.3 that the boundary homomorphism ∂' is given by $\partial'(f) = d[f, i_n]$. Because $\pi_{2n+1}(\mathbb{C}\mathrm{P}^n) = \mathbb{Z} \cdot h_n$ and $[h_n, i_n] = 0$ (by Proposition 2.4), ∂' is trivial. Hence $\tilde{ev}_*$ is surjective.

Next, assume $n \equiv d \equiv 0 \pmod 2$. Then, because the order of $[h_n, i_n]$ is two, $d[h_n, i_n] = 0$. Hence $\partial'(m \cdot h_n) = md[h_n, i_n] = 0$ and ∂' is trivial. So $\tilde{ev}_*$ is also surjective.

(ii) Finally, we assume $n \equiv 0 \pmod 2$ and that $d \equiv 1 \pmod 2$. Then using Proposition 2.4 as above, we can easily see that $\tilde{ev}_*$ is not surjective. □

Corollary 2.6. *If $d = 1$ and $n \geq 2$, there is a fibration sequence (up to homotopy),*

$$(**)_n \qquad S^{2n-1} \to \tilde{\mathrm{Hol}}_1(S^2, \mathbb{C}\mathrm{P}^n) \to S^{2n+1}.$$

In particular, $(**)_n$ has a section if and only if $n \equiv 1 \pmod 2$.

Proof. Since $\mathrm{Hol}_1^*(S^2, \mathbb{C}\mathrm{P}^n) \simeq S^{2n-1}$ ([3]), the assertion easily follows from the fibration sequence $(*)_n$ and Proposition 2.5. □

Now we can give the proofs of Theorem 1.3 and Corollary 1.4.

Proof of Theorem 1.3. The assertions (i), (ii) follow from Proposition 2.1, Corollary 2.2 and Proposition 2.5. □

Proof of Corollary 1.4. This also easily follows from Theorem 1.1, Proposition 2.1, Corollary 2.2 and Proposition 2.5. □

3. Homology of $\tilde{\mathrm{Hol}}_d(S^2, \mathbb{C}\mathrm{P}^n)$

In this section, we shall prove Proposition 1.5. Recall the following result.

Lemma 3.1 ([4]). (i) $H_*(\Omega^2 S^{2n+1}, \mathbb{Z}/2) \cong \otimes_{i\geq 1}\mathbb{Z}/2[x_{2^i n-1}] = \mathbb{Z}/2[x_{2n-1}, x_{4n-1}, x_{8n-1}, \ldots]$, *where x_k has degree k with $\beta(x_{2^{i+1}n-1}) = (x_{2^i n-1})^2$ for $i \geq 1$.*
(ii) *If $p \geq 3$ is an odd prime integer,*
$H_*(\Omega^2 S^{2n+1}, \mathbb{Z}/p) \cong E[x_{2n-1}] \otimes (\otimes_{i\geq 1} E[x_{2np^i-1}] \otimes \mathbb{Z}/p[x_{2np^i-2}])$,
where x_k has degree k with $\beta(x_{2np^i-1}) = x_{2np^i-2}$ for $i \geq 1$.

Proof of Proposition 1.5. Since the proof is similar, we only show the existence of the first isomorphism. It follows from the universal coeffcient theorem that it sufiices to show that there is an isomorphism

$$(\ddagger) \qquad H_*(\tilde{\mathrm{Hol}}_d(S^2, \mathbb{C}\mathrm{P}^n), A) \cong H_*(\tilde{\mathrm{Hol}}_d^*(S^2, \mathbb{C}\mathrm{P}^n), A) \otimes H_*(S^{2n+1}, A)$$

for $A = \mathbb{Q}$ or $\mathbb{Z}/p$ (p: any prime integer).

Because the proof of the case $A = \mathbb{Q}$ is easier, we shall show (‡) for $A = \mathbb{Z}/p$ (p: any prime). Consider the Serre spectral sequence of the fibration $(*)_n$,

$$E^2_{p,q} = H_p(S^{2n+1}, A) \otimes H_q(\mathrm{Hol}_d^*(S^2, \mathbb{C}\mathrm{P}^n), A) \Rightarrow H_{p+q}(\tilde{\mathrm{Hol}}_d(S^2, \mathbb{C}\mathrm{P}^n), A).$$

Since $\Omega^2_d \mathbb{C}\mathrm{P}^n \simeq \Omega^2 S^{2n+1}$, it follows from Theorem 1.1 that there is an isomorphism

$$H_k(\mathrm{Hol}^*_d(S^2, \mathbb{C}\mathrm{P}^n), A) \cong H_k(\Omega^2 S^{2n+1}, A) \qquad \text{for any } k < (2n-1)d.$$

Hence, if $d \geq 2$, $E^2_{*,2n} = 0$ by Lemma 3.1. If $d = 1$, it follows from $\mathrm{Hol}^*_1(S^2, \mathbb{C}\mathrm{P}^{n+1}) \simeq S^{2n-1}$ ([3]) that $E^2_{*,2n} = 0$. Because $E^2_{*,2n} = 0$ for any $d \geq 1$, $E^2_{**} = E^\infty_{**}$ and the assertion (‡) follows. □

4. The homogenous space structure

For each pair of integers (k, n) with $1 \leq k \leq n$ and an integer $d \geq 1$, let $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ denote the space consisting of all holomorphic maps $f\colon \mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^n$ of degree d. Now we shall study the case $d = 1$ carefully.

Let $1 \leq k \leq n$ be integers and consider the right U_{n+1}-action on $\mathbb{C}\mathrm{P}^n$ induced by matrix multiplication.

Define the map $\phi'_{k,n}\colon U_{n+1} \to \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ by

$$\phi'_{k,n}(A)([z_0 : z_1 : \cdots : z_k]) = [z_0 : z_1 : \cdots : z_k : 0 : 0 : \cdots : 0 : 0]A$$

for $([z_0 : z_1 : \cdots : z_k], A) \in \mathbb{C}\mathrm{P}^k \times U_{n+1}$.

Since two subgroups $U_{n-k} \subset \Delta_{k+1} \times U_{n-k} \subset U_{n+1}$ are fixed by this map, the map $\phi'_{k,n}$ induces the maps

$$\begin{cases} \phi''_{k,n}\colon W_{n+1,k+1} = U_{n+1}/U_{n-k} \to \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \\ \phi_{k,n}\colon X_{n+1,k+1} = U_{n+1}/(\Delta_k \times U_{n-k}) \to \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \end{cases}$$

such that the diagram

$$\begin{array}{ccc} U_{n+1} & \xrightarrow{\phi'_{k,n}} & \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \\ \downarrow & & \downarrow = \\ W_{n+1,k+1} = U_{n+1}/U_{n-k} & \xrightarrow{\phi''_{k,n}} & \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \\ \downarrow & & \downarrow = \\ X_{n+1,k+1} = U_{n+1}/(\Delta_{k+1} \times U_{n-k}) & \xrightarrow{\phi_{k,n}} & \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \end{array}$$

is commutative, where the left vertical maps are natural projections.

Now, we identify $S^2 = \mathbb{C}\mathrm{P}^1$ and consider the case $k = 1$. Recall the fibration $(\dagger)_n\colon \tilde{\mathrm{Hol}}_1(S^2, \mathbb{C}\mathrm{P}^n) \xrightarrow{p_n} \mathrm{Hol}_1(S^2, \mathbb{C}\mathrm{P}^n) \xrightarrow{\iota'} K(\mathbb{Z}, 2)$. Since $\iota' \circ \phi''_{1,n}$ is contained in $[W_{n+1,2}, K(\mathbb{Z}, 2)] \cong H^2(W_{n+1,2}, \mathbb{Z}) = 0$, $\iota' \circ \phi''_{1,n}$ is null-homotopic. Hence there is a lifting $\tilde{\phi}_{1,n}\colon W_{n+1,2} \to \tilde{\mathrm{Hol}}_1(S^2, \mathbb{C}\mathrm{P}^n)$ such that $p_n \circ \tilde{\phi}_{1,n} = \phi''_{1,n}$ (up to homotopy),

$$\begin{array}{ccccc}
W_{n+1,2} & \xrightarrow{=} & W_{n+1,2} & & \\
{\scriptstyle \tilde{\phi}_{1,n}}\downarrow & & {\scriptstyle \phi''_{1,n}}\downarrow & & \\
\tilde{\mathrm{Hol}}_1(S^2, \mathbb{C}\mathrm{P}^n) & \xrightarrow{p_n} & \mathrm{Hol}_1(S^2, \mathbb{C}\mathrm{P}^n) & \xrightarrow{\iota'} & K(\mathbb{Z}, 2).
\end{array}$$

REMARK. Because $H^2(X_{n+1,2}, \mathbb{Z}) \neq 0$, there is no lifting of $\phi_{1,n}$ to the space $\tilde{\mathrm{Hol}}_1(S^2, \mathbb{C}\mathrm{P}^n)$.

Lemma 4.1. *The diagram*

$$\begin{array}{ccc}
W_{n+1,2} & \xrightarrow{\tilde{\phi}_{1,n}} & \tilde{\mathrm{Hol}}_1(S^2, \mathbb{C}\mathrm{P}^n) \\
{\scriptstyle q_1}\downarrow & & {\scriptstyle \tilde{ev}}\downarrow \\
U_{n+1}/U_n & \underset{\simeq}{\xrightarrow{\beta_1}} & S^{2n+1}
\end{array}$$

is commutative up to homotopy, where q_1 and β_1 denote the natural projection and natural homeomorphism, respectively.

Proof. Using $h_n \circ \tilde{ev} = ev \circ p_n$ and the direct computation, we have $h_n \circ \beta_1 \circ q_1 = h_n \circ \tilde{ev} \circ \tilde{\phi}_{1,n}$ (up to homotopy). Moreover, because the sequence

$$\{*\} = [W_{n+1,2}, S^1] \to [W_{n+1,2}, S^{2n+1}] \xrightarrow{(h_n)_*} [W_{n+1,2}, \mathbb{C}\mathrm{P}^n]$$

is exact as a pointed set, $(h_n)_*$ is injective. Hence, $\beta_1 \circ q_1 = \tilde{ev} \circ \tilde{\phi}_{1,n}$ (up to homotopy). □

Proof of Theorm 1.6. We assume that $n \geq 2$ and we shall show that the two maps $\phi_{1,n}$ and $\tilde{\phi}_{1,n}$ are homotopy equivalences.

First, consider the map $\tilde{\phi}_{1,n}$. It follows from Lemma 4.1 that there is a homotopy commutative diagram

$$\begin{array}{ccc}
U_n/U_{n-1} & \underset{\simeq}{\xrightarrow{\beta_2}} & S^{2n-1} \\
{\scriptstyle q_2}\downarrow & & \downarrow \\
W_{n+1,2} = U_{n+1}/U_{n-1} & \xrightarrow{\tilde{\phi}_{1,n}} & \tilde{\mathrm{Hol}}_1(S^2, \mathbb{C}\mathrm{P}^n) \\
{\scriptstyle q_1}\downarrow & & {\scriptstyle \tilde{ev}}\downarrow \\
U_{n+1}/U_n & \underset{\simeq}{\xrightarrow{\beta_1}} & S^{2n+1}
\end{array}$$

where vertical sequences are fibrations, and q_2, β_2 denote natural projection and natural homeomorphism, respectively. Then it follows from the homotopy exact sequences of the fibrations that $\tilde{\phi}_{1,n}$ is a homotopy equivalence.

Next, we shall show that $\phi_{1,n}$ is a homotopy equivalence. Similarly as above, there is a homotopy commutative diagram

$$\begin{array}{ccc}
\Delta_2 & \xrightarrow[\simeq]{\beta} & S^1 \\
\downarrow & & \downarrow \\
W_{n+1,2} = U_{n+1}/U_{n-1} & \xrightarrow[\simeq]{\tilde{\phi}_{1,n}} & \widetilde{\mathrm{Hol}}_1(S^2, \mathbb{C}\mathrm{P}^n) \\
\downarrow & & \downarrow{\scriptstyle p_n} \\
X_{n+1,2} = U_{n+1}/(\Delta_2 \times U_{n-1}) & \xrightarrow{\phi_{1,n}} & \mathrm{Hol}_1(S^2, \mathbb{C}\mathrm{P}^n)
\end{array}$$

where vertical sequences are fibrations and β is a homeomorphism. Hence $\phi_{1,n}$ is also a homotopy equivalence. □

Proof of Corollary 1.7. Since $W_{3,2} \simeq SU_3$ and $W_{4,2} \simeq S^5 \times S^7$, the assertion easily follows. □

REMARK. In a subsequent paper, we would like to study the map $\phi_{k,n}$ and investigate the homotopy type of $\mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ explicitly for the case $2 \le k \le n$. In fact, we shall prove that $\phi_{k,n}$ is a homotopy equivalence for any $1 \le k \le n$ in [11].

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