

## VARIETIES FOR MODULES OVER A BLOCK OF A FINITE GROUP

Dedicated to Professor Yukio Tsushima on his Sixtieth Birthday

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### 1. Introduction

Let  $p$  be a prime number and let  $k$  be an algebraically closed field of characteristic  $p$ . Let  $G$  be a finite group and  $b$  a block of  $kG$ . In [10], M. Linckelmann introduces two notions: the *transfer maps* in Hochschild cohomology of symmetric algebras, and the *cohomology ring of the block  $b$* . Using the transfer map, he shows that the cohomology ring  $H^*(G, b, D_\gamma)$  of the block  $b$  is embedded into the Hochschild cohomology ring  $HH^*(kGb)$  of the block algebra  $kGb$ , which gives a block version of the well-known embedding from the usual cohomology ring  $H^*(G, k)$  into the Hochschild cohomology ring  $HH^*(kG)$  (see [10, Proposition 4.5]). Around the same time, M. Linckelmann [11] introduces a cohomological variety which is a block version of Carlson's module variety [4]. As is well known, for a finitely generated  $kG$ -module  $U$ , Carlson's variety  $V_G(U)$  is defined to be the maximal ideal spectrum of  $H^*(G, k)/I_G^*(U)$ , where  $I_G^*(U)$  is the annihilator of the action of  $H^*(G, k)$  on  $\text{Ext}_{kG}^*(U, U)$  induced by the cup product. On the other hand, for a bounded complex  $U$  of  $kGb$ -modules, Linckelmann's variety  $V_{G,b}(U)$  is defined to be the maximal ideal spectrum of  $H^*(G, b, D_\gamma)/I_{G,b,D_\gamma}^*(U)$ , where  $I_{G,b,D_\gamma}^*(U)$  is the kernel of the algebra homomorphism  $H^*(G, b, D_\gamma) \rightarrow \text{Ext}_{kGb}^*(U, U)$  induced from the above embedding  $H^*(G, b, D_\gamma) \rightarrow HH^*(kGb)$  (see Section 2 for details).

In this paper we study the variety  $V_{G,b}(U)$ . The following is the main result, which gives a characterization of the ideal  $I_{G,b,D_\gamma}^*(U)$  in terms of the cohomology ring of a defect group  $D$  of  $b$ .

**Theorem 1.1.** *Let  $b$  be a block of  $kG$ ,  $D_\gamma$  a defect pointed group of  $b$  and let  $i \in \gamma$  be a source idempotent of  $b$ . Let  $U$  be a bounded complex of  $kGb$ -modules. Then we have that*

$$I_{G,b,D_\gamma}^*(U) = H^*(G, b, D_\gamma) \cap I_D^*(iU).$$

*In particular if  $b$  is a nilpotent block of  $kG$ , then we have that  $V_{G,b}(U) = V_D(iU)$ .*

In [11], M. Linckelmann shows that for any bounded complex  $U$  of  $kGb$ -modules, there exists a finite surjective map  $V_{G,b}(U) \rightarrow V_G(U)$  (see Theorem 2.4 below). From Theorem 1.1, we can show the following reverse version of this result.

**Corollary 1.2.** *For any bounded complex  $U$  of  $kGb$ -modules, there exists a finite surjective map  $\iota_{D,b}: V_D(iU) \rightarrow V_{G,b}(U)$ . Further, the dimensions of  $V_D(iU)$  and  $V_{G,b}(U)$  coincide.*

In [11], Linckelmann also shows that the varieties  $V_{G,b}(U)$  are invariant under splendid stable and derived equivalences in the sense of Linckelmann (for the definition of these notions, see [12]). Applying Theorem 1.1 we consider the question whether the varieties  $V_{G,b}(U)$  are invariant under the Brauer correspondence between blocks  $b$  in  $kG$  with defect group  $D$  and blocks  $b_0$  in  $kN_G(D)$  with defect group  $D$ . As a partial answer, we have the following.

**Theorem 1.3.** *Suppose that  $D$  is abelian. Let  $U$  be an indecomposable  $kGb$ -module with vertex  $D$  and let  $W$  be the Green correspondent of  $U$  with respect to  $(G, D, N_G(D))$ . Then we have that  $V_{G,b}(U) = V_{N_G(D),b_0}(W)$ .*

Let  $Z$  be a central  $p$ -subgroup of  $G$  and set  $G/Z = \bar{G}$ . Then the natural  $k$ -algebra homomorphism  $kG \rightarrow k\bar{G}$  gives a one to one correspondence from all blocks  $b$  of  $kG$  onto all blocks  $\bar{b}$  of  $k\bar{G}$ . Applying Theorem 1.1 again, we have the following.

**Theorem 1.4.** *Let  $U$  be a bounded complex of  $k\bar{G}\bar{b}$ -modules. Regarding  $U$  as a complex of  $kGb$ -modules through  $kGb \rightarrow k\bar{G}\bar{b}$ , we can define an affine map  $\text{inf}_{\bar{b},b}^*: V_{G,b}(U) \rightarrow V_{\bar{G},\bar{b}}(U)$  induced from the inflation  $\text{inf}_{\bar{b},b}$ .*

Finally, we consider the inverse images of the affine maps above also. See Propositions 5.2 and 5.3 below.

All modules considered in this paper are assumed to be finitely generated left modules, unless otherwise stated. We end the introduction with fixing notations. Let  $H$  be a subgroup of a finite group  $G$ . For a group algebra  $RG$  over a commutative ring  $R$ ,  $(RG)_H$  denotes the regular  $RG - RH$ -bimodule and, similarly,  ${}_H(RG)$  denotes the regular  $RH - RG$ -bimodule. For the conjugation given by an element  $x$  of  $G$ , we use the left notation:  ${}^xH = xHx^{-1}$  and  ${}^xa = xax^{-1}$  for  $a \in RG$ . Further,  $c_x: H^*(H, k) \rightarrow H^*({}^xH, k)$  denotes the conjugation map given by  $x$ . For complexes  $X$  and  $Y$ ,  $X | Y$  means that  $X$  is isomorphic to a direct summand of  $Y$ .

## 2. Preliminaries

Let  $A$  and  $B$  be algebras over a commutative ring  $R$ . By an  $A - B$ -bimodule, we mean a bimodule on which the left and right actions of  $R$  coincide, in other words,

an  $A \otimes_R B^o$ -module ( $B^o$  is the  $R$ -algebra opposite to  $B$ ). For an  $R$ -module  $M$  and a complex  $C$  of  $R$ -modules, we set  $M^* = \text{Hom}_R(M, R)$  and  $C^* = \text{Hom}_R(C, R)$ , their  $R$ -duals. Let  $U$  be a right bounded complex of  $A$ -modules. We denote by  $\mathcal{P}(U)$  a projective resolution of  $U$ , that is,  $\mathcal{P}(U)$  is a right bounded complex of projective  $A$ -modules with a map of chain complexes  $\mathcal{P}(U) \rightarrow U$  which is an isomorphism on homology. Consider  $A$  as an  $A \otimes_R A^o$ -module (i.e.,  $A - A$ -bimodule). Further we also consider  $A$  as a complex with the degree 0 component  $A$  and all other components being zero. Then we denote by  $\mathcal{P}_A$  a projective resolution of the complex  $A$ . Let  $U$  and  $V$  be right bounded complexes of  $A$ -modules. Notice that  $\text{Ext}_A^n(U, V) = H^n(\text{Hom}_A(\mathcal{P}(U), V))$  by definition. Further, we have the following isomorphisms (see, e.g., [3, I, Section 2.7] or [8, Chapter 6]):

$$\begin{aligned} \text{Ext}_A^n(U, V) &= H^n(\text{Hom}_A(\mathcal{P}(U), V)) \cong H^n(\text{Hom}_A(\mathcal{P}(U), \mathcal{P}(V))) \\ &\cong \text{Hom}_{K(A)}(\mathcal{P}(U), \mathcal{P}(V)[n]), \end{aligned}$$

where  $K(A)$  is the homotopy category of complexes of  $A$ -modules and  $\mathcal{P}(V)[n]$  is the complex obtained by moving  $\mathcal{P}(V)$  to the left by  $n$  places. Suppose that the  $R$ -algebra  $A$  is projective as an  $R$ -module. Then we can define Hochschild cohomology  $HH^n(A)$  to be the cohomology  $H^n(\text{Hom}_{A \otimes A^o}(\mathcal{P}_A, A)) = \text{Ext}_{A \otimes A^o}^n(A, A)$ . So we have an isomorphism

$$HH^n(A) \cong \text{Hom}_{K(A \otimes A^o)}(\mathcal{P}_A, \mathcal{P}_A[n]).$$

Note that if homogeneous components of  $U$  are projective as  $R$ -modules and if  $A$  also is projective as an  $R$ -module, then we see from the Künneth theorem that  $\mathcal{P}_A \otimes_A U$  becomes a projective resolution of  $U$ . So we have an algebra homomorphism

$$\alpha_U: HH^*(A) \rightarrow \text{Ext}_A^*(U, U)$$

induced by the functor  $-\otimes_A U$ . Here, the homomorphism  $\alpha_U$  maps the homotopy class of a chain map  $\zeta: \mathcal{P}_A \rightarrow \mathcal{P}_A[n]$  to that of  $\zeta \otimes \text{Id}_U$  ( $n$  is a nonnegative integer).

Suppose that  $A$  and  $B$  are symmetric  $R$ -algebras from now and let  $X$  be a bounded complex of  $A - B$ -bimodules which are projective as left  $A$ -modules and as right  $B$ -modules. M. Linckelmann gives a graded  $R$ -linear map  $t_X: HH^*(B) \rightarrow HH^*(A)$  which is called the *transfer map* associated with  $X$  (see [10, Definition 2.9]). He then shows the following connection between transfer maps in Hochschild cohomology and ordinary cohomology of finite groups: it is known that there exists an embedding of  $H^*(G, R)$  into  $HH^*(RG)$ , that is, an injective graded  $R$ -algebra homomorphism

$$\delta_G: H^*(G, R) \rightarrow HH^*(RG)$$

induced by the diagonal induction functor  $\text{Ind}_{\Delta G}^{G \times G}$  (see [10, Proposition 4.5]). Through

these embeddings, the following diagrams are commutative

$$\begin{array}{ccc}
 H^*(G, R) & \xrightarrow{\text{res}_{G,H}} & H^*(H, R) \\
 \delta_G \downarrow & & \downarrow \delta_H \\
 HH^*(RG) & \xrightarrow{t_{H(RG)}} & HH^*(RH),
 \end{array}
 \qquad
 \begin{array}{ccc}
 H^*(H, R) & \xrightarrow{\text{tr}_{H,G}} & H^*(G, R) \\
 \delta_H \downarrow & & \downarrow \delta_G \\
 HH^*(RH) & \xrightarrow{t_{(RG)_H}} & HH^*(RG),
 \end{array}$$

where  $\text{res}_{G,H}$  and  $\text{tr}_{H,G}$  denote the restriction map and the transfer map on ordinary cohomology of finite groups, respectively. He also gives a notion of  $X$ -stable elements as follows.

DEFINITION 2.1 (Linckelmann [10, Definition 3.1]). An element  $[\zeta] \in HH^*(A)$  is said to be  $X$ -stable if there is  $[\tau] \in HH^*(B)$  such that the following diagram is homotopy commutative for any nonnegative integer  $n$

$$\begin{array}{ccc}
 \mathcal{P}_A \otimes_A X & \xrightarrow{\simeq} & X \otimes_B \mathcal{P}_B \\
 \zeta_n \otimes \text{Id}_X \downarrow & & \downarrow \text{Id}_X \otimes \tau_n \\
 \mathcal{P}_A[n] \otimes_A X & \xrightarrow{\simeq} & X \otimes_B \mathcal{P}_B[n],
 \end{array}$$

where  $\zeta_n$  and  $\tau_n$  are the components in degree  $n$  of  $\zeta$  and  $\tau$ , respectively, and where the horizontal arrows are given by the natural homotopy equivalences  $\mathcal{P}_A \otimes_A X \simeq X \otimes_B \mathcal{P}_B$  lifting the natural isomorphism  $A \otimes_A X \cong X \otimes_B B$ . We denote by  $HH^*_X(A)$  the set of  $X$ -stable elements in  $HH^*(A)$ .

Then, using the notion of  $X$ -stable elements, the transfer map  $t_X$  is characterized as follows [10, Lemma 3.4 and Theorem 3.6]. Let  $t_X^0: Z(B) \rightarrow Z(A)$  be the linear map obtained by the degree zero component of  $t_X$ , composing with the natural isomorphisms  $Z(B) \cong HH^0(B)$  and  $Z(A) \cong HH^0(A)$ . Set  $t_X^0(1_B) = \pi_X$ . Similarly, set  $t_{X^*}^0(1_A) = \pi_{X^*}$  for the transfer map  $t_{X^*}: HH^*(A) \rightarrow HH^*(B)$ . If  $[\zeta] \in HH^n(A)$  is  $X$ -stable and  $[\tau] \in HH^n(B)$  is the element corresponding to  $[\zeta]$  in Definition 2.1, then it holds that (i)  $t_X([\tau]) = \pi_X[\zeta]$ , (ii)  $[\tau]$  is the  $X^*$ -stable element and  $t_{X^*}([\zeta]) = \pi_{X^*}[\tau]$ .

From the above, if  $\pi_X$  is invertible, then we see that

$$T_X = (\pi_X)^{-1}t_X: HH^*_{X^*}(B) \rightarrow HH^*_X(A)$$

is a surjective graded  $R$ -algebra homomorphism. Further, if both  $\pi_X$  and  $\pi_{X^*}$  are invertible, then it holds that  $T_{X^*} \circ T_X([\tau]) = [\tau]$  for  $[\tau] \in HH^*(B)$  and that  $T_X \circ T_{X^*}([\zeta]) = [\zeta]$  for  $[\zeta] \in HH^*(A)$ .

For example, let  $X = {}_H(RG)$ . Then we have from [10, Proposition 4.8] that  $\text{Im}(\delta_G) \subseteq HH^*_{(RG)_H}(RG)$ . Further, we see from [10, Example 2.6 and Definition 3.1]

that  $\pi_{H(RG)} = 1_{RH}$ . Thus we see that  $t_{H(RG)}: HH^*_{(RG)H}(RG) \rightarrow HH^*_{H(RG)}(RG)$  is a surjective graded  $R$ -algebra homomorphism in the above commutative diagram for  $\text{res}_{G,H}$  and  $t_{H(RG)}$ .

Using Puig’s pointed group theory, M. Linckelmann defined cohomology rings of blocks. We recall here some definitions and results on Puig’s pointed group theory [15]. Let  $k$  be an algebraically closed field of prime characteristic  $p$ . For a  $p$ -subgroup  $P$  of  $G$ , let  $\text{Br}_P^G: (kG)^P \rightarrow kC_G(P)$  be the Brauer homomorphism of  $P$ , which is a surjective algebra homomorphism. A point of  $P$  on  $kG$  is a  $((kG)^P)^\times$ -conjugacy class  $\gamma$  of primitive idempotents in  $(kG)^P$  and if  $\text{Br}_P^G(\gamma) \neq 0$ ,  $\gamma$  is called a local point of  $P$  on  $kG$ . A block of  $kG$  is a primitive idempotent  $b$  in the center  $Z(kG)$ . A defect group  $D$  of the block  $b$  is a maximal  $p$ -subgroup of  $G$  such that  $\text{Br}_D^G(b) \neq 0$ . For a defect group  $D$  of  $b$ , there is a primitive idempotent  $i \in (kGb)^D$  such that  $\text{Br}_D^G(i) \neq 0$ . So the point  $\gamma$  that contains  $i$  is a local point of  $D$  on  $kG$  contained in the block  $b$ . The idempotent  $i$  is called a source idempotent of the block  $b$  and the pair  $(D, \gamma)$ , which we denote by  $D_\gamma$ , is called a defect pointed group of the block  $b$ . Since  $\text{Br}_D^G(i)$  is a primitive idempotent in  $kC_G(D)$ , there is a unique block  $e_D$  of  $kC_G(D)$  such that  $\text{Br}_D^G(i)e_D = \text{Br}_D^G(i)$ . The pair  $(D, e_D)$  is the maximal  $b$ -Brauer pair which corresponds to  $D_\gamma$ . In general, a Brauer pair of  $G$  is a pair  $(Q, f)$  where  $Q$  is a  $p$ -subgroup of  $G$  and  $f$  is a block of  $kC_G(Q)$ . For a Brauer pair  $(Q, f)$  and a maximal  $b$ -Brauer pair  $(D, e_D)$  corresponding to  $D_\gamma$ , if  $Q$  is a subgroup of  $D$  and  $f$  satisfies  $\text{Br}_Q^G(\gamma)f = \text{Br}_Q^G(\gamma)$ , then  $(Q, f) \leq (D, e_D)$ . It is known that for any subgroup  $Q$  of  $D$ , there is a unique block  $e_Q$  such that  $(Q, e_Q) \leq (D, e_D)$ . For more details, see [16, Section 40].

DEFINITION 2.2 (Linckelmann [10, Definition 5.1]). Let  $G$  be a finite group,  $b$  a block of  $kG$  and  $D_\gamma$  a defect pointed group of  $b$ . The cohomology ring of the block  $b$  of  $kG$  associated with  $D_\gamma$  is the subring  $H^*(G, b, D_\gamma)$  of  $H^*(D, k)$  which consists of all  $[\zeta] \in H^*(D, k)$  satisfying  $\text{res}_{x^{-1}D, Q} \circ c_{x^{-1}}([\zeta]) = \text{res}_{D, Q}([\zeta])$  for any subgroup  $Q$  in  $D$  and for any  $x \in G$  with  ${}^x(Q, e_Q) = ({}^xQ, {}^xe_Q) \leq (D, e_D)$ .

If  $D'_\gamma$  is another defect pointed group of  $b$ , then there exists  $g \in G$  such that  $D'_\gamma = {}^gD_\gamma$  (see [16, Proposition 40.13]), and we see that the conjugation map  $c_g$  induces an isomorphism  $H^*(G, b, D_\gamma) \cong H^*(G, b, D'_\gamma)$ .

With the notation of Definition 2.2, let  $i \in \gamma$ . Consider  $kGi$  as a  $kGb - kD$ -bimodule and  $ikG$  as a  $kD - kGb$ -bimodule. By [10, Theorem 5.6],  $\pi_{kGi}$  and  $\pi_{ikG}$  are invertible in  $Z(kGb)$  and  $Z(kD)$ , respectively. Thus  $T_{kGi} = (\pi_{kGi})^{-1}t_{kGi}: HH^*_{ikG}(kD) \rightarrow HH^*_{kGi}(kGb)$  is a graded  $k$ -algebra isomorphism (notice that  $(kGi)^* \cong ikG$  as  $kD - kGb$ -bimodules). Furthermore from [10, Corollary 3.8 and Proposition 5.4], we have an injective graded  $k$ -algebra homomorphism  $\delta_D: H^*(G, b, D_\gamma) \rightarrow HH^*_{ikG}(kD)$ , where  $\delta_D$  is the restriction of the diagonal embedding  $H^*(D, k) \rightarrow HH^*(kD)$  stated before. Therefore, we obtain an injective graded  $k$ -algebra homomor-

phism

$$T_{kGi} \circ \delta_D: H^*(G, b, D_\gamma) \rightarrow HH^*_{kGi}(kGb).$$

Let  $U$  be a finitely generated  $kG$ -module. In [4], J.F. Carlson introduces a notion of the variety of  $U$  as follows. There is a  $k$ -algebra homomorphism  $\gamma_U: H^*(G, k) \rightarrow \text{Ext}^*_{kG}(U, U)$  mapping  $[\zeta] \in H^*(G, k)$  to  $[\zeta] \cup I \in \text{Ext}^*_{kG}(U, U)$ , where  $\cup$  is a cup product and  $I$  is the identity element of  $\text{Ext}^*_{kG}(U, U)$ . Let  $I_G^*(U)$  be the kernel of  $\gamma_U$ . Then  $V_G(U)$ , the variety of  $U$ , is the maximal ideal spectrum of the quotient  $H^*(G, k)/I_G^*(U)$ . On the other hand,  $\mathcal{P}(k) \otimes_k U$  is a projective resolution of  $U$ , where  $\mathcal{P}(k)$  is a projective resolution of the trivial  $kG$ -module  $k$  (see e.g. [11, 2.9]). So we see that  $\gamma_U$  is induced by the functor  $-\otimes_k U$  through the isomorphisms  $H^n(G, k) \cong \text{Hom}_{K(kG)}(\mathcal{P}(k), \mathcal{P}(k)[n])$  and  $\text{Ext}^n_{kG}(U, U) \cong \text{Hom}_{K(kG)}(\mathcal{P}(k) \otimes_k U, \mathcal{P}(k) \otimes_k U[n])$ . Similarly, for any bounded complex  $U$  of  $kG$ -modules, the functor  $-\otimes_k U$  induces a  $k$ -algebra homomorphism

$$\gamma_U: H^*(G, k) \rightarrow \text{Ext}^*_{kG}(U, U).$$

We also write  $I_G^*(U)$  for the kernel of  $\gamma_U$ , and  $V_G(U)$  for the variety of  $U$ . As a remarkable fact in this direction, we have from [11, 2.9] that the above  $k$ -algebra homomorphism  $\gamma_U$  is equal to the composite of  $k$ -algebra homomorphisms

$$H^*(G, k) \xrightarrow{\delta_G} HH^*(kG) \xrightarrow{\alpha_U} \text{Ext}^*_{kG}(U, U).$$

For a bounded complex  $U$  of  $kGb$ -modules, M. Linckelmann [11] gives a notion of the variety of  $U$  associated with the block  $b$ .

DEFINITION 2.3 (Linckelmann [11, Definition 4.1]). Let  $G$  be a finite group,  $b$  a block of  $kG$ ,  $D_\gamma$  a defect pointed group of  $b$ , and  $i$  a source idempotent in  $\gamma$ . For any bounded complex  $U$  of  $kGb$ -modules, denote by  $I_{G,b,D_\gamma}^*(U)$  the kernel in  $H^*(G, b, D_\gamma)$  of the composite of  $k$ -algebra homomorphisms

$$H^*(G, b, D_\gamma) \xrightarrow{T_{kGi \circ \delta_D}} HH^*(kGb) \xrightarrow{\alpha_U} \text{Ext}^*_{kGb}(U, U)$$

and let  $V_{G,b}(U)$  be the maximal ideal spectrum of  $H^*(G, b, D_\gamma)/I_{G,b,D_\gamma}^*(U)$ . We also let  $V_{G,b}$  be the maximal ideal spectrum of  $H^*(G, b, D_\gamma)$ .

For another defect pointed group  $D'_\gamma$  of  $b$ , there exists  $g \in G$  such that  $c_g: H^*(G, b, D_\gamma) \cong H^*(G, b, D'_\gamma)$ , as stated above. So the isomorphism class of the variety  $V_{G,b}(U)$  does not depend on the choice of  $D_\gamma$ .

M. Linckelmann shows the following connection between varieties associated with blocks and Carlson’s module varieties. By the definition, it is clear that the restriction map  $\text{res}_{G,D}$  induces an algebra homomorphism  $\rho_b: H^*(G, k) \rightarrow H^*(G, b, D_\gamma)$ .

Let  $\rho_b^*: V_{G,b} \rightarrow V_G(k)$  be the affine map defined by  $\rho_b$ , where  $V_G(k)$  is the maximal ideal spectrum of  $H^*(G, k)$ .

**Theorem 2.4** (Linckelmann [11, Corollary 4.4]). *For any bounded complex  $U$  of  $kGb$ -modules, it holds that  $I_G^*(U) = \rho_b^{-1}(I_{G,b,D_\gamma}^*(U))$ . Thus,  $\rho_b^*: V_{G,b}(U) \rightarrow V_G(U)$  is a finite surjective map and the dimensions of  $V_{G,b}(U)$  and  $V_G(U)$  coincide. In particular, if  $b$  is the principal block of  $kG$ , then the above map is an isomorphism.*

### 3. Proof of Theorem 1.1 and its applications

In this section we give a proof of Theorem 1.1. Further, applying this theorem we consider the question whether the varieties  $V_{G,b}(U)$  are invariant under the block correspondence in Brauer’s first main theorem (with the assumption that defect groups are abelian). Theorem 1.1 is based on the following theorem which is the reverse version of Linckelmann’s theorem [11, Theorem 5.1].

**Theorem 3.1.** *Let  $A, B$  be symmetric algebras over a commutative ring  $R$ , and let  $X$  be a bounded complex of  $A$ - $B$  bimodules whose components are projective as left and right modules. If  $\pi_{X^*}$  is invertible in  $Z(B)$ , then for any bounded complex  $U$  of  $A$ -modules there is a commutative diagram of graded  $R$ -algebra homomorphisms*

$$\begin{array}{ccc}
 HH_X^*(A) & \xrightarrow{\alpha_U} & \text{Ext}_A^*(U, U) \\
 \downarrow T_{X^*} & & \downarrow \beta_{X^*,U} \\
 HH_{X^*}^*(B) & \xrightarrow{\alpha_{X^* \otimes_A U}} & \text{Ext}_B^*(X^* \otimes_A U, X^* \otimes_A U),
 \end{array}$$

where the horizontal maps are induced by the functors  $- \otimes_A U$  and  $- \otimes_B (X^* \otimes_A U)$ , respectively, and where the right vertical map is induced by the functor  $X^* \otimes_A -$ .

Theorem 3.1 implies the next proposition. With the notation in Section 2, let  $X = kGi$  regarded as a  $kGb - kD$ -bimodule, and so  $X^* = (kGi)^* \cong ikG$  as  $kD - kGb$ -bimodules. Further, for a bounded complex  $U$  of  $kGb$ -modules, let  $\gamma_{iU}: H^*(D, k) \rightarrow \text{Ext}_{kD}^*(iU, iU)$  be the  $k$ -algebra homomorphism induced by the functor  $- \otimes_k iU$ , where  $iU = ikG \otimes_{kGb} U$  is considered as a complex of  $kD$ -modules.

**Proposition 3.2.** *Let  $U$  be a bounded complex of  $kGb$ -modules. The following diagram of graded  $k$ -algebra homomorphisms is commutative.*

$$\begin{array}{ccc}
 H^*(G, b, D_\gamma) & \xrightarrow{\alpha_U \circ T_{kGi} \circ \delta_D} & \text{Ext}_{kGb}^*(U, U) \\
 \downarrow \iota & & \downarrow i \circ \text{res}_{G,D} \\
 H^*(D, k) & \xrightarrow{\gamma_{iU}} & \text{Ext}_{kD}^*(iU, iU),
 \end{array}$$

where the right vertical map is the composite of the restriction homomorphism and the projection from  $\text{Ext}_{kGb}^*(U, U)$  onto  $\text{Ext}_{kD}^*(iU, iU)$ , and the left vertical map is the inclusion map.

Proof. From [10, Theorem 5.6],  $\pi_{ikG}$  is invertible in  $Z(kD)$ . So by Theorem 3.1, we have that  $\alpha_{iU} \circ T_{ikG} = \beta_{ikG,U} \circ \alpha_U$ . Now, since  $\pi_{kGi}$  is also invertible,  $T_{kGi}$  and  $T_{ikG}$  are mutually inverse  $k$ -algebra isomorphisms from [10, Theorem 3.6]. Thus it follows that  $\alpha_{iU} = \beta_{ikG,U} \circ \alpha_U \circ T_{kGi}$ . That is, we have the following commutative diagram of graded  $k$ -algebra homomorphisms:

$$\begin{array}{ccc}
 HH_{kGi}^*(kGb) & \xrightarrow{\alpha_U} & \text{Ext}_{kGb}^*(U, U) \\
 \uparrow T_{kGi} & & \downarrow \beta_{ikG,U} \\
 HH_{ikG}^*(kD) & \xrightarrow{\alpha_{iU}} & \text{Ext}_{kD}^*(iU, iU).
 \end{array}$$

From this diagram, we can form the commutative diagram

$$\begin{array}{ccccc}
 H^*(G, b, D_\gamma) & \xrightarrow{\delta_D} & HH_{ikG}^*(kD) & \xrightarrow{\alpha_U \circ T_{kGi}} & \text{Ext}_{kGb}^*(U, U) \\
 \downarrow \iota & & \parallel & & \downarrow \beta_{ikG,U} \\
 H^*(D, k) & \xrightarrow{\delta_D} & HH_{ikG}^*(kD) & \xrightarrow{\alpha_{iU}} & \text{Ext}_{kD}^*(iU, iU).
 \end{array}$$

Now, it is clear that the functor  $ikG \otimes_{kGb} -$  gives the algebra homomorphism  $i \circ \text{res}_{G,D}$  and so  $\beta_{ikG,U} = i \circ \text{res}_{G,D}$ . Moreover, from [11, 2.9], the composite  $\alpha_{iU} \circ \delta_D$  is equal to the algebra homomorphism  $\gamma_{iU}$  given by the functor  $- \otimes_k iU$ . So the proposition follows. □

The following lemma is a block variety version of [3, II, Proposition 5.7.5] on Carlson’s varieties  $V_G(U)$ .

**Lemma 3.3.** *Let  $U_1$  and  $U_2$  be bounded complexes of  $kGb$ -modules. Then we have that  $I_{G,b,D_\gamma}^*(U_1 \oplus U_2) = I_{G,b,D_\gamma}^*(U_1) \cap I_{G,b,D_\gamma}^*(U_2)$ , and hence we have that  $V_{G,b}(U_1 \oplus U_2) = V_{G,b}(U_1) \cup V_{G,b}(U_2)$ .*



Proof. Notice that  $\mathcal{P}_{kGb} \otimes_{kGb} (U_1 \oplus U_2) \cong (\mathcal{P}_{kGb} \otimes_{kGb} U_1) \oplus (\mathcal{P}_{kGb} \otimes_{kGb} U_2)$ . Under this isomorphism, we have the decomposition  $\zeta \otimes \text{Id}_{U_1 \oplus U_2} = (\zeta \otimes \text{Id}_{U_1}) \oplus (\zeta \otimes \text{Id}_{U_2})$  for a chain map  $\zeta: \mathcal{P}_{kGb} \rightarrow \mathcal{P}_{kGb}[n]$ . Thus for the projection

$$\mu_i: \text{Ext}_{kGb}^*(U_1 \oplus U_2, U_1 \oplus U_2) \longrightarrow \text{Ext}_{kGb}^*(U_i, U_i) \quad (i = 1, 2),$$

it follows that  $\mu_i \circ \alpha_{U_1 \oplus U_2} = \alpha_{U_i}$  ( $i = 1, 2$ ), respectively. Then these commutations show that  $I_{G,b,D_\gamma}^*(U_1 \oplus U_2) \subseteq I_{G,b,D_\gamma}^*(U_1) \cap I_{G,b,D_\gamma}^*(U_2)$ . Moreover, if  $\zeta \otimes \text{Id}_{U_i}$  ( $i = 1, 2$ ) is homotopic to 0, respectively, then  $\zeta \otimes \text{Id}_{U_1 \oplus U_2}$  is also homotopic to 0. So the lemma follows.  $\square$

The following lemma is well-known. We include here a proof for completeness.

**Lemma 3.4.** *Let  $U$  be a bounded complex of  $kGb$ -modules,  $D_\gamma$  a defect pointed group of  $b$  and let  $i \in \gamma$ . Then  $U$  is isomorphic to a direct summand of  $kGi \otimes_{kD} iU$ .*

Proof. Since  $\text{Tr}_D^G(i) = \sum_{x \in [G/D]} xix^{-1}$  is invertible in  $Z(kGb)$  (see [10, Theorem 5.6]), we have  $\beta = (\text{Tr}_D^G(i))^{-1} \in Z(kGb)$ . Now, consider the chain map  $t: U \rightarrow kGi \otimes_{kD} iU$  consisting of  $kG$ -homomorphisms  $t_n: U_n \rightarrow kGi \otimes_{kD} iU_n$  defined by

$$u \longmapsto \sum_{x \in [G/D]} xi \otimes i\beta x^{-1}u \quad \text{for } u \in U_n,$$

where  $U_n$  is the degree  $n$  component of  $U$ . On the other hand, there exists a chain map  $s: kGi \otimes_{kD} iU \rightarrow U$  consisting of natural homomorphisms  $s_n: kGi \otimes_{kD} iU_n \rightarrow U_n$  induced by the action of  $kG$  on  $U_n$ . Then, since  $\text{Tr}_D^G(i\beta) = \text{Tr}_D^G(i)\beta = b$ , we have  $s_n \circ t_n(u) = u$  for  $u \in U_n$ . So the lemma follows.  $\square$

Proof of Theorem 1.1. From [11, Theorem 5.1], there exists a commutative diagram

$$\begin{CD} HH_{ikG}^*(kD) @>\alpha_{iU}>> \text{Ext}_{kD}^*(iU, iU) \\ @V T_{kGi} VV @VV \beta_{kGi,iU} V \\ HH_{kGi}^*(kGb) @>\alpha_{kGi \otimes_{kD} iU}>> \text{Ext}_{kGb}^*(kGi \otimes_{kD} iU, kGi \otimes_{kD} iU). \end{CD}$$

Now, from Lemma 3.4, we have the canonical projection  $\mu: \text{Ext}_{kGb}^*(kGi \otimes_{kD} iU, kGi \otimes_{kD} iU) \longrightarrow \text{Ext}_{kGb}^*(U, U)$ . Then from Lemma 3.3 it follows that  $\mu \circ \alpha_{kGi \otimes_{kD} iU} =$

$\alpha_U$ . Thus the above diagram induces the following commutative diagram:

$$\begin{array}{ccccc}
 H^*(D, k) & \xrightarrow{\delta_D} & HH_{ikG}^*(kD) & \xrightarrow{\alpha_{iU}} & \text{Ext}_{kD}^*(iU, iU) \\
 \uparrow \iota & & \parallel & & \downarrow \mu \circ \beta_{kGi, iU} \\
 H^*(G, b, D_\gamma) & \xrightarrow{\delta_D} & HH_{ikG}^*(kD) & \xrightarrow{\alpha_U \circ T_{kGi}} & \text{Ext}_{kGb}^*(U, U).
 \end{array}$$

We note that  $\alpha_{iU} \circ \delta_D = \gamma_{iU}$ . Hence, we have that  $H^*(G, b, D_\gamma) \cap I_D^*(iU) \subseteq I_{G, b, D_\gamma}^*(U)$ . Conversely, by Proposition 3.2 we have that  $I_{G, b, D_\gamma}^*(U) \subseteq H^*(G, b, D_\gamma) \cap I_D^*(iU)$ . Hence we have  $I_{G, b, D_\gamma}^*(U) = H^*(G, b, D_\gamma) \cap I_D^*(iU)$ . In particular, if  $b$  is a nilpotent block of  $kG$ , then  $H^*(G, b, D_\gamma) = H^*(D, k)$  (see [11, 3.6]) and so we have that  $I_{G, b, D_\gamma}^*(U) = I_D^*(iU)$  and  $V_{G, b}(U) = V_D(iU)$ . This completes the proof.  $\square$

We came to know during the circulation of this paper without the last section that essentially the same fact as Theorem 1.1 was shown in M. Linckelmann [13] and Corollary 3.5 below also was obtained in it.

As is well known, the nilradical  $\sqrt{0}$  of  $H^*(G, k)$  is the intersection of all maximal ideals of  $H^*(G, k)$  (see [14]). Recall that  $H^*(G, k)/\sqrt{0}$  is a finitely generated commutative  $k$ -algebra and  $H^*(D, k)$  is finitely generated as a module over  $H^*(G, k)$  (via the restriction map) (see [3, II] and [5]). Thus  $H^*(D, k)$  is also finitely generated (that is, Noetherian) as a module over  $H^*(G, b, D_\gamma)$  and so, by [3, II, Section 5.4] and [14, Section 9, Lemma 2], we have a finite surjective affine map  $\iota_{D, b}: V_D \rightarrow V_{G, b}$  induced by the inclusion map  $\iota: H^*(G, b, D_\gamma) \rightarrow H^*(D, k)$ , where  $V_D$  is the maximal ideal spectrum of  $H^*(D, k)$  and  $V_{G, b}$  is the maximal ideal spectrum of  $H^*(G, b, D_\gamma)$ . Then, Proposition 3.2 shows that for any bounded complex  $U$  of  $kGb$ -modules, we can define the finite affine map

$$\iota_{D, b}: V_D(iU) \longrightarrow V_{G, b}(U).$$

**Proof of Corollary 1.2.** By Theorem 1.1, it is clear that  $\iota_{D, b}$  is a surjective map. Further, Theorem 1.1 shows also that the Krull dimensions of  $H^*(D, k)/I_D^*(iU)$  and  $H^*(G, b, D_\gamma)/I_{G, b, D_\gamma}^*(U)$  coincide. So the dimensions of  $V_G(iU)$  and  $V_{G, b}(U)$  coincide.  $\square$

With the notation of Theorem 1.1, let  $U_0$  be a bounded complex of  $kD$ -modules such that  $U_0 \mid iU$  and  $U \mid kGi \otimes_{kD} U_0$ . Then, from the proof of Theorem 1.1 we have also

$$I_{G, b, D_\gamma}^*(U) = H^*(G, b, D_\gamma) \cap I_D^*(U_0).$$

Hence  $\iota_{D, b}: V_D(U_0) \rightarrow V_{G, b}(U)$  is also a finite surjective map.

**Corollary 3.5.** *Let  $U$  be an indecomposable  $kGb$ -module with vertex  $D$ . Suppose that the dimension of a source of  $U$  is not a multiple of  $p$ . Then we have that  $V_{G,b}(U) = V_{G,b}$ .*

*Proof.* We see from Lemma 3.4 that  $U \mid kGi \otimes_{kD} iU$ . Thus there exists an indecomposable  $kD$ -module  $U_0$  such that  $U_0 \mid iU$  and  $U \mid kGi \otimes_{kD} U_0$ , so that  $U_0$  is a source of  $U$ . By the assumption, we see that  $\sqrt{I_D^*(U_0)} = \sqrt{0}$  in  $H^*(D, k)$  from [3, II, Corollary 5.8.5]. Further, by the above it follows that  $\sqrt{I_{G,b,D_\gamma}^*(U)} = H^*(G, b, D_\gamma) \cap \sqrt{I_D^*(U_0)}$ . Thus we have that  $\sqrt{I_{G,b,D_\gamma}^*(U)} = \sqrt{0}$  in  $H^*(G, b, D_\gamma)$ , that is,  $V_{G,b}(U) = V_{G,b}$ . □

As is well known,  $b_0 = \text{Br}_D^G(b)$  is a block of  $kN_G(D)$  with the defect group  $D$  and is called the Brauer correspondent of  $b$ . Let  $D_{\gamma_0}$  be a defect pointed group of  $b_0$ , and let  $i_0 \in \gamma_0$ . Since  $\text{Br}_D^{N_G(D)}(i_0)$  is primitive in  $kC_G(D)$ , there is a primitive idempotent  $i \in (kG)^D$  such that  $i = i_0i = ii_0$  and  $\text{Br}_D^G(i) = \text{Br}_D^{N_G(D)}(i_0)$ . Then  $i$  belongs to the block  $b$ . Indeed,  $\text{Br}_D^G(ib) = \text{Br}_D^G(i) \text{Br}_D^G(b) = \text{Br}_D^{N_G(D)}(i_0) \text{Br}_D^{N_G(D)}(b_0) = \text{Br}_D^{N_G(D)}(i_0) \neq 0$ . Let  $\gamma$  be the  $((kG)^D)^\times$ -conjugacy class of  $i$ . Then we obtain a defect pointed group  $D_\gamma$  of  $b$ . Let  $(D, e_D)$  be the maximal  $b$ -Brauer pair corresponding to  $D_\gamma$ . Note that  $(D, e_D)$  is also the maximal  $b_0$ -Brauer pair corresponding to  $D_{\gamma_0}$ .

**Corollary 3.6.** *With the above notation, let  $b_0$  be the Brauer correspondent of  $b$  and let  $U$  be a bounded complex of  $kGb$ -modules. Suppose that  $D$  is abelian. Then we have  $I_{N_G(D),b_0,D_{\gamma_0}}^*(b_0U) \subseteq I_{G,b,D_\gamma}^*(U)$ , and hence we have that  $V_{G,b}(U) \subseteq V_{N_G(D),b_0}(b_0U)$ .*

*Proof.* Theorem 1.1 shows that  $I_{G,b,D_\gamma}^*(U) = H^*(G, b, D_\gamma) \cap I_D^*(iU)$  and  $I_{N_G(D),b_0,D_{\gamma_0}}^*(b_0U) = H^*(N_G(D), b_0, D_{\gamma_0}) \cap I_D^*(i_0U)$ . As in [10, 5.2.3], if  $D$  is abelian, then we have that  $H^*(G, b, D_\gamma) = H^*(D, k)^{N_G(D,e_D)/C_G(D)}$  and similarly  $H^*(N_G(D), b_0, D_{\gamma_0}) = H^*(D, k)^{N_G(D,e_D)/C_G(D)}$ . So  $H^*(G, b, D_\gamma) = H^*(N_G(D), b_0, D_{\gamma_0})$ . Moreover, from [3, II, Proposition 5.7.5] it follows that  $I_D^*(i_0U) \subseteq I_D^*(iU)$  since  $i_0U = iU \oplus (i_0 - i)U$ . Thus we have that  $I_{N_G(D),b_0,D_{\gamma_0}}^*(b_0U) \subseteq I_{G,b,D_\gamma}^*(U)$  and so  $V_{G,b}(U) \subseteq V_{N_G(D),b_0}(b_0U)$ . □

For Carlson’s module varieties, the following is known. Let  $P$  be an abelian Sylow  $p$ -subgroup of  $G$ , so that  $N_G(P)$  controls the  $p$ -fusion in  $G$ . Let  $U$  be an indecomposable  $kG$ -module with vertex  $P$  and let  $W$  be the Green correspondent of  $U$  with respect to  $(G, P, N_G(P))$ . Then it follows that  $V_G(U) \cong V_{N_G(P)}(W)$  (see [2, Theorem 2.26.9] and [3, I, Proposition 3.8.4]). Therefore, in this case, we see from Theorem 2.4 that if  $b$  is the principal block of  $kG$ , then  $V_{G,b}(U) \cong V_{N_G(P),b_0}(W)$  for any indecomposable  $kGb$ -module  $U$  with vertex  $P$  and the Green correspondent  $W$  of  $U$ .

Extending this fact to any block  $b$  of  $kG$ , we obtain Theorem 1.3 as follows from Lemma 3.7 below. Note that  $D$  is not necessarily abelian in Lemma 3.7.

With the above notation, let  $T = T(e_D)$  be the inertial group of  $e_D$  in  $N_G(D)$ , that is,  $T = \{x \in N_G(D); {}^x e_D = e_D\}$ . Notice that  $T = N_G(D, e_D) = N_G(D_\gamma)$  (see [16, Proposition 40.13]), and  $e_D$  is also a block of  $kT$ . Using the Clifford theory of blocks, we have the following.

**Lemma 3.7.** *Let  $U$  be an indecomposable  $kGb$ -module with vertex  $D$ . Let  $W$  be the Green correspondent of  $U$  with respect to  $(G, D, N_G(D))$ . Let  $i \in \gamma$  and  $i_0 \in \gamma_0$ .*

(i) *We can choose a source  $U_0$  of  $U$  such that  $U_0 \mid iU$  and  $U \mid kGi \otimes_{kD} U_0$ , and a source  $W_0$  of  $W$  such that  $W_0 \mid i_0W$  and  $W \mid kN_G(D)i_0 \otimes_{kD} W_0$ .*

(ii) *Let  $U_0$  and  $W_0$  be the sources satisfying the conditions of (i). Then there exists an element  $t \in T$  such that  $tU_0 \cong W_0$  as  $kD$ -modules.*

Proof. First of all, we note that if  $i$  and  $i'$  lie in  $\gamma$ , then  $iU \cong i'U$  and  $kGi \cong kGi'$  as  $kD$ -modules, and likewise for  $i_0 \in \gamma_0$ . Indeed, let  $i' = \tilde{z}i$  for an element  $z$  in  $((kG)^D)^\times$ . Then, mapping  $a \in iU$  to  $za \in i'U$ , we have the isomorphism  $iU \cong i'U$  as  $kD$ -modules. By applying this argument, we obtain all other isomorphisms. Thus, the choices of  $U_0$  and  $W_0$  in (i) depend only on the points  $\gamma$  and  $\gamma_0$ . As we have shown in Corollary 3.5, (i) now follows from Lemma 3.4.

Next we prove (ii). Suppose that  $i \in \gamma$  satisfies  $i = i_0i = ii_0$ . Now, since  $W$  is the Green correspondent of  $U$ , we have that  $b_0U \cong W \oplus W'$  and so  $i_0U \cong i_0W \oplus i_0W'$  where any indecomposable direct summand of  $W'$  (and so  $i_0W'$ ) does not have vertex  $D$ . Here, from  $iU \mid i_0U$ , we see  $U_0 \mid i_0U$ . Then since  $U_0$  has vertex  $D$ , we see  $U_0 \mid i_0W$ . Further, we can choose  $i_0 \in \gamma_0$  such that  $e_D i_0 = i_0 e_D = i_0$ . Indeed, let  $i'_0$  be a primitive idempotent in  $(kN_G(D))^D$  such that  $e_D i'_0 = i'_0 e_D = i'_0$  and  $\text{Br}_D^{N_G(D)}(i'_0) \neq 0$ . Let  $\gamma'_0$  be a point of  $D$  on  $kN_G(D)$  containing  $i'_0$ . Then from  $\text{Br}_D^{N_G(D)}(i'_0)e_D = \text{Br}_D^{N_G(D)}(i'_0)$ ,  $D_{\gamma'_0}$  is the defect pointed group of  $b_0$  corresponding to  $(D, e_D)$ , because the relation  $\text{Br}_D^{N_G(D)}(i'_0)e_D = \text{Br}_D^{N_G(D)}(i'_0)$  defines a bijection between maximal  $b_0$ -Brauer pairs and defect pointed groups of  $b_0$  (see [16, Proposition 40.13]). Thus we see that  $\gamma'_0 = \gamma_0$ . By this choice of  $i_0 \in \gamma_0$ , we have that  $i_0W \mid e_D W$ . So it holds that  $U_0 \mid e_D W$  and also  $W_0 \mid e_D W$ . Let  $e_D = e_T$  and  $N_G(D) = N$ . Using the Clifford theory, we show that the  $kT$ -module  $e_T W$  is indecomposable and has vertex  $D$ . Since  $b_0 = \sum_{x \in N/T} {}^x e_T$  and  ${}^x e_T {}^y e_T = 0$  for  $x \not\equiv y \pmod{T}$ , we have that  $W = b_0 W = kN \otimes_{kT} e_T W$  and so  $e_T W$  is indecomposable and vertices of  $e_T W$  contain  $D$ . On the other hand, since  $W \mid kN i_0 \otimes_{kD} W_0$  and  $e_T i_0 = i_0$ , we see that  $W \mid kN e_T \otimes_{kD} W_0$  so that  $W = b_0 W \mid b_0 kN e_T \otimes_{kD} W_0$ . Here since  $b_0 kN = \bigoplus_{(x,y) \in [N \times N/T \times T]} {}^x (kT e_T) {}^y^{-1}$ , we see that  $e_T (b_0 kN) e_T = kT e_T$ . Thus  $e_T W \mid e_T (b_0 kN) e_T \otimes_{kD} W_0 = kT e_T \otimes_{kD} W_0$  and so  $e_T W \mid kT \otimes_{kD} W_0$ . Therefore we see that  $e_T W$  has vertex  $D$ . Further, from  $W_0 \mid e_T W$ , we see that  $W_0$  is a source of  $e_T W$ . Now, we also have  $U_0 \mid e_T W$  and so we see that there is an element  $t \in T$  such that  $tU_0 \cong W_0$  as  $kD$ -modules. □

Proof of Theorem 1.3. Let  $U_0$  and  $W_0$  be the sources chosen in Lemma 3.7. Then, as noted preceding to Corollary 3.5, we see that  $I_{G,b,D_\gamma}^*(U) = H^*(G, b, D_\gamma) \cap I_D^*(U_0)$  and  $I_{N_G(D),b_0,D_{\gamma_0}}(W)^* = H^*(N_G(D), b_0, D_{\gamma_0}) \cap I_D^*(W_0)$ . Now, let  $t \in T$  be the element obtained in Lemma 3.7 and let  $c_t: H^*(D, k) \rightarrow H^*(D, k)$  be the conjugation map given by  $t$ . Then, since  $T = N_G(D, e_D)$ , we see from Definition 2.2 that  $c_t$  is the identity map on  $H^*(G, b, D_\gamma)$ . Further it is easy to see that  $c_t \circ \gamma_{U_0} = \gamma_{tU_0} \circ c_t$  (for  $\gamma_{U_0}$  and  $\gamma_{tU_0}$  see Section 2) and so we have that  $c_t(I_D^*(U_0)) = I_D^*(tU_0) = I_D^*(W_0)$ . Here, since  $D$  is abelian, we have that  $H^*(G, b, D_\gamma) = H^*(N_G(D), b_0, D_{\gamma_0})$  (see the proof of Corollary 3.6). Therefore it follows that  $I_{G,b,D_\gamma}^*(U) = c_t(I_{G,b,D_\gamma}^*(U)) = c_t(H^*(G, b, D_\gamma) \cap I_D^*(U_0)) = H^*(G, b, D_\gamma) \cap I_D^*(W_0) = H^*(N_G(D), b_0, D_{\gamma_0}) \cap I_D^*(W_0) = I_{N_G(D),b_0,D_{\gamma_0}}(W)$ , that is, we have that  $V_{G,b}(U) = V_{N_G(D),b_0}(W)$ .  $\square$

#### 4. Module varieties and quotient groups

Let  $G$  be a finite group and let  $Z$  be a central  $p$ -subgroup of  $G$ . We denote  $G/Z$  by  $\bar{G}$  and let  $f: kG \rightarrow k\bar{G}$  be the natural  $k$ -algebra homomorphism. As is well known,  $f$  gives a one to one correspondence from all blocks of  $kG$  onto all blocks of  $k\bar{G}$ . In this section we consider relations of the varieties under this correspondence. For a subgroup  $H$  of  $G$ , its image in  $\bar{G}$  is denoted by  $\bar{H}$ . In general, the mark  $\bar{\phantom{x}}$  will be attached to the quantities associated with  $\bar{G}$  and  $k\bar{G}$ . Let  $P$  be a  $p$ -subgroup of  $G$ . Consider  $k$ -algebra homomorphisms  $f^P: (kG)^P \rightarrow (k\bar{G})^{\bar{P}}$  and composite  $f_{C_G(P)}: kC_G(P) \rightarrow k\overline{C_G(P)} \rightarrow kC_{\bar{G}}(\bar{P})$  induced by  $f$ , where  $(kG)^P$  is the subalgebra of  $kG$  consisting of all  $P$ -fixed elements and  $(k\bar{G})^{\bar{P}}$  likewise. Then  $\text{Br}_P^G \circ f^P = f_{C_G(P)} \circ \text{Br}_P^G$  as is well known. Now, since  $Z$  is a central  $p$ -subgroup we see that  $\overline{C_G(P)}$  is a normal subgroup of  $C_{\bar{G}}(\bar{P})$  and  $C_{\bar{G}}(\bar{P})/\overline{C_G(P)}$  is a  $p$ -group. Indeed, for  $x \in G$ , let  $x = x_p x_{p'}$  (where  $x_p$  is the  $p$ -component and  $x_{p'}$  is the  $p$ -regular component of  $x$ ) and if  $\bar{x} \in C_{\bar{G}}(\bar{P})$ , then we see  $x_{p'} \in C_G(P)$  so that  $C_{\bar{G}}(\bar{P})/\overline{C_G(P)}$  is a  $p$ -group.

**Lemma 4.1.** *With the notation above, if  $i$  is a primitive idempotent in  $(kG)^P$ , then  $f^P(i)$  remains a primitive idempotent in  $(k\bar{G})^{\bar{P}}$ .*

Proof. We follow the terminology and arguments in Külshammer-Puig [9]. Let  $\tilde{f}^P$  and  $\widetilde{f_{C_G(P)}}$  be the exomorphisms determined by  $f^P$  and  $f_{C_G(P)}$ , respectively. From [9, Theorem 3.16], it follows that  $\tilde{f}^P$  is a strict semicovering for any  $p$ -subgroup  $P$  of  $G$  if and only if  $\widetilde{f_{C_G(P)}}$  is a strict semicovering for any  $p$ -subgroup  $P$  of  $G$ . Now, from [9, Example 3.9] we see that  $\widetilde{f_{C_G(P)}}$  is a strict semicovering for any  $P$ , since  $C_{\bar{G}}(\bar{P})/\overline{C_G(P)}$  is a  $p$ -group. From the fact that  $\tilde{f}^P$  is a strict semicovering, we see that  $f^P(i) \neq 0$  and that if  $n$  and  $m$  are the number of primitive idempotents decomposing unity elements of  $(kG)^P$  and  $(k\bar{G})^{\bar{P}}$ , respectively, then  $n = m$ . So  $f^P(i)$  is primitive.  $\square$

Let  $b$  be a block of  $kG$ ,  $D_\gamma$  a defect pointed group of  $b$ , and let  $i \in \gamma$ . We set  $f(b) = \bar{b}$ ,  $D/Z = \bar{D}$  and  $f^D(i) = \bar{i}$ . Then since  $Z$  is a central  $p$ -subgroup of  $G$ , we see that  $\bar{b}$  is a block of  $k\bar{G}$  and  $\bar{D}$  is a defect group of  $\bar{b}$ . Further,  $\bar{i}$  is primitive by Lemma 4.1. Also, since  $k\overline{C_G(D)} = k(C_G(D)/Z) \cong kC_G(D)/J(kZ)kC_G(D)$  (where  $J(kZ)$  is the radical of  $kZ$ ), we see that  $\text{Br}_{\bar{D}}^{\bar{G}} \circ f^D(i) = f_{C_G(D)} \circ \text{Br}_D^G(i) \neq 0$  and so  $\bar{i}$  is local. Therefore, let  $\bar{\gamma}$  be the  $((k\bar{G})^{\bar{D}})^\times$ -conjugacy class of  $\bar{i}$ , then  $\bar{D}_{\bar{\gamma}}$  is a defect pointed group of  $\bar{b}$ . Recall that  $H^*(G, b, D_\gamma)$  consists of all  $[\zeta] \in H^*(D, k)$  satisfying  $\text{res}_{x^{-1}D, Q} \circ c_{x^{-1}}([\zeta]) = \text{res}_{D, Q}([\zeta])$  for any subgroup  $Q$  of  $D$  and for any  $x \in G$  with  ${}^x(Q, e_Q) \leq (D, e_D)$ .

**Lemma 4.2.** *Let  $Z$  be a central  $p$ -subgroup of  $G$ . In the definition of  $H^*(G, b, D_\gamma)$ , we can assume that  $Q$  contains  $Z$ .*

Proof. Let  $(Q, e_Q)$  be a  $b$ -Brauer pair such that  $(Q, e_Q) \leq (D, e_D)$ . Since  $Z$  is a central  $p$ -subgroup, we have that  $\text{Br}_{QZ}^G = \text{Br}_Q^G$ . Thus,  $\text{Br}_{QZ}^G(i)e_Q = \text{Br}_Q^G(i)e_Q = \text{Br}_Q^G(i) = \text{Br}_{QZ}^G(i)$ , and so  $e_{QZ} = e_Q$  and  $(QZ, e_Q) \leq (D, e_D)$ . Also, for an element  $x \in G$ , it is clear that  ${}^x(Q, e_Q) \leq (D, e_D)$  if and only if  ${}^x(QZ, e_Q) \leq (D, e_D)$ . Further, if  $\text{res}_{x^{-1}D, QZ} \circ c_{x^{-1}}([\zeta]) = \text{res}_{D, QZ}([\zeta])$ , then we have that  $\text{res}_{x^{-1}D, Q} \circ c_{x^{-1}}([\zeta]) = \text{res}_{D, Q}([\zeta])$ . So we can replace  $Q$  by  $QZ$  in the definition.  $\square$

**Proposition 4.3.** *With the notation above, let  $[\bar{\zeta}]$  be an element of  $H^*(\bar{G}, \bar{b}, \bar{D}_{\bar{\gamma}})$ . Then  $\text{inf}_{\bar{D}, D}([\bar{\zeta}])$  belongs to  $H^*(G, b, D_\gamma)$ .*

Proof. Let  $(Q, e_Q)$  be a  $b$ -Brauer pair such that  $(Q, e_Q) \leq (D, e_D)$  and that  $Q$  contains  $Z$ . For the block  $e_Q$ , since  $Z$  is a central  $p$ -subgroup of  $C_G(Q)$ ,  $\overline{e_Q}$  is the block of  $k\overline{C_G(Q)}$ . Further, since  $C_{\bar{G}}(\bar{Q})/\overline{C_G(Q)}$  is a  $p$ -group, we have a unique block  $e_{\bar{Q}}$  of  $kC_{\bar{G}}(\bar{Q})$  which covers  $\overline{e_Q}$ . Then, since  $(Q, e_Q) \leq (D, e_D)$ , we have  $(\bar{Q}, e_{\bar{Q}}) \leq (\bar{D}, e_{\bar{D}})$ . Note that  $(\bar{D}, e_{\bar{D}})$  is the maximal  $\bar{b}$ -Brauer pair corresponding to  $\bar{D}_{\bar{\gamma}}$ . Indeed,  $e_{\bar{Q}} \overline{e_Q} = \overline{e_Q}$  and so it holds that  $\text{Br}_{\bar{Q}}^{\bar{G}}(\bar{i})e_{\bar{Q}} = \text{Br}_Q^G(i)\overline{e_Q} = \text{Br}_Q^G(i) = \text{Br}_{\bar{Q}}^{\bar{G}}(\bar{i})$ . Also, we see that if  ${}^x(Q, e_Q) \leq (D, e_D)$  for  $x \in G$ , then  ${}^x(\bar{Q}, e_{\bar{Q}}) \leq (\bar{D}, e_{\bar{D}})$ . Thus, for an element  $[\bar{\zeta}] \in H^*(\bar{G}, \bar{b}, \bar{D}_{\bar{\gamma}})$ , it follows that

$$\begin{aligned} \text{res}_{x^{-1}D, Q} \circ c_{x^{-1}}(\text{inf}_{\bar{D}, D}([\bar{\zeta}])) &= \text{res}_{x^{-1}D, Q} \circ \text{inf}_{x^{-1}\bar{D}, x^{-1}D} \circ c_{\bar{x}^{-1}}([\bar{\zeta}]) \\ &= \text{inf}_{\bar{Q}, Q}(\text{res}_{x^{-1}\bar{D}, \bar{Q}} \circ c_{\bar{x}^{-1}}([\bar{\zeta}])) \\ &= \text{inf}_{\bar{Q}, Q}(\text{res}_{\bar{D}, \bar{Q}}([\bar{\zeta}])) \\ &= \text{res}_{D, Q}(\text{inf}_{\bar{D}, D}([\bar{\zeta}])). \end{aligned}$$

So we conclude that  $\text{inf}_{\bar{D}, D}([\bar{\zeta}]) \in H^*(G, b, D_\gamma)$ .  $\square$

From the above proposition, we can define an inflation map  $\text{inf}_{\bar{b}, b}: H^*(\bar{G}, \bar{b}, \bar{D}_{\bar{\gamma}}) \rightarrow H^*(G, b, D_\gamma)$ . Then, since  $H^*(G, b, D_\gamma)/\sqrt{0}$  is a finitely gen-

erated commutative  $k$ -algebra (see [11, Theorem 4.2]), the inflation  $\text{inf}_{\bar{b},b}$  induces an affine map  $\text{inf}_{b,b}^*: V_{G,b} \rightarrow V_{\bar{G},\bar{b}}$ . Further, for any bounded complex  $U$  of  $k\bar{G}\bar{b}$ -modules, which may also be considered as a complex of  $kGb$ -modules through  $kGb \rightarrow k\bar{G}\bar{b}$ , we have Theorem 1.4.

Proof of Theorem 1.4. For the convenience of the proof, when we consider  $U$  as a complex of  $k\bar{G}\bar{b}$ -modules, we denote it by  $\bar{U}$ . Then Theorem 1.1 shows that  $I_{\bar{G},\bar{b},\bar{D}_\gamma}^*(\bar{U}) = H^*(\bar{G}, \bar{b}, \bar{D}_\gamma) \cap I_{\bar{D}}^*(\bar{i}\bar{U})$  and that  $I_{G,b,D_\gamma}^*(U) = H^*(G, b, D_\gamma) \cap I_D^*(iU)$ . Note that for the chain maps  $\zeta$  and  $\tau$  representing  $[\zeta] \in \text{Ext}_{kD}^n(\bar{i}\bar{U}, \bar{i}\bar{U})$  and  $[\tau] \in \text{Ext}_{kD}^n(iU, iU)$ ,  $\text{inf}_{\bar{D},D}([\zeta]) = [\tau]$  if and only if the following diagram is homotopy commutative:

$$\begin{array}{ccc} \mathcal{P}(iU) & \xrightarrow{\tau} & \mathcal{P}(iU)[n] \\ \downarrow & & \downarrow \\ \mathcal{P}(\bar{i}\bar{U}) & \xrightarrow{\zeta} & \mathcal{P}(\bar{i}\bar{U})[n], \end{array}$$

where the left vertical map is a chain map lifting the identity map  $iU \rightarrow \bar{i}\bar{U}$  and the right vertical map is its shift. It is easy to see that  $\text{inf}_{D,D} \circ \gamma_{\bar{i}\bar{U}} = \gamma_{iU} \circ \text{inf}_{D,D}$ , so that we have  $\text{inf}_{D,D}(I_D^*(\bar{i}\bar{U})) \subseteq I_D^*(iU)$  and so  $\text{inf}_{\bar{b},b}(I_{\bar{G},\bar{b},\bar{D}_\gamma}^*(\bar{U})) \subseteq I_{G,b,D_\gamma}^*(U)$ . Therefore we can define an affine map  $\text{inf}_{b,b}^*: V_{G,b}(U) \rightarrow V_{\bar{G},\bar{b}}(\bar{U})$ .  $\square$

### 5. Inverse images

Under the assumption that the defect groups are abelian, we consider the inverse images of the affine maps given in Corollary 1.2 and Theorem 1.4. Our consideration is based on Linckelmann’s following stratification theorem for block varieties of modules [13]. Let  $b$  be a block of  $kG$  and  $D_\gamma$  a defect pointed group of  $b$ . For any subgroup  $Q$  of  $D$ , the composite graded algebra homomorphism  $r_Q : H^*(G, b, D_\gamma) \xrightarrow{\iota} H^*(D, k) \xrightarrow{\text{res}_{D,Q}} H^*(Q, k)$  induces a finite affine map of varieties

$$r_Q^* : V_Q \rightarrow V_{G,b},$$

where  $\iota$  is the inclusion and  $\text{res}_{D,Q}$  is the restriction map. In particular,  $r_D^* : V_D \rightarrow V_{G,b}$  is finite surjective (this map is denoted by  $\iota_{D,b}$  in the previous sections). Let  $U$  be a finitely generated  $kGb$ -module and let  $i \in \gamma$ . Following [13], we now define the following subvarieties of  $V_Q$  and  $V_{G,b}$  :

$$\begin{array}{ll} V_Q^+ = V_Q - \cup_{R < Q} (\text{res}_{Q,R})^* V_R, & V_Q^+(iU) = V_Q(iU) \cap V_Q^+ \\ V_{G,Q} = r_Q^* V_Q, & V_{G,Q}^+ = r_Q^* V_Q^+ \\ V_{G,Q}(U) = r_Q^* V_Q(iU), & V_{G,Q}^+(U) = r_Q^* V_Q^+(iU) \end{array}$$

**Theorem 5.1** (Linckelmann [13, Theorem 4.2 and Proposition 4.3]). *Let  $U$  be a finitely generated  $kGb$ -module.*

(i) *The variety  $V_{G,b}(U)$  is the disjoint union of locally closed subvarieties  $V_{G,E}^+(U)$ , where  $E$  runs over the set of subgroups of  $D$  such that  $(E, e_E)$  runs over the set of representatives of the  $G$ -conjugacy classes of those  $b$ -Brauer pairs contained in  $(D, e_D)$  for which  $E$  is elementary abelian and  $C_D(E)$  is a defect group of the block  $e_E$ .*

(ii) *Let  $E$  be an elementary abelian subgroup of  $D$  such that  $C_D(E)$  is a defect group of  $e_E$ . The group  $W(E) = N_G(E, e_E)/C_G(E)$  acts on the variety  $V_E^+(iU)$ , and  $r_E^*$  induces an inseparable isogeny  $V_E^+(iU)/W(E) \rightarrow V_{G,E}^+(U)$ .*

(iii) *Suppose that  $U$  is indecomposable with  $D$  as a vertex and a source of dimension prime to  $p$ . Then, for any subgroup  $Q$  of  $D$  we have  $V_Q(iU) = V_Q$  and  $V_Q^+(iU) = V_Q^+$ . Further, for any subgroup  $Q$  of  $D$  we have  $V_{G,Q}(U) = V_{G,Q}$  and  $V_{G,Q}^+(U) = V_{G,Q}^+$ .*

It is known that a simple  $kG$ -module  $U$  in  $b$  of height 0 satisfies the condition in (iii) (see [7, Corollary 4.6]). So (i) and (ii) give a stratification of  $V_{G,b}$  also. Using the above theorem, we show the following which is a block variety version of Avrunin and Scott [1, Theorem 3.1].

**Proposition 5.2.** *Suppose that  $D$  is abelian. Let  $Q$  be a subgroup of  $D$  and let  $U$  be a finitely generated  $kGb$ -module. Then we have that*

$$V_Q(iU) = (r_Q^*)^{-1} V_{G,b}(U).$$

*Proof.* First we show two facts obtained from the assumption that  $D$  is abelian. Let  $(R, e_R)$  be a  $b$ -Brauer pair contained in  $(D, e_D)$ . Then  $e_D$  and  $e_R$  are Brauer correspondents of  $b$ , and further  $e_D$  is also a Brauer correspondent of  $e_R$ . Thus  $e_R$  has a defect group  $D$ , because  $e_D$  has the defect group  $D$ . So, for any  $(R, e_R) \leq (D, e_D)$ ,  $e_R$  satisfies the last condition of Theorem 5.1 (i) (that is,  $C_D(R) = D$  is a defect group of  $e_R$ ). The second assertion is the following. Let  $(R', e_{R'})$  also be a  $b$ -Brauer pair contained in  $(D, e_D)$ . Suppose that  ${}^x(R', e_{R'}) = (R, e_R)$  for some  $x \in G$ . Then, by the fusion theorem for Brauer pairs (see [16, Proposition 49.5 and Proposition 49.6]), there is  $y \in N_G(D, e_D)$  such that  ${}^y r' = {}^x r'$  for all  $r' \in R'$ . Thus, mapping  $a \in {}^x iU$  to  $yx^{-1}a \in {}^y iU$ , we have the isomorphism  ${}^x iU \cong {}^y iU$  as  $kR$ -modules. Further, since  $N_G(D, e_D) = N_G(D_\gamma)$ , there exists  $z \in ((kG)^D)^\times$  such that  ${}^y i = {}^z i$ . Thus, mapping  $b \in iU$  to  $zb \in {}^y iU$ , we have the isomorphism  $iU \cong {}^y iU$  as  $kD$ -modules. So  ${}^x iU \cong iU$  as  $kR$ -modules. Therefore the conjugation map  $c_x: H^*(R', k) \rightarrow H^*(R, k)$  induces an isomorphism of subvarieties  $V_{R'}(iU) \cong V_R({}^x iU) = V_R(iU)$ .

Clearly  $r_Q^* V_Q(iU) \subseteq V_{G,b}(U)$ . Conversely, let  $v \in (r_Q^*)^{-1} V_{G,b}(U)$ . Applying the Quillen stratification theorem (see e.g. [1, Theorem 2.2]) to  $V_Q$ , we can choose an elementary abelian  $p$ -subgroup  $E$  of  $Q$  and  $s \in V_E^+$  with  $(\text{res}_{Q,E})^*(s) = v$ . On the other hand, by Theorem 5.1 (i), we can choose an elementary abelian  $p$ -subgroup



$E'$  of  $D$  and  $s' \in V_{E'}^+(iU)$  with  $r_{E'}^*(s') = r_Q^*(v)$ . Then  $r_{E'}^*(s') = r_Q^*(v) = r_E^*(s)$ . Thus, by Theorem 5.1 (i) for  $V_{G,b}$ , we see that  $b$ -Brauer pairs  $(E, e_E)$  and  $(E', e_{E'})$  must be  $G$ -conjugate. Let  $(E, e_E) = {}^g(E', e_{E'})$  for an element  $g$  in  $G$ . Then since  ${}^g(E', e_{E'}) = (E, e_E) \leq (D, e_D)$ , we see from the definition of  $H^*(G, b, D_\gamma)$  that every  $[\zeta] \in H^*(G, b, D_\gamma)$  satisfies  $c_{g^{-1}} \circ \text{res}_{D,E}([\zeta]) = \text{res}_{D,E'}([\zeta])$  (equivalently,  $\text{res}_{{}^{g^{-1}}D,E'} \circ c_{g^{-1}}([\zeta]) = \text{res}_{D,E'}([\zeta])$ ). Thus we have that  $r_E^*(c_{g^{-1}}(s')) = r_{E'}^*(s') = r_E^*(s)$ . Here, note that  $c_{g^{-1}}^*: V_{E'} \rightarrow V_E$  is equal to the natural map induced by the conjugation map  $c_g: H^*(E', k) \rightarrow H^*(E, k)$ . We write  ${}^g s'$  for  $c_{g^{-1}}^*(s')$ . Since  ${}^g s'$  and  $s$  are contained in  $V_E^+$  and  $r_E^*({}^g s') = r_E^*(s)$  and further  $C_D(E) = D$  is a defect group of  $e_E$ , Theorem 5.1 (ii) for  $V_{G,b}$  shows that there exists  $h \in N_G(E, e_E)$  with  ${}^{hg} s' = s$ . Now, since  $D$  is abelian, we see from the fact stated above that the conjugation map  $c_{hg}$  induces an isomorphism  $V_{E'}(iU) \cong V_E(iU)$ . Thus  $s$  is contained in  $V_E(iU)$ . Therefore  $v = (\text{res}_{Q,E})^*(s)$  is contained in  $V_Q(iU)$ , so the proposition follows.  $\square$

Using the above proposition, we show the following, which is a block variety version of [6, Theorem 1].

**Proposition 5.3.** *With the notation in Section 4, suppose that  $D$  is abelian. Let  $U$  be a finitely generated  $k\bar{G}\bar{b}$ -module. Then we have that*

$$V_{G,b}(U) = (\text{inf}_{\bar{b},b}^*)^{-1} V_{\bar{G},\bar{b}}(U).$$

Proof. It is clear that there is a commutative diagram as follows:

$$\begin{array}{ccc} V_D & \xrightarrow{r_D^*} & V_{G,b} \\ \text{inf}_{\bar{D},D}^* \downarrow & & \downarrow \text{inf}_{\bar{b},b}^* \\ V_D & \xrightarrow{\bar{r}_D^*} & V_{\bar{G},\bar{b}} \end{array}$$

Then we have

$$(r_D^*)^{-1} \circ (\text{inf}_{\bar{b},b}^*)^{-1} V_{\bar{G},\bar{b}}(U) = (\text{inf}_{\bar{D},D}^*)^{-1} \circ (\bar{r}_D^*)^{-1} V_{\bar{G},\bar{b}}(U) = (\text{inf}_{\bar{D},D}^*)^{-1} V_D(\bar{i}U) = V_D(iU),$$

where the second equality holds by Proposition 5.2, and the third holds by [6, Theorem 1]. Thus, since  $r_D^*: V_D \rightarrow V_{G,b}$  and its restriction  $r_D^*: V_D(iU) \rightarrow V_{G,b}(U)$  are both surjective, it follows that  $(\text{inf}_{\bar{b},b}^*)^{-1} V_{\bar{G},\bar{b}}(U) = r_D^* V_D(iU) = V_{G,b}(U)$ . This completes the proof.  $\square$

REMARK 5.4. Suppose that  $b$  is the principal block of  $kG$ . Then we have from Theorem 2.4 that  $V_{G,b}(U) = (\text{inf}_{\bar{b},b}^*)^{-1} V_{\bar{G},\bar{b}}(U)$ , without the assumption that  $D$  is abelian. Indeed, from the relation  $\rho_b \circ \text{inf}_{\bar{G},G}^* = \text{inf}_{\bar{b},b}^* \circ \rho_b$ , we see that  $(\rho_b^*)^{-1} \circ (\text{inf}_{\bar{G},G}^*)^{-1} V_{\bar{G}}(U) = (\text{inf}_{\bar{b},b}^*)^{-1} \circ (\rho_b^*)^{-1} V_{\bar{G}}(U) = (\text{inf}_{\bar{b},b}^*)^{-1} V_{\bar{G},\bar{b}}(U)$ . Further, we see

from [6, Theorem 1] that  $(\rho_b^*)^{-1} \circ (\inf_{G,G}^*)^{-1} V_G(U) = (\rho_b^*)^{-1} V_G(U) = V_{G,b}(U)$ . So, it follows that  $V_{G,b}(U) = (\inf_{b,b}^*)^{-1} V_{\tilde{G},b}(U)$ .

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